Characterizing Group Behavior*

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Abstract

We study the demand function of a group of S members facing a global budget constraint. Any vector belonging to the budget set can be consumed within the group, with no restriction on the form of individual preferences, the nature of individual consumptions or the form of the decision process beyond efficiency. Moreover, only the group aggregate behavior, summarized by its demand function, is observable. We provide necessary and (locally) sufficient restrictions that fully characterize the group's demand function, with and without distribution factors. We show that the private or public nature of consumption within the group is not testable from aggregate data on group behavior. *Journal of Economic Literature* Classification Numbers: D11, D13, C65.

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1 Introduction

1.1 Individual demand and group demand

The study and characterization of market behavior is one of the goals of micro economic theory. Most existing results concentrate on two extreme cases. On the one hand, it has been known for at least one century that individual demand, as derived from the maximization of a single utility function under budget constraint, satisfies specific and stringent properties (homogeneity, adding up, Slutsky symmetry and negativeness). On the other hand, the main conclusion of the so-called Debreu-Mantel-Sonnenschein (henceforth DMS) literature[1] is that, if the number of agents is large enough, aggregate demand does not exhibit specific properties, except for the obvious ones (continuity, homogeneity, Walras Law). In other words, the standard assumptions of microeconomic theory generate considerable structure at the individual level, but this structure is essentially lost by (large) aggregation.

However, many interesting economic situations lie somewhere in-between these two polar situations. These are cases where the group under consideration includes more than one individual, but is not large enough for the aggregation results of DMS type to apply. For instance, standard demand theory uses data on households or families, most of which gather several individuals. The behavior of even large firms is routinely analyzed as stemming from the interaction of a small number of agents (management and unions, manager and shareholders, top manager and division heads, etc.), who bargain under some global financial constraint. The same remark obviously applies to committees, clubs, villages and other local organizations, who have also attracted much interest. In short, many economic decisions are made by small, multi-person groups.

The goal of the present paper is to provide a general characterization of group behavior in a market environment. In contrast with many existing works, we do not make particular assumptions on the details of the interaction or the nature of the decision process. Rather, following [10] and [7] among others, we simply extend the traditional individual rationality requirement to aggregate behavior, by assuming that outcomes are Pareto efficient - what can readily be called 'collective rationality'. We view efficiency as a natural assumption in many contexts, and as a natural benchmark in all cases. For instance, the analysis of household behavior often borrows the 'collective' point of view, where efficiency is the basic postulate. Other models, in particular in the literature on firm behavior[2], are based on cooperative game theory in a symmetric information context, where efficiency is paramount. The analysis of intra-group risk sharing, starting with Townsend's seminal paper [40], provides other interesting examples. Finally, even in the presence of asymmetric information, first best efficiency is a natural benchmark. For instance, a large part of the empirical literature on contract theory tests models involving asymmetric information

[1] See [36] for a general survey.

[2] See for instance the 'insider-outsider' literature, and more generally the models involving bargaining between the management and the workers (or the unions).

against the null hypothesis of symmetric information and first best efficiency[3]

The general problem we consider can be stated as follows. Take a group consisting of S members. The group is faced with a global budget constraint that limits its global consumption vector x; this constraint takes the usual market form $\pi' x = 1$ (where π is a vector of prices, and where total group income is normalized to one). Any demand vector belonging to the global budget set thus defined provides the group with an aggregate endowment, which will be consumed by the members. A crucial feature of our approach is that we do *not* restrict the form of individual preferences (except for the standard convexity assumptions) or the nature of individual consumptions. That is, we allow for public as well as private consumption, for intra-group production, and for any type of consumption externalities across group members. Also, we do *not* place any restriction on the form of the decision process except for efficiency. Finally, our approach does *not* require observability of individual consumptions or intra-group transfers. Only the group aggregate behavior, summarized by its demand function, is observable. The question we investigate throughout the paper is: *what restrictions (if any) on the aggregate demand function characterize the efficient behavior of the group?* And how do these restrictions vary with the size of the group?

Our work generalizes two existing lines of research.[4] The DMS tradition, on the one hand, has investigated the property of the aggregate demand of a 'small' economy (defined as an economy with less consumers than commodities). Diewert [23] and Geanakoplos and Polemarchakis [29] provide necessary conditions for a given function to be the aggregate excess demand of an economy of this kind. Our approach generalizes these results in two respects. First, the whole literature on aggregate demand only considers exchange economies where all commodities are privately consumed and no externalities are allowed for. Our framework is more general, as it allows for a much richer set of interactions. Secondly, these contributions only give necessary conditions, the sufficiency of which was still an open problem. The characterization we provide is necessary and sufficient.

Our paper can also be viewed as a direct generalization of Browning and Chiappori's [7] analysis of household behavior. Their framework is exactly as general as ours; in particular, they allow for public consumptions, household production and consumption externalities. Although most of their analysis deals with couples, they indicate how their results extend to larger groups. However, the purpose of Browning and Chiappori's contribution is to exhibit *necessary* conditions that have to be fulfilled by the demand of a group making efficient decisions. Whether these conditions are also sufficient is still an open question. In other words, what is missing is a generalization to the collective setting of the well-known *integrability* results in consumer theory, whereby any smooth function that satisfies homogeneity, adding up and Slustky conditions is indeed

[3] See [20] for a recent review.

[4] A third references is provided by the abstract characterization of group attitude toward risk, as pioneered by Wilson's contribution [41]. See Mazzocco ([32] and [33]) for recent advances in this direction.

the demand function stemming from the maximization of a well behaved utility function.

The goal of the present paper is precisely to fill this gap; i.e., to derive conditions that are sufficient for any smooth function to stem from some Pareto efficient decision process within a 'well behaved' group. We first show that the SR(1) condition in Browning-Chiappori (or its immediate generalization to our context, SR$(S-1)$) is also sufficient for local integration. That is, for any map ξ satisfying the SR$(S-1)$ condition, it is possible to find a group (i.e., S preferences, a production function and a decision rule) the aggregate demand of which locally coincides with ξ. We then investigate whether stronger assumptions on the nature of consumption could generate additional restrictions. We consider the two particular cases when the consumption is purely private and purely public, and then we treat the general case. We show that condition SR$(S-1)$ is still necessary and sufficient; an interesting consequence being that an assumption like privateness (or publicness) of individual consumptions is not testable from data on group behavior.

Finally, let us point out that all our results are local, that is, they hold in a sufficient small neighborhood of a given point. What happen if, for instance, condition SR$(S-1)$ holds globally we have not investigated, but our strong suspicion is that it would not imply that a decomposition similar to the one in Proposition 5 would hold globally.

1.2 Distribution factors

In many situations, the group's decision depends not only on prices, but also on factors that can affect the influence of various members on the decision process. Think, for instance, of the decision process as a bargaining game. Typically, the outcomes will depend on the members' respective bargaining positions. It follows that *any factor of the group environment* (EEPs in McElroy's [34] terminology) *that may influence the respective bargaining strengths will potentially affect the outcome.* Such effects are of course paramount, and their relevance is not restricted to bargaining in any particular sense. In general, group behavior depends not only on preferences and budget constraint, but also on the members' respective 'power', as summarized for instance by their Pareto weights. Any variable that changes the powers may have an impact on observed collective behavior.

In many cases, distribution factors are readily observable. To take a very basic example, think of the group as a small open economy with private consumption only (in the DMS tradition). Any efficient outcome is an equilibrium; and the particular equilibrium that will prevail depends typically on individual incomes (or endowments): initial incomes are distribution factors for the group under consideration. Other examples are provided by the literature on household behavior. In their study of household labor supply, Chiappori, Fortin and Lacroix [19] use the state of the marriage market, as proxied by the sex ratio by age, race and state, and the legislation on divorce as particular distribution factors affecting the intrahousehold decision process, hence its outcome (labor supplies in that case). They find, indeed, that any improvement in women's position (e.g., more favorable divorce laws, or excess 'supply' of males on the marriage market) significantly decreases (resp. increases) female (resp. male) labor supply. In a similar context, Rubalcava and Thomas [35] refer to the generosity of single parent benefits and reach identical conclusions. Thomas *et al.* [39], using an Indonesian survey, show that the distribution of wealth by gender at marriage - another candidate distribution factor - has a significant impact on children health in those areas where wealth remains under the contributor's control[5]. Duflo [24] has derived related conclusions from a careful analysis of a reform of the South African social pension program that extended the benefits to a large, previously not covered black population. She finds that the recipient's gender - a typical distribution factor - matters for the consequences of the transfers on children's health.

