An Example of Resonance Overlap

Jacques Féjoz*, Marcel Guardia†

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Abstract

Chirikov's celebrated criterion of resonance overlap has been widely used in celestial mechanics and Hamiltonian dynamics to detect global instability, but is rarely rigourous. We introduce two simple but fairly general Hamiltonian systems, each depending on two parameters measuring respectively the distance to resonance overlap and non-integrability. Within some thin region of the parameter plane, classical perturbation theory shows the existence of global instability and symbolic dynamics, thus illustrating Chirikov's criterion.

1 Heuristic introduction – resonance overlapping

Let $H : \mathbb{T} \times \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ be a time-dependent Hamiltonian of class $\mathcal{C}^\infty$, periodic in time, of the form

$$H(x, y, t) = H_0(y) + \varepsilon F(x, y, t). \tag{1}$$

When $\varepsilon = 0$, the time-one map ϕ of the flow of H is integrable and the level curves of the coordinate y are all invariant. Curves whose rotation number $H_0'(y)$ is rational or have good rational approximations disappear for generic Fourier coefficients of F, as Poincaré noticed [27]. In place of some of those resonant curves, periodic orbits originate, usually by elliptic/hyperbolic pairs. (More generally, non-smooth invariant graphs known as Aubry-Mather sets can be found as the support of minimizing measures.) For systems of one and a half degree of freedom, like ours (or two degrees of freedom), elliptic orbits are surrounded by elliptic "eyes", where some kind of stability prevails over long time intervals (see [3] and references

*CEREMADE/Université Paris-Dauphine & IMCCE/Observatoire de Paris, CNRS, PSL Research University, Paris, France, jacques.fejoz@dauphine.fr.

†Universitat Politècnica de Catalunya (Departament de Matemàtiques) & Barcelona Graduate School of Mathematics (BGSMATH), Spain, marcel.guardia@upc.edu.

therein). Simultaneously, as KAM theory proves, a positive Lebesgue measure of Diophantine invariant curves persist [21, 22]. As ε increases, more invariant curves disappear. (Some new invariant curves also show up, although these ones are harder to detect.) Persisting invariant curves are obstructions to large deviations in the y direction. Note that, in higher dimension, Lagrangian invariant tori do not separate the phase space anymore and allow the dynamics for slow, non-local instability, called *Arnold diffusion* [1, 7]. Arnold diffusion is notoriously difficult to show.

As ε keeps increasing, the seeming sizes of resonant eyes grow. As long as the separatrices of two resonances are well apart, invariant curves separating the two zones confine orbits on one side or the other, and the the system behaves as if the two resonances did not interact: in each zone, the dynamics is reasonably described by an integrable approximation retaining only the harmonics responsible for the opening of the relevant eye. Chirikov has conjectured that an orbit will start moving between two resonance eyes in a chaotic and unpredictable manner "as soon as these unperturbed resonances overlap" [6, 7]. For a modern reference, with applications to celestial mechanics, see Morbidelli's book [19, Chap. 6 and Section 9.2 in particular]. Indeed, as soon as the separatrices of the two resonances get close to each other, the dynamics is no more described by two adjacent one-resonance integrable models and, as Morbidelli puts it, "an initial condition in the overlapping region does not know which resonance it belongs to, and hesitates about which guiding trajectory it should follow". The criterion has been used for magnetically confined plasmas (as in Chirikov's initial work or Escande's review [11]), the Solar System (e.g. [20, 19, 24, 26]), space debris [4], transport and turbulence in fluid mechanics [9], as well as particle dynamics in accelerators, microwave ionization of Rydberg atoms, etc. (see [14] and references therein).

Defining the closeness of two resonance eyes, or their overlap, is not a simple matter, since generically separatrices split and thus do not precisely circumscribe an "eye". Physicists speak of a "stochastic layer" at the border, but little is really known about dynamics in this layer, apart from the horseshoe (a set of zero-measure) given by the Birkhoff-Smale theorem [23]. Moreover, there is a whole web of resonances, and, for each resonance, there are infinitely many ways to choose integrable approximations describing the opening of the corresponding eye. All this makes Chirikov's criterion imprecise. For a further analysis of why Chirikov's criterion fails in general, see for example [2, 5, 17].

