The N-body problem

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Summary.

We introduce the N-body problem of mathematical celestial mechanics, and discuss its astronomical relevance, its simplest solutions inherited from the two-body problem (called homographic motions and, among them, homothetic motions and relative equilibria), Poincaré's classification of periodic solutions, symmetric solutions and in particular choreographies such as the figure-eight solution, some properties of the global evolution and final motions, Chazy's classification in the three-body problem, some non-integrability results, perturbations series of the planetary problem and a short account on the question of its stability.

Contents

1. Introduction

The problem is to determine the possible motions of N point particles of masses m_1, ..., m_N, which attract each other according to Newton's law of inverse squares. The conciseness of this statement belies the complexity of the task. For although the one and two body problems were completely solved by the time of Newton by means of elementary functions, no similar solution to the N-body problem exists for $N \geq 3$.

The N-body problem is intimately linked to questions such as the nature of universal attraction and the stability of the Solar System. In the introduction of the *New Methods of Celestial Mechanics* [147], Poincaré suggested that it aims at solving "this major question to know whether Newton's law alone explains all astronomical phenomena". But since the N-body problem ignores such crucial phenomena as tidal forces and the effects of general relativity, this model is now known to be quite a crude approximation for our Solar System. So it is not useless in this introduction to give some brief account of how the N-body problem has become a central piece of celestial mechanics and remains so. For further background, we refer to Chapters 1 and 13 of this volume.

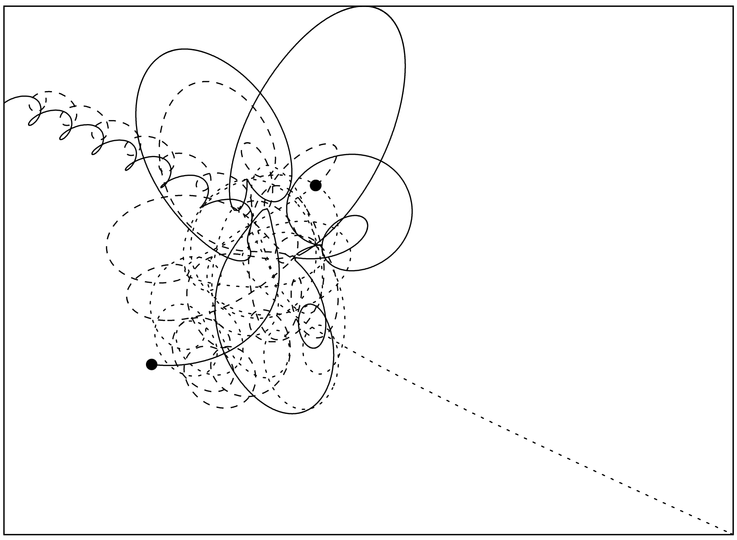

Figure 1. A solution of the plane three-body problem, starting from a double collision and leading to a hyperbolic escape

Hook's and Newton's discovery of universal attraction in the XVII century dramatically modified our understanding of the motion of celestial bodies [2, 10, 12, 137, 140]. This law masterly reconciles two seemingly contradictory physical principles: the principle of inertia, put forward by Galileo and Descartes in terrestrial mechanics, and the laws of Kepler, governing the elliptical motion of planets around the

Sun. In an additional *tour de force*, in his *Philosophiae naturalis principia mathematica*, Newton also estimated the first order effect on Mars of the attraction of other planets.

The unforeseen consequence of Hook's and Newton's discovery was to question the belief that the solar system be stable: it was no longer obvious that planets kept moving immutably, without collisions or ejections. And symmetrically, the question remained for a long time, whether universal attraction could explain the irregularities of motion, due to the mutual attraction of the various celestial bodies, observed in the past. A two-century long competition started between astronomers, who made more and more precise observations, and *geometers*, who had the status and destiny of Newton's law in their hands. Two main mysteries kept the mathematical suspense at its highest: the motion of the Moon's perigee, and the shift of Jupiter's and Saturn's longitudes, revealed by the comparison between the observations of that time and those which Ptolemy had recorded almost two thousand years earlier. The first computations of Newton, Euler and others were giving wrong results [37, 59]. But infinitesimal calculus was in its infancy and geometers, at first, lacked the necessary mathematical apparatus to understand the long-term influence of mutual attractions.

Regarding the Moon's perigee, Clairaut and d'Alembert understood that the most glaring discrepancy with observations could be explained by higher order terms [37, 59]. Thus the works of Euler, Clairaut, d'Alembert and others in the XVIII century constituted the Newtonian N-body problem as the description of solutions of a system of ordinary differential equations (see section 2 below). The problem was given a major impulse when Lagrange transformed mechanics and dynamics into a branch of mathematical analysis (e.g. [92]), laying the foundations of differential and symplectic geometry [176]. In his study of Jupiter's and Saturn's motions, Laplace found approximate evolution equations, describing the average variations of the elliptical elements of the planets. These variations are called *secular* because they can be detected only over a long time interval, typically of the order of a century (= *secular* in Latin). Laplace computed the secular dynamics at the first order with respect to the masses, eccentricities and inclinations of the planets. His analysis of the spectrum of the linearized vector field, at a time when this chapter of linear algebra did not exist, led him and Lagrange to a resounding theorem on the stability of the solar system, which entails that the observed variations in the motion of Jupiter and Saturn come from resonant terms of large amplitude and long period, but with zero average [93, p. 164]. We are back to a regular –namely, quasi-periodic (or *conditionally periodic*, according to the terminology of some authors)– model, however far it is conceptually from Ptolemy's ancient epicycle theory. Yet it is a mistake, which Laplace made, to infer the topological stability of the planetary system, since the theorem deals only with a truncated problem (see section 10, and Chapter 13 of this volume).

Around that time Euler and Lagrange found two explicit, simple solutions of the three-body problem, called *relative equilibria* because the bodies rigidly rotate around the center of attraction at constant speed [92]. These solutions, where each body moves as if it were attracted by a unique fictitious body, belong to a larger class of motions, called *homographic*, parametrized by the common eccentricity of bodies ; see section 4, and Chapter 5 of this volume. Some mathematical and more global questions started to compete with the purely initial astronomical motivations. Recently, many new periodic orbits have been found, which share some of the discrete symmetries of Euler's and Lagrange's orbits in the equal-mass problem; see [167], or, in this Chapter, section 6.

The theory of the Moon did not reach a satisfactory stage before the work of Adams and Delaunay in the XIX century. Delaunay carried out the Herculean computation of the secular dynamics up to the eighth order of averaging, with respect to the semi major axis ratio; as already mentioned, the secular dynamics is the slow dynamics of the elliptic elements of the Keplerian ellipses of planets and satellites. The first order secular Hamiltonian is merely the gravitational potential obtained by spreading the masses of planets and satellites along their Keplerian trajectories, consistently with the third Kepler law. Delaunay mentioned *un résultat singulier*, already visible in Clairaut's computation: according to the first order secular system, the perigee and the node describe uniform rotations, in opposite directions, with the *same* frequency [62]. This was to play a role later in the proof of Arnold's theorem (see Chapter 6 of this volume), although higher order terms of large amplitude destroy the resonance.

At the same time as Delaunay, Le Verrier pursued Laplace's computations, but questioned the astronomical relevance of his stability theorem. In the XIX century, after the failure of formal methods due to the irreducible presence of small denominators in perturbation series generally leading to their divergence, Poincaré has drawn the attention of mathematicians to qualitative questions, concerning the structure of the phase portrait rather than the analytic expression of particular solutions, of the N-body problem. In particular, Bruns [25, 88] and Poincaré in his epoch-making treatise *New Methods of Celestial Mechanics* [147] gave arguments against the existence of first integrals other than the energy and the angular momentum in the 3-body problem (see section 9 below).

Some facts like the anomalous perihelion advance of the planet Mercury could only be explained in 1915 by Einstein's theory of general relativity [101, 141]. Classical dynamics thus proved to be a limit case of, already inextricably complicated but simpler than, Einstein's infinite dimensional field equations.

On the positive side, Poincaré gave a new impulse to the perturbative study of periodic orbits. Adding to the work of Hill and cleverly exploiting the symmetries of the three-body problem, he found several new families, demanding a classification in terms of *genre*, *espèce* and *sorte* (genre, species and kind) [147, Chap. III]; see section 5 below. In the XX century, followers like Birkhoff, Moser, Meyer have developed a variety of techniques to establish the existence, and study the stability, of periodic solutions in the N-body problem, and more generally in Hamiltonian systems: analytic continuation (in the presence of symmetries, first integrals and other degeneracies), averaging, normal forms, special fixed point theorems, symplectic topology. Broucke, Bruno, Hénon, Simó and others have quite systematically explored families of periodic orbits, in particular in the Hill (or lunar) problem [1, 76, 82, 85, 115, 131, 132, 160, 152, 167, 182].

Regarding perturbation series, a stupendous breakthrough came from Siegel and Kolmogorov, who proved that, respectively for the linearization problem of a one-dimensional complex map and for the perturbation of an invariant torus of fixed frequency in a Hamiltonian system, perturbation series do converge, albeit non uniformly, under some arithmetic assumption saying that the frequencies of the motion are far from resonances [90, 164]. Siegel's proof overcomes the effect of small denominators by cleverly controlling how they accumulate, whereas Kolmogorov uses a fast convergence algorithm, laying the foundations for the so-called Kolmogorov-Arnold-Moser theory; see section 10, or Chapters 3, 6 and 13 of this volume. See [8, 22, 24, 48, 49, 72, 60, 143, 150, 162, 163] for further background, references and applications to celestial mechanics.

Two discoveries have led to another shift of paradigm. First, came the discovery of exoplanets in the early 1990's [161]. This confirmation of an old philosophical speculation has sustained the interest in extraterrestrial life. Many of these exoplanets have larger eccentricities, inclinations or masses (not to mention brown dwarfs), or smaller semi major axes, than planets of our solar system–and there seems to be billions of them in our galaxy alone. Are such orbital elements consistent with a stable dynamics? This wide spectrum of dynamical forms of behavior has considerably broadened the realm of relevant many-body problems in astronomy, and renewed interest in the global understanding of the many-body problems, far from the so-called planetary regime (with small eccentricities, inclinations and masses), and possibly with important tidal or more general dissipating effects (see [28]).

The second discovery is mathematical. Nearly all attempts to find periodic solutions of the N-body problem by minimizing the action functional had failed until recently because collisions might occur in minimizers, as Poincaré had pointed out [148]. Indeed, the Newtonian potential is weak enough for the Lagrangian action to be finite about collisions. In 1999 Chenciner-Montgomery overcame this difficulty and managed to prove the existence of a plane periodic solution to the equal-mass three-body problem, earlier found by Moore numerically, with the *choreographic* symmetry –a term coined by Simó, meaning that the bodies chase each other along the same closed curve in the plane. After this breakthrough, many symmetric periodic solutions have been found, theoretically and numerically. See section 6.

For most of the topics in this Chapter, it is only possible to outline major results. Further references with more precise statements and proofs, strongly recommended and which we have largely used, are mentioned at then end in the section *Related Chapters.*

2. Newton's equations and their symmetries

The motion of N bodies is assumed to be governed by Newton's equations

$$\ddot{x}_j = \sum_{k\neq j} m_k \frac{x_k - x_j}{\|x_k - x_j\|^3}, \quad j = 1, ..., N, \tag{1}$$

where $x_j \in E = \mathbb{R}^d$ is the position of the j-th body in the d-dimensional Euclidean space, $\ddot{x}_j$ its second time-derivative, m_j its mass, and $\|\cdot\|$ the Euclidean norm; the Euclidean scalar product of x_j and x_k will be denoted be $x_j \cdot x_k$. We have conveniently chosen the time unit so that the universal constant of gravitation, which is in factor of the right hand side, equals 1. The space dimension d is usually assumed less than or equal to three, but larger values may occasionally prove worth of interest.