In many contexts, the group demand is observable as a function of prices and distribution factors. It is intuitively clear that the presence of distribution factors should generate additional restrictions. This is actually known to be the case in particular situations. For instance, Browning and Chiappori [7] provide conditions of this type in the case of couples; these conditions have since then be empirically tested (and not rejected). Another illustration is provided by the small open economy context with private consumption, where Chiappori, Ekeland, Kubler and Polemarchakis ([18] and [30]) show that the knowledge of initial endowments leads to testable restrictions on the form of the aggregate excess demand and the equilibrium manifold. We extend these results by providing general necessary conditions that characterize the impact of distribution factors on group demand.

Section 2 describes the model and summarizes the existing results. The main findings are presented in Section 3, with the exception of distribution factors, that are considered in Section 4.

2 The model

2.1 Preferences

We consider a S person group. Total purchases are denoted by the vector $x \in \mathbb{R}^N$. Note that, formally, purchases could include leisure: the price vector would then include the wages (or virtual wages for non-participants). We allow for production within the group, so that x could be seen as the input of some production process described by a production set $\Gamma \subset R^{N+M}$, the output $y \in \mathbb{R}^M$ being consumed within the group. In the absence of intra-group production, the model still applies by taking Γ to be the diagonal in $R^N \times R^N$ (then $M = N$ and $x = y$).

As it will become clear, our results are more interesting when the number of purchased commodities N is 'large' (technically, when N is larger than the number of agents S in the group). In the alternative situation, some generalized

[5] See also Galasso [28] for a similar investigation.

version of the DMS theorem applies, and there are no restrictions of the group demand except for the obvious ones (continuity, adding up); we shall discuss this case in subsection 3.2. If, however, additional assumptions are made on the number of output consumed (specifically, if $M < S$), then stronger conditions apply; these will be considered in subsection 3.5.

Within the group, each output y^i can be consumed privately, publicly, or both. We denote private consumptions by $y_1^i, ..., y_S^i$, where the lower (resp. upper) index denotes a group member (resp. a commodity), and public consumption by Y; then

$$\left(-x, \sum y_s + Y\right) \in \Gamma$$

If, for instance, commodity i is purely public (resp. purely private), then $y_s^i = 0$ for all s (resp. $Y^i = 0$). In general, however, we allow the consumption of any given commodity to be partly private and partly public.

The group operates in a market environment; hence the input can be purchased at given (linear) prices. The budget constraint is thus:

$$\pi' x = 1 \tag{1}$$

where $\pi \in \mathbb{R}_{++}^N$ is the price vector. Note that the group's total income (or total expenditure for empirical purposes) is normalized to one: we (implicitly) assume that the group's behavior is zero-homogeneous in prices and income. A vector $(-x, \sum y_s + Y)$ is said to be *feasible* if (i) it belongs to Γ, and (ii) x satisfies $\pi' x \leq 1$.

Each member has her or his own preferences over the goods consumed in the group. In the most general case, each member's preferences can depend on other members' private and public consumptions; this allows for altruism, but also for externalities or any other preference interaction. Formally, member s's utility is of the form $U^s(y_1, ...y_S, Y)$, where U^s is concave, strictly increasing in (y_s, Y), twice differentiable in $(y_1, ...y_S, Y)$, with the matrix of second derivatives being positive definite everywhere. Henceforth, such function will be referred to as *strongly concave.*

In some sections, however, we further specify the form of preferences. We devote a particular attention to two benchmark cases. In both of them, we assume away production, so that $x_s = y_s$ and $X = Y$. The two alternative assumptions are either that all goods are publicly consumed (then preferences are $U^s(X)$), or that all goods are privately consumed, with no externalities except may be for altruism (then preferences are either egoistic, $U^s(x_s)$, or of the 'caring' type $W^s\left(U^1(x_1), ..., U^S(x_S)\right)$. Finally, we also consider the case of household production.

2.2 The decision process

We now consider the mechanism that the group uses to decide on what to buy. Note, first, that if the functions $U^1, ..., U^S$ represent the same preferences, we are back in the conventional 'unitary' model; then the common utility is maximized under the budget constraint.[6] Alternatively, we could assume that one of the partners imposes his/hers preferences and use the corresponding utility function in the traditional way; this also yields a unitary model. But these are highly specific assumptions. In general, the decisions process that takes place within the group is much more complex. Following the 'collective' approach, we postulate efficiency, as expressed in the following axiom :

Axiom 1 *The outcome of the group decision process is Pareto efficient; that is, for any price vector, the vector* $\left(\bar{x}, \bar{y}_1, ...\bar{y}_S, \bar{Y}\right)$ *chosen by the group is such that no other feasible vector* $(x, y_1, ...y_S, Y)$ *could make all members at least as well off, one member at least being strictly better off.*

Denote the vector of distribution factors by z, the vector of Pareto weights by $\mu = (\mu_1, ..., \mu_S)$ (with the normalization $\sum \mu_s = 1$), and let μ_{-S} denote the vector $(\mu_1, ..., \mu_{S-1})$. The axiom can be restated as follows: there exists S scalar functions $\mu_1(\pi, z) \geq 0, ..., \mu_S(\pi, z) \geq 0$, with $\sum \mu_s = 1$, such that $(x, \bar{y}_1, ...\bar{y}_S, \bar{Y})$ solves[7]:

$$\max_{x.y_1,...y_S,Y} \sum \mu_s(\pi, z)\, U^s(y_1, ...y_S, Y) \tag{P}$$

$$\left(-x, \sum y_s + Y\right) \in \Gamma \tag{2}$$

$$\pi' x \leq 1 \tag{3}$$

In what follows, we denote by $\xi(\pi, z) \in R^N$ the vector of the first N components in the solution $(x, \bar{y}_1, ...\bar{y}_S, \bar{Y})$ to program (P); remember that, according to our assumption, only $\xi(\pi, z)$ is observable by the outside observer.

As it is well-known, any point on the Pareto frontier can be obtained as a solution to a program of type (P) for some well-chosen μ. The Pareto vector μ summarizes the decision process. Take some given utility functions U^1,...,U^S. The budget constraint defines, for any price-income bundle, a Pareto frontier. From Axiom 1, the final outcome will be located on this frontier. Then μ determines the final location of the demand vector on this frontier. The vector μ has an obvious interpretation in terms of distribution of power. If one of the weights, μ_s, is equal to one, then the group behaves as though s is the effective dictator. For intermediate values, the group behaves as though each person s has some decision power, and the person's weight μ_s can be seen as an indicator of this power. It is important to note that the weights μ_s will in general depend on prices π and distribution factors z, since these variables may in principle influence the distribution of power within the group, hence the location of the final choice over the Pareto frontier. However, while prices enter both Pareto weights and the budget constraint, distribution factors matter only (if at all) through their impact on μ.

Following [7], we add some structure by assuming the following:

[6] This assumption is implicit in Becker's model of household production (see section 3.5 below).

[7] Note that, from the normalization, $\mu_S = 1 - \sum_{s=1}^{S-1} \mu_s$; in particular, both vectors μ and μ_{-S} vary within a $(S-1)$-dimensional plane.

Axiom 2 *The function $\mu_s(\pi, z)$ is continuously differentiable for $s = 1, ..., S$.*

Problem (P) can be rewritten as a two-stage optimization problem, namely:

$$\max_x \tilde{U}(x, \pi, z) \qquad \text{((P1))}$$
$$\pi' x \leq 1$$

where:

$$\tilde{U}(x, \pi, z) = \max_{y_1, \ldots y_S, Y} \left\{ \sum \mu_s(\pi, z) U^s(y_1, ..., y_S, Y) \mid \left(-x, \sum y_s + Y\right) \in \Gamma \right\} \qquad \text{((P2))}$$

Again, the solution $\bar{x} = \xi(\pi, z)$ to problem (P1) is observable, whereas the solution $\left(\bar{y}_1, ..., \bar{y}_S.\bar{Y}\right)$ to problem (P2) is not.

2.3 Necessary conditions without distribution factors

In the benchmark case where the group consists of only one person, the framework above boils down to the standard 'unitary' model of consumer theory, and demand is derived from the maximization of the person's utility function under budget constraint. As it is well known, this derivation implies restrictive properties on the form of the demand function: it is homogeneous and satisfies the Walras law, together with symmetry and negativeness of the Slutsky matrix. Moreover, these conditions are sufficient under mild smoothness conditions. A natural question is thus: how does this result extend to collective demand? That is, what conditions does (P) imply upon the form of $\xi(\pi, z)$?

