Introduction to KAM theory with a view to the three-body problem

Jacques Féjoz

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Abstract. The theory of *Kolmogorov*, *Arnold* and *Moser* consists of a set of results regarding the persistence of quasiperiodic solutions, primarily in Hamiltonian systems. We bring forward a "twisted conjugacy" normal form, due to Herman, which contains all the (not so) hard analysis. We focus on the real analytic setting. A variety of KAM results follows, including most classical statements as well as more general ones. This strategy makes it simple to deal with various kinds of degeneracies and symmetries. As an example of application, we prove the existence of quasiperiodic motions in the spatial lunar 3-body problem.

Bibliographical comments. For background in Hamiltonian systems, excellent references are the books of V. Arnold [4], V. Arnold-V. Koslov-A. Neishtadt [5], V. Guillemin-S. Sternberg [35], A. Knauf [42], K. Meyer-G. Hall [50], C. Siegel-J. Moser [74] or S. Sternberg [77].

In KAM theory, there exist many surveys, among which we recommend those of J.-B. Bost [9], L. Chierchia [16], J. Pöschel [61] or M. Sevryuk [71, 72, 73]. S. Dumas's book [20] is an interesting, historical account of the subject.

Several results emphasized in the present paper, including the twisted conjugacy theorem 1, were already proved in [26] in the smooth setting.

Keywords. Hamiltonian system, stability, Kolmogorov, Arnold, Moser, KAM theory, Herman, normal form, Rüssmann, invariant torus, quasiperiodic solution, Pyartli, Diophantine approximation, inverse function theorem, Nash-Moser theorem, frequency map, degeneracy, symmetry, three-body problem

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1 Twisted conjugacy normal form

Let $\mathcal{H}$ be the set of germs along $\mathrm{T}_0 = \mathbb{T}^n \times \{0\}$ of real analytic functions ("Hamiltonians") in $\mathbb{T}^n \times \mathbb{R}^n = \{(\theta, r)\}$. The Hamiltonian vector field associated with $H \in \mathcal{H}$ is

$$\begin{cases} \dot{\theta} = \partial_r H \\ \dot{r} = -\partial_\theta H. \end{cases}$$

For any given vector $\alpha \in \mathbb{R}^n$, let $\mathcal{K}(\alpha)$ be the set of Hamiltonians $K \in \mathcal{H}$ of the form

$$K = c + \alpha \cdot r + O(r^2),$$

for some (non fixed) $c \in \mathbb{R}$ and $O(r^2)$ depending on θ. It consists exactly of Hamiltonians for which T_0 is invariant ($\dot{r}|_{r=0} = 0$) and carries a linear flow with velocity α ($\dot{\theta}|_{r=0} = \alpha$).[1]

Let $\mathcal{G}$ be the set of germs along T_0 of exact symplectic real analytic isomorphisms of the form[2]

$$G(\theta, r) = \left(\varphi(\theta), (r + S'(\theta)) \cdot \varphi'(\theta)^{-1}\right),$$

where φ is an isomorphism of $\mathbb{T}^n$ fixing the origin and S is a function on $\mathbb{T}^n$ vanishing at the origin. The goal being to find invariant tori close to T_0 and carrying a linear flow of frequency α, φ allows us to make changes of coordinates at will on the Lagrangian torus T_0, while S allows us to bring back to the zero section any graph over T_0, of 0-average and sufficiently close to the zero section.

In the next theorem, we assume that α is *Diophantine*:

$$|k \cdot \alpha| \geq \frac{\gamma}{|k|^\tau} \quad (\forall k \in \mathbb{Z}^n \setminus \{0\})$$

for some fixed $\gamma, \tau > 0$; we have set $|k| = |k_1| + \cdots + |k_n|$. We will call $D_{\gamma,\tau}$ the set of such vectors. $D_{\gamma,\tau}$ is non empty if and only if $\tau \geq n - 1$ (Dirichlet's theorem) and, if $\tau > n - 1$, the complement of $D_{\gamma,\tau}$ within a ball has measure $O(\gamma)$, hence $\cup_\gamma D_{\gamma,\tau}$ has full measure [64].

1 Theorem (Herman)**.** *If $K^o \in \mathcal{K}(\alpha)$ and if $H \in \mathcal{H}$ is close enough to K^o, there is a unique $(K, G, \beta) \in \mathcal{K}(\alpha) \times \mathcal{G} \times \mathbb{R}^n$ such that*

$$H = K \circ G + \beta \cdot r. \tag{1}$$

We will prove theorem 1 in the next two sections.

The statement calls for some remarks.

[1] Recall that, in Dynamical Systems, a path $\gamma : \mathbb{R} \to X$ on a manifold is *quasiperiodic* (of some rank $\leq k$) if there exists $\omega \in \mathbb{R}^k$ and a map $\Gamma : \mathbb{T}^k \to X$ such that $\gamma(t) = \Gamma(t\omega)$. Provided γ is smooth enough, γ then admits a Fourier expansion of the form $\gamma(t) = \sum_{j \in \mathbb{Z}^k} \gamma_j e^{i2\pi\,(j\cdot\omega)\,t}$ (in some local coordinates in a tubular neighborhood of $\gamma(\mathbb{R})$).

[2] Conventionally, if f is a map from an open set U of a vector space E into another vector space F, we define $f'(x)$ as an element of $F \otimes E^*$ (as opposed to $E^* \otimes F$), and we write $f'(x) \cdot \xi$ for the contraction with a vector $\xi \in E$. Also, we identify $\mathbb{R}^n$ with its dual, so that $S'(\theta)$ may be imaged as the gradient of S.

- The frequency being a conjugacy invariant of quasi-periodic flows, the counter-term $\beta \cdot r$, which allows us to tune the frequency, is necessary. Yet it breaks the dynamical conjugacy between K and H and does not comply H with having an invariant torus, as K does. We will call this normal form a *twisted conjugacy.* The geometrical contents of the theorem is that locally the set of Hamiltonians possessing a α-quasiperiodic torus is a submanifold of finite codimension if α is Diophantine (it has infinite codimension if α is not). The counter-term is the finite dimensional obstruction to conjugacy to a Hamiltonian of $\mathcal{K}(\alpha)$, and can be imaged as a simple control to preserve a torus of the same frequency and cohomology class as that of K^o.
- In general, one cannot expect H to be of the form

  $$H = (K + \beta \cdot r) \circ G;$$

  this would show that having a Diophantine invariant torus is an open property, which is wrong, as the following example shows.

  Consider the Hamiltonian $H = \alpha \cdot r$, $\alpha \in \mathbb{R}^2$. All the tori $r = cst$ are invariant. By a first arbitrarily small perturbation, we may assume that α is resonant: $k \cdot \alpha = 0$ for some $k \in \mathbb{Z}^2 \setminus \{0\}$. Then add a resonant monomial:

  $$H = \alpha \cdot r - \epsilon \sin(2\pi k \cdot \theta).$$

  The vector field is

  $$\begin{cases} \dot{\theta} = \alpha \\ \dot{r} = 2\pi\epsilon \cos(2\pi k \cdot \theta)\, k. \end{cases}$$

  So, the solution through $(0, r)$ at time $t = 0$ is

  $$t \mapsto (t\alpha, r + 2\pi\epsilon t k).$$

  So, if $\epsilon > 0$, this solution is unbounded and prevents any invariant torus (among graphs over T_0) to exist.

Bibliographical comments. – The idea of proving the finite codimension of a set of conjugacy classes of a vector field has been imprecisely called the "method of parameters". It has been used fruitfully in a number of works, among which: Arnold's normal form of vector fields on the torus [2, 53, 81], Moser's normal form of vector fields [54] (which encompasses many natural subcases [11, 49, 79] but which has been surprisingly overlooked for 30 years), Chenciner's work on the bifurcation of elliptic fixed points [12, 13, 14] or Eliasson-Fayad-Krikorian's study of the neighborhood of invariant tori [23]. In KAM theory, it is very powerful, because it allows us to first prove a normal form theorem which does not depend on any non-degeneracy assumption, but which contains all the hard analysis; the remaining, finite dimensional problem is then to show that the frequency offset vanishes, using a non-degeneracy hypothesis. This last step was probably not well understood before the 80s [22, 67, 71]. The method fails for other kinds of dynamics than quasiperiodic one on the torus [32].

– The normal form of theorem 1, which was advertised by Herman in the 90s, in particular in his lectures on Arnold's theorem at the Dynamical System Seminar in Université Paris VII, can be seen as a particular case of Moser's normal form, when the vector field is Hamiltonian, then giving more precise information [27, 49]. (Surprisingly enough, M. Herman seemingly did not know Moser's normal form.) Also, Herman's normal form implies Arnold's normal form on the torus. A proof in the smooth category can be found in [26].

2 One step of the Newton algorithm

Let

$$\phi(x) = K \circ G + \beta \cdot r, \quad x = (K, G, \beta).$$

We want to solve the following equation between Hamiltonians:

$$\phi(x) = H, \tag{2}$$

for H close to $\phi(K^o, id, 0) = K^o$. The twisted conjugacy theorem thus reduces to prove that ϕ is invertible, keeping in mind that

– if ϕ is formally defined on the whole space $\mathcal{K}(\alpha) \times \mathcal{G} \times \mathbb{R}^n$, it is only if G is close enough to the identity, with respect to to the width of analyticity of K, that $\phi(K, G, \beta)$ is analytic on a neighborhood of T_0,

– equation (2) is really of interest to us only if it holds on a neighborhood of $G^{-1}(\mathrm{T}_0)$, a domain depending on the unknown G.

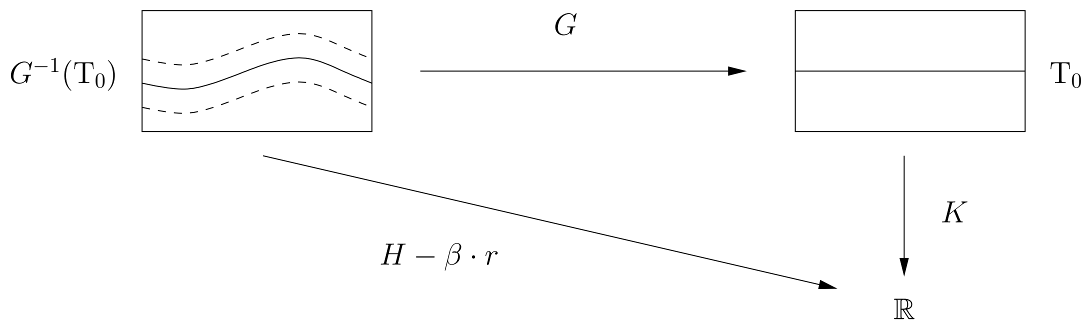


Figure 1:

Note that $\mathcal{H}$ and $\mathbb{R}^n$ are trivially vector spaces, while

- $\mathcal{K}(\alpha)$ is an affine space, directed by the vector space $\mathbb{R} + O(r^2)$
- and $\mathcal{G}$, while being a groupoid with semi-direct product law given by $G_2 \circ G_1 = (\varphi_2 \circ \varphi_1, (r + S_1' + S_2' \cdot \varphi_1') \cdot (\varphi_2 \circ \varphi_1)'^{-1})$, will rather be identified, locally in a neighborhood of the identity, to an open set of the affine space passing through the identity and directed by the linear space $\{(\varphi - \mathrm{id}, S)\}$, where $v = \varphi - \mathrm{id} : \mathbb{T}^n \to \mathbb{R}^n$ and $S : \mathbb{T}^n \to \mathbb{R}$ are analytic and vanish at the origin.

We will invert ϕ using the Newton algorithm, which consists in iterating the operator

$$f : x = (K, G, \beta) \mapsto \hat{x} = x + \phi'(x)^{-1}(H - \phi(x)). \tag{3}$$

Each step of the induction requires to invert the linearized operator $\phi'(x)$, not only at $x_0 = (K^o, \mathrm{id}, 0)$, but at some unknown x in the neighborhood of x_0, i.e. to solve the linearized equation

$$\phi'(K, G, \beta) \cdot (\delta K, \delta G, \delta\beta) = \delta K \circ G + K' \circ G \cdot \delta G + \delta\beta \cdot r = \delta H \tag{4}$$

where δH is the data, (K, G, β) is a parameter, and the unknowns are the "tangent vectors" $\delta K \in \mathbb{R} \oplus O(r^2)$, δG (geometrically, a vector field along G) and $\delta\beta \in \mathbb{R}^n$. Pre-composing with G^{-1} modifies the equation into an equation between germs along the standard torus T_0 (as opposed to the G-dependant torus $G^{-1}(\mathrm{T}_0)$):

$$\delta K + K' \cdot \dot{G} + \delta\beta \cdot r \circ G^{-1} = \dot{H}, \tag{5}$$

where we have set $\dot{G} = \delta G \circ G^{-1}$ (geometrically, a germ along T_0 of tangent vector field) and $\dot{H} = \delta H \circ G^{-1}$. It is a key point in measuring norms that we are intertested in the neighborhood of T_0 on one side of the conjugacy, and in the neighborhood of $G^{-1}(\mathrm{T}_0)$ on the other side.

Using the additional notations (in which $*_{\geq k}$ stands for a function in $O(r^k)$):

$$\begin{cases} K = c + \alpha \cdot r + Q(\theta) \cdot r^2 + K_{\geq 3} \\ \delta K = \delta c + \delta K_{\geq 2} \\ \dot{G} = (\dot{\varphi}, -r \cdot \dot{\varphi}' + \dot{S}') \\ \dot{H} = \dot{H}_0 + \dot{H}_1 \cdot r + \dot{H}_{\geq 2} \end{cases}$$

and the fact that

$$\delta\beta \cdot r \circ G^{-1} = \left(r \cdot \varphi' \circ \varphi^{-1} - S' \circ \varphi^{-1}\right) \cdot \delta\beta,$$

and identifying the Taylor coeeficients in equation (5) yield the following three equations:

$$\delta c + \dot{S}' \cdot \alpha - S' \circ \varphi^{-1} \cdot \delta\beta = \dot{H}_0 \tag{6}$$

$$\varphi' \circ \varphi^{-1} \cdot \delta\beta - \dot{\varphi}' \cdot \alpha + 2\, \dot{S}' \cdot Q = \dot{H}_1 \tag{7}$$

$$\delta K_{\geq 2} + \partial_\theta K \cdot \dot{\varphi} - r \cdot \dot{\varphi}' \cdot \partial_r K_{\geq 2} = \dot{H}_{\geq 2}. \tag{8}$$

The first equation aims at infinitesimally straightening the would-be invariant torus of $\dot{H}_0$, the second equation at straightening its dynamics, and the third at equating higher order terms. Due to the symplectic constraint, the first two equations are coupled (the Hamiltonian lift of the vector field $\dot{\varphi}$ having a non trivial component in the r-direction), whereas in the context of general vector fields the system is triangular. We will now show the existence of a unique solution to this system of equations, and derive estimates of the solution, within some appropriate functional setting.

Let

$$\mathbb{T}^n_s = \mathbb{T}^n \times i[-s,s]^n$$

be the complex extension of $\mathbb{T}^n$ of "width" s, and

$$|f|_s = \max_{\theta\in\mathbb{T}^n_s} |f(\theta)|$$

for functions f which are real holomorphic on the interior of $\mathbb{T}^n_s$ and continuous on $\mathbb{T}^n_s$; such functions form a space $\mathcal{A}(\mathbb{T}^n_s)$ which is Banach (there are other possible choices here). We extend this definition to vector-valued functions by taking the maximum of the norms of the componants (and, consistently, the ℓ^1-norm for "dual" integer vectors, e.g. $k\in\mathbb{Z}^n$). Similarly, let $\mathbb{R}^n_s$ be a complex neighborhood of the origin in $\mathbb{R}^n$ of "width" s:

$$\mathbb{R}^n_s = \{z\in\mathbb{C}^n,\ |z|\le s\},\quad |z| = \max(|z_1|,...,|z_n|).$$

We will call $\mathcal{A}(\mathbb{T}^n_s\times\mathbb{R}^n_s)$ the Banach space of functions which are continuous on $\mathbb{T}^n_s\times\mathbb{R}^n_s$ and real holomorphic on the interior.

