Global secular dynamics in the planar three-body problem

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Abstract. We use the global construction which was made in [6, 7] of the secular systems of the planar three-body problem, with regularized double inner collisions. These normal forms describe the slow deformations of the Keplerian ellipses which each of the bodies would describe if it underwent the universal attraction of only one fictitious other body. They are parametrized by the masses and the semi major axes of the bodies and are completely integrable on a fixed transversally Cantor set of the parameter space. We study this global integrable dynamics reduced by the symmetry of rotation and determine its bifurcation diagram when the semi major axes ratio is small enough. In particular it is shown that there are some new secular hyperbolic or elliptic singularities, some of whose do not belong to the subset of aligned ellipses. The bifurcation diagram may be used to prove the existence of some new families of 2-, 3- or 4-frequency quasiperiodic motions in the planar three-body problem [7], as well as some drift orbits in the planar n-body problem [8].

Keywords: three-body problem, secular system, averaging, KAM theorem, regularization, singularity

1. Introduction

In perturbative studies of the three-body problem, the dynamics is split into two parts: a fast, *Keplerian* dynamics, which describes the motion of the bodies along three ellipses as if each body underwent the attraction of only one fictitious center of attraction; and a slow, *secular* dynamics, which describes the deformations of these Keplerian ellipses, due to the fact that each body actually undergoes the attraction of the other two. This splitting is not unique and amounts to writing the Hamiltonian as the sum

$$F = F_{\text{Kep}} + F_{\text{per}}$$

of a Keplerian part F_{Kep}, that is the Hamiltonian of the uncoupled fictitious two-body problems, and a supposedly small perturbing function F_{per}, which determines the secular dynamics . If we want to preserve the symmetry of translations, the choice of the splitting boils down to that of only two 2-body problems.[1]

[1] Cf. Chap. II, first Vol. of Poincaré's *Leçons* [15].

In the 18th century, when Lagrange and Laplace tried to prove the stability of the system consisting of the Sun, Jupiter and Saturn, they introduced the *averaged system*.[2] Its Hamiltonian

$$\langle F \rangle = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2$$

is obtained by averaging the initial Hamiltonian along the Keplerian ellipses which are parametrized by the mean anomalies λ_1 and λ_2 of the two fictitious Kepler problems.

A striking feature of $\langle F \rangle$ is that it extends, where the eccentricity of the inner ellipse is $e_1 = 1$, to an analytic function, whereas the perturbing function itself extends by continuity to a function which is not even differentiable. In [6], I proved that the averaged system $\langle F \rangle$ actually agrees with the averaged system $\langle \mathcal{F} \rangle$ of the Hamiltonian $\mathcal{F}$ obtained from F by regularizing double inner collisions on some given energy manifold.

In turn, $\langle \mathcal{F} \rangle$ agrees on some transversally Cantor set[3] with the first of the normal forms which I denote $\mathcal{F}^n_{\mathrm{sec}}$, $n \geq 0$, which are obtained by trying to eliminate the fast angles from the regularized Hamiltonian $\mathcal{F}$, up to increasing orders of smallness.[4] These normal forms are the *secular systems* of the regularized three-body problem; they describe the slow deformations of the Keplerian ellipses. In the planar problem, they are completely integrable in the sense of Pöschel [16] on the transversally Cantor set where the regularized Keplerian frequencies satisfy some homogeneous diophantine conditions. More precisely, $\mathcal{F}^n_{\mathrm{sec}}$ is the sum of a *(Liouville-) integrable* part $\mathcal{F}^n_{\mathrm{int}}$ which does not depend on the mean anomalies, and of a *resonant part* $\mathcal{F}^n_{\mathrm{res}}$ whose infinite jet[5] vanishes along this transversally Cantor set; in particular,

$$\mathcal{F}^1_{\mathrm{int}} = \langle \mathcal{F} \rangle.$$

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[2] Cf. the *Averaging Principle*, in the Russian Encyclopaedia [2], Chap. 5, Section 1.1.

[3] In dynamics, Cantor sets occur mainly through *diophantine conditions* of the frequencies of quasiperiodic motions along invariant tori (cf. Section 2.2). Diophantine conditions are arithmetic conditions on the frequencies which ensure that the motion is evenly spread along the torus.

By definition, a subset K of a topological space is *Cantor* if each point in K is both an accumulation point (i.e. K is *perfect*) and a connected component (i.e. K is *totally disconnected*). A subset L is *transversally Cantor* if it is the topological product of a Cantor set K and a line segment.

[4] For an introduction to the theory of normal forms, see [2], Chap. 5, Section 2.