We first omit the distribution factors z and concentrate on price effects; this allows us to stress the links with the Slutsky conditions on the one hand, and with the DMS literature on aggregate demand on the other. We thus consider the group demand function $\xi(\pi)$ as a function of π only.

A first necessary condition can be found in [7]. It uses the Slutsky matrix defined from ξ by $S(\pi) = (D_\pi \xi)(I - \pi\xi')$. Note, incidentally, that $v' S(\pi) v = 0$ for all vectors $v \in \text{Span}\{\pi\}$

Proposition 3 *(The SR($S-1$) condition). If $\xi(\pi)$ solves problem (P), then the Slutsky matrix $S(\pi) = (D_\pi \xi)(I - \pi\xi')$ can be decomposed as:*

$$S(\pi) = \Sigma(\pi) + R(\pi) \qquad \text{(SR(S-1))}$$

where:

- *the matrix $\Sigma(\pi)$ is symmetric and satisfies $v'\Sigma(\pi) v = 0$ for all vectors $v \in \text{Span}\{\pi\}$, $v'\Sigma(\pi) v < 0$ for all vectors $v \notin \text{Span}\{\pi\}$*
- *the matrix $R(\pi)$ is of rank at most $S - 1$.*

Equivalently, there exists a subspace $\mathcal{E}(\pi)$ of dimension at least $N - (S-1)$ such that the restriction of $S(\pi)$ to $\mathcal{E}(\pi)$ is symmetric and negative, in the sense that $v'\Sigma(\pi) v = 0$ for all vectors $v \in \mathcal{E}(\pi) \cap \text{Span}\{\pi\}$, $v'\Sigma(\pi) v < 0$ for all vectors $v \in \mathcal{E}(\pi) - \text{Span}\{\pi\}$.

Proof. A detailed proof can be found in the original paper. However, it is important for our present purpose to see the core of the argument. This can be summarized as follows. Given $(\mu_1, ..., \mu_{S-1}) \in R^{S-1}$, with $\mu_s \geq 0$, and $\sum \mu_s = 1$, define:

$$\hat{U}(x, \mu_1, ..., \mu_S) = \max_{y_1,...y_S,Y} \left\{ \sum_s \mu_s U^s(y_1, ..., y_S, Y) \mid \left(-x, \sum y_s + Y\right) \in \Gamma \right\}$$

$$\hat{V}(\pi, \mu_1, ..., \mu_S) = \max_x \left\{ \hat{U}(x, \mu_1, ..., \mu_{S-1}) \mid \pi' x \leq 1 \right\}$$

and denote by $\hat{x} = \hat{\xi}(\pi, \mu_1, ..., \mu_S)$ the maximizer, so that $\pi' \hat{\xi}(\pi, \mu_1, ..., \mu_S) = 1$ and:

$$\hat{V}(\pi) = \hat{U}(\xi(\pi, \mu_1, ..., \mu_S), \mu_1, ..., \mu_S)$$

The map $\pi \to \hat{\xi}(\pi, \mu_1, ..., \mu_S)$ is the standard Marshallian demand associated to $x \to \hat{U}(x, \mu_1, ..., \mu_S)$, and as such it must satisfy Slutsky symmetry and negativeness[8]. In addition, it is related to ξ by $\xi(\pi) = \hat{\xi}(\pi, \mu_1(\pi), ..., \mu_S(\pi))$. It follows that

$$S(\pi) = \Sigma(\pi) + \sum_{s=1}^{S-1} a_s(\pi) b'_s(\pi) \tag{4}$$

where $S(\pi)$ is the Slutsky matrix associated to $\hat{\xi}(\bullet, \mu_1, ..., \mu_S)$, the matrix $\Sigma(\pi)$ has the standard Slutsky properties, and where a_s and b_s are vectors defined by:

$$a_s = D_{\mu_s} \hat{\xi} \quad \text{and} \quad b'_s = (D_\pi \mu_s)' \left(I - p\hat{\xi}' \right) \tag{5}$$

In particular, $a_s b'_s$ is of rank at most 1 for $s = 1, ..., S-1$, so that $R = \sum_{s=1}^{S-1} a_s b'_s$ is of rank at most $S-1$. Note, incidentally, that $v' R(\pi) v = 0$ for all vectors $v \in \text{Span}\{\pi\}$.

Finally, let $\mathcal{E}(\pi)$ be the space of vectors $v \in \mathbb{R}^N$ such that $v'R = 0$.Then $\dim \mathcal{E}(\pi) = N - \text{rank}(R) \geq N - (S-1)$, and for any $v, w \in \mathcal{E}(\pi)$,

$$v'Sw = v'\Sigma w = w'\Sigma v = w'Sv \quad \text{and}$$
$$v'Sv = v'\Sigma v \leq 0$$

which shows that the restriction of S to $\mathcal{E}(\pi)$ is symmetric and negative. ■

Interestingly enough, Proposition 3 implies that the behavior of a group reflects the number of decision makers: *how* you consume is how many you are. The proposition also shows that, in contrast with individual demand, the Slutsky matrix of a group need not be symmetric negative. However, there are restrictions on how symmetry and negativeness can be violated, and these restrictions are more stringent for smaller groups. Specifically, there exist a subspace of dimension at least $N-(S-1)$ on which symmetry and negativeness are preserved. The interpretation is the following. Changes in prices can have

[8] Note that the function $x \to \hat{U}(x, \mu_1, ..., \mu_S)$ is not concave, but the Slutsky relations express the maximization property of consumption rather than concavity of the utility function.

two effects. On the one hand, they modify the Pareto frontier, which will affect behavior even when μ is kept constant. However, such changes would satisfy Slutsky symmetry and negativeness. On the other hand, changing prices vary the μ, hence the location on the Pareto frontier. This effect is summarized by the matrix R. The conditions on the rank of R simply reflect the fact that the Pareto frontier is of dimension $S-1$.

How can a property like SR$(S-1)$ be tested? The answer, again, generalizes [7], and uses the antisymmetric part of S. Formally:

Proposition 4 *Let* $M(\pi) = S(\pi) - S(\pi)'$. *Then the antisymmetric matrix* $M(\pi)$ *is of rank at most* $2(S-1)$.

2.4 The sufficiency problem

2.4.1 The problem

In the remainder of the paper, we consider the converse properties. The question we are trying to answer is thus the following**:** *assume that some demand function* $\xi(\pi)$ *satisfies SR(S-1); is it possible to find S quasi-concave utility functions,* $U^1(y_1,...,y_S,Y),...,U^S(y_1,...,y_S,Y)$*, a production function* F*, and a vector function* $\mu(\pi)$ *such that* $\xi(\pi)$ *solves problem (P) ?*

In other words, we are looking for an equivalent, in the collective setting, to the *integrability* theorem in the unitary case, whereby Slutsky conditions (with homogeneity and adding up) are sufficient for the existence of a well-behaved utility function generating a given demand.

This is a difficult problem. It is important to note, first, that the decomposition SR$(S-1)$, if it exists, is not unique. To see why, consider the simple case where $S=2$ (then $S(\pi)=\Sigma(\pi)+ab'$ and $\mathcal{E}$ can be taken to be $[b]^{\perp}$). Replacing a with $a+tb$, with $t>0$ scalar, changes the decomposition to:

$$\begin{aligned} S(\pi) &= \Sigma(\pi) + ab' = \Sigma(\pi) + (a+tb)b' - tbb' \\ &= [\,\Sigma(\pi) - tbb'] + (a+tb)b' \end{aligned}$$

and the bracketed term is again a symmetric matrix, whose restriction to $[b]^{\perp}$ will be negative and definite on $[\pi]^{\perp}$. In fact, [7] show that a and b belong to $\operatorname{Im} M(\pi)$, where $M(\pi) = S(\pi) - S(\pi)'$. Conversely, for any two vectors $\alpha, \beta \in \operatorname{Im} M(\pi)$, there exist a scalar t and a symmetric $\tilde{\Sigma}(\pi)$ such that

$$S(\pi) = \tilde{\Sigma}(\pi) + t\alpha\beta' \tag{6}$$

2.4.2 The main mathematical tool

In the following, we will solve the sufficiency problem in various contexts. Our main mathematical tool will be the following:

Proposition 5 *Suppose $\xi(\pi)$ satisfies the Walras law $\pi'\xi(\pi) = 1$ and condition SR$(S-1)$ in some neighborhood of $\bar{\pi}$:*

$$S(\pi) = (D_\pi \xi)(\mathbf{I} - \pi\xi') = \Sigma(\pi) + \sum_{s=1}^{S-1} a_s(\pi) b'_s(\pi) \tag{7}$$

where $\Sigma(\pi)$ is symmetric, negative, and the vectors $\xi(\pi)$, $a_s(\pi)$ and $b_s(\pi)$ are linearly independent. Then there are positive functions $\lambda_s(\pi)$ and strongly concave functions $V^s(\pi), 1 \leq s \leq S$, such that the decomposition:

$$\xi(\pi) = \sum_{s=1}^{S} \lambda_s(\pi) D_\pi V^s(\pi)$$

holds true in some neighborhood of $\bar{\pi}$.

