Some applications of the Cartan-Kähler theorem to economic theory

Ivar Ekeland
CEREMADE and Institut de Finance
Université Paris-Dauphine
75775 Paris CEDEX 16

1 The economic problem.

The purpose of this short paper is to show a fundamental problem of economic theory gives rise to a system of nonlinear PDEs of the first order which - up to now - can only be solved by applying the celebrated Cartan-Kähler theorem on integral manifolds of exterior differential systems (see [3], [2]).

Following standard definitions in microeconomics (see [11] for an exhaustive review of the theory), an *economy* is described by N agents and K goods. Agents trade and consume, not individual goods, but *bundles*, each bundle being described by a point $x = \left(x^1, ..., x^K\right) \in R^K$, where x_k denotes the quantity of good k ; for instance, we do not eat of buy bread and butter separately, but sandwiches, which are certain bundles of bread and butter. There will also be a set of *prices* $p = (p_1, ..., p_K) \in R^K$. It the prevailing price system is p, the cost of bundle x is $p'x = \sum p_k x^k$.

Agent n is fully described by his *utility function* $U^n : R^K \to R$ and by his *initial endowment*; the latter can be given,

- either in real terms, namely a goods bundle $\omega_n \in R^K$, which the agent will trade at market prices
- or in monetary terms, namely a wealth $w_n \in R$ which the agent will spend

The utility function determines the preferences, and hence the behaviour, of the consumers: agent n prefers the bundle x to the bundle y iff $U^n(x) \geq U^n(y)$. Given his initial endowment, agent n chooses the bundle he prefers among all those he can afford. This leads to an optimization problem.

- in the case of *real endowments*, $\omega_n \in R^K$, the agent's problem is:

  $$\max_x U^n(x)$$
  $$p'x \leq p'\omega_i$$

  leading to a solution $\hat{x}_n(p)$, which is the agent's response to the set of prices p. The map

  $$z_n(p) = \hat{x}_n(p) - \omega_n$$

  is called the *excess demand* of agent n.

- In the case of *monetary endowments*, $w_n \in R$, the agent's problem is:

$$\max_x U^n(x)$$
$$p'x \leq w_n$$

leading to a solution $\bar{x}_n(p)$, which is called the *market demand* of agent n.

Suitable assumptions, mainly strict concavity of the utility functions, will ensure that the excess demand or market demand functions are well-defined, smooth, and satisfy the so-called Walras law, which simply expresses the fact that all available resources are spent towards acquiring the consumption bundle:

$$p'z_n(p) = 0 \text{ or } p'\bar{x}_n(p) = w_n$$

Typically, it will be assumed that the U^n are C^2, and that its second derivative is positive definite, so that the solutions $\hat{x}_n(p)$ or $\bar{x}_n(p)$ are unique, depend smoothly on the data, and activates the budget constraint.

In practice, the utility function of an individual cannot be observed, but his demand can. This leads to a very interesting question: suppose one observes the (excess of market) demand of an individual; does it arise from a maximization procedure similar to the one I just described, and if so, can one recover the utility function U^i from the data ? In the case of market demand, the answer has been known very early on, since the work of Antonelli [1], later rediscovered by Slutsky [10]. It turns out that, to arise from a maximization procedure, the market demand of an individual must satisfy a stringent set of conditions, and if it does so, then the corresponding preferences are fully determined.

However, individual demand is hard to observe. It is much easier to observe the aggregate demand of a large number of consumers, for instance by sifting through macroeconomic data. With the above notations, we define aggregate excess demand by:

$$Z(p) = \sum_{n=1}^{N} z_i(p)$$

and aggregate market demand by:

$$X(p) = \sum_{n=1}^{N} \bar{x}_i(p)$$

so that both satisfy the Walras law:

$$p'Z(p) = 0, \text{ and } p'X(p) = \sum w_n$$

In a celebrated series of papers, around 1975, Sonnenschein, Mantel and Debreu treated the case of excess demand. They proved that, if there are more agents than goods, $N \geq K$, then any function $Z(p)$ satisfying the Walras law can be written as $Z(p) = \sum z_i(p)$, where the $z_i(p)$ are individual excess demands (see [11], [9] and the references therein), and this result has been very influential in the development of economic theory.