Let

- $\mathcal{H}_s = \mathcal{H}\cap\mathcal{A}(\mathbb{T}^n_s\times\mathbb{R}^n_s)$ (endowed with the supremum norm $|\cdot|_s$)
- $\mathcal{K}_s(\alpha) = \mathcal{K}(\alpha)\cap\mathcal{A}(\mathbb{T}^n_s\times\mathbb{R}^n_s)\subset\mathcal{H}_s$.
- $\mathcal{G}^\sigma_s$ be the subset of $\mathcal{G}$ consisting of isomorphisms $G\simeq(\varphi,S)$ such that $\varphi-\mathrm{id}\in\mathcal{A}(\mathbb{T}^n_s,\mathbb{R}^n_1)$ and $S\in\mathcal{A}(\mathbb{T}^n_s,\mathbb{R}_1)$ and G is σ-polynomially-close to the identity, i.e.

$$|G-\mathrm{id}\,|_s \le C_{\mathcal{G}}\sigma^{k_{\mathcal{G}}} \tag{9}$$

  for some fixed $C_{\mathcal{G}}>0$ and $k_{\mathcal{G}}>0$ to be later determined.

2 Lemma (Linearized equation)**.** *If x is close enough to x^0, equation* (5) *posseses a unique solution $\dot{x} = (\delta K,\dot{G},\delta\beta)$. Moreover, there exist $C',\tau'>0$ such that, for all s,σ,*

$$|\dot{x}|_s \le \frac{C'}{\sigma^{\tau'}}|\dot{H}|_{s+\sigma},$$

where C' depends only on n, τ, provided K, G^{-1} and β are bounded on $\mathbb{T}^n_{s+\sigma}\times\mathbb{R}^n_{s+\sigma}$.

Proof. First assume that $\delta\beta\in\mathbb{R}^n$ is given with $|\delta\beta\cdot r\circ G^{-1}|\le Cst\,|\dot{H}|_{s+\sigma}$, and replace equation (7) by

$$\delta\hat{\beta} + \varphi'\circ\varphi^{-1}\cdot\delta\beta - \dot{\varphi}'\cdot\alpha + 2\,\dot{S}'\cdot Q = \dot{H}_1, \tag{10}$$

where $\delta\hat{\beta}\in\mathbb{R}^n$ is an additional unknown; as elsewhere in this proof, Cst stands for a constant, to which we do not want to give a consistant name, and which depends only on n, τ and $|(K-\alpha\cdot r, G^{-1}-\mathrm{id},\beta)|_{s+\sigma}$.

- Averaging equation (6) yields $\delta c = \int_{\mathbb{T}^n}\left(\dot{H}_0 + S'\circ\varphi^{-1}\cdot\delta\beta\right)\,d\theta$, hence

$$|\delta c|\le Cst\,|\dot{H}|_{s+\sigma}$$

- According to lemma 43, equation (6) has a unique solution $\delta\tilde{S}$ having zero average, with
$$|\delta\tilde{S}|_s \leq \frac{Cst}{\gamma\sigma^{\tau_c}}|\dot{H}_0|_{s+\sigma} \leq \frac{Cst}{\gamma\sigma^{\tau_c}}|\dot{H}|_{s+\sigma}.$$
Then the unique solution vanishing at the origin, $\delta S = \delta\tilde{S} - \delta\tilde{S}(0)$, satisfies the same estimate (up to an unessential factor 2 which we absorb in the constant).

Note that the estimates hold for all s, σ (at the expense of possibly having an infinite right hand side). We now proceed similarly with equation (10):

- The average yields $\delta\hat{\beta} = \int_{\mathbb{T}^n}\left(\dot{H}_1 - 2\dot{S}' \cdot Q - \varphi' \circ \varphi^{-1} \cdot \delta\beta\right)$, hence, using Cauchy's inequality,[3]
$$|\delta\hat{\beta}| \leq \frac{Cst}{\gamma\sigma^{\tau_c+1}}|\dot{H}|_{s+\sigma}.$$
- The average-free part determines $\delta\varphi$, with
$$|\dot{\varphi}|_s \leq \frac{Cst}{\gamma\sigma^{\tau_c}}\left(|\dot{H}_1|_{s+\sigma} + 2|\dot{S}' \cdot Q^o|_{s+\sigma}\right) \quad (\forall s, \sigma > 0).$$
Using Cauchy's inequality and the fact that Q^o is given, we see that
$$|\dot{\varphi}|_s \leq \frac{Cst}{\gamma^2\sigma^{\tau_c(\tau_c+1)}}|\dot{H}|_{s+\sigma},$$
where as before the constant depends only on n, τ, and $|Q^o|_{s+\sigma}$.

Equation (8) can then be solved explicitely:
$$\delta K_{\geq 2} = -\partial_\theta K^o \cdot \dot{\varphi} + 2r \cdot \dot{\varphi}' \cdot Q^o \cdot r + \delta H_{\geq 2},$$
and whence
$$|\delta K|_s \leq \frac{Cst}{\gamma^2\sigma^{\tau_c(\tau_c+1)+1}}|\delta H|_{s+\sigma}.$$

We have built a map $\delta\beta \mapsto \delta\hat{\beta}$ in the neighborhood of $\delta\beta = 0$. It is affine and, when φ is close to the identity, invertible. Thus there exists a unique $\delta\beta$ such that $\delta\hat{\beta} = 0$, which satisfies
$$\|\delta\beta\| \leq \frac{Cst}{\gamma\sigma^{\tau_c+1}}|\dot{H}|_{s+\sigma}.$$

The claim follows, with $\tau' = \tau_c(\tau_c + 1) + 1$ and $C' = Cst/\gamma^2$ for some constant Cst independant of γ.[4] □

[3] We use the ℓ^∞-norm on $\mathbb{R}^n$ (consistantly with the ℓ^1-norm already used on the dual space $\mathbb{Z}^n$).

[4] A better, but less transparent, choice of norms lets $C' = Cst/\gamma$ instead.

The lemma may be rephrased: the linear operator $\phi'(x)$ has a unique local inverse $\phi'(x)^{-1}$, with the given estimates.

Let

$$\delta x = \phi'(x)^{-1}(y - \phi(x)) \quad \text{and} \quad \hat{x} = \phi(x) = x + \delta x.$$

Taylor's formula says that

$$\begin{aligned} \phi(\hat{x}) &= \phi(x) + \phi'(x) \cdot \delta x + Q(x, \hat{x}), \quad Q(x, \hat{x}) = \int_0^1 (1-t)\phi''(x_t) \cdot (\delta x)^2 \, dt \\ &= y + Q(x, \hat{x}), \end{aligned}$$

where we have set $x_t = x + t\,\delta x$ $(0 \leq t \leq 1)$, hence

$$y - \phi(\hat{x}) = -Q(x, \hat{x}).$$

3 Lemma (Remainder)**.** *If* $|\dot{G}|_{s+\sigma} \leq \sigma/2$,

$$|Q(x, \hat{x}) \circ G^{-1}|_s \leq \frac{C''}{\sigma^{\tau''}} |\dot{x}|^2_{s+\sigma}.$$

Proof. Let $\delta^2\phi = \phi''(K, G, \beta) \cdot (\delta K, \delta G, \delta\beta)^2$. We have

$$\delta^2\phi = 2\delta K' \circ G \cdot \delta G + K'' \circ G \cdot (\delta G)^2,$$

hence

$$\delta^2\phi \circ G^{-1} = 2\delta K' \cdot \dot{G} + K'' \cdot \dot{G}^2,$$

so

$$|\delta^2\phi \circ G^{-1}|_s \leq \frac{Cst}{\sigma} |(\delta K, \dot{G})|^2_{s+\sigma}; \tag{11}$$

note here that $\delta^2\phi$ is computed in (K, G, β), and it is then pre-composed by G^{-1}.

Now, if $x_t = (K_t, G_t, \beta_t)$,

$$|Q(x, \hat{x}) \circ G^{-1}|_s \leq \int_0^1 \left| \left(\phi''(x_t) \cdot \delta x^2 \right) \circ G^{-1} \right|_s \, dt.$$

Since $|(\mathrm{id} + \dot{G})^{-1} - (\mathrm{id} - \dot{G})|_s \leq Cst |\dot{G}|^2_s$,

$$|Q(x, \hat{x}) \circ G^{-1}|_s \leq \int_0^1 \left| \left(\phi''(x_t) \cdot \delta x^2 \right) \circ G_t^{-1} \right|_{s + 2|\dot{G}|^2_{s+\sigma}} \, dt,$$

whence the wanted estimate, using (11). □

It remains to show that the iterated images

$$x_0 = (K^o, \mathrm{id}, 0), \quad x_{n+1} = f(x_n)$$

of the Newton map (3) are defined for $n \in \mathbb{N}$ and converge to some $(K, G, \beta) \in \mathcal{K}(\alpha) \times \mathcal{G} \times \mathbb{R}^n$ such that $H = K \circ G + \beta \cdot r$ in the neighborhood of $G^{-1}(\mathrm{T}_0)$, provided H is close enough to K^o. Namely, we will assume that $K^o \in \mathcal{K}_{s+\sigma}(\alpha)$, $H \in \mathcal{H}_{s+\sigma}$ for some fixed s, σ with $0 < s < s + \sigma \leq 1$, and

$$|H - K^o|_{s+\sigma} \leq \epsilon$$

for some $\epsilon > 0$. This is the goal of the next section.

3 Inverse function theorem

We first give an abstraction of our problem, and will afterwards show how it allows us to complete the proof of the twisted conjugacy theorem.

Let $E = (E_s)_{0<s<1}$ be a decreasing family of Banach spaces with increasing norms $|\cdot|_s$, and $\epsilon B^E_s = \{x \in E_s, |x|_s < \epsilon\}$, $\epsilon > 0$, be its balls centered at 0. Let (F_s) be an analogous family, and $\phi : \sigma B^E_{s+\sigma} \to F_s$, $s < s+\sigma$, $\phi(0) = 0$, be maps of class C^2, commuting with inclusions.

On account of composition operators, we will assume there are additional, deformed norms $|\cdot|_{x,s}$, $x \in \mathrm{Int}(sB^E_s)$, $0 < s < 1$, satisfying

$$|y|_{0,s} = |y|_s \quad \text{and} \quad |y|_{x',s} \le |y|_{x,s+|x'-x|_s},$$

and we will phrase our hypotheses on ϕ in terms of these norms.

Define

$$Q : \sigma B^E_{s+\sigma} \times \sigma B^E_{s+\sigma} \to F_s, \quad (x, \hat{x}) \mapsto \phi(\hat{x}) - \phi(x) - \phi'(x)(\hat{x} - x).$$

Assume that, if $x \in sB^E_{s+\sigma}$, the derivative $\phi'(x) : E_{s+\sigma} \to F_s$ has a right inverse $\phi'(x)^{-1} : F_{s+\sigma} \to E_s$, and

$$\begin{cases} |\phi'(x)^{-1}\eta|_s \le C'\sigma^{-\tau'}|\eta|_{x,s+\sigma} \\ |Q(x,\hat{x})|_{x,s} \le C''\sigma^{-\tau''}|\hat{x} - x|^2_{s+\sigma+|\hat{x}-x|_s} \quad (\forall s, \sigma, x, \hat{x}, \eta) \end{cases}$$

with $C', C'', \tau', \tau'' \ge 1$. Let $C := C'C''$ and $\tau := \tau' + \tau''$.

The important fact in the Newton algorithm below, is that the index loss σ can be chosen arbitrarily small, without s itself being small, provided the deformed norm substitutes for the initial norm of the spaces F_s. The initial norm $|\cdot|_s$ of F_s is here only for the practical purpose of having a fixed target space, to which perturbations belong.

4 Theorem. *ϕ is locally surjective and, more precisely, for any s, η and σ with $\eta < s$,*

$$\epsilon B^F_{s+\sigma} \subset \phi\left(\eta B^E_s\right), \quad \epsilon := 2^{-8\tau}C^{-2}\sigma^{2\tau}\eta.$$

In other words, ϕ has a right-inverse $\psi : \epsilon B^F_{s+\sigma} \to \eta B^E_s$.

Some numbers s, η and σ and $y \in B^F_{s+\eta}$ being given, let

$$f : \sigma B^E_{s+\eta+\sigma} \to E_s, \quad x \mapsto x + \phi'(x)^{-1}(y - \phi(x)).$$

Proof of the theorem. Now, let s, η and σ be fixed, with $\eta < s$ and $y \in \epsilon B^F_{s+\sigma}$ for some ϵ. We will see that if ϵ is small enough, the sequence $x_0 = 0$, $x_n := f^n(0)$ is defined for all $n \ge 0$ and converges towards some preimage $x \in \eta B^E_s$ of y by ϕ.

Let $(\sigma_n)_{n\ge0}$ be a sequence of positive real numbers such that $3\sum\sigma_n = \sigma$, and $(s_n)_{n\ge0}$ be the sequence decreasing from $s_0 := s + \sigma$ to s defined by induction by the formula $s_{n+1} = s_n - 3\sigma_n$.

Assuming the existence of $x_0, ..., x_{n+1}$, we see that $\phi(x_k) = y + Q(x_{k-1}, x_k)$, hence

$$x_{k+1} - x_k = \phi'(x_k)^{-1}(y - \phi(x_k)) = -\phi'(x_k)^{-1}Q(x_{k-1}, x_k) \qquad (1 \le k \le n).$$

Further assuming that $|x_{k+1} - x_k|_{s_k} \le \sigma_k$, the estimate of the right inverse and lemma 37 entail that

$$|x_{n+1} - x_n|_{s_{n+1}} \le c_n |x_n - x_{n-1}|_{s_n}^2 \le \cdots \le c_n c_{n-1}^2 \cdots c_1^{2^{n-1}} |x_1|_{s_1}^{2^{n-1}}, \quad c_k := C\sigma_k^{-\tau}.$$

The estimate

$$|x_1|_{s_1} \le C'(3\sigma_0)^{-\tau'}|y|_{s_0} \le C\sigma_0^{-\tau}\epsilon = c_0\epsilon$$

and the fact, to be checked later, that $c_k \ge 1$ for all $k \ge 0$, show :

$$|x_{n+1} - x_n|_{s_{n+1}} \le \left(\epsilon \prod_{k\ge 0} c_k^{2^{-k}}\right)^{2^n}.$$

Since $\sum_{n\ge 0} \rho^{2^n} \le 2\rho$ if $2\rho \le 1$, and using the definition of constants c_k's, we get a sufficient condition to have all x_n's defined and to have $\sum |x_{n+1} - x_n|_s \le \eta$:

$$\epsilon = \frac{\eta}{2} \prod_{k\ge 0} c_k^{-2^{-k}} = \frac{2\eta}{C^2} \prod_{k\ge 0} \sigma_k^{\tau 2^{-k}}. \tag{12}$$

Maximizing the upper bound of ϵ under the constraint $3\sum_{n\ge 0} \sigma_n = \sigma$ yields $\sigma_k := \frac{\sigma}{6} 2^{-k}$. A posteriori it is straightforward that $|x_{n+1} - x_n|_{s_n} \le \sigma_n$ (as earlier assumed to apply lemma 37) and $c_n \ge 1$ for all $n \ge 0$. Besides, using that $\sum k2^{-k} = \sum 2^{-k} = 2$ we get

$$\epsilon = \frac{\eta}{2} \prod_{k\ge 0} c_k^{-2^{-k}} = \frac{\eta}{2} \prod_{k\ge 0} \frac{1}{2^{\tau k 2^{-k}}} \left(\frac{1}{C}\left(\frac{\sigma}{6}\right)^{\tau}\right)^{2^{-k}} = \frac{\eta}{C^2}\left(\frac{\sigma}{12}\right)^{2\tau} > \frac{\sigma^{2\tau}\eta}{2^{8\tau}C^2},$$

whence the theorem. □

Remark. The two competing small parameters η and σ being fixed, our choice of the sequence (σ_n) maximizes ϵ for the Newton algorithm. It does not modify the sequence (x_k) but only the information we retain from (x_k).