The proof is given in Appendix 1: it is a consequence of the convex Darboux Theorem. Actually, there is more information to be derived from that result. The $(2S-1)$-dimensional subspace:

$$F(\pi) = \text{Span}\{\xi(\pi), a_s(\pi), b_s(\pi) \mid 1 \leq s \leq S-1\}$$

depends only on $\xi(\pi)$, and not on the particular choice of a_s and b_s in the decomposition (7). For every π and s we must have

$$DV^s(\pi) \in F(\pi)$$
$$D\lambda_s(\pi) \in F(\pi)$$

Finally, the $DV^s(\bar{\pi})$ can be chosen arbitrarily close to $\frac{1}{S}\xi(\bar{\pi})$, so that, if $\xi(\bar{\pi}) \in R^N_+$, the $V^s(\pi)$ can be chosen to be increasing in a neighborhood of $\bar{\pi}$

3 Characterization of group demands: sufficient conditions

We now prove that condition SR$(S-1)$ is sufficient for local integration in different contexts. We first consider the general framework described above, and its implications when the number of agents in the group is 'large'. Then we analyze the two benchmark cases of purely private and purely public consumptions; finally, we discuss the implications of intragroup production. Throughout this section we omit the distribution factors, and consider demands as functions of prices only; distribution factors will be considered in the next section.

3.1 The general case

We can state our first basic result:

Proposition 6 *Take any function* $\xi\colon \mathbf{R}^N \to R^N$*, with* $\pi'\xi(\pi) = 1$*, satisfying SR*$(S-1)$ *in some neighborhood of* $\bar{\pi}$*, and such that the Jacobian* $D_\pi \xi(\bar{\pi})$ *is invertible. Then there are* S *functions* $y_s(\pi)$ *and a function* $Y(\pi)$ *defined in some neighborhood of* $\bar{\pi}$*,* S *strongly concave functions* $U^s(y_1, ..., y_S, Y)$ *defined in some neighborhood of* $(y_1(\bar{\pi}), ..., y_S(\bar{\pi}), Y(\bar{\pi}))$*, a convex production set* Γ*, and* $S-1$ *functions* $\mu_s(\pi) \geq 0$ *with* $\sum \mu_s \leq 1$*, such that* $(\xi(\pi), y_1(\pi), ..., y_S(\pi), Y(\pi))$ *solves problem (P) in some neighborhood of* $\bar{\pi}$*.*

In particular, $\xi(\pi)$ solves problem (P1). Proposition 6 states that condition SR$(S-1)$ is sufficient for (local) integration; it thus generalizes the standard integration theorem of consumer theory.

In some cases, however, additional information is available *a priori* on the structure under consideration. It may be the case, for instance, that no production takes place within the group (then $x = y$), that all commodities are known to be privately (or publicly) consumed, so that utility functions are of the form $U_s(y_s)$ (or $U_s(Y)$), or that the number of outputs of the production process is known a priori.[9] In principle, these additional restrictions may narrow the scope of the integration result: although SR$(S-1)$ is sufficient for general integration, additional property may be required to recover individual utilities with specific characteristics. Conversely, if integrability is proved in any of these particular cases, then it obtains a fortiori in the general case and Proposition 6 is proved.

We investigate this issue in the two benchmark cases listed above. We show, in particular, that in the case of purely public or purely private goods without production, condition SR$(S-1)$ is sufficient for local integration. A surprising implication is that the publicness (or privateness) assumption is simply not testable per se; a test requires either stronger assumptions (e.g., exclusivity) or the presence of distribution factors. As we shall see, the presence of an intra-group production function is not testable either.

We do not provide a proof of Proposition 6, since it is an obvious consequence of Propositions 8 and 11 below. Throughout the remainder of this section, we specialize the notation for prices, using P for the price vector for public goods and p for the price vector for private goods.

3.2 The case of 'large' groups: extending DMS

Before considering the two benchmark cases, we briefly study a simple but interesting application of the general result. In Proposition 6, no assumption is made on the size S of the group, with respect to the number N of purchased commodities. Let us now consider the case of a 'large' group, in the sense that $S \geq N$. Then:

Proposition 7 *Assume that* $S \geq N$*. Take any smooth function* $\xi\colon \mathbf{R}^N \to R^N$ *defined in some neighborhood of* $\bar{\pi}$*, that satisfies the adding-up condition*

[9] A different line of investigation obtains when information is available (or assumptions are made) regarding the *nature of the decision process* (e.g., cooperative bargaining with known threat points). This direction will not be followed here; see [12] for a recent analysis.

$\pi'\xi(\pi) = 1$,*and such that the Jacobian* $D_\pi\xi(\bar{\pi})$ *is invertible. Then there are* S *functions* $y_s(\pi)$ *and a function* $Y(\pi)$ *defined in some neighborhood of* $\bar{\pi}$, S *strongly concave functions* $U^s(y_1, ..., y_S, Y)$ *defined in some neighborhood of* $(y_1(\bar{\pi}), ..., y_S(\bar{\pi}), Y(\bar{\pi}))$, *a convex production set* Γ, *and* $S-1$ *functions* $\mu_s(\pi) \geq 0$ *with* $\sum \mu_s \leq 1$, *such that* $(\xi(\pi), y_1(\pi), ..., y_S(\pi), Y(\pi))$ *solves problem (P) in some neighborhood of* $\bar{\pi}$.

Conversely, assume that $S < N$. *There exists an open set of smooth functions* $\xi\colon \mathbf{R}^N \rightarrow R^N$ *that satisfy the adding-up condition* $\pi'\xi(\pi) = 1$, *such that the Jacobian* $D_\pi\xi(\bar{\pi})$ *is invertible, and for which it is impossible to find* S *functions* $y_s(\pi)$ *and a function* $Y(\pi)$ *defined in some neighborhood of* $\bar{\pi}$, S *strongly concave functions* $U^s(y_1, ..., y_S, Y)$ *defined in some neighborhood of* $(y_1(\bar{\pi}), ..., y_S(\bar{\pi}), Y(\bar{\pi}))$, *a convex production set* Γ, *and* $S-1$ *function* $\mu_s(\pi) \geq 0$ *with* $\sum \mu_s \leq 1$, *such that* $(\xi(\pi), y_1(\pi), ..., y_S(\pi), Y(\pi))$ *solves problem (P) in some neighborhood of* $\bar{\pi}$.

Proof. We start with the first part of the Proposition. From Proposition 6, it is sufficient to show that when $S \geq N$, any smooth function $\xi\colon \mathbf{R}^N \rightarrow R^N$ defined in some neighborhood of $\bar{\pi}$, that satisfies the adding-up condition $\pi'\xi(\pi) = 1$,and such that the Jacobian $D_\pi\xi(\bar{\pi})$ is invertible must satisfy SR($S-1$). Take any matrix $\Sigma(\pi)$ that is symmetric, negative, definite on $[\pi]^\perp$ and null on Span(π). Define the matrix $R(\pi)$ as

$$R(\pi) = S(\pi) - \Sigma(\pi)$$

Then the rank of $R(\pi)$ is at most $N-1 \leq S-1$ (remember that $R(\pi).\pi = 0$), hence the conclusion.

Regarding the second statement, take any smooth function $\xi\colon \mathbf{R}^N \rightarrow R^N$ that satisfies the adding-up condition $\pi'\xi(\pi) = 1$, such that the Jacobian $D_\pi\xi(\bar{\pi})$ is invertible, and such that the matrix $S(\pi) = (D_\pi\xi)(I - \pi\xi')$ is positive definite on $[\pi]^\perp$ (i.e., it satisfies $v'S(\pi)v = 0$ for all vectors $v \in$ Span$\{\pi\}$, $v'S(\pi)v > 0$ for all vectors $v \notin$ Span$\{\pi\}$). Then there cannot exist a subspace $\mathcal{E}(\pi)$ of dimension at least $N - (S-1)$ such that the restriction of $S(\pi)$ to $\mathcal{E}(\pi)$ is symmetric and negative, hence the necessary condition $SR(S-1)$ is violated. ■

In words: when $S \geq N$, any smooth, locally regular function satisfying adding-up is the aggregate demand of a well chosen group using some well-chosen, efficient decision process. If, however, $S < N$, there exists an open set of smooth, locally regular functions satisfying adding-up that cannot be the aggregate demand of a well behaved group.