Following Lagrange, the equations can be written more concisely

$$\ddot{x} = \nabla U(x),$$

where $x = (x_1, ..., x_N) \in E^N$ is the configuration of the N points, U is the *force function* (opposite of the gravitational potential energy)

$$U(x) = \sum_{j<k} \frac{m_j m_k}{\|x_j - x_k\|},$$

and ∇U is the gradient of U with respect to the *mass scalar product* on E^N (in the sense that $\langle dU(x), \delta x\rangle = \nabla U(x) \cdot \delta x$ for all $\delta x \in E^N$), the mass scalar product itself being defined by

$$x \cdot x' = \sum_{1\leq j\leq N} m_j (x_j \cdot x'_j).$$

Introducing the *linear momentum* $y = (y_1, ..., y_N) \in E^N$, with components $y_j = m_j \dot{x}_j$, these equations can be put into Hamiltonian form ($|y| = \sqrt{y \cdot y}$ for $y \in E^N$) by saying that

$$\dot{x} = \partial_y H \quad \text{and} \quad \dot{y} = -\partial_x H, \quad \text{where} \quad H(x, y) = \frac{|y|^2}{2} - U(x).$$

As a particular case of the general equations of dynamics, the equations of the N-body problem are invariant by the Galilean group, generated by the following transformations:

- shift of time: $t' = t + \delta t$ ($\delta t \in \mathbb{R}$)
- shift of positions : $x'_j = x_j + \delta x_0$ ($\delta x_0 \in E$, $j = 1, ..., d$)
- space isometry: $x'_j = R x_j$ ($R \in O(E)$, $j = 1, ..., N$)
- shift of velocities (or boost): $\dot{x}'_j = \dot{x}_j + \delta \dot{x}_0$ ($\delta \dot{x}_0 \in E$, $j = 1, ..., N$).

The first three symmetries preserve the Hamiltonian and according to Noether's theorem, entail the existence of first integrals, respectively:

- the energy $H \in \mathbb{R}$
- the linear momentum $P = \sum_j y_j \in E$
- the angular momentum $C = \sum_j x_j \wedge y_j \in E \wedge E$ (a bivector, which identifies to a scalar when $d = 2$ and to a vector when $d = 3$)

The invariance by velocity shifts is associated with the first integral $\sum_j (m_j x_j - y_j t)$, which depends on time. But let us stick to autonomous vector fields and integrals. This invariance has the additional consequence that the dynamics does not depend on the fixed value of the linear momentum (put differently, this value can be ajusted arbitrarily by switching to an arbitrary inertial frame of reference), whereas the dynamics does depend on the fixed value of the angular momentum (see the paragraph below on the reduction of Lagrange): for example, as a lemma of Sundman will show in section 7, total collision may occur only if the angular momentum is zero. In the sequel, we will assume that the linear momentum is equal to zero whenever needed.

In addition to these Galilean symmetries, there is a much more specific *scaling invariance* due to the fact that the kinetic energy $|y|^2/2$ and the force function $U(x)$ are homogeneous of respective degrees 2 and -1: if $x(t)$ is a solution, so is $x_\lambda(t) = \lambda^{-2/3} x(\lambda t)$ for any $\lambda > 0$; see [34].

2.1. **Reduction of the problem by translations and isometries.** The invariance by translations and isometries can be used to reduce the number of dimensions of the N-body problem. The first complete reduction of the three-body problem was carried out by Lagrange [92]. Albouy-Chenciner generalized it for N bodies in $\mathbb{R}^d$, which we now outline [3]. This reduction has proved efficient in particular in the study of relative equilibria [3, 120] or, recently, for numerical integrators [66].

But before fleshing out this construction, let us mention that a somewhat less elegant reduction, known as the "reduction of the node", was later obtained by Jacobi [87] for three bodies, generalized by Boigey [19] for four bodies and Deprit [63] for an arbitrary number N of bodies. Jacobi's reduction has the disadvantage of breaking the symmetry between the bodies and of being rather specific (at least in its usual form) to the three-dimensional physical space. Yet it has proved more wieldly in perturbative problems. Using this reduction, Chierchia-Pinzari have managed to show that the planetary system is non-degenerate in the sense of Kolmogorov at the elliptic secular singularity (see Chapter 6 of this volume, and [50, 51]).

Recall that $E = \mathbb{R}^d$ is the Euclidean vector space where motion takes place. Thus the state space (combined positions and velocities of the N bodies) is $(E^N)^2 =$

$\{(x,\dot{x})\}$. Let $(e_1,...,e_N,\dot{e}_1,...,\dot{e}_N)$ be the canonical basis of $\mathbb{R}^{2N}$. The map

$$\left(E^N\right)^2 \to E\otimes\mathbb{R}^{2N}, \quad (x,\dot{x})\mapsto \xi = \sum_{1\le i\le N}(x_i\otimes e_i + \dot{x}_i\otimes\dot{e}_i)$$

is an isomorphism, which allows us to identify a state $(x,\dot{x})$ to the tensor ξ.

The space E acts diagonally by translations on positions:

$$x+\delta x_0 = (x_1+\delta x_0,...,x_N+\delta x_0), \quad \delta x_0\in E,$$

and similarly (but separately: Newton's equations are invariant by separate translations on positions and on velocities) on velocities. The isomorphism above induces an isomorphism

$$\left(E^N/E\right)^2 \to E\otimes\mathcal{D}^2$$

where $\mathcal{D}$ is what Albouy-Chenciner call the *disposition space* $\mathbb{R}^N/\mathrm{Vect}\,(1,...,1)$. The space $E\otimes\mathcal{D}^2$ represents states up to translations, which we will still denote by the letter ξ.

Let ϵ denote the Euclidean structure of E. Pulled-back by ξ, it becomes a symmetric tensor

$$\sigma = {}^t\xi\cdot\epsilon\cdot\xi \in \left(\mathcal{D}^2\right)^{\otimes 2},$$

which characterizes ξ up to the isometry $\iota=\xi\cdot\sigma^{-1/2}$ of E (otherwise said, $\xi=\iota\cdot\sigma^{1/2}$ is the standard polar decomposition). Hence the space $(\mathcal{D}^2)^{\otimes 2}$ represents states up to translations and isometries, called *relative states*.

For the sake of concreteness, write

$$\sigma = \begin{pmatrix}\beta & \gamma-\rho\\ \gamma+\rho & \delta\end{pmatrix}$$

the block decomposition of σ, where $\beta,\gamma,\delta,\rho\in\mathcal{D}^{\otimes 2}$ and ${}^t\gamma=\gamma$ and ${}^t\rho=-\rho$. The space $\mathcal{D}^* = \{v^*\in\mathbb{R}^{N*},\ v^*\cdot(1,...,1)=0\}$ having no canonical basis, consider instead the generating family of covectors $e^*_{ij}=e^*_j-e^*_i$ and $\dot{e}^*_{ij}=\dot{e}^*_j-\dot{e}^*_i$ in $\mathcal{D}^{2*}$, where $(e^*_1,...,e^*_N,\dot{e}^*_1,...,\dot{e}^*_N)$ is the canonical basis of $\mathbb{R}^{2N*}$, and

$$\xi\cdot e^*_{ij} = x_j - x_i \quad\text{and}\quad \xi\cdot\dot{e}^*_{ij} = \dot{x}_j-\dot{x}_i.$$

The blocks β, γ and δ being symmetric, they are determined by the identities

$$\begin{cases}\beta\cdot(e^*_{ij}\otimes e^*_{ij}) = \|x_j-x_i\|^2\\ \delta\cdot(\dot{e}^*_{ij}\otimes\dot{e}^*_{ij}) = \|\dot{x}_j-\dot{x}_i\|^2\\ \gamma\cdot(e^*_{ij}\otimes\dot{e}^*_{ij}) = (x_j-x_i)\cdot(\dot{x}_j-\dot{x}_i)\\ \rho\cdot(e^*_{ij}\otimes\dot{e}^*_{kl}) = \frac{1}{2}\left[(x_j-x_i)\cdot(\dot{x}_l-\dot{x}_k)-(x_l-x_k)\cdot(\dot{x}_j-\dot{x}_i)\right],\end{cases}$$

involving only scalar products of mutual distances and velocities.

But what does the equation of dynamics become in this framework? Let $e_G^* = \frac{1}{M}(m_1 e_1^* + \cdots + m_N e_N^*)$. The bilinear form on $\mathbb{R}^N$

$$\sum_{1\leq i\leq N} m_i(e_i^* - e_G^*) \otimes e_i^* = \frac{1}{M} \sum_{1\leq i<j\leq N} m_i m_j {e_{ij}^*}^{\otimes 2},$$

with $M = m_1 + \cdots + m_N$, descends to the quotient by $(1, ..., 1)$ and induces the *mass scalar product* μ on $\mathcal{D}$. Newton's equation then reads, in $E^* \otimes \mathcal{D}^*$,

$$\epsilon \cdot \ddot{x} \cdot \mu = dU, \quad U = \sum_{i<j} \frac{m_i m_j}{\|x_i - x_j\|},$$

provided x is thought of as an element of $E \otimes \mathcal{D}$ —an absolute configuration. The force function factorizes through relative positions: $U(x) = \hat{U}(\beta)$, for it depends only on mutual distances. Since $d\hat{U}$ is a linear form on the space of symmetric tensors of $\mathcal{D}^{\otimes 2}$, it is itself symmetric. Hence,

$$dU \cdot x' = d\hat{U} \cdot ({}^t x' \cdot \epsilon \cdot x + {}^t x \cdot \epsilon \cdot x') = 2\epsilon \cdot x \cdot d\hat{U} \cdot x',$$

and the equation becomes

$$\ddot{x} = 2x \cdot A,$$

provided we define the *Conley-Wintner endomorphism* of $\mathcal{D}^*$ as $A = d\hat{U} \cdot \mu^{-1}$. It is then straightforward to deduce the reduced equation:

$$\begin{cases} \dot{\beta} = 2\gamma \\ \dot{\gamma} = ({}^t A \cdot \beta + \beta \cdot A) + \delta \\ \dot{\delta} = 2({}^t A \cdot \gamma + \gamma \cdot A) - 2({}^t A \dot{\rho} - \rho \cdot A) \\ \dot{\rho} = {}^t A \cdot \beta - \beta \cdot A. \end{cases}$$

We have already defined the energy as

$$H = \frac{\|y\|^2}{2} - U,$$

which induces a function on the phase space $\mathcal{P} = (E \otimes \mathcal{D}) \oplus (E \otimes \mathcal{D}^*)$, whose first term corresponds to absolute position x (modulo translations) and second term corresponds to absolute linear momentum y (acting on absolute velocities, modulo translations). Let ω be the natural symplectic form on (the tangent space of) $\mathcal{P}$:

$$\omega \cdot (x, y) \otimes (x', y') = \epsilon \cdot \left(x \cdot {}^t y' - x' \cdot {}^t y \right).$$

The vector field $X = (\dot{x}, \dot{y})$ associated with Newton's equation in $\mathcal{P}$ is nothing else than the Hamiltonian vector field of H with respect to ω: $i_X \omega = dH$. The inverse of ω (as an isomorphism $\mathcal{P} \to \mathcal{P}^*$) is a Poisson structure π, which can be pulled back by the transpose of the quotient by isometries, to a degenerate Poisson structure $\bar{\pi} \in \mathcal{D}^{\otimes 2}$. The symplectic leaves of $\bar{\pi}$ are the submanifolds obtained by fixing the rank of σ and the conjugacy invariants of the endomorphism $\omega_{\mathcal{D}} \cdot \sigma$ (the invariants of the angular momentum), where ω_μ stands for the symplectic form $\omega_\mu \cdot (u, v) \otimes (u', v') = \mu \cdot (u \otimes v' - u' \otimes v)$ of $\mathcal{D}^2$.

3. Some limit problems of particular importance in astronomy

3.1. **The planetary problem.** In the Solar System, the mass of the Sun is significantly larger than that of the planets, since Jupiter, the heaviest planet, is about 1000 times smaller than the Sun. So it has become customary to call *Planetary Problem* the asymptotic case of the N-body problem when the attraction of one of the bodies, say, the first one of mass m_1, called the Sun, dominates the attraction of all the others, called the Planets. This is achieved by assuming that the mass ratios m_k/m_1 are small ($k = 2, ..., N$) and that mutual distances of planets are lower bounded. Let $\epsilon = \max_{2\le j\le N} m_j/m_1$ (so, $\epsilon \simeq 1/1000$ for our Solar System).