Key to instability is the destruction of invariant curves. The precise mechanism remains mysterious, despite extensive efforts (e.g. [13, 16]). One attempt to describe whether invariant curves persist or not, which has been quite successful for practical purposes, is Greene's criterion, which analyzes the stability of ac-

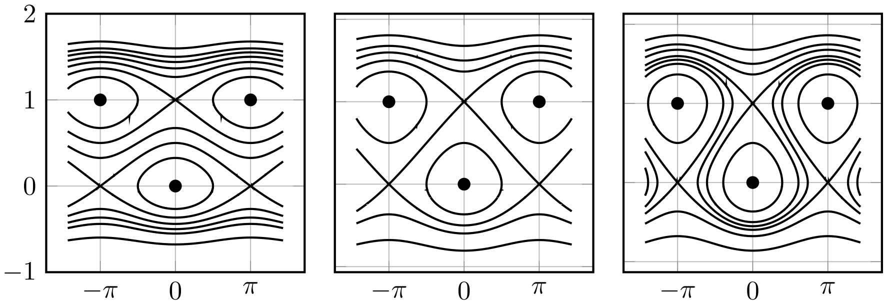

Figure 1: Level curves of h_ε for $\varepsilon = \frac{1}{2}, 1, \frac{3}{2}$

cumulating periodic orbits. This criterion has been partly justified [10, 12, 15]. Renormalization should also be a important tool for the full picture [5].

We will not address this difficult issue directly, but rather aim at illustrating Chirikov's criterion on a simple, rigorous example. Consider the Hamiltonian

$$h_\varepsilon(x, y) = \frac{y^2}{2} - \frac{y^3}{3} - \frac{\varepsilon}{12}\cos x \tag{2}$$

on $\mathbb{T} \times \mathbb{R}$, where ε is a real parameter. We view it as a modification of the twist Hamiltonian $y^2/2$ in the class of "classical" Hamiltonians (sums of a kinetic part depending only on y and a potential part depending only on x) involving only the lowest degree term and lowest order harmonic creating two resonance eyes close to each other. In this sense, h_ε is a simple but somewhat general model family whose perturbations are eligible to Chirikov's criterion of resonance overlapping. As a variant we also consider the doubly periodic Hamiltonian

$$h'_\varepsilon(x, y) = \cos y - \varepsilon \cos x \tag{3}$$

on $\mathbb{T}^2$, for which the instability is similar locally, but also more global due to the double periodicity.

Interestingly, the Hamiltonian h_0 (cubic in the actions) has a twistless curve (where the unperturbed frequency map $y \mapsto y(1-y)$ has a fold singularity). Greene's criterion has been applied to this twistless curve in the article [8]. Also, Hamiltonians similar to h'_ε have been studied notably by Zaslavsky in [28], as examples displaying "stochastic webs" with spatial patterns.

2 An illustration of the overlapping principle

We will now describe a case akin to the resonance overlap phenomenon for (time dependent perturbations of) the Hamiltonians (2) and (3), where one can quite

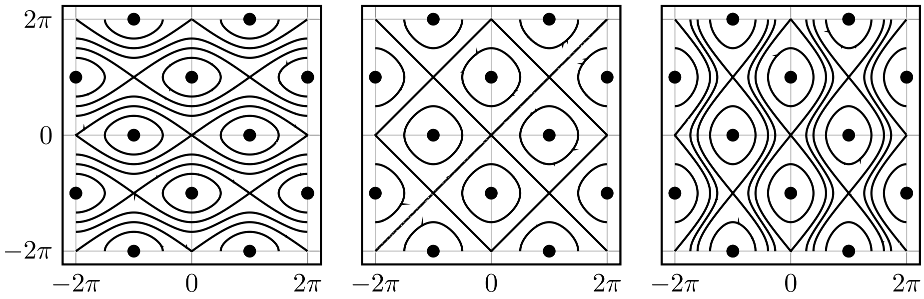

Figure 2: Level curves of h'_ε for $\varepsilon = \frac{1}{2}$, 1, $\frac{3}{2}$

explicitly see the transition from complete integrability to resonance overlapping. The key point in these examples is that the two parameters controlling the distance of resonance eyes and the non-integrability are decoupled. The same overlapping of resonances behavior would take place if $\cos x$ in (2) and (3) would be replaced by any potential $V(x)$ with unique non-degenerate maximum and minimum (a generic condition). We state the results just for the models (2) and (3) for the sake of simplicity.

The phase portraits of Hamiltonians h_ε and h'_ε defined above are shown in Figures 1 and 2 for different values of the parameter ε. The interesting bifurcation value for us will be $\varepsilon = 1$, so henceforth we will assume that $\varepsilon > 0$. Both Hamiltonians (2) and (3) are integrable but do not have global action-angle coordinates (as they would for $\varepsilon = 0$) –namely, they have separatrices which create "eyes" of width $O(\sqrt{\varepsilon})$ in the (x, y)-coordinates. Also, they possess hyperbolic critical points at

$$\begin{cases} ((2k+1)\pi, 0) \quad \text{and} \quad (2k\pi, 1) & \text{for } h_0 \\ (2k\pi, 2k'\pi) \quad \text{and} \quad ((2k+1)\pi, (2k'+1)\pi) & \text{for } h'_0 \end{cases} \tag{4}$$

for every $k, k' \in \mathbb{Z}$. The two Hamiltonians undergo a bifurcation (sometimes called a *heteroclinic reconnection*) when the energy levels of the two families of hyperbolic points coincide, namely for $\varepsilon = 1$.