[5] The *infinite jet* of a function at a point can be thought of as the formal Taylor series of the function at the given point, in some local coordinates.

Hence $\mathcal{F}^n_{\rm int}$ gives some formal quasiperiodic skeleton of $\mathcal{F}$. The purpose of this paper is to study the dynamics of $\mathcal{F}^n_{\rm int}$ as globally as possible in the direct product of the phase and parameter spaces.

In Sections 2 and 3, the setting and some definitions from [6, 7] are recalled, coping with the regularization of double inner collisions in the planar three-body problem, the construction of the (regularized) secular systems and the description of the secular space. In Section 4, we describe the topology of the reduction by the symmetry of rotations. In particular we show that the subset of aligned ellipses plays the role of a "real form" of the whole phase space. In Section 5, we list some discrete symmetries of the first and higher order secular systems. These symmetries allow to forsee the existence of all the singularities (i.e. fixed points) that will be proved to exist in the next section, and show that the real form of the secular space is an invariant submanifold of the averaged system. In Section 6, the singularities of the integrable part $\mathcal{F}^n_{\rm int}$ of the secular systems are described, assuming that the semi major axes ratio is small enough. In particular, $\mathcal{F}^n_{\rm int}$ has a hyperbolic singularity for which the inner ellipses are almost degenerate. In Section 7, we describe the singularities of the system $\mathcal{F}^n_{\rm int}$ reduced by the symmetry of rotation–that is, secular fixed points in a rotating frame of reference. In particular, we prove that the averaged system has some singularities which do not belong to the subset of aligned ellipses, contrary to what the study of the planetary and lunar regions alone had let think. A bifurcation diagram is eventually given in the five-dimensionnal parameter space, always assuming that the semi major axes ratio is small enough.

All the figures are drawings. Footnote explanations will hopefully help non-mathematician readers to better grasp the technical part of this paper.

Existing studies of the secular systems call for a few comments. Since Lagrange and Laplace introduced the averaged system, secular systems of the three-body problem have been extensively studied, but, for astronomical reasons, mainly in the neighborhood of the configuration where the three Keplerian ellipses are circular. By a simple symmetry argument (cf. § 2), this configuration is a singularity of the secular systems at any order. A number of results is collected in Tisserand's *Traité* [17]. In particular the averaged system $\langle F \rangle$ was shown there to have two fixed points in a rotating frame of reference, assuming that the angular momentum is large enough, that is, close enough to the tricircular configuration. These two singularities were used by Poincaré to find his periodic orbits of the second kind [14].

More recently, Lieberman [13] in the planar case and Jefferys and Moser [11] in the spatial case have found some new secular singularities, with a finite eccentricity or inclination; Lieberman's singularity is the continuation of one of those used by Poincaré. However their study remains local. For instance, we did not know whether all the singularities of the system reduced by the rotations were located on the submanifold of aligned ellipses. This was a natural hypothesis given the fixed points we knew the existence of. We will see that a global point of view in the parameter space will let us show that this hypothesis is wrong. Moreover, we will show the existence of some hyperbolic singularities which can be used for proving the existence of diffusion in the n-body problem [8].

Lidov and Ziglin [12] do have a global point of view in the spatial problem, although they restrict their study to a 2-dimensional submanifold of the parameter space. But the dynamics of only the first term of the averaged system is investigated. Besides, eventhough Lidov and Ziglin do not assume that the angular momentum is large, their study is not relevant when eccentricities get close to one. On the other hand, our global point of view in the phase space itself leads to both a better understanding of the bifurcation diagram of the secular systems, and the existence of some new families of quasiperiodic motions on invariant punctured tori in the planar three-body problem [4, 7].

2. Regularized System

2.1. Initial Hamiltonian

Consider three points of masses m_0, m_1 and m_2 undergoing gravitational attraction in the plane. By choosing a frame of reference, identify the physical plane to the complex plane $\mathbf{C}$, endowed in particular with its Euclidean norm $|\cdot|$. The phase space is the space

$$\left\{(q_j, p_j)_{0\leq j\leq 2} \in (\mathbf{C}\times\mathbf{C}^*)^3 \ \mid \forall\, 0 \leq j < k \leq 2,\ q_j \neq q_k\right\}$$

of position vectors q_j and linear momentum covectors p_j of the bodies. It is the open set of the cotangent bundle $T^*\mathbf{C}^3$ which is obtained by ruling out collisions. Hence it is naturally endowed with the symplectic form

$$\omega = \Re(dq_0 \wedge d\bar{p}_0 + dq_1 \wedge d\bar{p}_1 + dq_2 \wedge d\bar{p}_2),$$

where $\Re$ stands for the real part of a complex number. If the frame of reference is Galilean, the Hamiltonian is

$$\frac{1}{2}\sum_{0\leq j\leq 2}\frac{|p_j|^2}{m_j} - \gamma\sum_{0\leq j<k\leq 2}\frac{m_j m_k}{|q_j - q_k|},$$

where γ is the universal constant of gravitation. Thanks to the invariance of Newton's equations with respect to change of the time unit, we may suppose that $\gamma = 1$.