Then it is natural to use heliocentric coordinates

$$X_j = x_j - x_1, \quad j = 2, ..., N$$

(this change of coordinates amounts to switching to the non-inertial frame of reference defined by the center of mass of the Sun). The motion of the Sun can easily be recovered by assuming that the (fixed) center of mass

$$\frac{1}{\sum_j m_j} \sum_j m_j x_j,$$

which nearly agrees with the Sun x_1, is at the origin. In these coordinates Newton's equations read

$$\ddot{X}_j = -m_1 \left(\frac{X_j}{\|X_j\|^3} + \sum_{k\notin\{1,j\}} \frac{m_k}{m_1} \frac{X_k - X_j}{\|X_k - X_j\|} \right), \quad j = 2, ..., N.$$

Since the first term of the right hand side dominates all of the others, unsurprisingly the vector field appears as the perturbation of $N-1$ uncoupled Kepler problems, as if each planet underwent the only attraction of a fixed, fictitious center of attraction located at the origin.

The perturbative character of this system is perceived more easily in the Hamiltonian formalism, where all the dynamics is determined by a unique function, the Hamiltonian

$$H = \sum_j \frac{\|y_j\|^2}{2m_j} - \sum_{j<k} \frac{m_j m_k}{\|x_j - x_k\|};$$

Newton's equations are indeed equivalent to Hamilton's equations

$$\dot{x}_j = \partial_{y_j} H, \quad \dot{y}_j = -\partial_{x_j} H.$$

Symplectically lift the above heliocentric coordinates to the phase space by setting

$$X_1 = x_1, \quad X_j = x_j - x_1, \quad j = 2, ..., N$$

and

$$Y_0 = y_0 + \cdots + y_N, \quad Y_j = y_j, \quad j = 2, ..., N.$$

Since Y_0 is the total linear momentum, it can be assumed to be equal to 0 without loss of generality. Moreover, H being invariant by translations, it does not depend on X_1. Let us define the auxiliary masses $\mu_j^{-1} := m_j^{-1} + M_j^{-1} \simeq m_j^{-1}$ and $M_j = m_1 + \cdots + m_{j-1} \simeq m_1$ (as ϵ tends to 0). The obtained Hamiltonian, seen as a function of $(X_2, ..., X_N, Y_2, ..., Y_N)$:

$$H = \sum_{2\leq j} \left(\frac{\|Y_j\|^2}{2\mu_j} - \frac{\mu_j M_j}{\|X_j\|} \right) + \sum_{2\leq j<k} \left(-\frac{m_j m_k}{\|X_j - X_k\|} + \frac{Y_j \cdot Y_k}{m_0} \right),$$

is indeed the sum of a Keplerian part (itself the sum of $N-1$ uncoupled fictitious Kepler problems) and a *perturbing function*. In the limit when ϵ tends to zero, the Keplerian part is of the order of ϵ, whereas the perturbing function is of the order of ϵ^2, provided we assume that the linear momenta Y_j are of the order of ϵ —a natural hypothesis, if we recollect that $Y_j = m_j \dot{x}_j$.

During a fixed time interval, each planet describes an approximate Keplerian ellipse, with precision $O(\epsilon)$. But on a longer time interval, say of order $1/\epsilon$ or more, it is an extraordinarily complicated question to determine the effect of the mutual attractions of the planets. This long term dynamics is called *secular* and can be seen as a dynamical system of the space of $(N-1)$-uples of Keplerian ellipses with a fixed focus, which slowly deform and rotate in space.

Outside Keplerian resonances, the effect of the mutual attractions seems to average out and the secular dynamics is well described by simply averaging the perturbation in the phase space over the Keplerian tori of dimension $N-1$. Note however that the unperturbed, purely Keplerian, dynamics is dynamically degenerate because all its bounded orbits have the same number of frequencies as there are planets, whereas symplectic geometry would allow for quasiperiodic motions with $3N-4$ frequencies (see Chapters 3 and 6). This degeneracy is at the source of many difficulties in celestial mechanics, and is specific to the Newtonian potential $1/r$ and to the elastic potential r^2, as Bertrand's theorem asserts [11] (see [75] for an alternative proof from the point of view of normal forms).

If two or more planets have resonant Keplerian motions, then they will regularly find themselves in the same relative position. So, their mutual attraction, instead of averaging out, will tend to pile up. The obtained motions are generally unstable, and very different from Keplerian motions, but even more difficult to describe mathematically.

This problem will be taken up in section 10. For a further study using all the apparatus of perturbation theory of Chapter 3 and the assistance of computers, we refer to to Chapters 6 and 13.

3.2. **The Lunar and well-spaced problems.** In our Solar System, the ratio of the distances Moon-Earth and Moon-Sun is approximately 0.003. So the motion of the Moon can be studied by taking into account primarily the attraction of the Earth, and then the perturbation due to the Sun.

Hence there is another much studied, important, integrable limit case of the N-body problem. Without making any assumption on the masses (as opposed to the asymptotics of the planetary problem), let us assume that for all $j = 3, ..., N$ the body j is very far from the bodies $1, ..., j-1$. When $N = 3$, the system mimics the $Earth - Moon - Sun$ problem, and is thus called the *Lunar problem*, or *Hill's problem* due to the import work of the American mathematician and astronomer G. W. Hill [84]; it can also be a good model for a Sun-Jupiter-asteroid system, where the asteroid may belong for instance to the Asteroid Belt located between Mars and Jupiter.

Here it is even more of a necessity to switch to the Hamiltonian formalism because the integrable limit is subtler. Let $M_j = m_1 + \cdots + m_{j-1}$ be some new auxiliary masses. Define the symplectic *Jacobi coordinates* by referring each planet j to the center of mass of the Sun and the $j-2$ preceding inner planets:

$$X_1 = x_1, \quad X_j = x_j - \sum_{1<k<j} \frac{m_k}{M_{j-1}} x_k$$

and

$$Y_j = y_j + \frac{m_j}{M_j} \sum_{k>j} y_k.$$

This new change of coordinates reduces to switching to new frames of reference, which differ from one body to another!, but this is a strength of the Hamiltonian formalism to permit this, as long as the adequate changes are made for the linear momenta. See [149] or, more recently, [70], for a more thorough discussion of these coordinates. Additionally define the masses μ_j by the equality

$$\frac{1}{\mu_j} = \frac{1}{m_j} + \frac{1}{M_j}.$$

The Hamiltonian becomes

$$H = \sum_{1<k} \frac{\|Y_k\|^2}{2\mu_k} - \sum_{1<j<k} \frac{m_j m_k}{\|X_k + \sum_{j<a<k} \frac{m_a}{M_a} X_a - X_j\|}.$$

Under our assumption, $\|X_1\| \ll \|X_2\| \ll \cdots \ll \|X_n\|$. So, for the k-th body, the attraction of the bodies $X_{k+1}, ..., X_N$ is arbitrarily small. For a fixed k, we have

$$\sum_{j<k} \frac{m_j m_k}{\|X_k + \sum_{j<a<k} \frac{m_a}{M_a} X_a - X_j\|} \simeq \sum_{j<k} \frac{m_k M_{k-1}}{\|X_k - X_j\|},$$

whence the Keplerian approximation.

3.3. **The plane restricted three-body problem.** A class of simplified non-integrable (or so-believed) N-body problems is the following particular case. Newton's equations (1) have a regular limit when one or several masses tend to zero. At the limit, the zero-mass bodies are attracted by, without themselves influencing, the other bodies, the so-called *primaries*. (The limit thus violates the principle of action and reaction in its usual form.) Such limits are called *restricted* problems. Note that the mass ratios of the zero-mass bodies are lost in the limit. Among those problems, the simplest one is the restricted three-body problem, where one zero-mass body undergoes the attraction of two primaries revolving on a Keplerian ellipse. Even simpler is the invariant subproblem where all bodies move in the plane and the primaries have circular motions. This is the *circular plane restricted three-body problem.* It is a good model for the system consisting of the Earth and the Sun (primaries), and the Moon. We will now describe some of its first properties [18, 44, 57, 147].

Call $1-\mu$ and μ the masses of the Sun and the Earth, which then lie at points of coordinates $(-\mu, 0)$ and $(1-\mu, 0)$ in the frame rotating with the primaries.

The motion of the zero-mass is the solution of an autonomous system of differential equations of the first order in dimension four. (In contrast, the elliptic restricted three-body problem is non-autonomous.) Thus it has two degrees of freedom. Its Hamiltonian in the rotating frame, or

$$H = \frac{\dot{x}^2 + \dot{y}^2}{2} - V(x, y)$$

with

$$V(x, y) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2}},$$

is a first integral as always, and is classically called the *Jacobi constant.* It turns out that there are no other first integrals (this statement is only conjectural as such, but wildely believed true, and we refer to [129], or section 9, for more details), so that the system is non-integrable. Actually, much of the complexity which can be detected in the general N-body problem is already present in the circular plane restricted three-body problem.

The projection on the configuration plane of the constant energy hypersurface $H = h$ defines the region

$$\mathcal{H}_h = \{(x, y) \in \mathbb{R}^2, V(x, y) + h \geq 0\}$$

of possible motions, called the *Hill region.*

The simplest case is when $h \ll -1$. $\mathcal{H}_h$ then consists of two disks centered around each primary. The Moon is bounded to stay within one of those two disks –a stability theorem due to Hill.

Yet the conservation of the energy does not prevent the Moon from colliding with the closest primary. The Jacobi constant being fixed, these collisions can be regularized through the following two steps:

- Compactify the energy hypersurface (diffeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{R}^2$) by adding a circle at infinity corresponding to all the possible directions of arrival to and departure from collisions, into a space diffeomorphic to the real projective space $\mathbb{R}P^3$. This compactification is universal [106, 175].
- Slow the time down in order to keep the speed finite. Note that this can be done on a given energy hypersurface while preserving the Hamiltonian

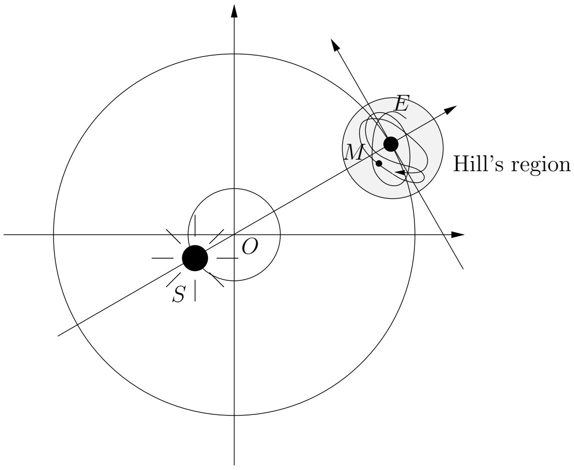

FIGURE 2. The Hill region around the Earth for very large negative Jacobi constant, in an inertial frame of reference

structure of the equations, by considering the Hamiltonian

$$\sqrt{(x-1+\mu)^2+y^2}(H-h)$$

instead of H (if one wants to regularize collisions with the Earth). When $H=h$, the time parametrization of orbits is changed but not the orbits themselves.

Conley [57] has shown that the problem then boils down to studying the first return map P of a global annulus of section A, which is diffeomorphic to $\mathbb{S}^1\times[-1,1]$ and transverse to the flow except on the boundary, which consists of two linked periodic orbits previously discovered by Hill.

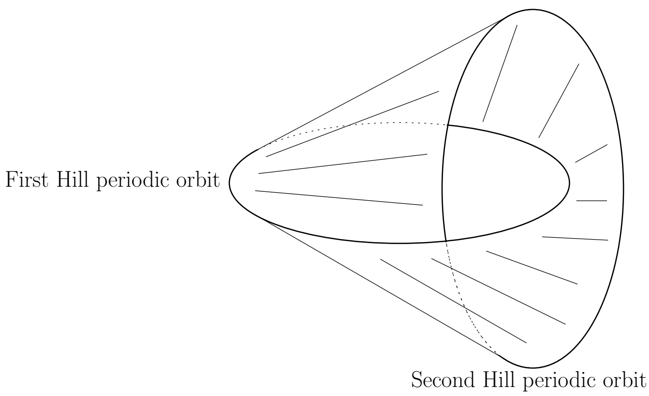

FIGURE 3. Annulus of section of the circular plane restricted three-body problem when $h \ll -1$

The annulus is the set of all possible positions of the perigees and the return map sends one position of the perigee to the next one. At the limit when the Sun in infinitely far away, P is an integrable conservative twist map of the annulus, i.e. a family of horizontal rotations by an angle which depends on the height in a monotone way. For the circular plane restricted three-body problem, P is a non-integrable conservative twist map, to which a huge mathematical apparatus applies: Poincaré's recurrence theorem, Birkhoff's fixed point theorem, Moser's invariant curve theorem and Aubry-Mather theory prove respectively the stability of motions in the sense of Poisson [11, Section 2.6.1], the existence of periodic motions of long period of the Moon around the Earth in the rotating frame, of quasi-periodic motions whose perigees have as envelope a smooth closed curve and of motions whose perigees have as envelope a Cantor set (closed curve with infinitely many holes). See [169] in particular. It is also possible to prove the existence of *stuttering orbits* as in figure 2, where the sign of the angular momentum changes infinitely many times but arbitrarily slowly, and the Moon undergoes infinitely many arbitrarily close encounters with the Earth [32].