This conclusion sheds a new light on the DMS literature. Its main conclusion, which was conjectured by Sonnenschein [37] and proved by Debreu [22] and Mantel [31] for excess demands and Chiappori and Ekeland [14] for market demands, can be summarized as follows: any function that satisfies trivial properties (continuity, adding up, homogeneity for excess demands) can be decomposed[10] as the aggregate demand of an economy with 'sufficiently many'

[10] at least locally in the case of market demands

agents. The 'economies' considered in this literature are very special cases of our 'groups': there is no production, all commodities are privately consumed, preferences are purely egoistic. Moreover, our efficiency assumption is satisfied by construction.[11] Hence the first part of Proposition 7 is an extension of Chiappori and Ekeland's market demand theorem to more general group behavior. This remark illustrates the difficulty of the integrability problem considered here: the more difficult version of the DMS theorem, i.e. the case of market demand, is an immediate corollary of Proposition 6.

From this perspective, moreover, the second part of Proposition 7 is in a sense more surprising. It states that extending the definition of an 'economy' (which, in the DMS literature, entails only private consumptions, egoistic preferences, no production,...) to a very general framework (allowing for production, externalities, public goods,...) does *not* change its basic conclusion, namely that in general N agents are *needed* to generate an arbitrary aggregate demand. Loosely speaking, allowing for much more general 'economies' does not make the decomposition any easier, at least insofar as efficiency is still postulated; the specifics of market equilibria (private consumptions, absence of externalities,...) play no significant role in the DMS result. In summary, the informal conclusion of the DMS literature, which could be stated as:

> 'any function satisfying the obvious restrictions can be decomposed as the aggregate demand of a market economy if and only if the number of agents is at least equal to the number of commodities'

should in fact be restated as follows:

> 'any function satisfying the obvious restrictions can be decomposed as the aggregate, Pareto efficient demand of group if and only if the number of agents is at least equal to the number of commodities'

3.3 Benchmark case 1 : no production, public consumption

We now study the specific benchmark case mentioned above. Let us first assume away intra-group production (so that x is both the household's market demand

[11] Consider a DMS economy with S agents and N goods plus a numeraire, in which agents can freely use their initial endowment of numeraire to purchase consumption goods (this is the 'market' demand case, as opposed to the excess demand case solved by Debreu, Mantel and others). This economy can equivalently be viewed as a production economy, in which consumption goods x are 'produced' from the numeraire according to the linear technology $p'x = Q$, where Q is the quantity of numeraire available in the economy (and initially owned by the agents). In this economy, assuming private consumptions and egoistic preferences, the set of Pareto efficient outcomes coincides with the set of competitive equilibria when initial endowments of numeraire. are varied. Given the linear technology, equilibrium prices must coincide with the existing prices p when the price of the numeraire is normalized to be one. In the end, the resulting market demand is the Pareto efficient outcome of a group which is a particular case of our setting.

and total consumption). Assume, moreover, that all goods are public within the group. So $x = Y$, $\pi = P$, and $y_s = 0$. Then:

Proposition 8 *Let there be given a function $Y(P)$, with $P'Y(P) = 1$, satisfying $SR(S-1)$ in some neighborhood of $\bar{P}$, and such that the Jacobian $D_P Y(\bar{P})$ is invertible. Then there are S strictly increasing, strongly concave functions $U^1(Y), ..., U^S(Y)$, defined in some neighborhood of $\bar{Y} = Y(\bar{P})$, and S nonnegative functions $\mu_1(P), ..., \mu_S(P)$, defined in some neighborhood of $\bar{P}$ and satisfying $\sum \mu_s = 1$, such that $Y(P)$ solves problem (P), i.e.,*

$$Y(P) = \arg\max \left\{ \sum_{s=1}^{S} \mu_s(P) U^s(Y) \mid P'Y \leq 1 \right\} \tag{8}$$

Proof. By the inverse function theorem, there is a well-defined function $P(Y)$ from some neighborhood of $\bar{Y}$ to some neighborhood of $\bar{P}$ which inverts the function $Y(P)$. We will show in Appendix A2 that the $P_n(Y), 1 \leq n \leq N$ meet the conditions of the concave Darboux theorem, so that we can write:

$$P(Y) = \sum_{s=1}^{S} \alpha_s(Y) D_Y U^s(Y)$$

with positive $\alpha_s(Y)$ and strictly concave $U^s(Y)$, in some neighborhood of $\bar{Y}$. Setting $A = \sum \alpha_s$ and $\beta_s = \alpha_s / A$, so that $\sum \beta_s = 1$, we rewrite this as:

$$P(Y) = A(Y) \sum_s \beta_s(Y) D_Y U^s(Y)$$

Set $\mu_s(P) = \beta_s(Y(P))$. The preceding inequality tells us that Y is the optimal solution of problem (P) if the prevailing price is $P(Y)$ and the Pareto weights are the $\mu_s(P(Y))$:

$$Y = \arg\max_y \left\{ \sum_{s=1}^{S} \mu_s(P(Y)) U^s(Y) \mid P(Y)' y \leq 1 \right\}$$

.Applying the inverse function theorem again, we find formula (8) ■

It is a by-product of our proof that inverse demand is characterized precisely in the same way as direct demand, namely:

Proposition 9 *A given, continuously differentiable inverse demand function $P(Y)$ satisfying $P(Y)'Y = 1$ can be written locally as:*

$$P(Y) = \sum_s \alpha_s(Y) D_Y U^s(Y)$$

with strictly concave $U^s(Y)$ and positive $\alpha_s(Y)$, if and only if the matrix

$$A(Y) = D_Y P(Y) . \left(I - Y P(Y)'\right)$$

can be written as the sum of a symmetric, negative matrix, and a matrix of rank at most $(S-1)$.

3.4 Benchmark case 2 : private goods only

Still assuming away production, we now turn to the other extreme, when all goods are private inside the group. One obvious example is the small open economy example discussed earlier: individuals are characterized by their own egoistic preferences $U^i(y_i)$, and one is looking for the equilibria of this economy, where prices are given exogenously. Note, however, that externalities of the 'caring' form can be introduced: if i's utility takes the form $W^i\left(U^1(y_1),...,U^S(y_S)\right)$, where W^i is increasing, any allocation that is efficient would also be efficient with egoistic preferences.

Technically, we restrict the model to $\pi = p, Y = 0$ and $\xi = y_1 + ... + y_S = y$. We then approach Pareto optimality in a different, although equivalent way : the location of the group's choice on the Pareto frontier will be characterized by a *sharing rule* $w(p)$ (instead of a weighting rule $\mu(p)$). More precisely, the individual consumptions $y_s(p)$ obtain as stated in the following result:

Proposition 10 *Assume that all goods are privately consumed and there is no consumption externality (except for caring). Assume that there exists a sharing rule* $w(p) = (w_1(p), ..., w_S(p))$ *,with* $w_s(p) \geq 0$ *and* $\sum w_s(p) = 1$*, such that the consumption of member* $s = 1, ..., S$ *is given by*

$$y_s(p) = \arg\max \{U^s(y) \mid p'y \leq w_s(p)\} \tag{9}$$

Then the resulting allocation $y_s(p), 1 \leq s \leq S$ *is Pareto optimal*

Proof. Assume not. Then, for a certain p, there is another allocation $z_z, 1 \leq s \leq S$, with $p' \sum z_s \leq 1$, such that $U^s(z_s) \geq U^s(y_s(p))$ for every s,which strict inequality holding for some s. By the maximization property (9) this implies that $p'z_s \geq w_s(p)$, which strict inequality holding for some s. Adding up, we find that $p' \sum z_s > 1$, contradicting the assumption. ■

Intuitively, there is an increasing correspondence between a member's share ρ_s and her Pareto weight μ_s: a member who has more weight in the decision process will be able to attract a larger fraction of the group income. However, shares are more convenient tools than Pareto weights, because they are expressed in monetary units and are independent of the cardinal representation of preferences. These advantages come however at a price: the sharing rule approach can only be adopted in the pure private goods case.