5 Exercise (End of proof of theorem 1). Complete the proof by checking that

– A similar statement as theorem 4 holds if ϕ is defined only on a ball of polynomial radius with respect to the width of analyticity (recall (9)).

– $|K_n|_{s_n}$, $|G_n^{-1}|_{s_n}$ and β_n are bounded along the induction (in order to justify the repeated use of estimates of lemmas 2 and 11). *Hint:* Use the fact that

$$G_{n+1}^{-1} = G_n^{-1} \circ (\mathrm{id} + \dot{G}_n)^{-1},$$

the estimate of $\dot{G}$ in the induction and the estimate of proposition 44.

6 Corollary. *The size of the allowed perturbation is polynomial in γ.*

7 Exercise. What is the domain of ψ in F_S? *Hint:* Optimize the function $\epsilon(\eta, \sigma)$ under the constraint $s + \sigma = S$.

Bibliographical comments. – The seeming detour through Herman's normal form reduces Kolmogorov's theorem to a functionally well posed inversion problem, as opposed to Zehnder's (remarkable) work [82, 83]. One may compare the present stragegy and Zehnder's in the following way. Inverting the operator

$$\phi : (K, G, \beta) \mapsto H = K \circ G + \beta \cdot r$$

(recall equation (2)) is equivalent to solving the implicit function

$$F(K, G, \beta; H) = K - (H - \beta \cdot r) \circ G^{-1} = 0.$$

But ϕ happens to be a local diffeomorphism, while $\partial F/\partial(K, G, \beta)$ is invertible in no neighborhood of $(K^o, id, 0)$. This is why Zehnder had to deal with approximate inverses. The draw back of focusing on the equation $\phi(K, G, \beta) = H$ is that we neeed it to be satisfied on a domain which depends on G.

As Zehnder, we have encapsulated the Newton algorithm in an abstract inverse function theorem, à la Nash-Moser. The algorithm indeed converges without very specific hypotheses on the internal structure of the variables (see exercise 5, though). At the expense of some optimality, ignoring this structure allows for simple estimates and control of the bounds, and for solving a whole class of analogous problems with the same toolbox (lower dimensional tori, codimension-one tori, Siegel problem, as well as some problems in singularity theory).

– The fast convergence of the Newton algorithm makes it possible to beat the effect of small denominators and other sources of loss of width of analyticity. Its has proved unreasonably efficient compared to other lines of proof in KAM theory, including direct proofs of convergence of perturbation series or proofs via renormalization. An algorithm alternative to Newtons's consists, at each step of the induction, in solving a (non-linear) finite dimentional approximation of the functional equation (2) using Ekeland's variational principle [21].

– The arithmetic condition is not optimal. Indeed, solving the exact cohomological equation at each step is inefficient because the small denominators appearing with intermediate-order harmonics deteriorate the estimates, whereas some of these harmonics could have a smaller amplitude than the error terms and thus would better not be taken care of. Even stronger, Rüssmann and Pöschel have noticed that at each step it is worth neglecting part of the low-order harmonics themselves (to some carefully chosen extent). Then the expense, a worse error term, turns out to be cheaper than that the gain –namely, the right hand side of the cohomological equation now has a smaller size over a larger complex extension. This makes it possible, with a slowly converging sequence of approximations, to show the persistence of invariant tori under some arithmetic condition which, in one dimension, is equivalent to Brjuno's condition [62]. Bounemoura-Fischler have found an interesting alternative proof [10].

– The analytic (or Gevrey) category is simpler than Hölder or Sobolev categories, in Nash-Moser theory, because the Newton algorithm can be carried out without intercalating smoothing operators (cf. [51, 70, 36, 9]). On the other hand, the analytic category is more complicated because of the absence of cut off functions, which forces us to pay attention to the domain of definition of the Hamiltonian more carefully (cf. [26]).

– The method of Jacobowitz [39] (see [52] also) in order to deduce an inverse function theorem in the smooth category from its analogue in the analytic category does not work directly, here. The idea would be to use Jackson's theorem in approximation theory to caracterize the Hölder spaces by their approximation properties in terms of analytic functions and, then, to find a smooth preimage x by ϕ of a smooth function y as the limit of analytic preimages x_j of analytic approximations y_j of y. However, in our inversion function theorem we require the operator ϕ to be defined only on balls $\sigma B_{s+\sigma}$ with shrinking radii when $s+\sigma$ tends to 0. This domain is too small in general to include all the analytic approximations y_j of a smooth y. Such a restriction is inherent in the presence of composition operators. The problem of isometric embeddings is simpler, from this viewpoint. Yet we *could* generalize Jacobowitz's proof at the expense of making additionnal hypotheses on the form of our operator ϕ, which would take into account the specificity of directions K and G, as well as of the real phase space and of its complex extension.

4 Local uniqueness and regularity

In the proof of theorem 4 we have built right inverses $\psi : \epsilon B^F_{s+\eta+\sigma} \to \eta B^E_{s+\eta}$, of ϕ, commuting with inclusions. The proof shows that ψ is continuous at 0; due to the invariance of the hypotheses of the theorem by small translations, ψ is locally continuous.

We further make the following two assumptions:

— The maps $\phi'(x)^{-1} : F_{s+\sigma} \to E_s$ are left (as well as right) inverses (in theorem 1 we have restricted to an adequate class of symplectomorphisms);

— The scale $(|\cdot|_s)$ of norms of (E_s) satisfies some interpolation inequality:

$$|x|^2_{s+\sigma} \leq |x|_s\, |x|_{s+\tilde\sigma} \quad \text{for all } s,\, \sigma,\, \tilde\sigma = \sigma\left(1+\frac{1}{s}\right)$$

(according to the sentence after the statement of corollary 46, this estimate is satisfied in the case of interest to us, since $\sigma + \log(1+\sigma/s) \leq \tilde\sigma$).

8 Lemma (Lipschitz regularity)**.** *If $\sigma < s$ and $y, \hat y \in \epsilon B^F_{s+\sigma}$ with $\epsilon = 2^{-14\tau} C^{-3} \sigma^{3\tau}$,*

$$|\psi(\hat y) - \psi(y)|_s \leq C_L |\hat y - y|_{\psi(y),s+\sigma}, \quad C_L = 2C'\sigma^{-\tau'}.$$

In particular, ψ is the unique local right inverse of ϕ, i.e. it is also the local left inverse of ϕ.

Proof. Fix $\eta < \zeta < \sigma < s$; the impatient reader can readily look at the end of the proof how to choose the auxiliary parameters η and ζ more precisely.

Let $\epsilon = 2^{-8\tau}C^{-2}\zeta^{2\tau}\eta$, and $y, \hat{y} \in \epsilon B^F_{s+\sigma}$. According to theorem 4, $x := \psi(y)$ and $\hat{x} := \psi(\hat{y})$ are in $\eta B^E_{s+\sigma-\zeta}$, provided the condition, to be checked later, that $\eta < s + \sigma - \zeta$. In particular, we will use a priori that

$$|\hat{x} - x|_{s+\sigma-\zeta} \leq |\hat{x}|_{s+\sigma-\zeta} + |x|_{s+\sigma-\zeta} \leq 2\eta.$$

We have

$$\begin{aligned} \hat{x} - x &= \phi'(x)^{-1}\phi'(x)(\hat{x} - x) \\ &= \phi'(x)^{-1}(\hat{y} - y - Q(x, \hat{x})) \end{aligned}$$

and, according to the assumed estimate on $\phi'(x)^{-1}$ and to lemma 37,

$$|\hat{x} - x|_s \leq C'\sigma^{-\tau'}|\hat{y} - y|_{x,s+\sigma} + 2^{-1}C\zeta^{-\tau}|\hat{x} - x|^2_{s+2\eta+|\hat{x}-x|_s}.$$

In the norm index of the last term, we will coarsely bound $|\hat{x} - x|_s$ by 2η. Additionally using the interpolation inequality:

$$|\hat{x} - x|^2_{s+4\eta} \leq |\hat{x} - x|_s|\hat{x} - x|_{s+\tilde{\sigma}}, \quad \tilde{\sigma} = 4\eta\left(1 + \frac{1}{s}\right),$$

yields

$$\left(1 - 2^{-1}C\zeta^{-\tau}|\hat{x} - x|_{s+\tilde{\sigma}}\right)|\hat{x} - x|_s \leq C'\sigma^{-\tau'}|\hat{y} - y|_{x,s+\sigma}.$$

Now, we want to choose η small enough so that

— first, $\tilde{\sigma} \leq \sigma - \zeta$, which implies $|\hat{x} - x|_{s+\tilde{\sigma}} \leq 2\eta$. By definition of $\tilde{\sigma}$, it suffices to have $\eta \leq \frac{\sigma-\zeta}{4(1+1/s)}$.

— second, $2^{-1}C\zeta^{-\tau}\,2\eta \leq 1/2$, or $\eta \leq \frac{\zeta^\tau}{2C}$, which implies that $2^{-1}C\zeta^{-\tau}|\hat{x} - x|_{s+\tilde{\sigma}} \leq 1/2$, and hence $|\hat{x} - x|_s \leq 2C'\sigma^{-\tau'}|\hat{y} - y|_{x,s+\sigma}$.

A choice is $\zeta = \frac{\sigma}{2}$ and $\eta = \frac{\sigma^\tau}{16C} < s$, whence the value of ϵ in the statement. □

9 Proposition (Smoothness). *For every $\sigma < s$, there exists ϵ, C_1 such that for every $y, \hat{y} \in \epsilon B^F_{s+\sigma}$,*

$$|\psi(\hat{y}) - \psi(y) - \phi'(\psi(y))^{-1}(\hat{y} - y)|_s \leq C_1|\hat{y} - y|^2_{s+\sigma}.$$

Moreover, the map $\psi' : \epsilon B^F_{s+\sigma} \to L(F_{s+\sigma}, E_s)$ defined locally by $\psi'(y) = \phi'(\psi(y))^{-1}$ is continuous and, if $\phi : \sigma B^E_{s+\sigma} \to F$ is C^k, $2 \leq k \leq \infty$, for all σ, so is $\psi : \epsilon B^F_{s+\sigma} \to E_s$.

Proof. Fix ϵ as in the previous proof and $y, \hat{y} \in \varepsilon B^F_{s+\sigma}$. Let $x = \psi(y)$, $\eta = \hat{y} - y$, $\xi = \psi(y+\eta) - \psi(y)$ (thus $\eta = \phi(x+\xi) - \phi(x)$), and $\Delta := \psi(y+\eta) - \psi(y) - \phi'(x)^{-1}\eta$. Definitions yield

$$\Delta = \phi'(x)^{-1}(\phi'(x)\xi - \eta) = -\phi'(x)^{-1}Q(x, x + \xi).$$

Using the estimates on $\phi'(x)^{-1}$ and Q and the latter lemma,

$$|\Delta|_s \leq C_1|\eta|^2_{s+\sigma'}$$

for some σ' tending to 0 when σ itself tends to 0, and for some $C_1 > 0$ depending on σ. Up the substitution of σ by σ', the estimate is proved.

The inversion of linear operators between Banach spaces being analytic, $y \mapsto \phi(\psi(y))^{-1}$ has the same degree of smoothness as ϕ'. □

10 Corollary. *If $\pi \in L(E_s, V)$ is a family of linear maps, commuting with inclusions, into a fixed Banach space V, then $\pi \circ \psi$ is C^1 and $(\pi \circ \psi)' = \pi \cdot \phi' \circ \psi$.*

This corollary is used with $\pi : (K, G, \beta) \mapsto \beta$ in the proof of theorem 1.

It will later be convenient to extend ϕ^{-1} to non-Diophantine vectors α. Whitney-smoothness is a criterion for such an extension to exist [76, 80].

Suppose $\phi(x) = \phi_\alpha(x)$ now depends on some parameter $\alpha \in B^\kappa$ (the unit ball of $\mathbb{R}^\kappa$),

— that the estimates assumed up to now are uniform with respect to α over some closed subset $\mathrm{D} \subset \mathbb{R}^\kappa$,

— and that ϕ is C^1 with respect to α.

We will denote ψ_α the parametrized version of the inverse of ϕ_α.

11 Proposition (Whitney-smoothness)**.** *If s, σ and ϵ are chosen like in proposition 9, the map $\psi : \mathrm{D} \times \epsilon B^F_{s+\sigma} \to E_s$ is C^1-Whitney-smooth and extends to a map $\psi : \mathbb{R}^n \times \epsilon B^F_{s+\sigma}$ of class C^1. If ϕ is C^k, $1 \le k \le \infty$, with respect to α, this extension is C^k.*

Proof. Let $y \in \epsilon B^F_{s+\sigma}$. If $\alpha, \alpha+\beta \in \mathrm{D}$, $x_\alpha = \psi_\alpha(y)$ and $x_{\alpha+\beta} = \psi_{\alpha+\beta}(y)$, we have

$$\phi_{\alpha+\beta}(x_{\alpha+\beta}) - \phi_{\alpha+\beta}(x_\alpha) = \phi_\alpha(x_\alpha) - \phi_{\alpha+\beta}(x_\alpha).$$

Since $\hat{y} \mapsto \psi_{\alpha+\beta}(\hat{y})$ is Lipschitz (lemma 8),

$$|x_{\alpha+\beta} - x_\alpha|_s \le C_L |\phi_\alpha(x_\alpha) - \phi_{\alpha+\beta}(x_\alpha)|_{s+\sigma},$$

and, since $\hat{\alpha} \mapsto \phi_{\hat{\alpha}}(x_\alpha)$ itself is Lipschitz, so is $\alpha \mapsto x_\alpha$.

Moreover, the formal derivative of $\alpha \mapsto x_\alpha$ is

$$\partial_\alpha x_\alpha = -\phi'_\alpha(x_\alpha) \cdot \partial_\alpha \phi(x_\alpha).$$

Expanding $y = \phi_{\alpha+\beta}(x_{\alpha+\beta})$ at $\beta = 0$ and using the same estimates as above, shows that

$$|x_{\alpha+\beta} - x_\alpha - \partial_\alpha x_\alpha \cdot \beta|_s = O(\beta^2)$$

when $\beta \to 0$, locally uniformly with respect to α. Hence $\alpha \mapsto x_\alpha$ is C^1-Whitney-smooth, and, similarly, C^k-Whitney-smooth if $\alpha \mapsto \phi_\alpha$ is.

Thus, by the Whitney extension theorem, the claimed extension exists. Note that Whitney's original theorem needs two straightforward generalizations to be applied here: ψ takes values in a Banach space, instead of $\mathbb{R}$ or a finite dimension vector space (see [33]); and ψ is defined on a Banach space, but the extension directions are of finite dimension. □

12 Exercise (Quasiperiodic time dependant perturbations). Let $\nu \in \mathbb{R}^m$ be fixed. Consider the subspace $\mathcal{H}_\nu$ of $\mathcal{H}$ (in dimension $2(n+m)$) consisting of Hamiltonians in

$$\mathbb{T}^{n+m} \times \mathbb{R}^{n+m} = \mathbb{T}^n_\theta \times \mathbb{T}^m_\psi \times \mathbb{R}^n_r \times \mathbb{R}^m_\Psi$$

of the form

$$\hat{H} = \nu \cdot \Psi + H,$$

where H does not depend on Ψ. Since the corresponding Hamiltonian vector field has the component

$$\dot{\psi} = \nu,$$

$\mathcal{H}_\nu$ may be imaged as the space of Hamiltonians on $\mathbb{T}^n \times \mathbb{R}^n$ with quasiperiodic time dependance. Show that, if $\hat{H} \in \mathcal{H}_\nu$ and $\hat{H} = \phi(K, G, (\beta, \beta'))$ (with $\beta \in \mathbb{R}^n$ and $\beta' \in \mathbb{R}^m$), then

$$\begin{cases} K \in \mathcal{H}_\nu \\ G \text{ leaves } \psi \text{ unchanged} \\ \beta' = 0. \end{cases}$$

Further question: develop the KAM theory below in this particular case.