When the Jacobi constant is not $\ll -1$ –which is the case for our Moon, the two connected components of the Hill region merge, and the dynamics is more complicated. In particular, there exist transit orbits which connect small neighborhoods of primaries [121]. Transit orbits can have important application in the design of low-cost space missions [58].

4. Homographic solutions

In this section we introduce the few explicitly known solutions of the N-body problem. We refer to section 5 of this volume for further study in the case $N = 3$, which is completely understood.

To begin with, there are some very special configurations $x \in E^N$, called *central configurations*. Although these configurations can lead to several remarkable kinds of motions according to the initial velocity of the bodies, one convenient way to characterize them is that whenever the bodies are left at rest ($\dot{x}(0) = 0$) in such a configuration $x(0)$, the subsequent solution is homothetic with respect to the center of mass $x_G = \frac{1}{M} \sum m_j x_j$:

$$x_j(t) - x_G(t) = \lambda(t) \left(x_j(0) - x_G(0)\right)$$

for some positive function of time $\lambda(t)$. When $\lambda \to 0$, the homothetic motion leads to a total collision. It is a striking theorem that actually any motion leading to a total collision is asymptotic to central configurations [35, 113, 157, 178], giving an additional importance to central configurations.

By differentiating the above equality twice with respect to time, we see that this condition is equivalent to saying that the accelerations be proportional to positions (relatively to the center of mass):

$$\sum_{k \neq j} m_j \frac{x_k - x_j}{\|x_k - x_j\|^3} = \lambda(0)\,(x_j - x_G)\,,$$

or,

$$\nabla U(x) = \lambda(0)(x - x_G).$$

Interpreting the scalar $\lambda(0)$ as a Lagrange multiplier, central configurations appear as the critical points of the restriction of the potential U to submanifolds of fixed *inertia* $I = \|x - x_G\|^2$.

Central configurations for $N = 3$ were known to Euler in the collinear case [68] and to Lagrange in the general case [92]. In particular, Lagrange has shown that for any masses there is a central configuration where the bodies are at the vertices of an equilateral triangle. A trivial generalization of the Lagrange central equilibrium for N equal masses is the regular N-gon.

A theorem of Moulton describes collinear central configurations. Namely, the latter correspond exactly to the numberings of the point masses on the line. So, there are $N!/2$ of them for N bodies [183].

But the general determination of central configurations turned out a difficult problem, still largely open today, despite some spectacular advances such as in [4] for $N = 5$. It is not even known whether there are finitely many central configurations for any given number $N \geq 6$ of bodies [29], even in the plane [172].

Along a homothetic motion, the force undergone by each body j is the same force as if all the other bodies were replaced by a fictitious body located at their center of mass. Each body thus follows a motion of an associated 2-body problem. There are other motions sharing this very specific property, together with the fact that at all times the configuration is central. Along such a solution, the bodies' configuration changes by similarities in E –the solution is thus called *homographic*– and the bodies have a Keplerian motion, with some common eccentricity $e > 0$.

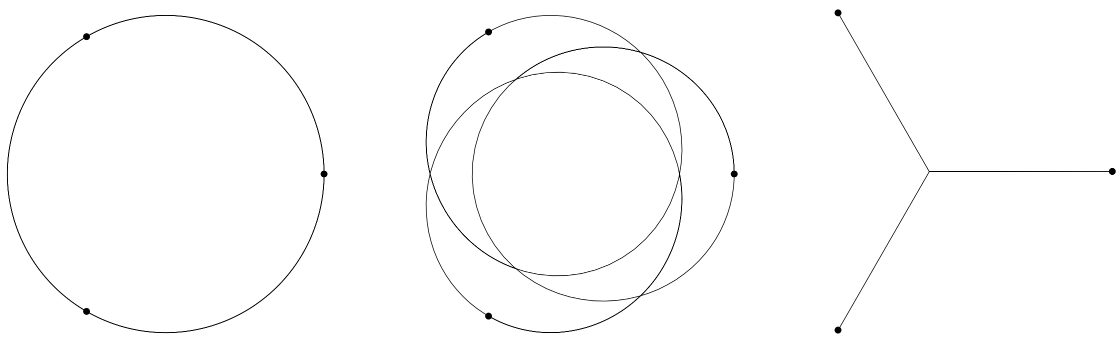

FIGURE 4. Three elements of the homographic family of the equilateral triangle with equal masses. From left to right: eccentricity = 0.0 (relative equilibrium), 0.5 and 1 (homothetic triple collision)

Among homographic motions, homothetic ones correspond to the particular case where $e = 1$. The other most interesting case is $e = 0$. Corresponding solutions, along which mutual distances are constant and the configuration remains congruent to itself, are called *relative equilibria*. In a frame of reference rotating at the adequate (necessarily constant) velocity, relative equilibria become fixed points and thus, in the absence of absolute fixed points in the N-body problem, play the role of organizing centers of the dynamics.

5. Periodic solutions

Next to homographic solutions, periodic solutions are among the simplest ones of the N-body problem. They have the additional advantage of being abundant: that periodic orbits are dense among bounded motions, as Poincaré conjectured, is still an open and highly plausible conjecture (see [151] for C^1-generic Hamiltonian systems and [79] for the restricted three-body problem with small mass ratio for the primaries). Also, periodic orbits play a decisive role in Poincaré's proof on non-integrability (see section 9 below). Poincaré famously commented:

> On peut alors avec avantage prendre [les] solutions périodiques comme première approximation, comme *orbite intermédiaire* [...]. Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable. [147, § 36]
> (One can then advantageously take periodic solutions as first approximation, as *intermediate orbit*. [...] What makes periodic orbits so valuable is that they are the only breach, so to speak, through which we can try to enter a place up to now deemed unapproachable.)

While proving the existence of a wealth of periodic orbits in the planetary problem, Poincaré felt the need to classify this zoology in sorts, genres and species. They are found by means of continuation arguments, either from the Keplerian approximation (where the mutual attraction of planets is neglected) or from the first order secular dynamics (corresponding to a first order normal form along Keplerian tori).

- Solutions of the *first sort* have zero inclination (which means that the bodies move in a plane) and the eccentricities of the planets are small. In the limit where the masses vanish, the orbits are circular, with rationally dependent frequencies.

  Solutions of the *second sort* still have zero inclination but finite eccentricities; in the limit one gets elliptic motions with the same direction of major semi-axes and conjunctions or oppositions at each half-period.

  In solutions of the *third sort*, eccentricities are small but inclinations are finite and the limit motions are circular but inclined.

- Solutions of the *second genre* are today called subharmonics: they are associated with and close to a given T-periodic solution of one of the three previous sorts and their period is an integer multiple of T.
- Solutions of the *second species* are particularly interesting: in the limit of zero masses, the planets follow Keplerian orbits until they collide and shift to another pair of Keplerian ellipses. See [107] for a complete proof of existence in the plane, symmetric case. In the spatial case, a full symbolic dynamics of such almost collision orbits has been constructed by Bolotin and MacKay: it implies the existence of solutions with an erratic diffusion of the angular momentum and a much slower one of the Jacobi constant [20, 21]. This is a beautiful example of the "breach" described by Poincaré, where periodic solutions are used to build solutions of a complicated kind.

For a recent account of the known periodic orbits of the three-body problem, we refer to the book of Meyer [115].

6. Symmetric periodic solutions

We will now focus on orbits which, in addition to being periodic, display discrete symmetries. A wealth of such orbits have been found theoretically in the past decade, following the foundational paper of Chenciner and Montgomery [45]. These orbits are usually found by means of variational methods, although occasionally some shooting or other method can be used.

A solution $x(t)$ ($t_0 \leq t \leq t_1$) of Newton's equations can be viewed as a critical point, in the class of paths with fixed ends, of the *(Lagrangian) action* functional

$$A(x) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt,$$

where L is the *Lagrangian* of the N-body problem, defined by

$$L(x, \dot{x}) = \sum_{1 \leq j \leq n} \frac{1}{2} m_j \|\dot{x}_j\|^2 + U(x).$$

According to the celebrated Maupertuis principle, a solution of energy (or Hamiltonian) h can also be viewed, up to a time reparametrization, as a geodesic of the Jacobi metric

$$ds^2 = 2(h + U) \left(\sum_{1 \leq j \leq N} m_j \|\dot{x}_j\|^2 \right) dt^2.$$

(This beautiful idea of incorporating the forces into the geometry can rightfully be seen as one of the key ideas having later led to the theory of General Relativity.) If $h \geq 0$, since $U > 0$, the Jacobi metric is non-degenerate and the study of its geodesics thus reduces to a standard problem of Riemannian geometry. Yet the case of negative energy is more interesting: in section 7, we will see that all the recurrent behavior, and periodic orbits in particular, can exist only if $h < 0$, in which case the motion takes place within Hill's region

$$\mathcal{H}_h := \{x \in E^N,\ U(x) + h \geq 0\}$$

(already mentioned in the particular case of Hill's problem in section 3) and the Jacobi metric is degenerate on the boundary of $\mathcal{H}_h$.

Finding periodic geodesics on a Riemannian manifold as length minimizers within a fixed non-trivial homotopy or homology class is commonplace. In contrast, all attempts to apply the same strategy to the three-body problem had failed because collisions might occur in minimizers, as Poincaré had pointed out [148]. Indeed, the Newtonian potential is weak enough for the Lagrangian action to be finite around collisions; this specific issue can be circumvented conveniently but somehow artificially by considering stronger potentials of interaction of the particles than the Newtonian potential.

Considering spaces of loops with large symmetry groups has permitted to rule out collisions without changing the potential of interaction. Using variational methods, in 1999 Chenciner-Montgomery managed to prove the existence of a plane periodic solution to the equal-mass three-body problem, whose symmetry group is a 12th-order subgroup of the symmetry group of the Lagrange equilateral triangle. In particular the bodies chase each other along a closed curve—such solutions have been named *choreographies* by Simó. This solution being eight-shaped, it has been called the *Eight* [45, 36, 124]. It had earlier been found numerically by Moore [125].

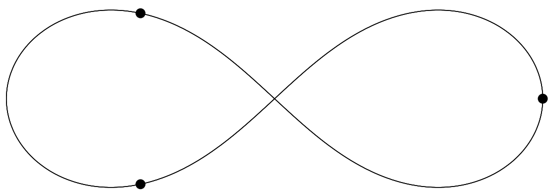

FIGURE 5. The Eight

Since then, Simó has searched the phase space for solutions with various symmetry groups quite systematically, and found a whole wealth of them [167]. Theoretical works, in particular from S. Terracini and her students, have also shown the existence of a large number of symmetric periodic orbits which minimize the Lagrangian action within their symmetry class [15, 14, 76]. And Marchal, helped by Chenciner, remarkably brought the first general answer to the question of collisions:

Theorem (Marchal-Chenciner [36, 104]). *Minimizers of the Lagrangian action (among all fixed-end paths) are collision-free.*

Marchal-Chenciner's theorem thus shows a subtle difference between Cauchy and Dirichlet boundary conditions in the many-body problem, and its proof gives the most powerful method to date for showing that minimizers avoid collisions.

Ferrario and Terracini have found the general, equivariant version of this theorem, thereby establishing the existence of a host of infinite families of choreographies [76].

At the Saari conference in 1999, Marchal realized that the Eight could be related to the equilateral triangle relative equilibria, through a Lyapunov family of spatial orbits, periodic in a rotating frame [41, 43, 103]. This family has been named $\mathcal{P}_{12}$, after the order of its symmetry group (figure 6); see a description of the beginning of this Lyapunov family on the side of the Lagrange relative equilibrium in Marchal's book [102].