We can now state the main result of this subsection:

Proposition 11 *Let* $y(p)$*, with* $p'y(p) = 1$*, be a given function satisfying* $SR(S-1)$ *in some neighborhood of* $\bar{p}$*. Then there are* S *strictly increasing, strictly concave functions* $U^s(y)$ *, defined in some neighborhood of* $y = y(\bar{p})$*, and a sharing rule* $y_s(p), 1 \leq s \leq S$*, defined in some neighborhood of* $\bar{p}$*,such that*

$$y(p) = \sum_{s=1}^{S} y_s(p)$$

with $y_s(p)$ *being given by formula (9).*

Proof. By Proposition 5, we find that:

$$y(p) = -\sum_s \alpha_s(p)\, D_p V^s$$

for some positive α_s and strongly convex V^s. For each s, define a function $w_s(p)$ by

$$w_s(p) = p'\left(D_p w_s - \alpha_s(p)\, D_p V^s\right) \tag{10}$$

This is a linear first-order partial differential equation for $w_s(p)$. Note that the sum $w(p) = \sum w_s(p)$ satisfies a similar equation:

$$w = p' D_p w + p' y = p' D_p w + 1 \tag{11}$$

which has the obvious solution $w(p) = 1$.

Equation (10) can be solved by the method of characteristics[12]. It follows that $w_s(p)$ can be prescribed arbitrarily on the affine hyperplane H defined as the set of p where $\bar{p}'(p - \bar{p}) = 0$ (technically speaking, this is a non-characteristic hypersurface, at least in some neigborhood of $\bar{p}$). We choose $w_s(p) = 1/S$ on H. It follows that $w = \sum w_s = 1$ on H, and since w satisfies equation (11), it follows that $\sum w_s(p) = 1$ everywhere. As a consequence, we have:

$$\sum D_p w_s = 0$$

Now define:

$$y_s(p) = D_p w_s - \alpha_s(p)\, D_p V^s \tag{12}$$

We have:

$$\begin{aligned} p' y_s(p) &= w_s(p) \quad 1 \le s \le S \\ \sum_s y_s(p) &= y(p) \end{aligned}$$

We now have to show that the $y_s(p)$ solve the consumer's problem. For each s, consider the function:

$$U^s(y) = \min_p \left\{ V^s(p) \mid p' y \le w_s(p) \right\} \tag{13}$$

[12] The method of characteristics consists of considering the flow:

$$\frac{dp}{dt} = p$$

in R^N, the solutions of which are given by $p(t) = p(0)\, e^t$, and to note that the function $\bar{w}(t) = w(p(t))$ solves the differential equation

$$\bar{w}(t) = \frac{d\bar{w}}{dt}(t) - \alpha_s(p(t))\, D_p V^s(p(t))$$

on R. This determines the solution $w(p)$ on each trajectory of the flow. See for instance [26] for details.

Note that, by the envelope theorem, U^s is differentiable, and $D_y U^s(y_s(p))$ is proportional to p. Now, equation (12) is the optimality condition for this problem. Since V^s is strongly convex, this condition is sufficient, so that:

$$U^s(y_s(p)) = V^s(p) \tag{14}$$

Now set:

$$W^s(p) = \sup_y \{U^s(y) \mid p'y \leq w_s(p)\} \tag{15}$$

We have $W^s(p) \geq U^s(y_s(p)) = V^s(p)$. On the other hand, for every y such that $p'y \leq w_s(p)$, we have $U^s(y) \leq V^s(p)$. Taking the supremum with respect to all such y, we get $W^s(p) \leq V^s(p)$. Finally $W^s = V^s$, and equation (15) becomes:

$$V^s(p) = \max_y \{U^s(y) \mid p'y \leq w_s(p)\} = U^s(y_s(p))$$

which tells us that $y_s(p)$ solves the consumer's problem for the utilities $U^s(y)$ and the sharing rule $w_s(p)$.

It remains to show that the U^s are quasi-concave, at least in some neighborhood of $\bar{p}$. To do this, pick y_1 and y_2 and a number a such that $U^s(y_1) \geq a$ and $U^s(y_2) \geq a$. We have:

$$U^s\left(\frac{y_1+y_2}{2}\right) = \min_p \left\{V^s(p) \mid p'\left(\frac{y_1+y_2}{2}\right) \leq w_s(p)\right\}$$

Now, if $\frac{1}{2}p'y_1 + \frac{1}{2}p'y_2 \leq w_s(p)$, then we must have $p'y_i \leq w_s(p)$ for $i = 1$ or $i = 2$. Hence:

$$\left\{p \mid p'\left(\frac{y_1+y_2}{2}\right) \leq w_s(p)\right\} \subset \{p \mid p'y_i \leq w_s(p)\} \cup \{p \mid p'y_2 \leq w_s(p)\}$$

$$U^s\left(\frac{y_1+y_2}{2}\right) \geq \min_{i=1,2} \{V^s(p) \mid p'y_i \leq w_s(p)\} = \min_{i=1,2} U^s(y_i) = a$$

So the U^s are differentiable and quasi-concave. It is well known that the same preferences can be represented by concave functions, which concludes the proof. ■

3.5 The case of intragroup production

Finally, how are the previous results modified when intragroup production is allowed for? A first remark is that the previous results can readily be extended unless specific constraints are introduced on the production process, and specifically on the number M of commodities actually consumed. Indeed, we have the following result:

Proposition 12 *Let $x(p)$, with $p'x(p) = 1$, be a smooth function satisfying SR$(S-1)$ in some neighborhood of $\bar{p}$. Then there exist:*

- *S vectors $x_i(p)$ defined in some neighborhood of $\bar{p}$, such that $\sum_{i=1}^{S} x_i(p) = x(p)$*

- *S strictly increasing, strictly concave production functions $f_i(x)$, each defined in some neighborhood of $\bar{x}_i = x_i(\bar{p})$,*

- *S strictly concave utility functions $U^s(y)$*

- *and S scalar functions $\mu_1(\pi, z) \geq 0, ..., \mu_S(\pi, z) \geq 0$, with $\sum \mu_s = 1$,*

such that $(x, \bar{y}_1, ...\bar{y}_S, \bar{Y})$ solves problem (P) for the production set Γ defined by the production functions $f_1, ..., f_S$.

Proposition 12 is an immediate corollary of Proposition 11, after a simple reinterpretation. Specifically, for any $x(p)$ satisfying SR($S-1$) in some neighborhood of $\bar{p}$, we know that there exist S utility functions U^i and a decomposition $x(p) = \sum_{i=1}^{S} x_i(p)$ solving problem (P). Now, the trick is simply to reinterpret U^i as a production function (producing some 'commodity' $y_i = U^i(x_i)$, and to define S utilities $\tilde{u}^1, ..., \tilde{u}^S$ by, say, $u^s(y^1, ..., y^S) = \sqrt{y^i}$ (i.e., consumer i only consumes commodity i for all $i = 1, ..., S$).

This construct illustrates the fact that when studying the necessity and sufficiency of conditions $SR(S-1)$, there is no basic difference between economies with and without productions. In particular, whether production is taking place (or not) within the group is not testable from data on the group's aggregate behavior.

It should however be added that, in many economic models, additional restrictions are introduced on the production process, and these conditions drastically alter the integrability conditions. Consider, for instance, Becker's celebrated model of domestic production.[13] In its simplest version (which is used in particular to prove the well-known 'Rotten Kid' theorem), the model assumes that a *single* commodity y is produced through the domestic production function, then distributed across the members of the household. It is easy to see that, in this context, whatever the number of individuals in the household, the household aggregate demand will satisfy Slustky symmetry and negativeness.[14] Hence a Beckerian household will always behave as a single individual. Obviously, this property is entirely due to assumption of a single domestic commodity; in essence, the latter imposes that individuals have identical (ordinal) preferences over the bundles purchased, a strong hypothesis indeed.

[13] See [2] for a presentation.

[14] Pareto efficiency requires that, whatever the sharing, the household produces the maximum quantity of the domestic commodity compatible with the budget constraint. I.e., its behavior can be described by the program:

$$\max_x f(x)$$
$$\pi' x \leq 1$$

where f is the domestic production function. Clearly, the resulting demand $x(\pi)$ has the propseties of an individual demand function.

More generally, consider a S person group that buys some input vector $x \in \mathbb{R}^N$, which is used to produce M outputs that are privately consumed by the members. Then one can readily show the following result: whenever $M < S < N$, the group's aggregate demand $x(p)$ must locally satisfy conditions $SR(M-1)$, which are more restrictive than $SR(S-1)$. In our perspective, however, assumptions on the number of commodities internally produced and consumed by the household may be hard to justify, if only because intra-household consumptions are assumed unobservable.