13 Exercise (Control & persistence of tori). If H is close to an integrable Hamiltonian $K^o = K^o(r)$, show that there is a smooth integrable Hamiltonian $\beta = \beta(r)$ such that for every R such that T_0 is a (γ, τ)-Diophantine invariant torus of K^o, $H - \beta(r) \cdot r$ has an invariant torus carrying a quasiperiodic dynamics with the same frequency.

Hint. Apply the twisted conjugacy theorem to each $H(R + \cdot, \cdot)$, with $R \in \mathbb{R}^n$ close to 0 such that the torus $r = R$ is Diophantine for K^o and, using propositions 11, extend the so-obtained function $R \mapsto \beta(R)$ as a smooth function.

Bibliographical comments. – It is possible to prove that ψ is C^1 without the asumption that $\phi'(x)$ has unique (or right) inverse, just by patterning [70, p. 626]). Yet the proof simplifies and the estimates improve under the combined two additional asumptions. In particular, the existence of a right inverse of $\phi'(x)$ makes the inverse ψ unique and thus allows to ignore the way it was built (a posteriori regularity result).

– Latzutkin understood, in the case of the standard map, the fundamental importortance of Whitney-smoothness of the invariant circles with respect the rotation number. This is a key point in the method of parameter.

– The dependance is of Gevrey class [60], but we do not need it here.

5 Conditional conjugacy

We now move to a *conditional conjugacy*, the common ground of invariant tori theorems of later sections.

Let

$$\mathcal{K}_s = \cup_{\alpha \in \mathbb{R}^n} \mathcal{K}_s(\alpha) = \left\{ c + \alpha \cdot r + O(r^2),\ c \in \mathbb{R}, \alpha \in \mathbb{R}^n \right\}$$

be the set of Hamiltonians on $\mathbb{T}^n_s \times \mathbb{R}^n_s$ for which T_0 is invariant and quasi-periodic, with unprescribed frequency.

14 Theorem (Conditional conjugacy)**.** *For every $K^o \in \mathcal{K}_{s+\sigma}(\alpha^o)$ with $\alpha^o \in D_{\gamma,\tau}$, there is a germ of smooth map*

$$\Theta : \mathcal{H}_{s+\sigma} \to \mathcal{K}_s \times \mathcal{G}_s, \quad H \mapsto (K_H, G_H), \quad K_H = c_H + \alpha_H \cdot r + O(r^2),$$

at $K^o \mapsto (K^o, \mathrm{id})$ such that the following implication holds:

$$(\forall H)\ \alpha_H \text{ Diophantine} \Longrightarrow H = K_H \circ G_H$$

and (K_H, G_H) is unique in $\mathcal{K} \times \mathcal{G}$.

Proof. Denote ϕ_α the operator we have been denoting ϕ –because the frequency α was fixed while we now want to vary it. Define the map

$$\begin{array}{rlcl} \hat{\Theta}: & \mathrm{D}_{\gamma,\tau} \times \mathcal{H}_{s+\sigma} & \to & \mathcal{K}_s \times \mathcal{G}_s \times \mathbb{R}^n \\ & (\alpha, H) & \mapsto & \hat{\Theta}_\alpha(H) := (\phi_\alpha)^{-1}(H) = (K, G, \beta) \end{array}$$

locally in the neighborhood of (α^o, K^o). Since ϕ is infinitely differentiable, by proposition 11 there exist a C^∞-extension

$$\hat{\Theta} : \mathbb{R}^n \times \mathcal{H}_{s+\sigma} \to \mathcal{K}_s \times \mathcal{G}_s \times \mathbb{R}^n.$$

Write $K^o = \alpha^o \cdot r + \hat{K}$, $\hat{K} = c + O(r^2)$. In particular, since

$$\phi_\alpha(K^o + (\alpha - \alpha^o) \cdot r, \mathrm{id}, \alpha^o - \alpha) \equiv K^o$$

locally for all $\alpha \in \mathbb{R}^n$ close to α^o we have

$$\hat{\Theta}(\alpha, K^o) = (K^o, \mathrm{id}, \beta), \quad \beta(\alpha, K^o) = \alpha^o - \alpha.$$

In particular,

$$\frac{\partial \beta}{\partial \alpha} = -\,\mathrm{id}$$

and, by the implicit function theorem, locally for all H there exists a unique $\hat{\alpha}$ such that $\beta(\hat{\alpha}, H) = 0$. We conclude by letting $\Theta(H) = \hat{\Theta}(\hat{\alpha}, H)$. □

The so-defined vector α_H, which is called a *frequency vector* of H, is unique when belonging to $D_{\gamma,\tau}$. It depends Gevrey-smoothly on H (i.e. their partial derivatives of order r behave like positive powers of $r!$), as discovered by Popov [60], but not analytically (except for a family of integrable Hamiltonians). (For our purpose, the Lipschitz regularity would suffice, in conjunction with the Lipschitz inverse function theorem. For the sake of simplicity, we stick to the C^1 class.)

6 Invariant torus with prescribed frequency

The first invariant torus theorem will be a trivial corollary of the conditional conjugacy theorem. Consider a smooth family $(K^o_t)_{t \in \mathbb{B}^\kappa}$ of Hamiltonians in some

$\mathcal{K}_s$. Each K^o_t is of the form $K^o_t = c^o_t + \alpha^o_t \cdot r + O(r^2)$. The frequency map of the family is

$$\alpha^o : \mathbb{B}^\kappa \mapsto \mathbb{R}^n, \quad t \mapsto \alpha^o_t.$$

In this section, we will describe the simplest case, where (the derivative of) α^o has rank n, which, by the submersion theorem, implies that α^o is onto, stably with respect to smooth C^1-perturbations. In celestial mechanics, the parameter t may be masses, semi major axes, eccentricities, inclinations, energy, angular momentum, etc.

Now, let (H_t) be a smooth family of Hamiltonians in $\mathcal{H}_s$ such that, for each t, H_t is close enough to K_t (a condition that we will not repeat in each statement).

15 Theorem. *If the frequency map α^o is a local submersion (that is, of rank n) and if $\alpha^o(0) \in D_{\gamma,\tau}$, there exists a unique $t \in \mathbb{B}^n$ such that H_t has an invariant torus with frequency $\alpha^o(0)$. Moreover, the subset formed by the values of $t \in \mathbb{B}^n$ for which H_t has an invariant torus has positive Lebegue measure.*

Proof. According to the conditional conjugacy theorem, the family (H_t) has some frequency map α which is C^∞-close to α^o. So α itself is a local C^∞-submersion and attains $\alpha^o(0)$. Besides, as soon as $\alpha_t \in D_{\gamma,\tau}$, H_t has an invariant torus, which happens for a subset $\mathcal{B} \subset \mathbb{B}^n$ of positive Lebesgue measure. □

16 Remark. The first part of the conlusion holds under the topological hypothesis that α^o has non-zero degree, a remark which applies if α^o has a ramification point, for example.

Poincaré [59] introduced the following two transversality conditions (he was considering the particular case, considered next, where t is the action variable, $K^o = K^o(r)$ and $\alpha = \partial_r K^o(r)$).

17 Definition. The Hamiltonian family (K^o_t) is

- *isochronically non-degenerate* if the frequency map has rank n
- *isoenergetically non-degenerate* if the map

  $$\mathbb{B}^\kappa \to \mathbb{R} \times \mathbb{P}(\mathbb{R}^n), \quad t \mapsto ([\alpha^o_t], c^o_t)$$

  (where $[\alpha^o_t]$ stands for the homogeneous class of α^o_t) has rank n.

(Neither condition implies the other.)

18 Exercise (Variants of theorem 15). Prove the following two variants.

– Isoenergetic theorem: If the family (K_t) is isoenergetically non-degenerate, and if the frequency vector α^o belongs to $D_{\gamma,\tau}$, there exists $t \in \mathbb{B}^n\kappa$ such that H_t has an invariant torus of energy c^o_0 and frequency class $[\alpha^o_0]$. Moreover, the subset formed by the values of $t \in \mathbb{B}^n$ for which H_t has an invariant torus of energy c^o_0 has positive $(n-1)$-dimensional Lebesgue measure.[5]

[5]There is no intrinsic $(n-1)$-dimensional Lebesgue measure, but the fact of having positive $(n-1)$-dimensional Lebesgue measure *is* intrinsic.

– "Iso first integral" theorem: More generally, assume that for all t, f_t is an $\mathbb{R}^\lambda$-valued first integral of K^o_t and H_t (e.g., with $\lambda = 2$, f_t may stand for the energy and the angular momentum of a mechanical system in the plane) and that the frequency vector α^o belongs to $D_{\gamma,\tau}$ The function f_t must be constant on T_0 and we call $f_t(\mathrm{T}_0)$ this constant. If the map

$$\mathbb{B}^\kappa \to \mathbb{R} \times \mathbb{P}(\mathbb{R}^n), \quad t \mapsto (f_t(\mathrm{T}_0), [\alpha^o_t])$$

has maximal rank n, there exists $t \in \mathbb{B}^\kappa$ such that H_t has an invariant torus on which $f_t = f_0(\mathrm{T}_0)$ and with frequency class $[\alpha^o_0]$. More strongly, if the map

$$\mathbb{B}^\kappa \to \mathbb{R} \times \mathbb{R}^n, \quad t \mapsto (f_t(\mathrm{T}_0), \alpha^o_t)$$

has maximal rank n, for every $t^o \in \mathbb{B}^\kappa$ close to 0 there exists $t \in \mathbb{B}^\kappa$ such that H_t has an invariant torus with $f_t = f_0(\mathrm{T}_0)$ and frequency vector α^o_0.

(For a first integral associated with a non-Abelian symmetry, see section 8).

We now turn to Kolmogorov's theorem, which corresponds to the particular case where the family (H_t) is obtained by mere translation of some initial Hamiltonian $H \in \mathcal{H}$, in the direction of actions: $H_t(\theta, r) = H(\theta, t + r)$, $t \in \mathbb{B}^n$. Call c^o and Q^o the constant and quadratic parts of some $K^o \in \mathcal{K}(\alpha^o)$:

$$K^o = c^o + \alpha^o \cdot r + Q^o(\theta) \cdot r^2 + O(r^3).$$

19 Theorem (Kolmogorov)**.** *If the frequency vector α^o belongs to $D_{\gamma,\tau}$ and if the quadratic form $\int_{\mathbb{T}^n} Q^o(\theta)\, d\theta$ is non-degenerate, there exists a unique $R \in \mathbb{R}^n$ such that $G^{-1}(\mathrm{T}_0) + (0, R)$ is an α^o-quasiperiodic invariant torus of H. Moreover, the invariant tori of H form a set of positive Lebesgue measure in the phase space.*

Proof. Let F be the analytic function taking values among symmetric bilinear forms, which solves the cohomological equation $L_{\alpha^o} F = Q^o - \int_{\mathbb{T}^n} Q^o\, d\theta$ (use lemma 43), and ψ be the germ along T_0 of the (well defined) time-one map of the flow of the Hamiltonian $F(\theta) \cdot r^2$. The map ψ is symplectic and restricts to the identity on T_0. At the expense of substituting $K^o \circ \psi$ and $H \circ \psi$ for K^o and H respectively, one can thus assume that

$$K^o = c^o + \alpha^o \cdot r + Q^1 \cdot r^2 + O(r^3), \quad Q^1 := \int_{\mathbb{T}^n} Q^o(\theta)\, d\theta.$$

The germs so obtained from the initial K^o and H are close to one another.

Consider the family of trivial perturbations obtained by translating K^o in the direction of actions:

$$K^o_R(\theta, r) := K^o(\theta, R + r), \quad R \in \mathbb{R}^n,\ R \text{ small},$$

and its approximation obtained by truncating the first order jet of K^o_R along T_0 from its terms $O(R^2)$:

$$\hat{K}^o_R(\theta, r) := (c^o + \alpha^o \cdot R) + \big(\alpha^o + 2Q^1 \cdot R\big) \cdot r + O(r^2) = K^o_R + O(R^2).$$

For the Hamiltonian $\hat{K}^o_R$, T_0 is invariant and quasiperiodic of frequency $\alpha^o + 2Q^1 \cdot R$. The first assertion then follows from theorem 15.

What has been done for the torus of frequency α^o can more generally be done for all tori of Diophantine frequency. What remains to be proved is that the collection of perturbed invariant tori has positive measure. Using the map Θ of the conditional conjugacy theorem, now define

$$(K_R, G_R) = \Theta(H_R),$$

with

$$\begin{cases} K_R = c_R + \alpha_R \cdot r + O(r^2) \\ G_R(\theta, r) = (\varphi_R(\theta), (r + S'_R(\theta)) \cdot \varphi'_R(\theta)^{-1}) \end{cases}$$

locally in the neighborhood of $R = 0$, say for $\|R\| < R_0$. Let

$$\mathcal{R} = \{R \in \mathbb{R}^n,\ \|R\| < R_0,\ \alpha_R \in D_{\gamma,\tau}\}.$$

As soon as $R \in \mathcal{R}$, $G_R^{-1}(\mathrm{T}_0)$ is invariant for H_R, hence

$$\mathcal{T}_R = G_R^{-1}(\mathrm{T}_0) + (0, R) = \{(\theta, R - S'_R(\theta)),\ \theta \in \mathbb{T}^n\}$$

is invariant for H. Because of proposition 11, $\mathcal{T}_R$ depends Whitney-smoothly on $R \in \mathcal{R}$. Thus the diffeomorphisms which straighten all the $\mathcal{T}_R$'s individually may be glued together, by Whitney's extension theorem and the last assertion follows. □

20 Remark (Measure of the set of tori). Due to the estimate of the inverse function theorem 4, if $\gamma \ll 1$, the allowed size of $|H - K^o|_s$ (for some $s > 0$) is polynomial in γ (of degree 4). One can actually show that it is $|H - K^o|_s = O(\gamma^2)$ [61]. In other words, for a given H, a torus with frequency vector in $D_{\gamma,\tau}$ is preserved for some $\gamma = O(\sqrt{\epsilon})$, and, as a classical estimate of the measure of the complement of Diophantine vectors shows, the measure of the complement of the invariant tori is of order $O(\gamma) = O(\sqrt{\epsilon})$.

Once one has one invariant torus, it is straightforward to obtain a set of positive measure of invariat tori, as the proof above has shown. (This was not so at the level of generality of theorem 15. Why? If $t_1, t_2 \in \mathcal{B}$, the invariant tori of H_{t_1} and H_{t_2} may meet. In Kolmogorov's theorem, the parameter being the cohomology class of the tori, this cannot happen.) We will see in the next section that a much weaker transversality condition is sufficient for locally finding a positive measure of tori. Yet, in the absence of any transversality hypothesis, the question of the accumulation of a quasiperiodic invariant torus by other quasiperiodic solutions, and their measure, is the subject of Herman's conjecture [23].