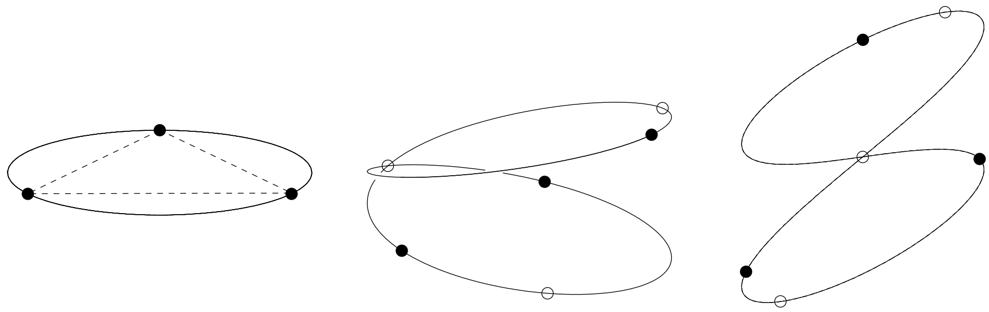

FIGURE 6. The P_{12} family, interpolating from the Lagrange relative equilibrium to the figure-eight solution of Chenciner-Montgomery

In fact, such a connection between relative equilibria and symmetric periodic orbits is a very general phenomenon, bringing light to the family tree of all the newly discovered periodic orbits [40, 42]. The second example is that of the Hip-Hop solution of Chenciner-Venturelli [46], which is similarly related to the square relative equilibrium (figure 7).

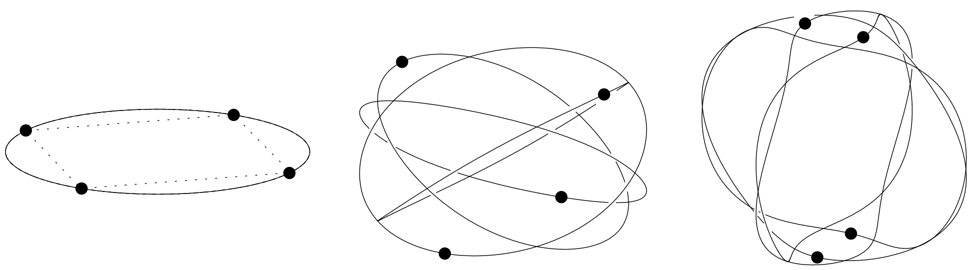

FIGURE 7. The Hip-Hop family, interpolating from the square relative equilibrium to the Hip-Hop solution of Chenciner-Venturelli

Relative equilibria themselves and, in turn, symmetric periodic solutions, have become intermediate orbits in the neighborhood of which local perturbation theory can be applied [43], with the possibility from this starting point to vary masses and explore the existence of asymmetric periodic or quasiperiodic solutions.

At this point, let us also mention some interesting solutions recently found by minimizing the Lagrangian action, without symmetry arguments, or not even periodic: they are the retrograde and prograde periodic orbits for various choices of masses [31], and the (non periodic) *brake solutions*, which end at the Hill boundary $\partial\mathcal{H}_h$, hence with zero-angular momentum, and for which a kind of symbolic dynamics exists [122].

7. Global evolution, collisions and singularities

7.1. Sundman's inequality.

The following functions play a fundamental role:

$$I = x \cdot x, \quad J = x \cdot \dot{x}, \quad K = \dot{x} \cdot \dot{x},$$

where we recall that $x \cdot y$ denotes the mass scalar product defined in section 2. They are respectively the *moment of inertia*, the half of its time derivative, and twice the kinetic energy. In particular,

$$H = \frac{K}{2} - U.$$

Note that

$$\dot{J} = \frac{\ddot{I}}{2} = \dot{x} \cdot \dot{x} + x \cdot \ddot{x} = \dot{x}^2 + x \cdot \nabla U(x).$$

Since U is homogeneous of degree -1, $x \cdot \nabla U(x) = -U(x)$. Hence, we get:

Proposition (Lagrange-Jacobi identity)**.**

$$\dot{J} = K - U = 2H + U = \frac{K}{2} + H.$$

This identity has important consequences. For example:

Corollary . *If the recurrent set is non empty, the energy is negative.*

Proof. Since $U > 0$, if $H \geq 0$ then J is increasing along trajectories (in other words, it is a strict Lyapunov function), preventing any recurrence in the dynamics. □

In particular, periodic or quasiperiodic behavior exists only for negative energies. We will see below that even bounded motions cannot take place if the energy is non negative.

Recall that we have defined the angular momentum as the bivector

$$C = \sum_{1 \leq j \leq N} m_j x \wedge \dot{x}.$$

For example, in dimension $d = 2$, C identifies to a scalar and the norm of C is merely the absolute value of this scalar. In dimension $d = 3$, C identifies to a vector of $\mathbb{R}^3$ by the formula

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and the norm of C is just the Euclidean norm of the latter vector.

Proposition (Sundman inequality [178]).

$$IK - J^2 \geq \|C\|^2.$$

Proof. The norm of the angular momentum is

$$\|C\| = (\mathcal{J}x) \cdot \dot{x},$$

where $\mathcal{J}$ is the possibly degenerate complex structure of E^N associated with C (first let $\mathcal{J}_0 = \sqrt{-C^2}^{-1} C$ be the associated complex structure on E, and call $\mathcal{J}$ the structure on E^N acting diagonally by $\mathcal{J}_0$ on each factor E; cf. [3]). Let $X = x/\|x\|$. The inequality we have to prove becomes:

$$\dot{x}^2 \geq (X \cdot \dot{x})^2 + ((\mathcal{J}X) \cdot \dot{x})^2 .$$

Since X and $\mathcal{J}X$ are orthogonal with respect to the mass metric, the right hand side is the square of the norm of the projection of $\dot{x}$ on the possibly degenerate complex line generated over $\mathbb{R}$ by X and $\mathcal{J}X$. So the inequality follows from the Cauchy-Schwarz inequality. □

Sundman's fundamental inequality is better understood having in mind the *Saari decomposition* of the velocity [3, 156, 158], into three components which are orthogonal to each other with respect to the mass metric:

$$\dot{x} = \dot{x}_h + \dot{x}_r + \dot{x}_d,$$

where

- $\dot{x}_h = J^2/I$ is the *homothetic velocity*, proportional to x
- $\dot{x}_r = \|C\|^2$ is the *rotational velocity*, corresponding to solid body rotation
- $\dot{x}_d$ is the *deformation velocity*, the remainder.

Sundman's inequality gives a lower bound on the kinetic energy, consisting in neglecting the deformation velocity.

Now, let

$$S = I^{-1/2}(J^2 + \|C\|^2) - 2I^{1/2}H$$

be the *Sundman function*. Using the facts that $\dot{I} = 2J$ and $\dot{J} = K/2 + H$, we see that its time derivative is

$$\dot{S} = I^{-3/2} J \left(IK - J^2 - \|C\|^2 \right).$$

Thus Sundman's inequality is equivalent to saying that S and I are both increasing, or both decreasing.

7.2. **Collisions and singularities.** Let $x(t)$ be a solution of the N-body problem. It has a *collision* at some time t_0 if at the limit when t tends to t_0 from below the positions of two or more bodies agree: $x_i(t_0^-) = x_j(t_0^-)$ for some $i \neq j$. The two extreme cases are when the collision is *binary*, if the mutual distance $\|x_i - x_j\|$ of only two bodies tend to zero, and when it is *total*, if all mutual distances tend to zero, or, equivalently, if the inertia I tends to zero.

In the two-body problem, collisions are only possible if the angular momentum is zero. Sundman's inequality expressed in terms of the Sundman function allows us to generalize this property to the N-body problem:

Lemma (Sundman)**.** *In case of a total collision, necessarily* $C = 0$*.*

Proof. Look at what happens before, but close to, a collision. Certainly, U tends to infinity, and so does $\ddot{I} = 4H + 2U$. Moreover, if the collision is total, $I > 0$ tends to 0. Since $\ddot{I}$ tends to infinity, $\ddot{I} > 0$, so $\dot{I} = 2J$ increases. So $\dot{I} < 0$ (otherwise I could not tends to 0). So, $I > 0$ decreases to 0, $J < 0$ increases to 0 and $K = 2H + 2U$ tends to $+\infty$.

Since I decreases, so does the Sundman function S. But in the expression

$$S = I^{-1/2}(J^2 + \|C\|^2) - 2I^{1/2}H,$$

if $C \neq 0$, the dominating term is $I^{-1/2}\|C\|^2$, which is increasing. Hence, $C = 0$. □

Triple collisions of the three-body problem have been regularized by blow-up by McGehee, and have thus become a privileged place to study the three-body problem. In particular, Moeckel has uncovered the complicated dynamical behavior which occurs in their neighborhood; their see [113, 118] and references therein.

We have seen that periodic and quasiperiodic solutions can exist only if the energy is negative. More generally, we have the following criterion for *stable* solutions.

Theorem (Jacobi)**.** *If a motion is bounded ($|x_j| \leq c$ for all j and all t, for some constant c) and bounded away from collisions ($\|x_j(t) - x_k(t)\| \geq c$ for all $j \neq k$ and all t, for some constant c), the energy is negative.*

Indeed, stable solutions in the sense of the statement are defined for all times because velocities too are bounded (because $\dot{x} \cdot \dot{x} = K = 2(H + U)$). Moreover, as in the corollary above, if the energy is ≥ 0, the inertia is strictly convex and thus cannot be bounded simultaneously below and above, which proves the theorem.

However, the criterion that the energy should be negative for stability is not sufficient, by far, as soon as the number of bodies is larger than 2. Indeed, it can be shown that close to a triple collision one of the bodies assumes an arbitrarily large velocity. This makes it possible, even with negative energy, for a body to escape to infinity while two other bodies asymptotically describe a Keplerian ellipse [113, 180].

Binary collisions are regularizable in the sense that, if t_0 is the collision time, as shown by Sundman the configuration $x = (x_1, ..., x_N)$ is locally a holomorphic function of the uniformizing variable

$$v(t) = \int_{t_0}^{t} \frac{dt}{\|r_i(t) - r_j(t)\|},$$

where i and j are the indices of the two colliding bodies. In the two-body problem, this variable is the true anomaly. In the N-body problem, this variable is equivalent to the true anomaly of the two colliding bodies, since in the neighborhood of the collision the two colliding bodies behave nearly as if they did not feel the influence of the other bodies. Since $t(v) - t_0$ has a zero at $v = 0$ of multiplicity 3, the configuration is also a holomorphic function of $(t - t_0)^{1/3}$. So it has a unique real analytic extension with respect to this variable, hence a unique Puiseux extension with respect to t, past t_0. In order to regularize all binary collisions simultaneously, the variable

$$v(t) = \int_{t_0}^{t} U(x(t))\, dt$$

should be used instead. Simultaneous double collisions (involving two or more pairs of bodies) are more complicated; some of them are regularizable though, but with a finite degree of differentiability, at least in some cases [109].

In the three-body problem, always assuming that the angular momentum is non-zero, Sundman has shown that solutions $x(v)$ are real holomorphic in a uniform strip $|\mathrm{Im}\, v| < \delta$ containing the real axis. Following Poincaré [144, 145], map this strip to the unit disc $\mathbb{D} : |\omega| < 1$ by the transformation

$$v \mapsto \omega = \frac{e^{\pi v/2\delta} - 1}{e^{\pi v/2\delta} + 1},$$

which sends the real axis onto the real line segment $]-1, 1[$. The configuration x becomes a holomorphic function on the unit disc $\mathbb{D}$. Hence all solutions of the three-body problem with non-vanishing angular momentum can be represented by series of ω converging uniformly over any compact subset of the open unit disk. These series of Sundman (later generalized to the N-body problem by Wang [181]) positively answer a question asked in an international competition honoring the 60th birthday of Oscar II, king of Sweden and Norway [13, 16, 64]. Yet, they converge so slowly that they have been completely useless, for practical computations as for any theoretical purpose.

Multiple collisions are more complicated than binary ones. Indeed, Chazy has shown that there exist solutions leading to a triple collision at time $t = 0$ which are infinite series of the form

$$x(t) = t^{2/3} \sum_{n \geq 0} a_n (-t)^{\alpha n}$$

for some real number α which is a non-constant algebraic function of the masses of the bodies [29]. These solutions were found as heteroclinic solutions positively asymptotic to the homothetic total collision, $-\alpha < 0$ being a characteristic exponent of the homothetic solution. When α is irrational, such a series has an isolated logarithmic singularity at $t = 0$; it has infinitely many analytic branches after the collision, but none of them is real, thus preventing any attempt to extend such a solution analytically past the collision.