4 Distribution factors: necessary conditions

We finally consider the case where distribution factors are observable. Then new conditions are generated, that are worked out below.

From Proposition 3, if the function $\xi(\pi, z)$ solves problem (P), then the function $\xi(., z)$ satisfies condition SR$(S-1)$, and its Slutsky matrix $S(\pi, z)$ can be decomposed as

$$S(\pi, z) = \Sigma(\pi, z) + R(\pi, z)$$

where Σ is symmetric, negative and R is of rank at most $S-1$. In what follows, we consider the general case where R is of rank exactly $S-1$.

In the presence of distribution factors, their first-order effect on demand is summarized by the matrix

$$Z(\pi, z) = D_z\xi(\pi, z) = (\partial\xi_i/\partial z_k)_{i,j}$$

Let us first assume that there are 'enough' distribution factors, in the sense that their number d is at least $S-1$. The following necessary condition generalizes a result derived in Browning-Chiappori (1994) for two-person groups:

Proposition 13 *The rank of $Z(\pi, z)$ is at most $S-1$. If* rank$Z = S-1$*, so that $d \geq S-1$), denote by $\mathcal{F}(\pi, z)$ be the space of vectors $v \in \mathbb{R}^N$ such that $v'Z(\pi, z) = 0$. Then the restriction of the Slutsky matrix S to $\mathcal{F}(\pi, z)$ is symmetric, negative, and definite on $[\pi]^\perp$.*

Proof. We use the notations of Proposition 3. Setting $\xi(\pi, z) = \hat{\xi}(\pi, \mu_{-S}(\pi, z))$ we have:

$$Z = D_z\xi = D_{\mu_{-S}}\hat{\xi}.D_z\mu_{-S}$$

Since the matrix $D_{\mu_{-S}}\hat{\xi}$ has $S-1$ columns, its rank is at most $S-1$. This proves that rank$Z \leq S-1$. From Proposition 3, we know that

$$R = D_{\mu_{-S}}\hat{\xi}.B'$$

where B is a $N \times (S-1)$ matrix, the columns of which are the vectors b_s. Now, let v be such that $v'Z = 0$. If rank$Z = S-1$, we must have rank$D_{\mu_{-S}}\hat{\xi} = S-1$ and $v'D_{\mu_{-S}}\hat{\xi} = 0$, so that $v'R = 0$. Rank conditions imply that the subspaces $\mathcal{F}$ and $\mathcal{E}$ coincide, and the conclusion follows from Proposition 3. ■

The first part of Proposition 13 states that if there are more than $S-1$ distribution factors, their impact on demand must be linearly dependent. This is a direct generalization of previous results obtained by [4] in the case of couples ($S=2$), where several distribution factors are shown to have proportional impacts on the demands for various commodities (formally, if $d \geq 2$, then the ratio $(\partial\xi_i/\partial z_k) / (\partial\xi_i/\partial z_l)$ does not depend on i). In general, the impact of distribution factors can be at most $S-1$-dimensional. The interpretation is that distribution factors can only change the location of the outcome on the Pareto frontier, and the latter is of dimension (at most) $S-1$.

The second part of Proposition 13 refines the SR($S-1$) condition, in the sense that the restriction of S must be negative symmetric on a subspace that is fully identified from the knowledge of the matrix Z. In particular, this result establishes a relationship between the impact of distribution factors and that of prices. The intuition, again, is that both the violations of Slutsky symmetry and negativeness, on the one hand, and the effect of distribution factors on the other hand, operate through a similar channel, namely the induced variations of the Pareto weights μ. In the case $S=2$, this result has been tested by Browning and Chiappori [7] on consumption data, and by Chiappori, Fortin and Lacroix [19] on labor supply. Both works fail to reject the predictions.

We have a similar result in the case of a 'small' number of distribution factors, and it is proved in the same way:

Proposition 14 *Assume that* $\operatorname{rank} Z = d < S-1$*. Then there exist* $N-S-1$ *linearly independent vectors* $\{v_1, ..., v_{N-S-1}\}$ *such that (i)* $v_s'.Z = 0$ *for* $s = 1, ..., N-S-1$*, and (ii) the restriction of the Slutsky matrix* S *to the span of* $\{v_1, ..., v_{N-S-1}\}$ *is symmetric negative.*

5 Conclusion

The main conclusion of the paper is clear: in a market environment, collective rationality has strong testable implications on group behavior, provided that the size of the group is small enough with respect to number of observed variables. This result has a known flavor: the DMS literature has generated similar conclusions regarding aggregate demand in the particular case where all consumptions are private, so that the group's behavior can be seen as the aggregate demand of an exchange economy. Our findings are that the conclusions generated by the existing literature are in fact much more robust, and apply whenever the group's behavior satisfies a basic efficiency property. It is Pareto efficiency, not market equilibrium, that drives the properties of aggregate behavior. In addition, we have identified properties which are not only necessary, but also (locally) sufficient. In other words, they provide a complete (local) characterization of efficient group behavior for any group size. A by-product of this conclusion is that it is possible, from the knowledge of the group's aggregate demand, to compute the number of 'true' decision makers within the group: if aggregate demand satisfies our SR($S-1$) property, then the group behaves *as if* there were exactly S independent decision makers.[15]

Our second result deals with the testability of two particular cases. We find that whenever a demand function is compatible with collective rationality, it is compatible with collective rationality with purely private consumption, and also with collective rationality with purely public consumption; moreover, it is also compatible with the presence and/or the absence of intragroup production. It follows that when the only data available are at the group level, neither the nature of consumption nor the presence of production are testable unless additional assumptions are made.

Finally, several recent papers have studied, from an empirical point of view, the impact of 'distribution factors', defined as variables that may influence the group's behavior only through their impact on the decision process. Our paper provides a theoretical underpinning to these approaches, that generalizes previous results.

Several questions remain open at this stage. An obvious one is identification: to what extent is it possible to recover the fundamentals (preferences, production technology, decision process) from observed behavior. Quite obviously, the most general version of the model (with intra-group production and arbitrary preferences) is testable but not identifiable, in the sense that a continuum of structurally different models typically generate the same demand function. In other words, identification requires additional assumptions. These problems are analyzed in a companion paper ([15]).

A Appendix

A.1 Proof of Proposition 5

A first step is to express condition SR($S-1$) in the language of differential forms. We refer to [16] for an introduction to differential calculus, and to [9], [1] and [8] for a detailed treatment.

Introduce the differential one-form

$$\omega(\pi) = \sum \xi^j(\pi)\, d\pi_j.$$

Taking the exterior differential yields:

$$d\omega = \sum \left(\frac{\partial \xi^j}{\partial \pi_i} - \frac{\partial \xi^i}{\partial \pi_j} \right) d\pi_i \wedge d\pi_j \tag{16}$$

Introduce the vector field:

$$\Pi(\pi) = \sum \pi_i \frac{\partial}{\partial \pi_i} \tag{17}$$

[15] The exact interpretation of this statement is of course delicate, since it is related to both preferences and the decision process; for instance, two decision makers with identical preferences will always count as one. For an empirical test of this conclusion, see [21].

so that the Walras law becomes $\omega(\Pi) = \sum_{j=1}^{N} \xi^j \pi_j = 1$.

Denote by $(d\omega)^S$ the exterior product $d\omega \wedge ... \wedge d\omega$ with S terms. It is a differential $2S$-form.

Lemma 15 *The following conditions are equivalent:*

(a) *The Slutsky matrix $S(\pi)$ decomposes as $S = \Sigma + \sum_{s=1}^{S-1} a_s (b_s)'$, with Σ symmetric.*

(b) $\omega \wedge d\omega^S = 0$

(c) *There exists one-forms $\gamma, \alpha_s, \beta_s$ such that:*

$$d\omega = \sum_{s=1}^{S-1} \alpha_s \wedge \beta_s + \omega \wedge \gamma \tag{18}$$

Proof. By inspection, (c) implies (b). The converse holds as well; we refer to Bryant et al. (1991), Prop. I.1.6. or to Ekeland and Nirenberg (2003), Lemma 2.