21 Exercise. Instead of applying theorem 15, complete the proof of theorem 19 using the twisted conjugacy theorem.

Hint. The twisted conjugacy normal form of $\hat{K}^o_R$ with respect to the frequency α is

$$\hat{K}^o_R = \left(\hat{K}^o_R - \hat{\beta}^o_R \cdot r\right) \circ \mathrm{id} + \hat{\beta}^o_R \cdot r, \quad \hat{\beta}^o_R := 2Q^1 \cdot R.$$

By assumption the matrix $\frac{\partial \hat{\beta}^o}{\partial R}\Big|_{R=0} = 2Q^1$ is invertible and the map $R \mapsto \hat{\beta}^o(R)$ is a local diffeomorphism. Now, there is an analogous map $R \to \beta(R)$ for H_R, which is a small C^∞-perturbation of $R \mapsto \hat{\beta}^o(R)$, and thus a local diffeomorphism, with a domain having a lower bound locally uniform with respect to H. Hence if H is close enough to K^o there is a unique small R such that $\beta = 0$. For this R the equality $H_R = K \circ G$ holds, hence the torus obtained by translating $G^{-1}(\mathrm{T}_0)$ by R in the direction of actions is invariant and α-quasiperiodic for H.

Bibliographical comments. – Kolmogorov's celebrated theorem initiated KAM theory. Claims that Kolmogorov's proof was incomplete are unfounded in view of the breakthrough: the supposedly missing arguments in Kolmogorov's paper bear upon to Cauchy's inequality and elementary harmonic analysis [43, 17, 28]. Kolmogorov actually gave these details in Moscow's seminar, as Arnold and Sinaï have testified. Arnold later gave an alternative proof. Arnold's statement is equivalent to Kolmogorov's, despite the superficial difference of looking to all neighboring tori at a time. Arnold additionally payed attention to how far H can be from K^o, as the torsion gets close to degenerate [3].

– The remark that parameters are not necessarily action variables adds some flexibility for finding invariant tori, e.g. in the work of Zhao L. [85, 86]. Another example is an analogue of Arnold's theorem where one would be allowed to tune not only the semi major axes but also the masses of the planets.

7 Invariant tori with unprescribed frequencies

There is a KAM theory which assumes only a much weaker non-degeneracy condition as above. Let $\alpha : \mathbb{B}^\kappa \to \mathbb{R}^n$ be a smooth map.

22 Definition. The frequency map α is *skew*[6] if its image is nowhere locally contained in a vector hyperplane.

23 Lemma (Rüssmann [65, 66])**.** *If α is skew and analytic, for all $t \in B^\kappa$ there exist $r \in \mathbb{N}_*$ and $j_1, ..., j_r \in \mathbb{N}^\kappa$ such that*

$$\mathrm{Vect}\left(\partial^{j_1}\alpha_t, ..., \partial^{j_r}\alpha_t\right) = \mathbb{R}^n; \qquad (13)$$

the integer $\max_i |j_i|$ *(where $|j_i|$ is the length of j_i) is called the* index of degeneracy *at t. Conversely, if there exists $t \in B^\kappa$, $r \in \mathbb{N}_*$ and $j_1, ..., j_r \in \mathbb{N}^\kappa$ such that* (13) *holds, α is skew.*

The property of being skew is a very weak transversality condition. It is of crucial interest that κ may be smaller than n.

24 Example. The monomial curve

$$\alpha : t \in [0,1] \mapsto (1, t, ..., t^{n-1}) \in \mathbb{R}^n$$

[6] The terminology we have chosen here is not standard. Related (but not always equivalent) conditions have been called *essentially non planar*, *non planar*, *Rüssmann-non-degenerate*, *weakly non-degenerate*, *curved* etc.

skew. Indeed, with the convention that $1/n! = 0$ if $n \in \mathbb{Z}^-$, the matrix

$$\left(\alpha, \alpha', \cdots, \alpha^{(n-1)}\right) = \left(\frac{(j-1)!}{(j-i)!} t^{j-i}\right)_{1 \leq i,j \leq n}.$$

has rank n.

See [69] for a comparison with a dozen conditions which have been used in KAM theory. Here we content ourselves with the following examples, showing in particular that being skew is implied by the traditional conditions of isochronic or isoenergetic non-degeneracy.

25 Example. – If α is isochronically non-degenerate, at every point $t \in B^\kappa$ its local image is an open set of $\mathbb{R}^n$, so α is skew, with index of degeneracy equal to 1.

– Suppose that t is the action variable, $H = H(r)$ and $\alpha = \partial_r H(r)$. If then H is isoenergetically non-degenerate, its frequency map is skew. Indeed, since the determinant of the "bordered torsion"[7] $\begin{pmatrix} \alpha' & \alpha \\ {}^t\alpha & 0 \end{pmatrix}$ is non zero, α' must have rank $n-1$, the bordered torsion is equivalent to

$$\begin{pmatrix} \bar{\tau} & 0 & \bar{\alpha} \\ 0 & 0 & \beta \\ {}^t\alpha & \beta & 0 \end{pmatrix}$$

with $\det \bar{\tau} \neq 0$ and $\beta \in \mathbb{R}$, hence the borderd torsion has determinant $-\beta^2 \det \bar{\tau}$, hence $\beta \neq 0$, hence H is skew with index of degeneracy equal to 1.

26 Example (L. Chierchia)*.* The integrable Hamiltonian defined over $\mathbb{T}^4 \times \mathbb{R}^4$ by

$$H = \frac{1}{4} r_1^4 + \frac{1}{2} r_1^2 r_2 + r_1 r_3 + r_4$$

is isochronically and isoenergetically degenerate, but its frequency, as a function of the action r_1, is skew at $(r_1, 0, 0, 0)$, $r_1 \neq 0$.

We now take up hypotheses of the beginning of section 6, i.e. we consider a smooth family $(K_t^o)_{t \in \mathbb{B}^\kappa}$ of Hamiltonians in $\mathcal{K}$. Each K_t^o is of the form $K_t^o = c_t^o + \alpha_t^o \cdot r + O(r^2)$. The (analytic) frequency map of the family is

$$\alpha^o : \mathbb{B}^\kappa \mapsto \mathbb{R}^n, \quad t \mapsto \alpha_t^o.$$

Let (H_t) be a smooth family of Hamiltonians in $\mathcal{H}_s$ such that, for each t, H_t is close enough to K_t. The conditional conjugacy theorem yields a smooth frequency map $t \mapsto \alpha_t$ of H which is C^∞-close to $t \mapsto \alpha_t^o$.

27 Proposition (Rüssmann [69])**.** *If α^o is skew, there exists $\mu \in \mathbb{N}_*$ (an affine function of the index of degeneracy of α^o) such that if α is C^μ-close to α^o,*

$$\mathrm{Leb}\,\{t \in \mathbb{B}^\kappa,\ \alpha_t \notin D_{\gamma,\tau}\} \leq C\gamma^{1/\mu}.$$

[7] Poincaré calls this square matrix the "bordered Hessian" of H [59].

From the proof of the proposition, it is not hard to see how these estimate deteriorate when there are several time scales (a situation otherwise called *properly degenerate*).

28 Corollary. *Under the hypotheses of proposition 27, if we split* α *into* $\alpha = (\hat{\alpha}, \check{\alpha}) \in \mathbb{R}^{\hat{n}} \times \mathbb{R}^{\check{n}}$*,* $\hat{n} + \check{n} = n$*, then*

$$\text{Leb}\,\{t \in \mathbb{B}^{\kappa},\ (\hat{\alpha}_t, \epsilon\check{\alpha}_t) \notin D_{\gamma,\tau}\} \le C\left(\frac{\gamma}{\epsilon}\right)^{1/\mu}$$

for some affine function μ *of the index of degeneracy.*

An immediate consequence is the following theorem.

29 Theorem. *If the frequency map* α^o *is skew, there is a subset* $\mathcal{T} \subset \mathbb{B}^{\kappa}$ *of positive Lebesgue measure such that, for all* $t \in \mathcal{T}$*,* H_t *has a Diophantine quasiperiodic invariant torus.*

30 Remark (Size of the allowed perturbation). In applications, it is often the case that there are several time scales. For example, in the planetary 3-body problem the dynamics splits into a fast Keplerian dynamics and a slow secular dynamics. If one wants to apply KAM theory, it is then crucial to know the size of the allowed perturbation in terms of these time scales. The relevant estimates may be established along the following lines.

Consider for example the case of a frequency curve $\alpha = (\hat{\alpha}, \check{\alpha}) : I \mapsto \mathbb{R}^n = \mathbb{R}^{\hat{n}} \times \mathbb{R}^{\check{n}}$, $t \mapsto (\hat{\alpha}(t), \check{\alpha}(t))$, assumed skew at some $t_0 \in I$. Then, after corollary 28, if we want to have some measure estimates which are uniform with respect to small ϵ, we need to choose $\gamma = O(\epsilon^N)$ for some N large enough. Last, due to the estimate of the inverse function theorem 4, if $\gamma \ll 1$, the allowed size of $|H - K^o|_s$ (for some $s > 0$) is polynomial in γ, hence in ϵ. (One can show that $|H - K^o|_s = O(\gamma^2)$ is enough for the conclusion to hold [61].)

The analogue of Kolmogorov's theorem for the weak transversality condition of being skew is the following. Consider one Hamiltonian $K \in \mathcal{K}(\alpha^o)$ for some $\alpha^o \in DH_{\gamma,\tau}$ (with γ small enough and τ large enough) and one Hamiltonian $H \in \mathcal{H}$ close to $\mathcal{K}$. Upon putting K^o under normal form at some high enough order, theorem 14 gives the existence of a frequency map $r \mapsto \alpha^o_r$ of K.

31 Theorem (Rüssmann)**.** *If the frequency map* α^o *is skew, the invariant tori of* H *form a set of positive Lebesgue measure in the phase space.*

The proof mimicks the second part of the proof of Kolmogorov's theorem.

Bibliographical comments. The theory of Diophantine approximations on manifolds was initiated by the works of Arnold and his students; see [7, 41, 63, 75]. It has later been used in dynamical systems, e.g. in [15, 6, 5, 22, 56, 57, 67, 68, 69].

8 Symmetries

This section consists in a remark regarding Hamiltonian systems invariant under a Hamiltonian group action. The natural way to find invariant tori is to apply KAM theory to the symplectically reduced system. Here, we explain how to take advantage of the symmetries "upstairs", avoiding to carry out explicit computations on the quotient.

Let (X, ω) be a symplectic real analytic manifold of dimension $2n$ and G a compact group, acting analytically on X in a Hamiltonian way, freely and properly. Call $2m$ the (neccessarily even) corank of G.

Let T_0 be a Lagrangian embedded real analytic torus of X and $K_t^o : X \mapsto \mathbb{R}$, $t \in \mathbb{B}^\kappa$, be a smooth family of G-invariant real analytic functions (Hamiltonians) for which T_0 is invariant, quasiperiodic of frequency vector $\alpha_t^o \in \mathbb{R}^n$.

The main example is a rotation-invariant mechanical system. The condition of being skew is always violated, because one frequency (corresponding in the phase space to the two directions of non trivial rotations of the angular momentum vector) vanishes identically. One can get rid of this degeneracy by fixing the direction of the angular momentum (see [84]). The remaining invariance by rotations around the direction of the angular momentum adds some flexibility for checking the transversality condition, since the harmonics which are not invariant have zero Fourier coefficient. What follows is an abstraction of this situation.

32 Lemma. *The image of the frequency map $t \mapsto \alpha_t^o$ lies in a subspace of $\mathbb{R}^n$ of codimension m.*

Proof. Let $\mathcal{T}$ be a maximal torus of G; its codimension is $2m$. Let μ be the moment map, thought of as a map $X \to \mathfrak{g}$, and $\mathfrak{t}_+$ be the positive Weyl chamber of $\mathcal{T}$. Guillemin and Sternberg have noticed that $X_+ = \mu^{-1}(\mathfrak{t}_+)$ is a symplectic, codimension-$2m$, real analytic submanifold of X, and a section of the G-action [34]. The velocity vector on T_0 is tangent to X_+, so $\mathrm{T}_0 \cap X_+$ is an invariant torus, whose ergodic components are isotropic (see appendix A), hence of dimension at most $n - m$. □

Let $\mathcal{T}$ be a maximal torus as in the proof above, of Lie algebra $\mathfrak{t} = \mathbb{R}^k$ (k thus being the rank of G). Let $\tau : X \to \mathbb{R}^k$ be its moment map. Consider the amended Hamiltonian

$$\hat{K}_{t,u}^o = K_t^o + u \cdot \tau, \tag{14}$$

depending on parameters $t \in \mathbb{B}^\kappa$ and $u \in \mathbb{R}^k$. By Lagrangian intersection theory, it has the same ergodic Lagrangian invariant tori as K_t^o, and the frequency vector of T_0 is changed into

$$\hat{\alpha}_{t,u}^o = \alpha_t^o + u \cdot \tau_1,$$

where $\tau_1 \in M_{k,n}(\mathbb{R})$ is defined by τ's Taylor expansion at $r = 0$:

$$\tau = \tau_0 + \tau_1 \cdot r + O(r^2).$$

Call $\mathrm{Vect}\,\tau_1$ the subspace of $\mathbb{R}^n$ spanned by the k row-vectors of τ_1. This is the subspace of frequencies which may be attained by tuning the parameter u.

Rather then repeating the whole theory in the G-invariant setting, we merely adapt four chief statements, according to the following array of hypotheses:

	Submersive frequency	Skew frequency
Partially reduced system	1	3
Fully reduced system	2	4

33 Theorem (G-invariant KAM theorem)**.** *1. If the frequency map $\alpha^o : \mathbb{R}^\kappa \to \mathbb{R}^n$ has rank $\geq n-m$ at 0, there exists t such that H_t has an invariant torus of frequency α^o_0. Besides, the subset formed by the values of $t \in \mathbb{B}^n$ for which H_t has an invariant torus has positive Lebegue measure.*

2. *If the amended frequency map $\hat{\alpha}^o : \mathbb{R}^\kappa \times \mathbb{R}^k \to \mathbb{R}^n/\mathrm{Vect}\,\tau_1$, has rank $\geq n-m$ at $(0,0)$, there exists t such that H_t has an invariant torus of frequency α^o_0* $(\mathrm{mod}\ \mathrm{Vect}\,\tau_1)$*. Besides, the subset formed by the values of $t \in \mathbb{B}^n$ for which H_t has an invariant torus has positive Lebegue measure.*

3. *If the image of the amended frequency map does not lie in any plane of codimension $> m$ in $\mathbb{R}^n$, for a subset of $t \in \mathbb{B}^\kappa$ of positive Lebesgue measure, H_t has a rank-$(n-m)$ quasiperiodic invariant torus.*

4. *If the image of the amended frequency map does not lie in any plane of codimension $> m$ in $\mathbb{R}^n/\mathrm{Vect}\,\tau_1$, for a subset of $t \in \mathbb{B}^\kappa$ of positive Lebesgue measure, H_t has a rank-$(n-m)$ (posibly non minimal) quasiperiodic invariant torus.*

If the parameter is the translation in the direction of the action variable r, one could further infer the existence of a subset of the phase space and of positive Lebesgue measure, consisting of invariant tori, as in section 6, using an argument which we will not repeat here.

Items 1 and 3 yield minimal tori. Items 2 and 4 yield strictly more tori, foliated into minimal invariant subtori of codimenion from 0 to k. Determining this codimension requires to compute the frequencies of the lift of the $\mathcal{T}$-action, which boils down to a quadrature, along the lines of the standard theory of symplectic reduction.

Proof. First restrict to the symplectic manifold X_+, which has dimension $2(n-m)$ (this restriction has sometimes been called *partial reduction*). Items 1 and 3 of the statement follow from theorems 15 and 29 respectively. Now, restrict to a regular level of μ and quotient by $\mathcal{T}$. The reduced Hamiltonian system of K^o_t has frequency the equivalence class of α^o_t modulo $\mathrm{Vect}\,\tau_1$. So, resonance hyperplanes in the partially reduced phase space which are broken by $u \cdot \tau_1$ project to zero in the reduced system. Assertions 2 and 4 thus follow from theorems 15 and 29, this time applied this time to the fully reduced system. □

Use will be made of this theorem in our study of the three-body problem.

Bibliographical comments. The idea of amending the Hamiltonian goes back to Poincaré when he would look to the three-body problem in a rotating frame of reference in order to break some degeneracies in his search for periodic orbits [59]. The role of partial reduction (consisting in fixing only the direction of the angular momentum) was put forward in [48].