A related question is to determine the nature of singularities of the N-body problem i.e., in which circumstance a solution might not extend past some finite time t_0. As Painlevé noticed in 1895 [139], for $N > 3$ a singularity need not correspond to non-regularizable collisions (although at least binary collisions must occur). The terminology is unfortunate since it has become customary to call such a singularity a *non-collision singularity*, although what is really meant is "a singularity whose collisions are binary". H. von Ziepel and H. J. Sperling have shown that such a non-collision singularity is equivalent to the motion in physical space becoming unbounded in finite time [177]. J. N. Mather and R. McGehee have established the existence of an uncountable Cantor set of solutions in the four-body problem, whose configuration becomes unbounded in finite time, with binary collisions accumulating at the singularity [112]. In the solutions they construct, a small mass body oscillates between a binary and a fourth particle. The commuting particle each time encounters the binary close to a triple collision, and is ejected with a larger and larger velocity, before overtaking and colliding with the fourth particle. Then it rebounds from the fourth particle with the elastic behavior characteristic of a binary collision, and so on. Saari has shown however that non-collision singularities of the four-body problem are improbable in both the sense of Lebesgue measure and of Baire category [155].

Finally, J. Xia has proved the existence of solutions with non-collision singularities in a symmetric 5-body problem, containing no binary collision prior to the singularity. McGehee's technique of blowing up the collision singularities was the major tool [184]; see also the paper of Gerver [78].

8. Final motions in the three-body problem

In the two-body problem, let $r = x_2 - x_1 \in E$. Every non-collision solution is defined for all times and belongs to one of the following three kinds according to its asymptotic behavior as $t \to +\infty$:

(1) elliptic (eccentricity < 1): $0 < \liminf r \leq \limsup r < +\infty$
(2) parabolic (eccentricity $= 1$): $\lim r = \infty$ and $\lim \dot{r} = 0$
(3) hyperbolic (eccentricity > 1): $\lim r = \infty$ and $\lim |\dot{r}| = c > 0$.

Moreover, the behavior in the past is analogous.

In the three-body problem, the classification gets more complicated and the symmetry of the past and future fails. Let $r_k = x_j - x_i \in E$ when $i < j < k$, with the cyclic convention that $1 < 2 < 3 < 1$, so that among other things we have $r_1 + r_2 + r_3 = 0$.

Theorem (Chazy [30]; see also [11, Chap. 2])**.** *Every non-collision solution of the three-body problem belongs to one of the following seven kinds, as $t \to +\infty$:*

(1) H *(hyperbolic)*

$$\lim r_i = \infty, \quad \lim |\dot{r}_i| = c_i > 0 \quad \textit{for all } i$$

(2) HP_k *(hyperbolic-parabolic)*

$$\lim r_i = \infty, \quad \lim |\dot{r}_i| = c_i > 0, \quad \lim |\dot{r}_k| = 0 \quad \textit{for all } i \neq k$$

(3) HE_k *(hyperbolic-elliptic)*

$$\lim r_i = \infty, \quad \lim |\dot{r}_i| = c_i > 0, \quad \sup |r_k| < \infty \quad \textit{for all } i \neq k$$

(4) PE_k *(parabolic-elliptic)*

$$\lim r_i = \infty, \quad \lim \dot{r}_i = 0, \quad \sup |r_k| < \infty \quad \textit{for all } i \neq k$$

(5) P *(parabolic)*

$$\lim |r_i| = \infty, \quad \lim \dot{r}_i = 0 \quad \textit{for all } i$$

(6) B *(bounded)*

$$\sup |r_i| < \infty \quad \textit{for all } i$$

(7) O *(oscillating)*

$$\limsup_t \sup_i |r_i| = \infty, \quad \liminf_t \sup_i |r_i| < \infty.$$

The terminology "elliptic/parabolic/hyperbolic" refers to the corresponding motions in the two-body problem. Also, note that for instance in the final motion HP_k, the constraint $r_1 + r_2 + r_3 = 0$ entails that two mutual distances have a hyperbolic behavior, and one a parabolic behavior.

Chazy knew examples of all those different kinds of motions, except for motions oscillating both in the past and in the future, whose existence was proved by Sitnikov in 1959 [133, 170, 99]. Alekseev has summarized the current state of knowlege of the various cases [5, 11], which we reproduce here. Columns correspond to the asymptotic behavior in the future (+), and rows in the past (−).

The first case is for positive energy:

	H^+	HE_i^+
H^-	Lagrange, 1772, isolated examples Chazy, 1922, measure > 0	*Partial capture* Measure > 0 Shmidt, 1947 (numerical) Sitnikov, 1953 (qualitative)
HE_j^-	*Complete dispersal* Measure > 0	$i = j$: Birkhoff, 1927 (measure > 0) $i \neq j$: *Exchange*, measure > 0 Bekker, 1920 (numerical) Alekseev, 1956 (qualitative)

The table below treats of the case of negative energy:

	HE_i^+	B^+	O^+
HE_j^-	$i = j$: Birkhoff, 1927 (measure > 0) $i \neq j$: *Exchange* Measure > 0 Bekker, 1920 (numerical) Alekseev, 1956 (qualitative)	*Complete capture* Chazy, 1929 and Merman, 1954 (measure $= 0$) Littlewood, 1952 Alekseev, 1968 ($\neq \emptyset$)	Chazy, 1929 and Merman, 1954, measure $= 0$ Alekseev, 1968, $\neq \emptyset$
B^-	*Partial dispersal* Symmetric to $HE_j^- \cap B^+$	Euler, 1772 Lagrange, 1772 Poincaré, 1892 Arnold, 1963, measure > 0	Littlewood, 1952, measure $= 0$ Aleskseev, 1968, $\neq \emptyset$
O^-	Symmetric to $HE_j^- \cap OS^+$	Symmetric to $B^- \cap OS^+$	Sitnikov, 1959, $\neq \emptyset$ Measure ?

There are conditions on the masses in several cases, which are not explicitely mentionned.

Some references which have not yet been cited are the works of Alekseev [5], Arnold [7], Birkhoff [18], Littlewood [98], Merman [114], Sitnikov [171, 170], cosmologist Shmidt [173, 174].

The partition given by this classification of the twelve-dimensional phase space of the three-body problem is poorly understood (see [11, 2.4.1]), except maybe in the isoceles invariant subproblem [168].

Chazy's classification emphasizes the positions and it is unknown whether there is a similar classification with respect to velocities. For example, in the three-body problem, there exist bounded motions (with positive measure) whose velocities oscillate [44, 71].

Some of the results of Birkhoff and Sundman have been generalized to the N-body problem par Marchal-Saari [105]. Other related results are: the existence of bounded motions in the planetary regime, according to Arnold's theorem (section 10, chap. 6 of this volume, and [7, 72]), and the fact that any two configurations of the N-body problem have at least one collision-free connection, due to the theorem of Marchal-Chenciner cited in section 6).

9. Non integrability

Every intuition we have of the N-body problem indicates that it is not integrable for $N \geq 3$. We now review some non-integrability results, which are both more difficult and weaker than what one would wish.

Bruns has proved the non-existence of first integrals which are algebraic with respect to positions and momenta, and which differ non-trivially from the classical first integrals [25, 88]. The result holds for any number $N \geq 3$ of bodies and any choice of the masses. Painlevé later showed that that it is enough to suppose algebraicity only in the momenta [139].

Poincaré proved that, in the N-body problem with $N \geq 3$, there is no new first integral which is analytic with respect to the elliptic elements and to the small masses (or even admit a formal expansion in the masses with analytic coefficients) [147, Volume II]: “Le problème [...] n’admet pas d’autre intégrale uniforme que celles des forces vives et des aires” (the problem has no uniform integral other than the energy and the angular momentum) [147, Chap. V, § 85]. Poincaré’s impossibility theorem, however beautiful it is, does not preclude that there could exist a first integral which depends only smoothly on the masses (see [183, p. 241] for a criticism of the shortcomings of the result)!

The strategy of proof of Poincaré is the following. Call H the Hamiltonian of the planetary problem and F a first integral. Expanding the equation $\{H, F\} = 0$ with respect to the small parameter ϵ (mass ratio) and then expanding the coefficients themselves in Fourier series, Poincaré shows that many Fourier coefficients of F must vanish at some well chosen resonances of H. Again, one of the difficulties is the dynamical degeneracy of H, whose limit when ϵ tends to 0 does not depend on all of the action variables.

Poincaré also uncovered the splitting of separatrices of a hyperbolic equilibrium point and the resulting entanglement (the interesting story of Poincaré’s mistake in the first version of his memoir [146] for king Oscar II, which later led him to this discovery, is told in [13, 16, 64]):

> On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n’est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous les problèmes de Dynamique où il n’y a pas d’intégrale uniforme et où les séries de Bohlin sont divergentes.
>
> [...] Cette remarque est de nature à nous faire comprendre [...] combien les transcendantes qu’il faudrait imaginer pour résoudre [le problème des trois corps] diffèrent de toutes celles que nous connaissons. [147, § 397–398]
>
> (One is struck by the complexity of this figure, which I will not even try to draw. Nothing is more appropriate to give an idea of the complexity of the three-body problem, and, in general, of those dynamical systems which do not have uniform integrals and where Bohlin series diverge.
>
> [...] This remark should make us understand to what extent the transcendants which we would have to imagine, to solve [the three-body problem], depart from all those we know.)

Figure 8 gives an idea of this figure which Poincaré did not dare to draw, for the standard map introduced by Chirikov [52, 96] as a universal model of chaotic layers around a separatrix of a non integrable twist map. In two dimensions, this kind of dynamical non-integrability implies the non-existence of additional first integrals, as explained by Moser [133]. See how the circular restricted three-body problem reduces to such a map in section 3.3.

Figure 8. The entanglement of the invariant curves of the primary hyperbolic fixed point (center of the figure) of the standard map $(x', y') = (x + y', y + \epsilon \sin 2\pi x)$, $\epsilon = 0.3$.

More recently, the non-existence of additional meromorphic integrals in the neighborhood of well chosen particular solutions has been proved by studying the monodromy group of the variational equation of a periodic solution; see [89, 185] for example. The method has been successfully applied for example to the Lagrange parabolic solution of the three-body problem in one of the results of Tzygvintsev [179], using Ziglin's and Morales-Ramis theories, for fixed masses. Some refinements lead to differential Galois theory [126, 127], the Galois group being an extension of the monodromy group. Recently, Combot has significantly generalized the results regarding celestial mechanics [53, 55, 56]. Higher order variational equations seem to give additional information in undetermined cases [54, 56, 129, 128, 110]. The main limitation of the method is that it is local in the neighborhood of some solution, and that *a priori* there could exist an additional meromorphic first integral in a domain of the phase space which is bounded by a natural frontier and which does not contain the studied periodic orbits.

10. Long term stability of the planetary system

We take up the discussion on the planetary system, as introduced at the beginning of section 3; further references for this section are Chapters 6 and 13 of this volume, as well as [11, Chap. 6] and [7, 27, 33, 39, 49, 72, 147].

Let $(\lambda_j, \Lambda_j, \xi_j, \eta_j, p_j, q_j)_{2\leq j\leq N}$ be the Poincaré coordinates of the planets. These coordinates are analytic and symplectic in the neighborhood of circular, horizontal, Keplerian motions. The Keplerian part of the Hamiltonian, describing the revolution of planets around the Sun without the mutual attraction of planets, is

$$H_{Kep} = \sum_{2\leq j\leq N} -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2},$$

while the perturbing function is an implicit, transcendent, ϵ-small function

$$\epsilon H_{pert}(\lambda, \Lambda, \xi, \eta, p, q)$$

of all the coordinates (where λ stands for $(\lambda_2, ..., \lambda_N)$, etc.), so that

$$H = H_{Kep}(\Lambda) + \epsilon H_{pert}(\lambda, \Lambda, \xi, \eta, p, q).$$

Averaging theory shows that, over a time interval of length $1/\epsilon$ and outside Keplerian resonances, the dynamics is well described by the Hamiltonian obtained from H by averaging the perturbing part with respect to the fast angles $\lambda = (\lambda_2, ..., \lambda_N)$.