For $0 < \varepsilon < 1$, the separatrices attached to the hyperbolic critical points are graphs over the x direction. This implies that there are invariant curves separating the saddles having different y components. At the bifurcation value, the net of separatrices changes its topology by creating heteroclinic connections between saddles with different y component. In particular, all smooth invariant curves separating the two resonance eyes of h_ε break down, and the Hamiltonian h'_ε does not have any invariant smooth curve over the x- or y-axes.

Consider the perturbed Hamiltonians

$$\begin{cases} h_{\varepsilon,\mu}(x,y,t) & = \frac{y^2}{2} - \frac{y^3}{3} - \varepsilon \cos x + \mu f(x,y,t) \\ h'_{\varepsilon,\mu}(x,y,t) & = \cos y - \varepsilon \cos x + \mu f(x,y,t) \end{cases} \tag{5}$$

where $f(x,y,t)$ is a $\mathcal{C}^\infty$ time-periodic perturbation, 2π-periodic in t (and possibly, but not necessarily in x and y). The parameter ε measures the size of resonant eyes, while μ measures the distance to the integrable approximations h_ε and h'_ε. A Hamiltonian of the form (1) is obtained by taking $\mu \lesssim \varepsilon$. The goal of the present work is to illustrate Chirikov's criterion within a thin region in the parameter plane, defined by $|\varepsilon - 1| \lesssim \mu$.

For μ small enough, $h_{\varepsilon,\mu}$ and $h'_{\varepsilon,\mu}$ have hyperbolic periodic orbits ε-close to the saddles of h_ε and h'_ε respectively. Melnikov Theory [18] implies that the separatrices of h_ε and h'_ε usually break down.

Lemma 1. *Fix $\varepsilon > 0$. For a generic f there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ all the separatrices of the Hamiltonians h_ε and h'_ε break down and the resulting invariant manifolds intersect transversally.*

Note that this result is significantly different for $\varepsilon \neq 1$ and $\varepsilon = 1$ since the separatrices for h_ε are different in both cases. Moreover, since we are interested in ε close to 1 and depending on μ, one can also prove the following more precise lemma, which is also a consequence of Melnikov Theory (*ibid.*).

Lemma 2. *Fix $\delta > 0$ small. For a generic f there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ and $\varepsilon \in [1-\delta, 1+\delta]$ all the separatrices of the Hamiltonians h_ε and h'_ε break down and the corresponding invariant manifolds intersect transversally.*

As a consequence, for $\varepsilon > 0$ (dependent or independent of μ) all homoclinic separatrix connections associated with the periodic orbits of $h_{\varepsilon,\mu}$ and $h'_{\varepsilon,\mu}$ split. Nevertheless, it says nothing about possible heteroclinic connections between different periodic orbits.

Proposition 1. *There exist constants $C_1, C_2, \mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$,*

1. *for $\varepsilon < C_1 < 1$, heteroclinic connections between the periodic orbits μ-close to the the saddles* (4) *with different y-component is not possible.*

2. *for $\varepsilon > 1 - C_2\mu$, $h_{\varepsilon,\mu}$ possesses transversal heteroclinic connections between the same periodic orbits as there are for the case $\varepsilon = 1$ and $\mu = 0$.*

Statement 1 is a direct consequence of KAM Theorem. Indeed, the Hamiltonian $h_{\varepsilon,\mu}$ possesses KAM invariant curves which are a graph over the base $y = 0$ and "separate" the saddles in (4) with different y-component. Statement 2 of this proposition is a direct consequence of Lemma 2 and classical perturbative arguments.

This result can be seen as the process of overlapping resonances in a non-integrable Hamiltonian system. In Regime 1 KAM curves prevent overlapping between the considered resonances: dynamics is confined between the invariant curves. In contrast, in Regime 2 all KAM curves break down and there is overlapping between resonances.

Note that here we are only considering the strongest resonances, corresponding to $\dot{x} = 0$. Certainly, the Hamiltonians $h_{\varepsilon,\mu}$ and $h'_{\varepsilon,\mu}$ possess many more at $\dot{x} \in \mathbb{Q}$ but they are much weaker (so we would need μ much smaller in order to apply the same arguments).

Proposition 1 has several consequences. Constant $C_2 > 0$ refers to Proposition 1.