Surprisingly, the corresponding problem for market demand remained open until very recently; it is only in 1995 that Chiappori and Ekeland [5] proved a similar result. Their proof uses the Cartan theory of exterior differential systems, and the Cartan-Kähler theorem.

2 The mathematical problem

Define, for $1 \geq n \geq N$, the indirect utility function V^n by:

$$\begin{aligned} V^n(p) &= \max_x \{U^n(x) \mid p'x \leq w_n\} \\ &= U^n(\bar{x}(p)) \text{ with } p'\bar{x}_n(p) = w_n \end{aligned}$$

There is a one-to-one correspondence between V^n and U^n, and we can recover one from the other. If U^n is quasi-concave, then V^n is quasi-convex, and conversely. By the Lagrange multiplier rule, there exists some $\lambda^n(p) > 0$ such that:

$$\begin{aligned} V^n(p) &= \max_x \{U^n(x) + \lambda^n(p)(w_n - p'x) \mid p'x \leq w_n\} \\ &= U^n(\bar{x}(p)) + \lambda^n(p)(w_n - p'\bar{x}_n(p)) \end{aligned}$$

Differentiating this identity with respect to p, most terms cancel, and we are left with:

$$DV^n(p) = -\lambda^n(p)\,\bar{x}_n(p)$$

So the vector field $\bar{x}_n = -DV^n/\lambda^n$ must be collinear to a gradient, a very strong condition. Unfortunately, we do not observe the individual demands $\bar{x}_n$, only their aggregate. Summing up, we get:

$$X(p) = -\sum_{n=1}^{N} \frac{DV^n(p)}{\lambda^n(p)} \tag{1}$$

In addition, we adjoin the condition $p'\bar{x}_n(p) = w_n$, which yields:

$$p'DV^n(p) = -\lambda^n(p)\,w_n \tag{2}$$

We now state the mathematical problem to be solved: given $X(p)$, satisfying $p'X(p) = \sum w_n$, find convex functions $V^1, ..., V^N$ and positive functions $\lambda^1, ..., \lambda^N$ satisfying (1) and (2).

Computing the $\lambda^n$from (2) and writing the result in (1) , we get:

$$\sum_{n=1}^{N} \frac{DV^n(p)}{p'DV^n(p)} w_n = -X(p) \tag{3}$$

which is a system of K nonlinear equations of the first order for N functions of K variables.

The following result, which is due to Chiappori and Ekeland (see [5]), states that this system can be solved locally, provided the right-hand side $X(p)$ is real analytic:

Theorem 1 *Assume $N \geq K$. Consider some open set $\mathcal{U}$ in $R^K \diagdown \{0\}$, and some analytic map $X : \mathcal{U} \to R^K$ such that $p'X(p) = \sum w_n$. For all $\bar{p} \in \mathcal{U}$ and for all $(\bar{x}_1, ..., \bar{x}_N) \in R^{NK}$ and $(\lambda_1, ..., \lambda_N) \in R^N$ that satisfy $\sum \bar{x}_n = X(\bar{p})$ and $\bar{\lambda}^n > 0$, there exists real-analytic functions V^n and $\lambda^n, 1 \leq n \leq N$, defined on some neighourhood $\mathcal{N}$ of $\bar{p}$, such that:*

1. *$\forall n,\ D_p V^n(\bar{p}) = -\bar{\lambda}^n \bar{x}_n,\ \lambda^n(\bar{p}) = \bar{\lambda}^n$,*
2. *$\forall n,\ D^2_{pp} V^n(\bar{p})$ is positive definite*
3. *$(V^n, \lambda^n), 1 \leq n \leq N$, solve (1) and (2)*

As a consequence, we extend the Mantel-Sonnenschein-Debreu result to market demand:

Corollary 2 *Assume $N \geq K$. Consider some open set $\mathcal{U}$ in $R^K \diagdown \{0\}$, and some analytic map $X : \mathcal{U} \to R^K$ such that $p'X(p) = \sum w_n$. For all $\bar{p} \in \mathcal{U}$ and for all $(\bar{x}_1, ..., \bar{x}_N) \in R^{NK}$ and $\left(\bar{\lambda}^1, ..., \bar{\lambda}^N\right) \in R^N$ that satisfy $\sum \bar{x}_n = X(\bar{p})$, $\bar{p}' x_n(\bar{p}) = w_n$ and $\bar{\lambda}^n > 0$, there exists real-valued functions U^n and $\lambda^n, 1 \leq n \leq N$, such that:*