This averaged Hamiltonian is the so-called (first order) *secular Hamiltonian*

$$H_{sec} = H_{Kep} + \epsilon \int_{\mathbb{T}^n} H_{pert}\, d\lambda.$$

Since it does not depend on the Keplerian angle λ, the conjugate action Λ is a first integral. So, H_{sec} can be thought of as a Hamiltonian on the secular space $\mathbb{R}^{4(N-1)} = \{(\xi, \eta, p, q)\}$, with parameter Λ. Since

$$\Lambda_j = \mu_j \sqrt{M_j a_j}, \quad j = 2, ..., N,$$

(see the definition of the fictitious masses μ_j, M_j in section 2), this means that the semi major axes are constant, for the averaged Hamiltonian; this is the first stability theorem of Lagrange.

The orgin $\xi = \eta = p = q = 0$ corresponds to circular, horizontal, direct Keplerian ellipses of fixed semi major axes. As some symmetry argument shows, it is a critical point of the secular system. It proves an elliptic fixed point. Hence, for the linearized vector field, the origin is stable; this is the second stability theorem of Lagrange and Laplace [93, p. 164]. As Poincaré noticed, those stability theorems prove stability only for approximate equations.

Moreover, the Hamiltonian has the following remarkable expansion.

Theorem (Lagrange-Laplace)**.** *Let $m = (m_2, ..., m_N)$, $a = (a_2, ..., a_N)$, $\xi = (\xi_2, ..., \xi_N)$, $\eta = (\eta_2, ..., \eta_N)$, $p = (p_2, ..., , p_N)$ and $q = (q_2, ..., q_N)$. There are two symmetric bilinear forms $Q_h = Q_h(m, a)$ and $Q_v = Q_v(m, a)$ on the tangent space at the origin of the secular space, respectively called* horizontal *and* vertical*, which depend on the masses and semi major axes analytically, and such that*

$$H_{sec} = C_0(m, a) + Q_h \cdot \left(\xi^2 + \eta^2\right) + Q_v \cdot \left(p^2 + q^2\right) + O(4),$$

with

$$\begin{cases} Q_h \cdot \xi^2 &= \displaystyle\sum_{2 \le j < k \le N} m_j m_k \left(C_1(a_j, a_k) \left(\frac{\xi_j^2}{\Lambda_j} + \frac{\xi_k^2}{\Lambda_k} \right) + 2C_2(a_j, a_k) \frac{\xi_j \xi_k}{\sqrt{\Lambda_j \Lambda_k}} \right) \\ Q_v \cdot p^2 &= \displaystyle\sum_{2 \le j < k \le N} -m_j m_k C_1(a_j, a_k) \left(\frac{p_j}{\sqrt{\Lambda_j}} - \frac{p_k}{\sqrt{\Lambda_k}} \right)^2 \end{cases}$$

and the C_j's themselves are explicit linear combinations of the Laplace coefficients.

The masses and semi major axes being fixed, let $\rho_h, \rho_v \in \mathrm{SO}(n)$ be diagonalizing transformations of Q_h and Q_v :

$$\rho_h^* Q_h = \sum_{2 \le j \le N} \sigma_j \, d\xi_j^2 \quad \text{and} \quad \rho_v^* Q_v = \sum_{2 \le j \le N} \varsigma_j \, dp_j^2, \quad \sigma_1, ..., \sigma_n, \varsigma_1, ..., \varsigma_n \in \mathbb{R}.$$

The map $\rho : (\xi, \eta, p, q) \mapsto (\rho_h \cdot \xi, \rho_h \cdot \eta, \rho_v \cdot p, \rho_v \cdot q)$ is symplectic and we have

$$H_{sec} \circ \rho = C_0 + \sum_{2 \le j \le N} \sigma_j \left(\xi_j^2 + \eta_j^2\right) + \sum_{2 \le j \le N} \varsigma_j \left(p_j^2 + q_j^2\right).$$

This first order normal form of the planetary system is the starting point to study the long term evolution of the system. The next step consists in building higher order integrable normal forms. Among the several kinds of formal perturbations series (see Chapter 3 of this volume), the series of *Lindstedt*, *von Zeipel* and *Birkhoff* are the most interesting ones. They are at the core of all stability studies, including KAM and Nekhoroshev theories. We end this chapter by a short account on these series. In the case of the planetary problem, the series of Lindstedt and von Zeipel are obtained by eliminating the fast, Keplerian angles λ, after which the series of Birkhoff at the elliptic secular singularity is obtained by factorizing the secular Hamiltonian (composed with ρ) through functions of the $\xi_j^2 + \eta_j^2$ and $p_j^2 + q_j^2$, wherever possible.

10.1. **Linstedt and von Zeipel series.** Consider a Hamiltonian on $\mathbb{T}^n \times \mathbb{R}^n = \{(\theta, r)\}$ $(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n)$,

$$H(\theta, r) = H_0(r) + \epsilon H_1(\theta, r) + \epsilon^2 H_2(\theta, r) + \cdots ,$$

which depends analytically on some small parameter ϵ and whose value when $\epsilon = 0$ is an integrable Hamiltonian $H_0(r)$. Poincaré described the basic procedure to eliminate the angle θ from H, outside resonances of H_0, using a formal symplectic diffeomorphism ϕ. He attributed the procedure to Lindstedt, and called it a *new method* [147, 97], as opposed to *old* ones usde by Delaunay, Bohlin and others (see [61] for instance).

A formal normalization ϕ can be built as the time-one map of the flow of an auxiliary ϵ-small Hamiltonian vector field (Poincaré rather used a generating function, but this is slightly more complicated in a perturbative setting):

$$\phi = \exp X_K, \quad K = \epsilon K_1(\theta, r) + \epsilon^2 K_2(\theta, r) + \cdots .$$

The formal pull-back of H by ϕ is

$$H \circ \phi = H + X_K \cdot H + \frac{1}{2} X_K^2 \cdot H + \cdots ,$$

where the Hamiltonian vector field X_K of K can be seen as a derivative operator or, using that $X_{K_j} \cdot H_0 = -X_{H_0} \cdot K_j$ (= the Poisson bracket of K_j and H_0),

$$H \circ \phi = H_0 + \epsilon \left(H_1 - X_{H_0} \cdot K_1\right) + \epsilon^2 \left(H_2 + X_{K_1} \cdot H_1 + \frac{1}{2} X_{K_1}^2 \cdot H_0 - X_{H_0} \cdot K_2\right) + \cdots .$$

One would like to find successively K_1, K_2, ..., so that each term of $H \circ \phi$ of given degree ≥ 1 in ϵ does not depend on θ anymore. We get a triangular infinite system of linear partial differential equations on $\mathbb{T}^n$, parameterized by actions:

$$\begin{aligned} X_{H_0} \cdot K_1 &= \{H_1\} := H_1 - \int_{\mathbb{T}^n} H_1 \, d\theta \\ X_{H_0} \cdot K_2 &= \left\{H_2 + X_{K_1} \cdot H_1 + \frac{1}{2} X_{K_1}^2 \cdot H_0\right\} \\ &\cdots \end{aligned}$$

where $X_{H_0} = \alpha \cdot \partial_\theta$, with $\alpha := (\partial_{r_1} H_0, \cdots, \partial_{r_n} H_0)$. As expanding in Fourier series shows, the first equation has formal solutions outside resonances $k \cdot \alpha = 0$, $k \in \mathbb{Z}^n$ (recall that α depends on actions). Choose for example the solution with zero average:

$$K_1(r) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{\hat{H}_1^k}{i2\pi\, k \cdot \alpha}, \quad H_1 = \sum_{k \in \mathbb{Z}^n} \hat{H}_1^k \, e^{i2\pi k \cdot \theta}.$$

The *small denominators* $k \cdot \alpha$ may prevent the Fourier series of $K_1(r)$ from converging. Intuitively, this mean that close to resonances the long term effect of the perturbation, instead of averaging out, will pile up. Outside resonances, the Fourier coefficients of K_1 are well defined. Then the second equation can be solved similarly. But since the right hand side is now a formal Fourier series, K_2 is a formal Fourier series whose coefficients themselves are formal Fourier series, etc.

In order to avoid convergence problem for constructing Lindtedt series up to some finite order, since the Fourier coefficients of analytic functions decay exponentially fast, one can truncate the Fourier series of the right hand sides of the equations, at some high order tending to infinity when ϵ tends to 0. This is done at the expense of loosing analyticity of expansions at $\epsilon = 0$. But then the equations involve only finitely many small denominators, which can be avoided on an open set in the space of actions. A way to make sense of the exact construction among formal Fourier series, is to restrict to Diophantine frequency vectors and build the infinite jet of the series (by formally differentiating the equations in the direction transverse to the Lagrangian tori) on this Cantor set. The so-obtained coefficients then extend to the whole phase space, due to a theorem of Whitney [33]. The *series of von Zeipel* generalize those of Lindstedt when the unperturbed Hamiltonian H_0 does not depend on all the action variables, which is called a *proper degeneracy*. Only the fast angles can then be eliminated. Thus von Zeipel series are a parameterized version of the Lindstedt series. This generalization is obviously needed in the planetary problem since the Keplerian part depends only on Λ.

The next question to arise is that of the convergence of the Lindstedt (or von Zeipel) series $H \circ \phi$, and of the normalization ϕ itself (of course, the convergence of ϕ implies that of $H \circ \phi$). The answer is not straightforward. Examples where the normalization diverges occur as a byproduct of the constructions of Anosov-Katok [6]. Here is a very simple example.

Example ([11]) On $\mathbb{T}^2 \times \mathbb{R}^2 = \{(\theta, r)\}$, consider the Hamiltonian

$$H = \alpha_1 r_1 + \alpha_2 r_2 + \epsilon \left(r_1 + \sum_{k \in \mathbb{Z}^2} a_k \sin(k \cdot \theta) \right),$$

where $a_k = \exp(-\|k\|)$ and α is Diophantine: $|k \cdot \alpha| \geq \gamma \|k\|^{-\tau}$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$ for some $\gamma, \tau > 0$. The angle θ rotates with constant frequency $\alpha_\epsilon = (\alpha_1 + \epsilon, \alpha_2)$: $\theta(t) = \theta(0) + t\alpha_\epsilon$. There are arbitrarily small values of ϵ such that α_ϵ is resonant:

$(\alpha_1 + \epsilon)/\alpha_2 = p/q \in \mathbb{Q}$. Then we have

$$\dot{r}_j = -\epsilon \sum_k k_j a_k \cos(k \cdot \theta), \quad j = 1, 2.$$

Terms such that $k \cdot \alpha_\epsilon \neq 0$ have zero average. But others, namely those for which k is of the form $k = \kappa(-q, p)$ for $\kappa \in \mathbb{Z}$ have a constant contribution, so that r goes to infinity (if by chance this constant contribution is zero, one can slightly modify one coefficient a_k to obtain the wanted behavior). On the other hand, if the Lindstedt series and the corresponding transformation converged, the action r would undergo only bounded oscillations. So the Lindstedt transformation diverges (whether the Lindsted series itself diverges, does not follow from the given simple argument).

The above examples are not generic, for they are perturbations of degenerate Hamiltonians. But Poincaré proved that divergence *is* generic (in a somewhat topological sense; in the closely related work by Siegel mentioned below, on the generic divergence of Birkhoff normalization series, the notion of genericity has a more metric flavor) [147, Chap. XIII]. His argument roughly goes as follows. If a Lindstedt normalization converges for some value of the action r, the torus $\mathbb{T}^n \times \{r\}$ is invariant and quasiperiodic for $H \circ \phi$. Its frequency is $\alpha_\epsilon(r) = \partial_r(H \circ \phi)(r) = \partial_r H_0 + \cdots$. The unperturbed frequency $\alpha_0(r)$ was chosen non-resonant, but, for abitrarily small $\epsilon > 0$, the perturbed frequency α_ϵ is resonant. Hence, the invariant torus is foliated by lower dimensional invariant tori. Such a resonant torus is non generic. So, generically Lindstedt normalizations diverge. Poincaré also proved that Lindstedt normalizations in the three-body problem diverge generally.