Corollary 1. *For $\varepsilon > 1 - C_2\mu$ and $\mu > 0$ small enough, both $h_{\varepsilon,\mu}$ and $h'_{\varepsilon,\mu}$ have a compact invariant subset carrying symbolic dynamics with random excursions in the y direction, of amplitude uniform with respect to both μ and ε.*

Indeed, consider for example the four saddle points $(\pm\pi, \pm\pi)$ of h'_ε. In the neighborhood of these equilibria, a classical construction leads to a subshift of four symbols by using the concatenation of heteroclinic connections through $(0, 0)$, $(\pm 2\pi, 0)$ and $(0, \pm 2\pi)$. This subshift is not a full shift, since for instance one cannot go directly from (neighborhoods of) (π, π) to $(-\pi, -\pi)$ without passing through neighborhoods of either the other two, but these obvious obstructions are the only obstructions. One actually gets subshifts of arbitrarily many symbols by considering neighborhoods of correspondingly many saddles.

In Regime 1 of Proposition 1, one also certainly has symbolic dynamics but it is confined in the vertical direction by the KAM curves.

The Lambda lemma [25] implies the following for $h'_{\varepsilon,\mu}$.

Corollary 2. *Let $y_+ > y_-$, $\mu > 0$ small enough and $\varepsilon > 1 - C_2\mu$. The Hamiltonian $h'_{\varepsilon,\mu}$ has orbits which travel from $y = y_-$ to $y = y_+$. Moreover, one can achieve such transition in time $T \sim (y_+ - y_-)\,|\ln \mu|$.*

This behavior is not possible in Regime 1 due to KAM curves. Note that this behavior is still not possible for the Hamiltonian $h_{\varepsilon,\mu}$ for $\mu > 0$ small enough and $\varepsilon > 1 - C_2\mu$ since it has invariant curves surrounding the two overlapped resonances.

3 Numerics

So-described instabilities easily show numerically; see Figure 3. Despite the exponential divergence of solutions, approximate computations in the above two regimes are justified by the Lambda lemma, which entails that computed pseudo-orbits are shadowed by true orbits of $h'_{\varepsilon,\mu}$.

Conflicts of interest

The authors have none.

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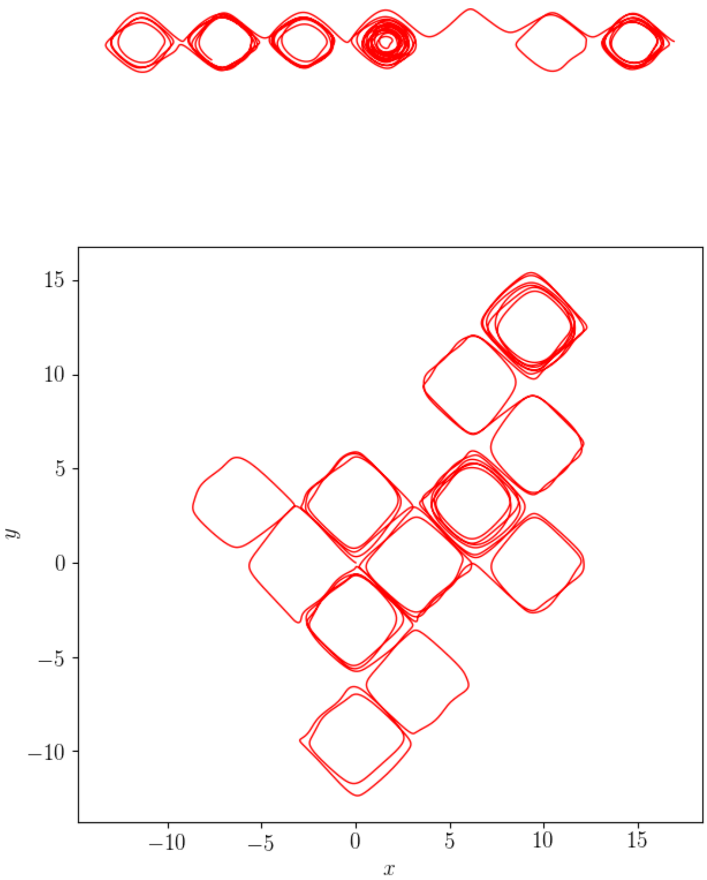

Figure 3: Examples of unstable orbits of $h'_{\varepsilon,\mu}(x, y, t) = \cos(y) - \varepsilon \cos(x) + \mu \cos(x + 2y + t)$, for $t \in [0, 500]$. On the top, $\varepsilon = 0.8$, $\mu = 0.1$, initial condition close to $(0, 10^{-1})$. On the bottom, $\varepsilon = 0.99$, $\mu = 0.1$, initial condition close to $(0, 10^{-2})$.

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