9 Lower dimensional tori

In this section, we sketch the theory for lower dimensional invariant tori. Some additional details may be found in [26].

Two integers $n \geq 1$ and $m \geq 0$ being fixed, let $\mathcal{H}$ be the set of germs along $\mathrm{T}_0 = \mathbb{T}^n \times \{0\} \times \{0\}$ of real analytic functions (Hamiltonians) in the phase phase

$$\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^m = \{(\theta, r, z = (x, y))\}.$$

A Hamiltonian $H \in \mathcal{H}$ defines a germ of vector field

$$\begin{cases} \dot\theta = \partial_r H \\ \dot r = -\partial_\theta H, \end{cases} \qquad \begin{cases} \dot x = \partial_y H \\ \dot y = -\partial_x H. \end{cases}$$

Let $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$. Split the integer m into $m = m' + m''$ (m' and m'' will respectively be the numbers of hyperbolic and elliptic directions), and let Q_β be the matrix

$$Q_\beta = 2\pi \operatorname{Diag}(\beta_1, ..., \beta_m, -\beta_1, ..., -\beta_{m'}, \beta_{m'+1}, ..., \beta_m) \in \mathrm{M}_{2m}(\mathbb{R}).$$

Define $\mathcal{K}(\alpha, \beta)$ as subset of $\mathcal{H}$ of Hamiltonians of the form

$$\begin{aligned} K &= c + \alpha \cdot r + \frac{1}{2} Q_\beta \cdot z^2 + O(r^2, rz, z^3) \\ &= \sum_{j=1}^{n} \alpha_j r_j + \pi \sum_{j=1}^{m'} \beta_j (x_j^2 - y_j^2) + \pi \sum_{j=m'+1}^{m} \beta_j (x_j^2 + y_j^2) + O(r^2, rz, z^3) \end{aligned}$$

where c is some (non-fixed) real number.

In the following definitions, maps are all real analytic. Let $\mathrm{B}^1(\mathbb{T}^n)$ be the group of exact 1-forms on $\mathbb{T}^n$, $\mathcal{D}$ be the group of isomorphisms of $\mathbb{T}^n$ fixing the origin,

$$Sp_{2m} = \left\{\psi \in M_{2m}(\mathbb{R}),\ {}^t\psi J \psi = J\right\}, \quad J = \begin{pmatrix} 0 & -\mathrm{id}_{\mathbb{R}^m} \\ \mathrm{id}_{\mathbb{R}^m} & 0 \end{pmatrix},$$

be the symplectic group,

$$\mathcal{A}_*(\mathbb{T}^n, Sp(2m)) = \{\exp \Delta\psi \in \mathcal{A}(\mathbb{T}^n, Sp(2n)), \Delta\psi \in \mathcal{A}_*(\mathbb{T}^n, sp_{2m})\}$$

be the image by the exponential of the subspace

$$\mathcal{A}_*(\mathbb{T}^n, sp_{2m}(\mathbb{R})) =$$

$$\left\{\psi \in \mathcal{A}(\mathbb{T}^n, sp_{2m}(\mathbb{R})),\ {}^t\psi = \psi \text{ and } \int_{\mathbb{T}^p} \psi_{jj}(\theta)\, d\theta = 0,\ j = 1, ..., 2m\right\}. \tag{15}$$

Let now

$$\mathcal{G} = B^1(\mathbb{T}^n) \times \mathcal{A}(\mathbb{T}^n, \mathbb{R}^{2m}) \times \mathcal{D}_* \times \mathcal{A}_*(\mathbb{T}^n, Sp(2m)).$$

Let $a = (\theta, r, z) = (\theta, r, x, y) \in \mathbb{T}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q$ and $G = (\rho, \zeta, \varphi, \psi) \in \mathcal{G}$. If ψ is C^0-close to the constant map $\theta \mapsto \mathrm{id}_{\mathbb{R}^{2q}}$, there exists a unique $\dot\psi \in \mathrm{C}^\infty_*(\mathbb{T}^p, \mathrm{sp}(2q))$ such that $\psi = \exp\dot\psi$. Let

$$\begin{cases} \rho(a) &= (\theta, r + \rho, z) \\ \zeta(a) &= (\theta, r + R_\zeta, z + \zeta(\theta)) \\ \varphi(a) &= (\varphi(\theta), {}^tD\varphi(\theta)^{-1} \cdot r, z) \\ \psi(a) &= (\theta, r + S_\psi \cdot z^2, \psi(\theta) \cdot z), \end{cases}$$

with

$$\begin{cases} R_\zeta &= -J \cdot ((z + \zeta/2) D\zeta) \\ S_\psi \cdot z^2 = & \frac{1}{2} \int_0^1 \left(\exp(t\dot\psi) \cdot z\right)^2 \cdot \left(J \cdot D\dot\psi\right) dt, \end{cases} \tag{16}$$

and then

$$G(a) = \psi(\varphi(\zeta(\rho(a)))). \tag{17}$$

This defines an exact symplectomorphism.

The generalized twisted conjugacy theorem is as follows.

Assume (α, β) is Diophantine in this sense: for every $k \in \mathbb{Z}^n$, $l' \in \mathbb{Z}^{m'}$ and $l'' \in \mathbb{Z}^{m''}$ such that $|l'|, |l''| = 1$ or 2,

$$\begin{cases} |k \cdot \alpha| \geq \frac{\gamma}{|k|^\tau} & \text{(if } k \neq 0) \\ |l' \cdot \beta'| \geq \gamma & \\ |k \cdot \alpha + l'' \cdot \beta''| \geq \frac{\gamma}{(|k|+1)^\tau} & \textit{(Melnikov condition)} \end{cases} \tag{18}$$

34 Theorem. *If $H \in \mathcal{H}$ is close enough to some $K^o \in \mathcal{K}(\alpha, \beta)$, there exists a unique $(K, G, \hat\alpha, \hat\beta) \in \mathcal{K}(\alpha, \beta) \times \mathcal{G} \times \mathbb{R}^n \times \mathbb{R}^m$ close to $(K^o, \mathrm{id}, 0, 0)$ such that*

$$H = K \circ G + \hat\alpha \cdot r + \frac{1}{2} Q_{\hat\beta} \cdot z^2.$$

We will skip the proof here. It only combines the same formal ideas as in the smooth category [26] and the inverse function theorem of section 3.

The theory for lower dimensional tori unwinds as in the Lagrangian case. Of course, there is no direct analogue of Kolmogorov's theorem if $m'' > 0$, since there are not enough action variables to control all of the tangent and normal frequencies. Let us merely give one statement, corresponding to theorem 29.

Consider a smooth family $(K^o_t)_{t \in \mathbb{B}^\kappa}$ of Hamiltonians in some $\mathcal{K}_s$, defining a frequency map

$$(\alpha^o, \beta^o) : \mathbb{B}^\kappa \mapsto \mathbb{R}^n, \quad t \mapsto (\alpha^o_t, \beta^o_t).$$

Let (H_t) be a smooth family of Hamiltonians in $\mathcal{H}_s$ such that, for each t, H_t is close enough to K_t.

35 Theorem. *If the frequency map* (α^o, β^o) *is skew, there is a subset* $\mathcal{T} \subset \mathbb{B}^\kappa$ *of positive Lebesgue measure such that, for all* $t \in \mathcal{T}$, H_t *has a Diophantine quasiperiodic invariant torus.*

Bibliographical comments. – It was a big surprise when H. Eliasson proved an invariant torus theorem for normaly elliptic tori [22], due to the problem of the lack of parameters.

– Bourgain later proved that it suffices to assume $|l| = 1$ in the Melnikov condition. The proof is more difficult since one cannot straighten the normal dynamics of the torus, so the linearized equations are not diagonal anymore in Fourier space [58].

10 Example in the spatial three-body problem

The Hamiltonian of the three-body problem is

$$H = \sum_{0 \le j \le 2} \frac{\|p\|^2}{2m_j} - \sum_{0 \le j < k \le 2} \frac{m_j m_k}{\|q_j - q_k\|},$$

where $q_j \in \mathbb{R}^3$ is the position of the j-th body and $p_j \in \mathbb{R}^3$ is its impulsion. We will prove the existence of quasiperiodic motions in the *hierarchical* (or *lunar*) *regime*, where two bodies (say, q_0 and q_1) revolve around each other while the third body revolves, far away, around the center of mass of the two primaries. Another classical perturbative regime would have been the *planetary regime*, where there is no asumption on the distances of the bodies, but two masses (planets) are assumed small with respect to the remaining one (Sun).

36 Theorem. *There exist a set of initial conditions of positive Lebesgue measure leading to quasiperiodic solutions, arbitrarily close to Keplerian, coplanar, circular motions, with semi major axis ratio arbitrarily small.*

The hurried reader may simplify the following discussion by focusing on the plane invariant subproblem.

Let $(Q_0, Q_1, Q_2, P_0, P_1, P_2)$ be the Jacobi coordinates, defined by:

$$\begin{cases} Q_0 = q_0 \\ Q_1 = q_1 - q_0 \\ Q_2 = q_2 - \sigma_0 q_0 - \sigma_1 q_1, \end{cases} \qquad \begin{cases} P_0 = p_0 + p_1 + p_2 \\ P_1 = p_1 + \sigma_1 p_2 \\ P_2 = p_2, \end{cases}$$

where $1/\sigma_0 = 1 + m_1/m_0$ and $1/\sigma_1 = 1 + m_0/m_1$. P_0 is the total linear momentum, which can be assumed equal to 0 without loss of generality. Besides, H does not depend on Q_0. So, (Q_1, Q_2, P_1, P_2) is a symplectic coordinate system on the phase space reduced by the symmetry of translation, and the equations read

$$\begin{cases} \dot{Q}_i = \partial_{P_i} H \quad (i = 1, 2) \\ \dot{P}_i = -\partial_{Q_i} H. \end{cases}$$

A direct computation shows that

$$H = \sum_{1\le i\le 2} \frac{\|P_i\|^2}{2\mu_i} - \sum_{0\le i<j\le 2} \frac{m_i m_j}{\|q_i - q_j\|},$$

with

$$M_0 = m_0, \quad M_1 = m_0 + m_1 \quad \text{and} \quad \frac{1}{\mu_i} = \frac{1}{M_{i-1}} + \frac{1}{m_i}.$$

One can split H into two parts

$$H = \mathrm{Kep} + \mathrm{Rem}$$

where

$$\mathrm{Kep} = \sum_{1\le i\le 2} \left(\frac{\|P_i\|^2}{2\mu_i} - \frac{\mu_i M_i}{\|Q_i\|} \right)$$

is a sum of two uncoupled Kepler problems, and

$$\mathrm{Rem} = \frac{\mu_2 M_2}{\|Q_2\|} - \frac{m_0 m_2}{\|q_2 - q_0\|} - \frac{m_1 m_2}{\|q_2 - q_1\|}$$

is the remainder.

Let us assume that the two terms of Kep are negative so that each body Q_i under the flow of Kep describes a Keplerian ellipse. Let $(\ell_i, L_i, g_i, G_i, \theta_i, \Theta_i)_{i=1,2}$ be the associated Delaunay coordinates. These coordinates are symplectic and analytic over the open set where motions are non-circular and non-horizontal [29]; since we will precisely be interested in a neighborhood of circular coplanar motions, these variables are only intermediate coordinates for computations. One shows that

$$\mathrm{Kep} = -\sum_{i=1,2} \frac{\mu_i^3 M_i^2}{2L_i^2}.$$

The Keplerian frequencies[8] are

$$\kappa_i = \frac{\partial\,\mathrm{Kep}}{\partial L_i} = \frac{\mu_i^3 M_i^2}{L_i^3} = \frac{\sqrt{M_i}}{a_i^{3/2}},$$

so that the *Keplerian frequency map*

$$\kappa : (L_1, L_2) \mapsto (\kappa_1, \kappa_2)$$

is a diffeomorphim $(\mathbb{R}_+^\times)^2 \to (\mathbb{R}_+^\times)^2$. Due to the fact that the Keplerian part depends only on 2 of the action variables, solutions of the Keplerian approximation are quasiperiodic with atmost 2 independant frequencies. This degeneracy has been interpreted as a hidden $SO(4)$-symmetry for each planet, whose momentum map is given partly by the eccentricity vector. How the Keplerian ellipses slowly rotate and deform will be determined by mutual attractions. This degeneracy is specific to the Newtonian and elastic potentials, as Bertrand's theorem asserts [8].

[8]Traditionally given the ununderstandable name of *mean motions*.

In the hierarchical regime ($a_1 \ll a_2$) the dominating term of the remainder is

$$\text{Main} := -\mu_1 m_2 P_2(\cos\theta)\frac{\|X_1\|^2}{\|X_1\|^3}, \tag{19}$$

with $P_2(c) = \frac{1}{2}(3c^2-1)$ (second Legendre polynomial) and $\theta = \widehat{Q_1, Q_2}$. Since the Keplerian frequencies satisfy $\kappa_1 \gg \kappa_2$, we may average out the fast, Keplerian angles ℓ_1 and ℓ_2 successively, thus without small denominators [25, 40]. The *quadrupolar Hamiltonian* is

$$\text{Quad} = \int_{\mathbb{T}^2} \text{Main}\ \frac{d\ell_1\, d\ell_2}{4\pi^2}; \tag{20}$$

It is the dominating interaction term which rules the slow deformations of the Keplerian ellipses. It naturally defines a Hamiltonian on the space of pairs of Keplerian ellipses with fixed semi major axes, which is called the *secular space.*

After reduction by the symmetry of rotations (e.g. with Jacobi's reduction of the nodes, which consists in fixing the angular momentum vector, say, vertically, and quotienting the so-obtained codimension-3 Poisson submanifold by rotations around the angular momentum), the secular space has 4 dimensions, with coordinates (g_1, G_1, g_2, G_2) outside coplanar or circular motions.

37 Lemma. *The quadrupolar system* Quad *is integrable.*

Indeed, it happens that Quad does not depend on the argument g_2 of the pericenter of the outer ellipse (but the next higher order term, the "octupolar term", does), thus proving its integrability:

$$\text{Quad} = -\frac{\mu_1 m_2 a_1^2}{8a_2^3\,(1-e_2^2)^{3/2}} \left[\begin{array}{l} (15e_1^2\cos^2 g_1 - 12e_1^2 - 3)\,\sin^2(i_2 - i_1) \\ +3e_1^2 + 2 \end{array} \right], \tag{21}$$

where i_j is the inclination of the ellipse of Q_j with respect to the Laplace plane (e.g. [46]); the Hamiltonian in the plane problem is simply obtained by letting $i_1 = i_2 = 0$.

We now need to estimate the frequencies and the torsion of the quadrupolar system, somewhere in the secular space. Lidov-Ziglin [46] have established the bifurcation diagram of the system, and proved the existence of 5 regimes in the parameter space, according to the number of equilibrium points of the reduced quadrupolar system. Here, for the sake of simplicity we will localize our study in some regular region (i.e. a region with a uniform action-angle coordinate system). We will focus on the neighborhood of the origin of the secular space, i.e. circular horizontal Keplerian ellipses. See [45] for more details on the computations.