Theorem 3 1. *U_n is defined on some convex neighbourhood V_n of $\bar{x}_n$ where it is analytic and strictly quasi-concave,*

2. *the λ^n are all defined on some neighourhood $\mathcal{N}$ of $\bar{p}$, where there are analytic and positive,*
3. *$\forall n,\ x_n(\bar{p}) = \bar{x}_n,\ \lambda^n(\bar{p}) = \bar{\lambda}^n$,*
4. *for all $p \in \mathcal{N}$, we have:*

$$
\begin{aligned}
& p' x_n(p) = w_n \ \forall n, \\
& X(p) = \sum_{n=1}^{N} x_n(p), \\
& \frac{\partial U^n}{\partial x^k}(x_n(p)) = \lambda^n(p)\, p_k \ \forall n, k, \\
U^n(x_n(p)) \quad = \quad & \max\{U^n(x) \mid x \in \mathrm{V}_n,\ p'x \leq w_n\} \ \forall n.
\end{aligned}
$$

The proof of the theorem will be described in the next section. Note that it is not known whether one can solve the system (1), (2) when the right-hand side $X(p)$ is C^{∞} instead of analytic. A simple case is the system:

$$\begin{aligned}\frac{u_x}{u_z}+\frac{v_x}{v_z} &= f(x,y,z)\\ \frac{u_y}{u_z}+\frac{v_y}{v_z} &= g(x,y,z)\end{aligned}$$

to be solved for two functions $u(x,y,z)$ and $v(x,y,z)$ of three variables, with f and g given and C^{∞}. We have been investigating this system for some time without success.

3 Proof.

Introduce the space:

$$\begin{aligned}E &= R^{K}\times R^{N}\times R^{NK}\\ &= \left\{\left(p_k,\mu^n,\Delta_{n'}^{k'}\right) \mid 1\le k,k'\le K, 1\le n,n'\le N\right\}\end{aligned}$$

In this space, consider the submanifold $\mathcal{M}$ defined by the equations:

$$\sum \mu^n\Delta_n^k = X^k(p)\ \forall k, \tag{4}$$

$$\sum p_k\Delta_n^k = \frac{1}{\mu^n}\ \forall n. \tag{5}$$

Sums are carried over repeated indices. The equations are independent, so that $\mathcal{M}$ is a submanifold of codimension $(N+L)$. In $\mathcal{M}$ (and *not* in E) we consider the exterior differential system (EDS):

$$\sum d\Delta_n^k\wedge dp_k = 0\ \forall n, \tag{6}$$

$$dp_1\wedge \dots\wedge dp_K = 0 \tag{7}$$

This EDS is equivalent to the system (1),(2). An integral manifold of (6),(7) is the graph of a map $\left(\Delta_n^k(p),\mu^n(p)\right), 1\le n\le N$; relations (6) mean that $\Delta_n^k(p)=\frac{\partial V^n}{\partial p_k}$ for some function V^n, and setting $\lambda^n=-1/\mu^n$ we get (1) and (2) from (4) and (5). We now apply the Cartan-Kähler theorem, bearing in mind that we seek convex V^n and negative μ^n .

The system is obviously closed. The next step is to find integral elements. This is done in the standard fashion, by writing:

$$d\mu^n = \sum m^{nk}dp_k \tag{8}$$

$$d\Delta_n^k = \sum \delta_n^{kk'}dp_{k'} \tag{9}$$

and substituting in (4),(5) and (6). The latter gives:

$$\delta_n^{kk'} = \delta_n^{k'k}$$

and the other two give:

$$\begin{aligned}\sum \Delta_n^k m^{nk'} + \sum \mu^n \delta_n^{kk'} &= \frac{\partial X^k}{\partial p_{k'}} \ \forall k, k' \qquad (10)\\ \sum (\mu^n)^2 \Delta_n^k + \sum (\mu^n)^2 p_{k'} \delta_n^{kk'} &= -m^{nk} \ \forall k, n \qquad (11)\end{aligned}$$

This is a system of $K^2 + KN$ equations for the $NK + NK(K+1)/2$ variables $m^{nk}, \delta_n^{kk'}$. We seek a solution such that the N matrices $\delta_n^{kk'}$ are symmetric, of course, but also *positive definite*: once the integral manifold is found, we will have $\Delta_n^k(p) = \frac{\partial V^n}{\partial p_k}$, and (9) then gives:

$$\delta_n^{kk'} = \frac{\partial^2 V^n}{\partial p_k \partial p_{k'}} (\bar{p})$$

Having $\delta_n^{kk'}$ positive definite ensures that V^n is convex in a neighbourhood of $\bar{p}$.