Let $(P_j, Q_j)_{j=0,1,2}$ be the Jacobi coordinates of the three bodies.[6] The phase space reduced by translations can be identified to the open set of $T^*\mathbf{R}^4$ which is described by the Jacobi coordinates $(P_j, Q_j)_{j=1,2}$ outside collisions. If the frame of reference is attached to the center of mass, i.e. if $P_0 = 0$, and if $Q_2 \neq 0$, the reduced Hamiltonian can be written as

$$F = F_{\mathrm{Kep}} + F_{\mathrm{per}},$$

where F_{Kep} and F_{per} are defined by

$$F_{\mathrm{Kep}} = \frac{|P_1|^2}{2\mu_1} + \frac{|P_2|^2}{2\mu_2} - \frac{\mu_1 M_1}{|Q_1|} - \frac{\mu_2 M_2}{|Q_2|}$$

and

$$F_{\mathrm{per}} = -\mu_1 m_2 \left[\begin{array}{l} \dfrac{1}{\sigma_0}\left(\dfrac{1}{|Q_2 - \sigma_0 Q_1|} - \dfrac{1}{|Q_2|}\right) \\ +\dfrac{1}{\sigma_1}\left(\dfrac{1}{|Q_2 + \sigma_1 Q_1|} - \dfrac{1}{|Q_2|}\right) \end{array} \right]$$

with the fictitious masses themselves defined by

$$\begin{cases} M_1 = m_0 + m_1 \\ \dfrac{1}{\mu_1} = \dfrac{1}{m_0} + \dfrac{1}{m_1} \end{cases} \quad \text{and} \quad \begin{cases} M_2 = M_1 + m_2 \\ \dfrac{1}{\mu_2} = \dfrac{1}{M_1} + \dfrac{1}{m_2} \end{cases}$$

and the adimensional coefficients σ_0 and σ_1 by

$$\frac{1}{\sigma_0} = 1 + \frac{m_1}{m_0} \quad \text{and} \quad \frac{1}{\sigma_1} = 1 + \frac{m_0}{m_1}.$$

For a discussion of these choices, see Poincaré's *Leçons* [15] (Chap. II, first Vol.), and, for a closer point of view, my paper [7]. For instance, the heliocentric splitting, which is used by Lieberman [13], is not relevant in the neighborhood of double inner collisions $Q_1 = 0$ because in the neighborhood of such collisions the heliocentric perturbing function is not uniformly small.

The Hamiltonian F_{Kep} is the *Keplerian Hamiltonian*. We will exclusively pay attention to bounded motions and their perturbations. Then F_{Kep} is the completely integrable Hamiltonian of two fictitious bodies

[6] Cf. Chap. II, first Vol. of the *Leçons* [15]. By definition, $Q_0 = q_0$, $Q_1 = q_1 - q_0$ and Q_2 give the position of the third body with respect to the center of mass of the other two. Then the P_js are given by the contragredient map of $(q_j)_{j=0,1,2} \mapsto (Q_j)_{j=0,1,2}$.

of masses μ_1 and μ_2 which revolve along ellipses around a fixed center of attraction, without mutual interaction.

The Hamiltonian F_{per} is the *perturbing function*. It is real analytic outside collisions of the bodies *and* outside collisions of the fictitious body Q_2 with the center; the latter restriction is not bothering insofar as we will suppose that the ellipse which is described by Q_2 is the outer ellipse.

2.2. Regularization of double inner collisions

Let us restrict ourselves to pairs of elliptic motions such that the two ellipses do not meet one another. Moser's and Levi-Civita's regularizations of the two-body problem easily extend to a regularization of double inner collisions in the three-body problem [6], of which we now give a short account.

Since we want to rule out collisions involving the outer body, the relevant part of the phase space is diffeomorphic to

$$T^*\left(\mathbf{C}\setminus 0\right)\times\Big(T^*\left(\mathbf{C}\setminus 0\right)\setminus\left(\left(\mathbf{C}\setminus 0\right)\times\mathbf{R}\right)\Big)\simeq\left(T^*\left(\mathbf{C}\setminus 0\right)\right)\times\mathbf{X}_2,$$

where $\mathbf{X}_2\simeq\mathbf{R}\times\mathbf{S}^1\times\mathbf{R}^2\times\mathbf{S}^0$ is the phase space of the outer body; the factor $\mathbf{S}^0$ corresponds to the two possible ways the outer body can move around the inner ellipse.