But Poincaré could not preclude that Lindstedt series and normalizations sometimes converge, non uniformly (notations in the quotation have been changed):

> Nous avons reconnu que les équations canoniques [...] peuvent être satisfaites formellement par des séries de la forme
>
> $$\begin{cases} \theta_i = \theta_i^0 + \epsilon\theta_i^1 + \epsilon^2\theta_i^2 + ..., \\ r_i = r_i^0 + \epsilon r_i^1 + \epsilon^2 r_i^2 + ..., \end{cases}$$
>
> où les θ_i^k et les r_i^k sont des fonctions périodiques des quantités
>
> $$w_i = \alpha_i t + \varpi_i, \quad (i = 1, 2, ..., n),$$
>
> [de quoi] nous avons tiré
>
> $$r_i^k = \sum \frac{B \sin(k_1 w_1 + k_2 w_2 + ... + k_n w_n + h)}{k_1\alpha_1^0 + k_2\alpha_2^0 + ... + k_n\alpha_n^0} + A_0.$$
>
> [Cette] série converge-t-elle absolument et uniformément ? [... À] deux degrés de liberté, les séries ne pourraient-elles pas, par exemple, converger quand r_1^0 et r_2^0 ont été choisis de telle sorte que le rapport $\frac{\alpha_1}{\alpha_2}$ soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport $\frac{\alpha_1}{\alpha_2}$ est assujetti à une autre condition

analogue à celle que je viens d'énoncer un peu au hasard) ? [147, §§ 146–149]

(We have realized that canonical equations [...] can be satisfied formally by series of the form

$$\begin{cases} r_i = r_i^0 + \epsilon r_i^1 + \epsilon^2 r_i^2 + ..., \\ \theta_i = \theta_i^0 + \epsilon\theta_i^1 + \epsilon^2\theta_i^2 + ..., \end{cases}$$

where θ_i^k and r_i^k are functions of the quantities

$$w_i = \alpha_i t + \varpi_i, \quad (i = 1, 2, ..., n).$$

From this we have inferred

$$r_i^k = \sum \frac{B\sin(k_1 w_1 + k_2 w_2 + ... + k_n w_n + h)}{k_1 n_1^0 + k_2 n_2^0 + ... + k_n n_n^0} + A_0.$$

Does [this] series converge absolutely and uniformly? [With] two degrees of freedom, couldn't it happen that the series converge when r_1^0 and r_2^0 have been so chosen that the ratio $\frac{\alpha_1}{\alpha_2}$ be rational and its square on the contrary be irrational (or so that the ratio $\frac{\alpha_1}{\alpha_2}$ satisfy another condition, analogous to the one I have just stated a bit randomly) ?)

Considering the unreasonable consequences of uniform convergence, in terms of existence of periodic orbits at resonances, he speculated:

Les raisonnements de ce Chapitre ne permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable. [*ibid.*]

(The arguments in this Chapter do not make it possible to assert that this fact will not occur. All I can say is that it is most unlikely.)

Hill and Weierstrass queried Poincaré's arguments against the convergence of Lindstedt series [13, 83]. Despite their own failure to show the convergence of those perturbation series, later events proved their case.

The stupendous breakthrough came more than half a century later from Kolmogorov (in addition to Siegel, as mentioned in the introduction). There is a variant of Lindstedt series in which one seeks invariant tori with incommensurable frequencies fixed beforehand. Kolmogorov proved that for the perturbation of an invariant torus of fixed frequency in a Hamiltonian system, some perturbation series does converge, albeit non uniformly, assuming in particular that the fixed frequency is Diophantine [90]:

$$|k_1\alpha_1 + \cdots + k_n\alpha_n| \geq \frac{\gamma}{(|k_1| + \cdots + |k_n|)^\tau} \quad (\gamma, \tau > 0).$$

This arithmetic condition had already occured in the work of Kryloff-Bogoliuboff (see [91] for references). Kolmogorov uses Newton's algorithm in a functional space of infinite dimension and finds quasi-periodic invariant tori by a limiting process.

The fast convergence of the algorithm beats the effect of resonances, one of the main ideas which laid the foundations for the so-called Kolmogorov-Arnold-Moser theory; see Chapter 3 of this volume for a precise statement.

A subtlety is that there are infinitely many normalizing transformations, leading to a unique Lindstedt normal form. This point was clarified by Moser, who showed (in a more general setting but always with fixed frequencies) that the convergence of Kolmogorov's normalizing transformation entails the convergence of the normalizing transformation associated to Lindstedt series [130, Section 4].

10.2. **Birkhoff series.** Recall that the secular system has an elliptic fixed point at the origin of the Poincaré coordinates. More generally, consider a Hamiltonian in $\mathbb{R}^{2n}$ of the form

$$H = H_2 + H_3 + \cdots$$

where H_i is homogeneous of total degree i and where the quadratic part is itself of the form

$$H_2 = \frac{1}{2}\left(\alpha_1(x_1^2 + y_1^2) + \cdots + \alpha_n(x_n^2 + y_n^2)\right).$$

If the frequency $\alpha = (\alpha_1, ..., \alpha_n)$ is non resonant: $k \cdot \alpha \neq 0$ (for all $k \in \mathbb{Z}^n \setminus \{0\}$), in a construction similar to Lindstedt series, one can eliminate the angle and find a formal symplectic diffeomorphism ϕ of $\mathbb{R}^{2n}$ such that $H \circ \phi$ is a formal power series in the variables $\rho_i = \frac{1}{2}(x_i^2 + y_i^2)$ [18]:

$$H \circ \phi = \sum_{k \geq 1} \mathcal{H}_k(\rho), \quad \deg \mathcal{H}_k = k.$$

In case of resonances, there remain resonant terms in the normal form, which is not integrable in general. We will not dwell on resonances here, for which we rather refer to the very interesting discussion of [11] and references therein. So, we assume that the frequency vector is non resonant.

If the normalization converges, so does the normal form, and the Hamiltonian is integrable. (Ito has proved that the converse is also true [86].) Various studies, starting with Poincaré's, showed with increasing strength that the normalization is generically divergent; see the book of Siegel-Moser [166, Chap. 30]. One of the strongest result on divergence was proved by Siegel in 1954 and showed the generic divergence of the normalization, the quadratic part of the Hamiltonian being fixed but otherwise arbitrary [165]. Somehow taking up Poincaré's idea for Lindstedt series, Siegel showed that convergence would imply the existence of families of periodic solutions having arbitrarily large period and lying in a neighborhood of the origin, and that existence of such a family implies an infinite countable number of independant analytic conditions, defining a set of first category in the space of all families of coefficients of Hamiltonians. On the other hand, Bruno and Rüssmann proved the convergence of the normalization when the normal form is quadratic and the eigenvalues satisfy Bruno's arithmetic condition [23, 154].

As Eliasson pointed out, whether there exist divergent Birkhoff normal forms (as opposed to divergent normalization series) remained an unsettled question for an inordinate amount of time [67], as in any other normal form problem for that matter. Only recently did Gong give examples, apparently, of diverging Birkhoff normal forms, for some given quadratic part satisfying some arithmetic condition, stronger than being Liouville. The proof consists in carefully controlling the accumulation of small denominators appearing when eliminating non-resonant terms [80]. Besides, Perez-Marco proved the remarkable dichotomy that, for a given non-resonant quadratic part, the set of Hamiltonians having a convergent Birkhoff normal form is either full or pluripolar [142]. This shows that, for the quadratic parts which Gong has taken care of, Hamiltonians with converging Birkhoff normal form are exceptional. Another case (for a hyperbolic fixed point) is filled by Hamiltonians recently constructed by Saprykina, whose normal form is quadratic (hence convergent, of course) but which are non integrable, and thus for whom no symplectic normalization converges [159].

10.3. **Stability and instability.** Arnold proved a degenerate version of Kolmogorov's celebrated theorem, and deduced the existence of a set of positive measure of almost plane and almost circular quasi-periodic solutions when the masses of the planets are small enough [7]. There are several degeneracies in this problem. The most important one comes from the fact that all the bounded orbits of the Kepler problem—a problem with two degrees of freedom—are periodic, which is a very specific feature of the Newtonian potential in $1/r$ (and of the elastic potential in r^2, and only them, according to a theorem of Bertrand). Arnold's proof was complete only for the plane two-planet problem, due to degeneracies. Some kind of non degeneracy is necessary for KAM tori to persist locally (see examples above), although some this is not true globally [69]. In the spatial case, an unforeseen and mysterious resonance is present. Namely, the trace of the linearized secular system is always zero, identically with respect to the semi major axes: keeping the same notations as at the beginning of this section,

$$\sum_{2\leq j\leq N} (\sigma_j + \varsigma_j) = 0.$$

As previously mentioned, this fact was actually known to Delaunay in the three-body problem. But it holds in general in the N-body planetary problem. This was first noticed by Herman who, in a series of lectures in the 1990's, sketched a complete and more conceptual proof of Arnold's theorem. Chierchia-Pinzari later proved in general [49] that Herman's resonance disappears when one reduces the problem by the rotational symmetry, as Robutel had proved in the three-body problem, using a computer. The whole subject is described in Chapter 6 of this volume.

Arnold's theorem hardly applies to our solar system. There is a first difficulty with the upper value of the small parameter ϵ. A similar issue occurs when semi-classical analysts let the Planck constant tend to zero. Hénon noticed that, without any additional care, the first proofs of Kolmogorov's theorem show the existence of invariant tori only for a derisory ϵ of the order of 10^{-300} [81]! However, Robutel has shown numerically that some significant parts of the solar system, in particular of the system consisting of the Sun, Jupiter and Saturn [94, 153], display a quasi-periodic behavior. Also, Celletti–Chierchia [26, 27] and Locatelli-Giorgilli [100] have proved quantitative versions of the KAM theorem, which they have applied to the systems Sun–Jupiter–asteroid Victoria and Sun–Jupiter–Saturn; these applications are assisted by computer symbolic processors, requiring in the second case the manipulation of series of ten million terms. Whether bounded motions form a set of positive Lebesgue measure for all ϵ—and not only for $\epsilon \ll 1$—remains a completely open problem.

Another matter for discontent when applying KAM theory to astronomy, is that the set of KAM invariant tori in phase space fill a transversely Cantor set, parametrized by Diophantine frequencies, which is topologically meager. Given the approximation which is made by substituting the Newtonian planetary system to the real solar system, whether the planet's mean motions are Diophantine or not, is not a question with any straightforward meaning. Incidentally, Molchanov has speculated on the opposite hypothesis that these mean motions could be totally periodic [123]. Hence the direct conclusion of Arnold's theorem over an infinite time interval, is illusory in astronomy. Yet KAM theory provides a fundamental conceptual tool in the study of conservative systems since, as is wildly believed, the conclusion of invariant tori theorems holds under much weaker hypotheses than current theoretical proofs require. To paraphrase Poincaré, quasi-periodic orbits too are part of the breach.

A related and more realistic theorem by Nekhoroshev [134], asserts that in the neighborhood of KAM quasi-periodic solutions motions are stable over an exponentially long time interval with respect to the small parameter. By applying a theorem of this type, Niederman has shown the stability of a solar system with two planets having small masses, not quite equal to but much closer to realistic values of Jupiter and Saturn [138]. In order to describe the slow evolutions more accurately, Neishtadt has developed the probabilistic theory of adiabatic invariants [136], and extended related results to non-Hamiltonian perturbations [135].

Over the centuries, geometers have spent an inordinate amount of time and energy proving stronger and stronger stability theorems for dynamical systems more or less closely related to the solar system. It was a huge surprise when the numerical computations of Laskar showed that over the life span of the Sun, or even over a few hundred million years, collisions and ejections of inner planets are probable (see [95] for a recent account). Our solar system is now wildly believed unstable. Works of Sitnikov and Alekseev [133], Moeckel [117, 118], Simó-Stuchi [169], Galante-Kaloshin [77] and Féjoz-Guardià-Kaloshin-Roldán [74], among others (see [11] for other references), show the complexity of the simplest non-integrable N-body problem, the restricted three-body problem. Arnold's diffusion and the general mechanisms of instability in large dimension are still to be understood, despite significant progress [8, 17, 47, 65, 108, 111, 119] (again, see [11] for more references).

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Related Chapters

Related references outside this volume are the books of Arnold [9], Arnold-Kozlov-Neishtadt [11], Moser [133], Siegel and Moser [166], Wintner [183], or, at a lesser level of difficulty, of Meyer-Hall [116], and the article of Chenciner in *Scholarpedia* [38]. The author has also used parts of his Mémoire d'Habilitation [73].

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