Back to the equations (10) and (11). Substituting the second into the first, we eliminate m^{nk} and get an equation for $\delta_n^{kk'}$ only:

$$\sum \mu^n \delta_n^{kk'} - \sum (\mu^n)^2 p_{k''} \delta_n^{k'k''} \Delta_n^k = \text{ rhs } \forall k, k' \qquad (12)$$

where rhs stands for some right-hand side which we do not care to write down. This is now a system of K^2 linear equations in $NK(K+1)/2$ unknowns (bearing in mind that $\delta_n^{kk'} = \delta_n^{k'k}$), and we want a solution which makes all the matrices δ_n positive definite.

To do this, we will show that the kernel of this system contains a family $\bar{\delta}_n, 1 \leq n \leq N$, where all the $\bar{\delta}_n$ are positive definite. If the family δ_n^0, is any solution of (12), then the family $\delta_n = \delta_n^0 + a\bar{\delta}_n$, is also a solution, and for $a > 0$ large enough it will be positive definite. So our next step is investigate the homogeneous system:

$$\sum \mu^n \delta_n^{kk'} - \sum (\mu^n)^2 p_{k''} \delta_n^{k'k''} \Delta_n^k = 0 \ \forall k, k' \qquad (13)$$

and to show that it has a solution $\delta_n, 1 \leq n \leq N$, with all the δ_n positive definite.

Let us rewrite this system as a relation between matrices. Call δ_n the matrix $\left(\delta_n^{kk'}\right)$, and set $\gamma_n^k = \sum p_{k'} \delta_n^{kk'}$. Call γ_n and ζ_n the vectors with components $\gamma_n^{k'}$ and $(\mu^n)^2 \Delta_n^k$. Equation (13) then can be rewritten as:

$$\sum \mu^n \delta_n - \sum \gamma_n \zeta_n = 0 \qquad (14)$$

where $\gamma_n \Delta_n$ must be understood as a rank one matrix. Since the δ_n are symmetric, so is the first term in this equation, and therefore the sum $\sum_n \gamma_n \zeta_n$

must also be symmetric. By a celebrated lemma of Elie Cartan, this means that there is a symmetric $(N \times N)$ matrix A such that $\gamma_n = A\zeta_n$ for every n.

In conclusion, there is a symmetric matrix $A = \left(\alpha^{nn'}\right)$ such that:

$$\sum p_{k'}\delta_n^{k'k} = (\mu^n)^2 \sum \alpha^{nn'} \Delta_{n'}^k \ \forall k, n \tag{15}$$

and writing this back into (13), we get:

$$\sum \mu^n \delta_n^{kk'} = \sum \alpha^{nn'} (\mu^n)^2 \left(\mu^{n'}\right)^2 \Delta_n^k \Delta_{n'}^{k'} \ \forall k, k' \tag{16}$$

Let us rewrite this system in matrix terms again. The family $\delta_n, 1 \leq n \leq N$, of symmetric matrices, solves (13) if and only if it satisfies the system:

$$\delta_n p = \sum \alpha^{nn'} \left(\mu^{n'}\right)^2 \Delta_{n'} \ \ \forall n \tag{17}$$

$$\sum \mu^n \delta_n = \sum \alpha^{nn'} (\mu^n)^2 \left(\mu^{n'}\right)^2 \Delta_n \left(\Delta_{n'}\right)' \tag{18}$$

for some symmetric matrix $A = \left(\alpha^{nn'}\right)$. One can check that the last equation is compatible with the preceding ones. Indeed, the first N equations give:

$$\sum \mu^n \delta_n p = \sum \mu^n \alpha^{nn'} \left(\mu^{n'}\right)^2 \Delta_{n'}$$

and the last one:

$$\sum \mu^n \delta_n p = \sum \alpha^{nn'} (\mu^n)^2 \left(\mu^{n'}\right)^2 \Delta_n \left(\Delta_{n'} p\right),$$

but since $\Delta_{n'} p = 1/\mu^{n'}$ by (5), and $\alpha^{nn'} = \alpha^{n'n}$, the two relations coincide.