Writing (a) into the definition of $d\omega$, we get:

$$d\omega = \sum_{i,j} \left\{ \sum_s \left(a_s^i b_s^j - a_s^j b_s^j \right) + \left(\left[\sum_k \frac{\partial \xi^j}{\partial \pi_k} \pi_k \right] \xi^i - \left[\sum_k \frac{\partial \xi^i}{\partial \pi_k} \pi_k \right] \xi^j \right) \right\} d\pi_i \wedge d\pi_j \tag{19}$$

$$= \sum_s \sum_{i,j} \left(a^i b^j - a^j b^j \right) d\pi_i \wedge d\pi_j + \omega \wedge \gamma, \text{ where } \gamma = \left[\sum \frac{\partial \xi^i}{\partial \pi_k} \pi_k \right] d\pi_i \tag{20}$$

$$= \sum_{s=1}^{S-1} \alpha_s \wedge \beta_s + \omega \wedge \gamma, \text{ where } \alpha_s = \sum a_s^i d\pi_i \text{ and } \beta_s = \sum b_s^i d\pi_i \tag{21}$$

and hence $\omega \wedge d\omega^S = 0$. So (a) implies (b)

We now show that (c) implies (a). Applying the vector field Π defined by (17) to both sides of the preceding equation, and using the Walras law, we get:

$$d\omega(\Pi, \bullet) = \sum \alpha_s(\Pi)\beta_s - \sum \beta_s(\Pi)\alpha_s + \gamma - \gamma(\Pi)\omega \tag{22}$$

which defines γ in terms of $d\omega(\Pi, \bullet)$, the α_s, and the $\beta_s, 1 \leq s \leq S-1$. Replacing γ by its value, we get:

$$d\omega = \sum_s \alpha_s \wedge \beta_s + \omega \wedge \left(d\omega(\Pi, \bullet) - \sum_s \alpha_s(\Pi)\beta_s + \sum_s \beta_s(\Pi)\alpha_s \right) \tag{23}$$

$$= \sum_s (\alpha_s - \alpha_s(\Pi)\omega) \wedge (\beta_s - \beta_s(\Pi)\omega) + \omega \wedge d\omega(\Pi, \bullet) \tag{24}$$

Differentiate the Walras law:

$$\sum \frac{\partial \xi^j}{\partial \pi_i} \pi_j + \xi_i = 0 \tag{25}$$

and substitute into the definition (16) of $d\omega$. We get:

$$d\omega(\Pi, u) = \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} (\pi_i u_j - \pi_j u_i) = \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} \pi_i u_j + \sum_i \xi_i u_i$$

so that:

$$d\omega(\Pi, \bullet) = \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} \pi_i d\pi_j + \sum_i \xi_i d\pi = \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} \pi_i d\pi_j + \omega \tag{26}$$

Substituting in formula (24) yields finally:

$$d\omega = \sum (\alpha - \alpha(\Pi)\omega) \wedge (\beta - \beta(\Pi)\,\omega) + \omega \wedge \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} \pi_i d\pi_j$$

Writing this coordinatewise, that is, using formula (16), and setting

$$\bar{\alpha}_s = \alpha_s - \alpha_s(\Pi)\omega = \sum_i a_s^i d\pi_i$$
$$\bar{\beta}_s = \beta_s - \beta_s(\Pi)\,\omega = \sum_i b_s^i d\pi_i$$

we get:

$$\frac{\partial \xi^j}{\partial \pi_i} - \frac{\partial \xi^i}{\partial \pi_j} = \sum_{s=1}^{S-1} \left(a_s^i b_s^j - a_s^j b_s^i \right) + \xi^i \sum_k \frac{\partial \xi^j}{\partial \pi_k} \pi_k - \xi^j \sum_k \frac{\partial \xi^i}{\partial \pi_k} \pi_k$$

which is precisely condition (a). ■

Here is the concave Darboux Theorem, as stated in [25]:

Theorem 16 *A necessary and sufficient condition for a one-form ω with C^1 coefficients to be decomposable as $\omega = \sum \lambda_s dV^s$, $1 \leq s \leq S$, in some neighborhood of $\bar{\pi}$, for some positive functions λ_s and some strongly concave functions V^s, is that there exists a decomposition of $d\omega$ as:*

$$d\omega = \sum_{s=1}^{S-1} \alpha_s \wedge \beta_s + \omega \wedge \gamma \tag{27}$$

valid in some neighborhood of $\bar{\pi}$, such that the $(\omega, \gamma, \alpha_1, \beta_1, ..., \alpha_{S-1}, ...\beta_{S-1})$ are linearly independent and the bilinear form

$$(\Omega\zeta, \eta) = \sum_{i,j} \frac{\partial \omega^i}{\partial \pi_j} (\bar{\pi})\, \zeta_i \eta_j$$

is symmetric and negative definite on $[\mathrm{E}(\bar{\pi})]^{\perp}$*, where:*

$$\mathrm{E}(\pi) = \mathrm{Span}\{\omega, \beta_1, ...\beta_{S-1}\}$$

In addition, we have the following properties:

- $\mathrm{E}(\pi) \subset \mathrm{F}(\pi)$*, where the* $2S$*-dimensional subspace* $\mathrm{F}(\pi)$ *spanned by* $\alpha_1, ..., \alpha_{S-1}, \beta_1, ..., \beta_{S-1}, \omega$*, depends only on* ω*, and not on the choice of the decomposition (27),*
- *the* $dV^s(\bar{\pi})$ *can all be chosen arbitrarily close to* $\frac{1}{S}\omega(\bar{\pi})$
- *the functions* V^s *can be chosen to be strictly increasing in a neighborhood of* $\bar{\pi}$*.*

If we omit the requirement that the λ_s be positive and the V^s be convex, then the requirement on Ω is dropped, and this reduces to an old theorem of Darboux (see [9] or [8]). As stated, the result is due to Ekeland and Nirenberg ([25]). It was also proved by Chiappori and Ekeland ([13]) in the case when the $\omega^n(\pi)$ are real analytic. There is, of course, a convex Darboux theorem along the same lines. Recall that, by a strongly convex function, we mean a function which is C^2, concave, increasing with respect to each variable, and with a positive definite matrix of second derivatives. By Lemma 15, condition (27) means that the Slutsky matrix S associated with ξ decomposes as $S = \Sigma + \sum_{s=1}^{S-1} a_s (b_s)'$, with Σ symmetric.

Now apply the concave Darboux Theorem to $\omega = \sum \xi^n d\pi_n$. The subspace $\mathrm{E}(\pi)$ then is generated by ξ and the $a_s, 1 \leq s \leq S$. Saying that Ω is symmetric and negative definite on $[\mathrm{E}(\bar{\pi})]^{\perp}$ means that the Jacobian matrix $D_\pi \xi$ is symmetric and negative definite on $[\mathrm{E}(\bar{\pi})]^{\perp}$. This is equivalent to

$$D_\pi \xi = \Sigma + \gamma \xi' + \sum_{s=1}^{S-1} a_s (b_s)'$$

where Σ is symmetric, negative definite.By Lemma 13, we have $\gamma^j = \sum_{i,j} \frac{\partial \xi^j}{\partial \pi_i} \pi_i$, so that $D_\pi \xi - \gamma \xi'$is the Slutsky matrix. So we get precisely condition SR($S-1$)

A.2 Proof of proposition 8

Consider the differential one-form ω^* and ω defined in respective neigborhoods of $\bar{P}$ and $\bar{Y} = Y(\bar{P})$ by:

$$\omega^* = \sum P_k(Y) dY^k \text{ and } \omega = \sum Y^k(P) dP_k$$

and the associated Jacobian matrices:

$$\Omega = \left(\frac{\partial Y^i}{\partial P_j}\right)_{i,j} \text{ and } \Omega^* = \left(\frac{\partial P^i}{\partial Y_j}\right)_{i,j}$$

By assumption, $Y(P)$ satisfies condition SR$(S-1)$, so that we can write:

$$d\omega = \sum_{s=1}^{S-1} \alpha_s \wedge \beta_s + \omega \wedge \gamma$$

the restriction of Ω to $[\text{Span}\{\omega, \alpha_1, ...\alpha_{S-1}\}]^{\perp}$ is symmetric and negative definite. By the Walras law, we have $\omega + \omega^* = d(P'X) = 0$, and hence:

$$d\omega^* = -d\omega = -\sum_{s=1}^{S-1} \alpha_s \wedge \beta_s - \omega \wedge \gamma = -\sum_{s=1}^{S-1} \alpha_s \wedge \beta_s + \omega^* \wedge \gamma \qquad (28)$$

Since Ω and $-\Omega^*$ are inverse of each other, so that the restriction of Ω^* to $[\text{Span}\{\omega^*, \alpha_1, ...\alpha_{S-1}\}]^{\perp}$ must also be symmetric and negative definite. It follows from lemma 15 that $P(Y)$ also satisfies SR$(S-1)$.

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