38 Lemma (Lagrange, Laplace)**.** *The first quadrupolar system has a degenerate elliptic singularity at the origin of the secular space, whose normal frequency vector is*

$$\alpha_{\text{Quad}}(0) = -\frac{3a_1^2}{4a_2^3}\begin{pmatrix} \Lambda_1^{-1} \\ \Lambda_2^{-1} \\ -\Lambda_1^{-1} - \Lambda_2^{-1} \\ 0 \end{pmatrix}.$$

Proof. The following steps lead to the wanted expansion of Quad (20):

- Using elementary geometry, express $\cos\theta_{12}$ in terms of the elliptic elements and the true anomalies. Then substitute the variable u_1 for v_1, using the relations

$$\cos(v_1) = \frac{a_1}{\|X_1\|}(\cos u_1 - e_1) \quad \text{and} \quad \sin v_1 = \frac{a_1}{\|X_1\|}\sqrt{1-e_1^2}\sin u_1.$$

- Multiply Main by the Jacobian of the change of angles

$$\frac{d\ell_1\, d\ell_2}{du_1\, dv_2} = \frac{\|X_1\|}{a_1}\,\frac{\|X_2\|^2}{a_2\sqrt{1-e_2^2}}.$$

- In the integrand of (20) with $i=2$, express the distances to the Sun in terms of the inner eccentric anomaly u_1 and outer true anomaly v_2:

$$\|X_1\| = a_1(1-e_1\cos u_1) \quad \text{and} \quad \|X_2\| = \frac{a_2(1-e_2^2)}{1+e_2\cos v_2},$$

  and expand at the second order with respect to eccentricitites and inclinations (even powers vanish; the fourth order will be needed later).

- The obtained expression is trigonometric polynomial in u_1 and v_2. Average it.

- Switch to the Poincaré coordinates $(\xi_j, \eta_j, p_j, q_j)$, which are symplectic and analytic in the neighborhood of circular horizontal ellipses, and are defined by the relations

$$\begin{cases} \xi_j + i\eta_j = \sqrt{2L_j}\sqrt{1-\sqrt{1-e_j^2}}\, e^{-i(g_j+\theta_j)} \\ p_j + iq_j = \sqrt{2L_j}\sqrt{\sqrt{1-e_j^2}(1-\cos i_j)}\, e^{-i\theta_j}; \end{cases}$$

  here we use the notation $\Lambda_j = L_j = \mu_j\sqrt{M_j a_j}$.

The invariance of Quad by horizontal rotations entails that, as proved by Lagrange and Laplace, there exist two quadratic forms Q_h and Q_v (indices h and v here stand for “horizontal” and “vertical”) on $\mathbb{R}^2$ such that

$$\text{Quad} = -\frac{3a_1^2}{8a_2^3}\left(Q_h(\xi) + Q_h(\eta) + Q_v(p) + Q_v(q) + O_4(\xi,\eta,p,q)\right).$$

The computation shows that

$$\begin{cases} Q_h(\xi) = \frac{\xi_1^2}{L_1} + \frac{\xi_2^2}{L_2} \\ Q_v(p) = -\frac{p_1^2}{L_1} - \frac{p_2^2}{L_2} + \frac{2p_1p_2}{\sqrt{L_1L_2}} \end{cases}$$

The horizontal part is already in diagonal form. The vertical part Q_v is diagonalized by the orthogonal operator of $\mathbb{R}^2$

$$\rho = \frac{1}{\sqrt{\Lambda_1+\Lambda_2}} \begin{pmatrix} \sqrt{\Lambda_2} & \sqrt{\Lambda_1} \\ -\sqrt{\Lambda_1} & \sqrt{\Lambda_2} \end{pmatrix}.$$

This operator of $\mathbb{R}^2$ lifts to a symplectic operator

$$\begin{aligned}\tilde\rho\ &: (x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)\\ &\mapsto (\xi_1,\eta_1,\xi_2,\eta_2,p_1,q_1,p_2,q_2) = (x_1,y_1,x_2,y_2,p_1,q_1,p_2,q_2)\end{aligned}$$

with

$$(p_1,p_2) = \rho\cdot(x_3,x_4) \quad \text{and} \quad (q_1,q_2) = \rho\cdot(y_3,y_4).$$

and

$$\mathrm{Quad} = -\frac{a_1^2}{4a_2^3}\left(\begin{array}{c} 1+\frac{3}{2\Lambda_1}(x_1^2+y_1^2)+\frac{3}{2\Lambda_2}(x_2^2+y_2^2)- \\ \frac{3}{2}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(x_3^2+y_3^2)+O_4(x,y) \end{array}\right), \tag{22}$$

thus showing that the origin is elliptic (as mentioned above), and degenerate (since there is no term in $x_4^2+y_4^2$).

Switching (outside the origin) to symplectic polar coordinates $(\tilde\varphi_j,\tilde r_j)_{j=1,\dots,4}$ defined by

$$x_j+iy_j = \sqrt{2\tilde r_j}\,e^{-i\tilde\varphi_j},$$

one gets the wanted expression of $\alpha_{\mathrm{Quad}}(0) = \frac{\partial\,\mathrm{Quad}}{\partial\tilde r}(0)$. □

It is an exercise (e.g. using generating functions) to check that all the changes of coordinates we have made on the secular space lift to changes of coordinates in the full phase space, up to adequately modifying the mean longitude. This does not change the Keplerian frequencies.

The quadrupolar frequency vector $\alpha_{\mathrm{Quad}}(0)$ calls for some comments:

- Due to the $SO(3)$-symmetry, rotations of the two inner ellipses around a horizontal axis leave Quad invariant. Hence the infinitesimal generators of such rotations (last two columns of the matrix $\tilde\rho$ in the proof of lemma 38) span an eigenplane of the quadratic part of (22), with eigenvalue 0. This explains for the vanishing last component of the normal frequency vector (for all r's for that matter):

$$\alpha_{\mathrm{Quad}}(0)_4 = 0 \qquad (\forall \Lambda_1,\Lambda_2). \tag{23}$$

- Unexpectedly, the sum of the frequencies vanishes:

$$\sum_{1\le j\le 4}\alpha_{\mathrm{Quad}}(0)_j = 0 \qquad (\forall\Lambda_1,\Lambda_2). \tag{24}$$

- The local image of the map $(\Lambda_1, \Lambda_2) \mapsto \alpha_{\text{Quad}}(0)$ thus lies in a 2-plane of $\mathbb{R}^4$ but in no line, since the map

$$(\Lambda_1, \Lambda_2) \mapsto -\frac{3a_1^2}{4a_2^3} \begin{pmatrix} \Lambda_1^{-1} \\ \Lambda_2^{-1} \end{pmatrix} = -\frac{3}{4} \frac{M_2^3}{M_1^2} \frac{\mu_2^6}{\mu_1^4} \begin{pmatrix} \Lambda_1^3 \Lambda_2^{-6} \\ \Lambda_1^4 \Lambda_2^{-7} \end{pmatrix}$$

  is a diffeomorphism. Hence, additional resonances may always be removed by slightly shifting Λ_1 and Λ_2.

39 Proposition. *The local image of the frequency map*

$$(\mathbb{R}_*^+)^2 \to \mathbb{R}^6, \quad (a_1, a_2) \mapsto \alpha = (\kappa_1, \kappa_2, \alpha_{\text{Quad}}(0))$$

is contained in the codimension-2 subspace

$$\alpha_6 = 0, \quad \alpha_3 + \alpha_4 + \alpha_5 = 0 \tag{25}$$

but in no subspace of larger codimension.

Proof. What remains to be checked is the second, negative assertion, i.e. that the frequency map

$$\begin{aligned} \tilde{\alpha} : (a_1, a_2) \mapsto \tilde{\alpha} &= (\kappa_1, \kappa_2, \alpha_{\text{Quad}}(0)_1, \alpha_{\text{Quad}}(0)_2) \\ &= \left(c_1 a_1^{-3/2}, c_2 a_2^{-3/2}, c_3 a_1^{3/2} a_2^{-3}, c_4 a_1^2 a_2^{-7/2} \right) \end{aligned}$$

is skew, wherethe c_i's depend only on the masses. Restricting to for example to the curve $a_2 = a_1^3$, one gets a frequency vector whose components are Laurent monomials in $\sqrt{a_1}$, with no two components of the same degree. Such a curve is skew according to example 24 (using the fact that extracting components of the monomial curve will preserve the skew property). □

Resonances (23) and (24) a priori prevent from eliminating resonant terms in the Lindstedt (or Birkhoff) normal form of Quad, and from applying theorem 29. And they will not disappear by adjusting the Λ_j's. But, as the following lemma shows, there are no resonant terms at the second order.

Let

$$\mathcal{L}(2) = \left\{ (\Lambda_1, \Lambda_2) \in \mathbb{R}^2, \ \forall |k| \leq 4, \ k_1 \Lambda_1^{-1} + k_2 \Lambda_2^{-1} \neq 0 \right\}$$

be the open set of values of (Λ_1, Λ_2) for which the horizontal first quadrupolar frequency vector satisfies no resonance of order ≤ 4.

40 Lemma. *If the parameters* (Λ_1, Λ_2) *belong to* $\mathcal{L}(2)$, Quad *has a non-resonant Lindstedt normal form at order* 2 *i.e., there exist coordinates* $(\varphi_j, r_j)_{j=1,\dots,4}$, *tangent to* $(\tilde{\varphi}_j, \tilde{r}_i)_{j=1,\dots,4}$, *such that*

$$\text{Quad} = \text{cst} + \alpha_{\text{Quad}}(0) \cdot r + \frac{1}{2} \tau_{\text{Quad}} \cdot r^2 + O(r^3).$$

Besides, the torsion is

$$\tau_{\mathrm{Quad}} = \left(\begin{array}{ccc|c} & & & 0 \\ & \bar\tau_{\mathrm{Quad}} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

with $\bar\tau_{\mathrm{Quad}} =$

$$-\frac{a_1^2}{8a_2^3}\frac{1}{\Lambda_1^2\Lambda_2^2}\begin{pmatrix} -6\,\Lambda_2^2 & 18\,\Lambda_1\,\Lambda_2 & -24\,\Lambda_2^2 - 18\,\Lambda_1\,\Lambda_2 \\ 18\,\Lambda_1\,\Lambda_2 & 24\,\Lambda_1^2 & -18\,\Lambda_1\,\Lambda_2 - 24\,\Lambda_1^2 \\ -24\,\Lambda_2^2 - 18\,\Lambda_1\,\Lambda_2 & -18\,\Lambda_1\,\Lambda_2 - 24\,\Lambda_1^2 & 6\,\Lambda_2^2 + 18\,\Lambda_1\,\Lambda_2 + 6\,\Lambda_1^2 \end{pmatrix}.$$

Proof. We carry out the same computation as in the proof of lemma 38, now up to the order 2 in the $\tilde r_j$'s. The truncated expression is a trigonometric polynomial in the angles $\tilde\varphi_j$, of degree ≤ 4. Eliminating non-resonant monomials, i.e. functions of $k \cdot \varphi$ with $k \cdot \alpha_{\mathrm{Quad}}(0)$, is a classical matter. Two kinds of terms cannot be eliminated by averaging:

- Monomials in the angle $4\tilde\varphi_4$. Such monomials actually cannot occur in the expansion, due to the invariance by rotations (they would not satisfy the d'Alembert relation [48]).
- Monomials in $\tilde\varphi_1 + \tilde\varphi_2 + \tilde\varphi_3 + \tilde\varphi_4$. Such monomials happen not to occur.

A direct computation leads to the given expression of the torsion τ_{Quad}. □

Note that the torsion τ_{Quad}, as a function of Λ_1 and Λ_2, extends analytically outside $\mathcal{L}(2)$ (as often do first order normal forms). This allows us to define the *quadrupolar frequency map*

$$\alpha_{\mathrm{Quad}} : r \mapsto \alpha_{\mathrm{Quad}} + \frac{1}{2}\tau_{\mathrm{Quad}} \cdot r,$$

a first order approximation of the normal frequencies.

41 Proposition. *The first quadrupolar frequency map has constant rank* 3 *and, in restriction to the symplectic submanifold obtained by fixing vertically the direction of the angular momentum, is a local diffeomorphism.*

Proof. For this lemma, we denote by $C = (C_x, C_y, C_z) \in \mathbb{R}^3$ the angular momentum of the first two planets. The submanifold $\mathcal{V}$ of vertical angular momentum, has equation $C_x = C_y = 0$. It is a symplectic, codimension-2 submanifold, transverse to the Hamiltonian vector fields X_{C_x} and X_{C_y} of C_x and C_y. Since it is invariant by the flow of Quad, its tangent space has equations, in the coordinates $(x_j, y_j)_{j=1,\dots,4}$ of the proof of lemma 38, $x_4 = y_4 = 0$. So, the upper left 3×3 submatrix $\tilde\tau_{\mathrm{Quad}}$ of τ_{Quad} is the Hessian of the restriction of Quad to $\mathcal{V}$. In order to conclude, one merely needs to notice that the determinant of the torsion $\tilde\tau_{\mathrm{Quad}}$:

$$\det \tilde\tau_{\mathrm{Quad}} = -\frac{a_1^2}{a_2^3}\frac{27}{64\Lambda_1^2\Lambda_2^2}\left(39\Lambda_1^2 + 39\Lambda_1\Lambda_2 + 4\Lambda_2^2\right)$$

is non-zero. □

So, Quad (adequately truncated) has a non-degenerate quasiperiodic dynamics in the three degrees of freedom corresponding to coordinates $(\psi_j, s_i)_{j=1,2,3}$.

End of the proof of theorem 36. We would like to prove the persistence of some of the invariant tori of our normal form, which have frequencies of the following order (assuming $a_1 = O(1)$ and $a_2 \to \infty$):

$$\alpha = O(1, a_2^{-3/2}, a_2^{-3}, a_2^{-7/2}, a_2^{-3}, 0).$$

The conlusion thus follows from either of the three arguments below:

- the first item of theorem 33 (using proposition 41)
- the second item of theorem 33 (using again proposition 41), which yields not only the precedingly found Diophantine tori but also resonant tori (which induce Diophantine tori after reduction by the symmetry of rotation)
- the third item of theorem 33 (using proposition 39, for which the computation of the torsion is not needed, at the expanse of deteriorating measure estimates, had we computed such estimates)
- the fourth item of theorem 33 (using again proposition 39).

In three cases, Diophantine invariant tori of Kep + Quad (either at the partially reduced level or at the fully reduced one). Locally they will have positive measure provided $\gamma = O(a_2^{-7/2})$. Theorem 33 applies with a perturbation of the size $|H - K^o| = O(\gamma^N)$ for some N (remark 30). Thus the theorem really applices to the perturbation of the normal form of the Hamiltonian of order $\sim 7N/2$ in $1/a_2$. $\square$

Bibliographical comments. – The discovery of the eccentricity vector is often wrongly attributed to Runge and Lentz [1].

– For an anachronistic proof of Bertrand's theorem using Kolmogorov's theorem, see [30].

– Lemma 37 is obvious in the plane problem, where the analoguous reduction leads to a 2-dimensional reduced secular space, with coordinates (g_1, G_1) (see [24, 47]). This is less so in space. Harrington noticed it only after having carried out the computation [37]. Lidov-Ziglin [46] called this a "happy coincidence", and indeed this invariance allowed them to study the bifurcation diagram of the quadrupolar Hamiltonian Quad. This was also crucial in various studies [40, 85, 86].

– Among the many accounts of the work of Lagrange and Laplace (comprising lemma 38), we refer to [78, 44, 31].

– Resonance (24) of order 3 was known to Clairaut, noticed by Delaunay as *un résultat singulier* [19], and discovered by Herman in the general n-planet problem.

– In the proof of lemma 40, it is a happy coincidence that resonant terms associated with the second resonance actually do not occur at our order of truncation. Malige has computed that higher degree resonant monomials occur, starting at order 10 [48].

– Chierchia-Pinzari [18] have generalized the first proof of theorem 36 above to the case of an arbitrary number of planets. Féjoz's strategy in [26] corresponds to the second proof above.

A Isotropy of invariant tori

Let (X, ω) be a symplectic manifold, (φ_t) a symplectic flow and T be an minimal quasiperiodic invariant embedded torus for (φ_t).