We end up with an interesting mathematical question: given $(N+1)$ points $x_1, ..., x_N$ and y in R^K, given a positive definite matrix Q, does there exist N positive definite matrices $M_1, ..., M_N$ such that $M_n y = x_n$ and $\sum M_n = Q$? Note that there are obvious necessary conditions, namely that $\sum x_n = Qy$ and $(x_n, y) > 0 \ \forall n$. In our case, x_n is the right-hand side of (17), $y = p$ and Q is the right-hand side of (18); in the paper [5] we solved that particular case, but since then, professor Inchtchakov ([4]) and professor SanMartin ([8]), independently of each other, have solved the general case.

Let us state and prove Inchtchakov's result. Without loss of generality, assume $Q = I$, and consider the quadratic form:

$$(Cz, z) = \sum \frac{(x_n, z)^2}{(x_n, y)} - (z, z)$$

which vanishes on $y = 0$.

Lemma 4 *Assume $(x_n, y) \neq 0$ for all n. Then a necessary and sufficient condition for the existence of positive definite matrices M_n such that $M_n y = x_n$ for all n and $\sum M_n = I$ is that $\sum x_n = y$, $(x_n, y) > 0$ for all n, and $(Cz, z) < 0$ for all z not collinear with y.*

Proof. Let us first prove necessity. Consider the quadratic form:

$$(C_n z, z) = \frac{(M_n y, z)^2}{(M_n y, y)} = \frac{(x_n, z)^2}{(x_n, y)}$$

By Cauchy-Schwarz, since M_n is positive definite, we have $(C_n z, z) < (M_n z, z)$ unless z is collinear to y. Adding up, we find $\sum (C_n z, z) < (z, z)$,as announced.

Conversely, assume $\sum x_n = y$, $(x_n, y) > 0$ for all n, and $(Cz, z) < 0$ for all z not collinear with y. Define

$$\begin{aligned} B_n z &= \frac{(x_n, z)}{(x_n, y)} x_n, \ 1 \leq n \leq N \\ B_0 &= I - \sum B_n \end{aligned}$$

Then $B_n y = x_n$ and $B_0 y = y - \sum x_n = 0$. Note also that:

$$\begin{aligned} (C_n z, z) &= \frac{(x_n, z)^2}{(x_n, y)} = (B_n z, z) \geq 0 \\ (Cz, z) &= \sum (B_n z, z) - (z, z) = -(B_0 z, z) \end{aligned}$$

Now set $M_n = B_n + \frac{1}{N} B_0$. We have:

$$\begin{aligned} M_n y &= B_n y + \frac{1}{N} B_0 y = x_n + 0 \\ \sum M_n &= \sum B_n - B_0 = I \\ (M_n z, z) &= (B_n z, z) + \frac{1}{N}(-B_0 z, z) \end{aligned}$$

In the last equation, both terms on the right-hand side are positive semidefinite, and the second one vanishes only when z is collinear to y. The M_n have the desired properties, and the proof is concluded.■ ■

Once a positive definite family $\left(\bar{\delta}_1, ..., \bar{\delta}_N\right)$ is found in the kernel, we take any solution $\left(\delta_1^0, ..., \delta_N^0\right)$of (12) and we consider $\delta_n = \delta_n^0 + a\bar{\delta}_n$ for large $a > 0$. This will then also be positive definite, and provides us with the integral element we are looking for.

There remains the last step of the Cartan-Kähler procedure, and this is to prove that every point p is *ordinary* in the sense of Elie Cartan. This, as always, is very delicate, and we refer the reader to the paper [5] for the computations. Let me just mention the conclusions, for the case $K = N$. The manifold $\mathcal{M}$ has dimension $N^2 + 1$. The codimension of the bundle of integral elements in the corresponding Grassmannian is $N^2 (N-1)/2$. The Cartan characters are $c_n = nN$ for $n \leq N$, so that:

$$c_0 + ... + c_{N-1} = N(0 + 1 + ... + (N-1)) = NN(N-1)/2$$

and the Cartan criterion is satisfied, so that the Cartan-Kähler theorem applies.

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