Let L.C. be the two-sheeted (symplectic) covering of Levi-Civita[7], defined as the product of the cotangent map of $z\mapsto z^2$ by $id_{\mathbf{X}_2}$:

$$\begin{array}{rlcl}\text{L.C.}: & T^*(\mathbf{C}\setminus 0)\times\mathbf{X}_2 & \longrightarrow & T^*(\mathbf{C}\setminus 0)\times\mathbf{X}_2\\ & ((z,w),x_2) & \longmapsto & ((Q_1,P_1),x_2)=\left(\left(z^2,\dfrac{w}{2\bar z}\right),x_2\right).\end{array}$$

For any real number $f>0$, the Hamiltonian

$$|z|^2\text{L.C.}^*(F+f)=|z|^2(F\circ\text{L.C.}+f)$$

$$=\frac{|w|^2}{8\mu_1}+\left(f+\frac{|P_2|^2}{2\mu_2}-\frac{\mu_2M_2}{|Q_2|}\right)|z|^2-\mu_1M_1+|z|^2\text{L.C.}^*F_{\text{per}}$$

extends to an $\mathbf{R}$-analytic Hamiltonian on $T^*\mathbf{C}\times\mathbf{X}_2$. Let f_1 be the function

$$f_1:=f+\frac{|P_2|^2}{2\mu_2}-\frac{\mu_2M_2}{|Q_2|};$$

on the manifold of constant energy $\text{L.C.}^*(|Q_1|(F_{\text{Kep}}+f))=0$, it is the opposite of the energy of the inner body. Also, it is a first integral of

[7] *Two-sheeted* means that every point (Q_1,P_1) has two preimages.

$|z|^2\mathrm{L.C.}^*(F_{\mathrm{Kep}}+f)$. Thus $|z|^2\mathrm{L.C.}^*(F_{\mathrm{Kep}}+f)$ is the skew-product of a rotator (outer body) slowed down by a pair of $(1,1)$-resonant harmonic oscillators (inner body).

The point $z=w=0$ from the phase space can be ignored, because it corresponds to an infinite energy for the initial problem. The phase space is then diffeomorphic to

$$(T^*\mathbf{C})\setminus 0\times\mathbf{X}_2\simeq\mathbf{S}^3\times\mathbf{R}\times\mathbf{X}_2.$$

Since the L.C. mapping is a two-sheeted covering, the pull-backs by L.C. of all the initial observables (e.g. Q_1, the slowed-down perturbing function $|Q_1|F_{\mathrm{per}}$, etc.) descend through the antipodal mapping of the sphere,

$$\mathbf{S}^3\times\mathbf{R}\times\mathbf{X}_2\overset{(z,w,x_2)\sim(-z,-w,x_2)}{\longrightarrow}\mathbf{X}_1\times\mathbf{X}_2=\mathrm{SO}_3\times\mathbf{R}\times\mathbf{X}_2,$$

where $\mathbf{X}_1$ stands for the (regularized) phase space of the inner body.[8] Moreover, if some pull-back by L.C. extends to an analytic function on $\mathbf{S}^3\times\mathbf{R}\times\mathbf{X}_2$, then the induced observable iteslf is analytic on $\mathbf{X}=\mathbf{X}_1\times\mathbf{X}_2$. We will generally denote the latter by the name of the initial observable (e.g. Q_1, etc.).

The phase spaces of the two bodies, $\mathbf{X}_1=\mathrm{SO}_3\times\mathbf{R}\simeq T^1\mathbf{S}^2\times\mathbf{R}$ (where $T^1\mathbf{S}^2$ denotes the circle bundle of the 2-sphere) and $\mathbf{X}_2=\mathbf{S}^1\times\mathbf{R}^3\times\mathbf{S}^0$, can be thought of as $\mathbf{S}^1$-bundles over $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{R}^3\times\mathbf{S}^0$ respectively.

DEFINITION 1. *The* (regularized) total Hamiltonian

$$\mathcal{F}=|Q_1|(F+f),$$

the Keplerian Hamiltonian

$$\mathcal{F}_{\mathrm{Kep}}=|Q_1|(F_{\mathrm{Kep}}+f)$$

and the perturbing function

$$\mathcal{F}_{\mathrm{per}}=|Q_1|F_{\mathrm{per}}$$

on

$$\mathbf{X}=(\mathrm{SO}_3\times\mathbf{R})\times(\mathbf{R}^3\times\mathbf{S}^1\times\mathbf{S}^0)$$

will be the direct images by the antipodal mapping, of the extensions of L.C.* $(|Q_1|(F+f))$, L.C.* $(|Q_1|(F_{\mathrm{Kep}}+f))$ *and* L.C.* $(|Q_1|F_{\mathrm{per}})$.