42 Lemma. *If ω is exact, T is isotropic.*

Proof. We may assume that $\varphi_t(\theta) = \theta + t\alpha$ ($t \in \mathbb{R}$, $\theta \in \mathbb{T}^n$) for some non-resonant vector α. Let

$$\nu = \sum_{i<j} \nu_{ij}(\theta)\, d\theta_i \wedge d\theta_2$$

be the 2-form induced by ω on T. Since (φ_t) preserves ω, for all t we have

$$\nu_{ij}(\theta + t\alpha) = \nu(\theta).$$

Since the flow on T is minimal, ν_{ij} is constant. By integrating with respect to θ_i and θ_j, this constant must be zero (more learnedly: according to the Hodge theorem, the zero 2-form is the unique harmonic representative of the cohomology class). □

If ω is not exact, the conclusion may be wrong. M. Herman has even constructed codimension-2 minimal invariant tori, in such a robust manner that this disproved the quasi-ergodic hypothesis [81].

B Two basic estimates

The following lemma is used in two instances in the proof of lemma 2, as well as in the proof of Kolmogorov's theorem 19.

43 Lemma (Cohomological equation)**.** *Let s and σ be given in $]0, 1]$. If $g \in \mathcal{A}(\mathbb{T}^n_{s+\sigma})$, there exists a unique function $f \in \mathcal{A}(\mathbb{T}^n_s)$ of 0-average such that*

$$L_\alpha f = g - \int_{\mathbb{T}^n} g(\theta)\, d\theta,$$

and there exists a $C_c = C_c(n, \tau)$ such that, for any s, σ:

$$|f|_s \leq C_c \gamma^{-1} \sigma^{-\tau_c} |g|_{s+\sigma}, \quad \tau_c = \tau + n.$$

Proof. Up to substituting $g - \int_{\mathbb{T}^n} g$, we may assume that g has zero average. Then, let $g(\theta) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} g_k\, e^{i\,2\pi k\cdot\theta}$ be the Fourier expansion of g. The unique formal solution to the equation $L_\alpha f = g$ is given by $f(\theta) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{g_k}{i\,2\pi\, k\cdot\alpha}\, e^{i\,2\pi\, k\cdot\theta}$.

Since g is analytic, its Fourier coefficients decay exponentially: we find

$$|g_k| = \left| \int_{\mathbb{T}^n} g(\theta)\, e^{-ik\cdot\theta}\, \frac{d\theta}{2\pi} \right| \le |g|_{s+\sigma} e^{-|k|(s+\sigma)}$$

by shifting the torus of integration to a torus $\operatorname{Im}\theta_j = \pm(s+\sigma)$ (the sign depending on the sign of k_j). Using this estimate and replacing the small denominators $k\cdot\alpha$ by its Diophantine lower bound, we get

$$\begin{aligned} |f|_s &\le \frac{|g|_{s+\sigma}}{\gamma} \sum_k |k|^\tau\, e^{-|k|\,\sigma} \\ &\le \frac{2^n |g|_{s+\sigma}}{\gamma} \sum_{\ell\ge 1} \binom{\ell+n-1}{\ell} \ell^\tau\, e^{-\ell\sigma} \le \frac{4^n |g|_{s+\sigma}}{\gamma\,(n-1)!} \sum_\ell (\ell+n-1)^{\tau+n-1}\, e^{-\ell\,\sigma}, \end{aligned}$$

where, as a change of variable and a rough approximation show, the latter sum is bounded by

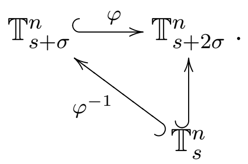
$$\int_1^\infty (\ell+n-1)^{\tau+n-1} e^{-(\ell-1)\sigma}\, d\ell < \sigma^{-\tau-n} e^{n\sigma} \int_0^\infty \ell^{\tau+n-1} e^{-\ell}\, d\ell$$

Hence f belongs to $\mathcal{A}(\mathbb{T}^n_s)$ and satisfies the wanted estimate. □

Bibliographical comments. The estimate has been obtained by bounding the terms of Fourier series one by one. In a more careful estimate, one should take into account the fact that if $|k\cdot\alpha|$ is small, then $k'\cdot\alpha$ is not so small for neighboring k''s. This allows to find the optimal exponent of σ, making it independant of the dimension [52, 64].

We have also used this inverse function theorem. Recall that we have set $\mathbb{T}^n_s := \{\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n,\ \max_{1\le j\le n} |\operatorname{Im}\theta_j| \le s\}$.

44 Proposition. *Let $v \in \mathcal{A}(\mathbb{T}^n_{s+2\sigma}, \mathbb{C}^n)$, $|v|_{s+2\sigma} < \sigma$. The map $\operatorname{id} + v : \mathbb{T}^n_{s+2\sigma} \to \mathbb{R}^n_{s+3\sigma}$ induces a map $\varphi : \mathbb{T}^n_{s+2\sigma} \to \mathbb{T}^n_{s+3\sigma}$ whose restriction $\varphi : \mathbb{T}^n_{s+\sigma} \to \mathbb{T}^n_{s+2\sigma}$ has a unique right inverse $\phi^{-1} : \mathbb{T}^n_s \to \mathbb{T}^n_{s+\sigma}$:*



Furthermore,

$$|\varphi^{-1} - \operatorname{id}|_s \le |v|_{s+\sigma}$$

and, provided $2\sigma^{-1}|v|_{s+2\sigma} \le 1$,

$$|(\varphi^{-1})' - \operatorname{id}| \le 2\sigma^{-1}|v|_{s+2\sigma}.$$

Proof. Let $\Phi : \mathbb{R}^n_{s+2\sigma} \to \mathbb{R}^n_{s+3\sigma}$ be a continuous lift of $\mathrm{id} + v$ and $k \in M_n(\mathbb{Z})$, $k(l) := \Phi(x+l) - \Phi(x)$. Denote by $p : \mathbb{R}^n_s := \mathbb{R}^n \times i[-s,s]^n \to \mathbb{T}^n_s$ the universal covering of $\mathbb{T}^n_s$.

1. *Injectivity of* $\Phi : \mathbb{R}^n_{s+\sigma} \to \mathbb{R}^n_{s+2\sigma}$. Suppose that $x, \hat{x} \in \mathbb{R}^n_{s+\sigma}$ and $\Phi(x) = \Phi(\hat{x})$. By the mean value theorem,

$$|x - \hat{x}| = |v(p\hat{x}) - v(px)| \leq |v'|_{s+\sigma}|x - \hat{x}|,$$

and, by Cauchy's inequality,

$$|x - \hat{x}| \leq \frac{|v|_{s+2\sigma}}{\sigma}|x - \hat{x}| < |\hat{x} - x|,$$

hence $x = \hat{x}$.

2. *Surjectivity of* Φ*:* $\mathbb{R}^n_s \subset \Phi(\mathbb{R}^n_{s+\sigma})$. For any given $y \in \mathbb{R}^n_s$, the contraction

$$f : \mathbb{R}^n_{s+\sigma} \to \mathbb{R}^n_{s+\sigma}, \quad x \mapsto y - v(x)$$

has a unique fixed point, which is a pre-image of y by Φ.

3. *Injectivity of* $\varphi : \mathbb{T}^n_{s+\sigma} \to \mathbb{T}^n_{s+2\sigma}$. Suppose that $px, p\hat{x} \in \mathbb{R}^n_{s+\sigma}$ and $\varphi(px) = \varphi(p\hat{x})$, i.e. $\Phi(x) = \Phi(\hat{x}) + \kappa$ for some $\kappa \in \mathbb{Z}^n$. That k be in $GL(n, \mathbb{Z})$, follows from the invertibility of Φ. Hence, $\Phi\left(x - k^{-1}(\kappa)\right) = \Phi(\hat{x})$, and, due to the injectivity of Φ, $px = p\hat{x}$.

4. *Surjectivity of* $\varphi : \mathbb{T}^n_s \subset \varphi(\mathbb{T}^n_{s+\sigma})$. This is a trivial consequence of that of Φ.

5. *Estimate on* $\psi := \varphi^{-1} : \mathbb{T}^n_s \to \mathbb{T}^n_{s+\sigma}$. Note that the wanted estimate on ψ is in the sense of $\Psi := \Phi^{-1} : \mathbb{R}^n_s \to \mathbb{R}^n_{s+\sigma}$. If $y \in \mathbb{R}^n_s$,

$$\Psi(y) - y = -v(p\Psi(y)),$$

hence $|\Psi - \mathrm{id}\,|_s \leq |v|_{s+\sigma}$.

6. *Estimate on* ψ'. We have $\psi' = \varphi'^{-1} \circ \varphi$, where $\varphi'^{-1}(x)$ stands for the inverse of the map $\xi \mapsto \varphi'(x) \cdot \xi$. Hence

$$\psi' - \mathrm{id} = \varphi'^{-1} \circ \varphi - \mathrm{id},$$

and, under the assumption that $2\sigma^{-1}|v|_{s+2\sigma} \leq 1$,

$$|\psi' - \mathrm{id}\,|_s \leq |\varphi'^{-1} - \mathrm{id}\,|_{s+\sigma} \leq \frac{|v'|_{s+\sigma}}{1 - |v'|_{s+\sigma}} \leq \frac{\sigma^{-1}|v|_{s+2\sigma}}{1 - \sigma^{-1}|v|_{s+2\sigma}} \leq 2\sigma^{-1}|v|_{s+2\sigma}.$$

□

C Interpolation of spaces of analytic functions

In this section we prove some Hadamard interpolation inequalities, which are used in 4.

Recall that we denote by $\mathbb{T}^n_{\mathbb{C}}$ the infinite annulus $\mathbb{C}^n/2\pi\mathbb{Z}^n$, by $\mathbb{T}^n_s$, $s > 0$, the bounded sub-annulus $\{\theta \in \mathbb{T}^n_{\mathbb{C}},\ |\mathrm{Im}\,\theta_j| \leq s,\ j = 1...n\}$ and by $\mathbb{D}^n_t$, $t > 0$, the polydisc $\{r \in \mathbb{C}^n,\ |r_j| \leq t,\ j = 1...n\}$. The supremum norm of a function $f \in \mathcal{A}(\mathbb{T}^n_s \times \mathbb{D}^n_t)$ will be denoted by $|f|_{s,t}$.

Let $0 < s_0 \leq s_1$ and $0 < t_0 \leq t_1$ be such that

$$\log \frac{t_1}{t_0} = s_1 - s_0.$$

Let also $0 \leq \rho \leq 1$ and

$$s = (1-\rho)s_0 + \rho s_1 \quad \text{and} \quad t = t_0^{1-\rho} t_1^{\rho}.$$

45 Proposition. *If* $f \in \mathcal{A}(\mathbb{T}^n_{s_1} \times \mathbb{D}^n_{t_1})$,

$$|f|_{s,t} \leq |f|^{1-\rho}_{s_0,t_0}\, |f|^{\rho}_{s_1,t_1}.$$

Proof. Let $\tilde{f}$ be the function on $\mathbb{T}^n_{s_1} \times \mathbb{D}^n_{t_1}$, constant on $2n$-tori of equations $(\mathrm{Im}\,\theta, r) = cst$, defined by

$$\tilde{f}(\theta, r) = \max_{\mu,\nu \in \mathbb{T}^n} \left| f\left((\pm\theta_1 + \mu_1, ..., \pm\theta_n + \mu_n), \left(r_1\, e^{i\nu_1}, ..., r_n\, e^{i\nu_n}\right)\right)\right|$$

(with all possible combinations of signs). Since $\log|f|$ is subharmonic and $\mathbb{T}^{2n}$ is compact, $\log\tilde{f}$ too is upper semi-continuous. Besides, $\log\tilde{f}$ satisfies the mean inequality, hence is plurisubharmonic.

By the maximum principle, the restriction of $|f|$ to $\mathbb{T}^n_s \times \mathbb{D}^n_t$ attains its maximum on the distinguished boundary of $\mathbb{T}^n_s \times \mathbb{D}^n_t$. Due to the symmetry of $\tilde{f}$:

$$|f|_{s,t} = \tilde{f}(is\epsilon, t\epsilon), \quad \epsilon = (1, ..., 1).$$

Now, the function

$$\varphi(z) := \tilde{f}(z\epsilon, e^{-(iz+s)}t\epsilon)$$

is well defined on $\mathbb{T}_{s_1}$, for it is constant with respect to $\mathrm{Re}\,z$ and, due to the relations imposed on the norm indices, if $|\mathrm{Im}\,z| \leq s_1$ then $|e^{-(iz+s)}t| \leq e^{s_1-s}t = t_1$.

The estimate

$$\log\varphi(z) \leq \frac{s_1 - \mathrm{Im}\,z}{s_1 - s_0}\varphi(s_0 i) + \frac{\mathrm{Im}\,z - s_0}{s_1 - s_0}\varphi(s_1 i)$$

trivially holds if $\mathrm{Im}\,z = s_0$ or s_1, for, as noted above for $j = 1$, $e^{s_j - s}t = t_j$, $j = 0, 1$. But note that the left and right hand sides respectively are suharmonic and harmonic. Hence the estimate holds whenever $s_0 \leq \mathrm{Im}\,z \leq s_1$, whence the claim for $z = is$. □

Recall that we have let $\mathrm{T}^n_s := \mathbb{T}^n_s \times \mathbb{D}^n_s$, $s > 0$, and, for a function $f \in \mathcal{A}(\mathrm{T}^n_s)$, let $|f|_s = |f|_{s,s}$ denote its supremum norm on T^n_s. As in the rest of the paper, we now restrict the discussion to widths of analyticity ≤ 1.

46 Corollary. *If* $\sigma_1 = -\log\left(1 - \frac{\sigma_0}{s}\right)$ *and* $f \in \mathcal{A}(\mathrm{T}^n_{s+\sigma_1})$,

$$|f|_s^2 \leq |f|_{s-\sigma_0} |f|_{s+\sigma_1}.$$

In 4, we will use the equivalent fact that, if $\tilde{\sigma} = s + \log\left(1 + \frac{\sigma}{s}\right)$ and $f \in \mathcal{A}(\mathrm{T}^n_{s+\tilde{\sigma}})$,

$$|f|_{s+\sigma}^2 \leq |f|_s |f|_{s+\tilde{\sigma}}.$$

Proof. In proposition 45, consider the following particular case :

- $\rho = 1/2$. Hence
$$s = \frac{s_0 + s_1}{2} \quad \text{and} \quad t = \sqrt{t_0 t_1}.$$
- $s = t$. Hence in particular $t_0 = s\, e^{s_0 - s}$ and $t_1 = s\, e^{s_1 - s}$.

Then
$$|f|_s^2 = |f|_{s,s}^2 \leq |f|_{s_0,t_0} |f|_{s_1,t_1}.$$

We want to determine $\max(s_0, t_0)$ and $\max(s_1, t_1)$. Let $\sigma_1 := s - s_0 = s_1 - s$. Then $t_0 = s\, e^{-\sigma_1}$ and $t_1 = s\, e^{\sigma_1}$. The expression $s + \sigma - s e^{\sigma}$ has the sign of σ (in the relevant region $0 \leq s + \sigma \leq 1$, $0 \leq s \leq 1$); by evaluating it at $\sigma = \pm\sigma_1$, we see that $s_0 \leq t_0$ and $s_1 \geq t_1$.

Therefore, since the norm $|\cdot|_{s,t}$ is non-decreasing with respect to both s and t,

$$|f|_s^2 \leq |f|_{t_0,t_0} |f|_{s_1,s_1} = |f|_{t_0} |f|_{s_1}$$

(thus giving up estimates uniform with respect to small values of s). By further setting $\sigma_0 = s - t_0 = s\,(1 - e^{-\sigma_1})$, we get the wanted estimate, and the asserted relation between σ_0 and σ_1 is readily verified. □

Bibliographical comments. – The obtained inequalities generalize the standard Hadamard inequalities. They are optimal and show that the convexity of analytic norms is twisted by the geometry of the phase space. See [55, Chap. 8] for more general but worse inequalities.

– Interpolation inequalities in the analytic category do not depend on regularizing operators as they do in the Hölder or Sobolev cases. See, e.g. [38, Theorem A.5] or [36].

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