[8] In other words, observables such as $Q_1\circ\mathrm{L.C.}$ take the same value on any two points of the type (z,w,x_2) and $(-z,-w,x_2)$.

A direct and important consequence of the Leibniz rule[9] is that on the energy surface $\mathcal{F} = 0$, outside collisions, the Hamiltonian vector fields associated to F and $\mathcal{F}$ define the same field of oriented straight lines.

The perturbing function equals

$$\mathcal{F}_{\text{per}} = -\mu_1 m_2 |Q_1| \left[\begin{array}{l} \frac{1}{\sigma_0}\left(\frac{1}{|Q_2 - \sigma_0 Q_1|} - \frac{1}{|Q_2|}\right) \\ +\frac{1}{\sigma_1}\left(\frac{1}{|Q_2 + \sigma_1 Q_1|} - \frac{1}{|Q_2|}\right) \end{array} \right],$$

while the Keplerian Hamiltonian is described in the next paragraph.

2.3. Keplerian dynamics

We recall angle-action variables for the Keplerian Hamiltonian $\mathcal{F}_{\text{Kep}}$. After lemma 2.1 of [6][10]:

LEMMA 1. *There are an immersed submanifold $\mathbf{T}^2 \times \mathbf{R}^2 \times \mathbf{X}_2$ of full Liouville-measure in $\mathbf{X}$, and symplectic coordinates*

$$((l_1, L_1, g_1, G_1), (\lambda_2, \Lambda_2, \xi_2, \eta_2))$$

on each of its two connected components, such that

$$\begin{cases} (l_1, g_1, \lambda_2) \in \mathbf{T}^3 \\ L_1, \Lambda_2 > 0 \\ |G_1| < L_1 \quad \textit{and} \quad 0 \le \xi_2^2 + \eta_2^2 < 4L_1, \end{cases}$$

and the Keplerian part is

$$\mathcal{F}_{\text{Kep}} = L_1 \sqrt{\frac{2f_1(\Lambda_2)}{\mu_1}} - \mu_1 M_1,$$

where f_1 may be thought of as a function of Λ_2:

$$f_1(\Lambda_2) = f - \frac{\mu_2^3 M_2^2}{2\Lambda_2^2}.$$

Besides, the position of the inner body in the physical plane is explicitly given by

$$Q_1 = \frac{e^{ig_1}}{\sqrt{2\mu_1 f_1(\Lambda_2)}} \left(-\sqrt{L_1^2 - G_1^2} + L_1 \cos l_1 + iG_1 \sin l_1 \right),$$

[9] If X_H denote the Hamiltonian vector field associated to a Hamiltonian H,

$$X_{\mathcal{F}} = |Q_1| X_F + \mathcal{F} X_{|Q_1|};$$

hence, when $\mathcal{F} = 0$ the two vector fields $X_{\mathcal{F}}$ and X_F are colinear.

[10] The notations are slightly changed, because here we consider only the regularized problem. The proof is elementary.

and G_1 is the angular momentum of the inner body.

Hence the Keplerian dynamics is a direct product, which allows for the following definition:

DEFINITION 2. *The* (regularized) Keplerian action *of the* 2*-torus on* $\mathbf{X}$ *is the* $\mathbf{T}^2$*-action induced by the dynamics of* $\mathcal{F}_{\text{Kep}}$.

Since the slow-down function $|Q_1|$ only depends on the inner body, orbits of the outer body are unchanged by the regularization and their projections on the physical space thus are ellipses. From the construction of the coordinates [6], the variables $(\lambda_2, \Lambda_2, \xi_2, \eta_2)$ of the outer body are Poincaré's coordinates, except the angle λ_2, which substitutes for the mean longitude and which depends on l_1.

Moreover, in the latter lemma, the formula expressing Q_1 as a function of the angle-action coordinates shows that the orbits of the fictitious inner body in the physical plane (coordinate Q_1) are ellipses not only on the regularized submanifold of constant energy $\mathcal{F}_{\text{Kep}} = 0$, but everywhere. Incidentally, this formula also proves that on $\mathcal{F}_{\text{Kep}} = 0$ the coordinates (l_1, L_1, g_1, G_1) agree with the Delaunay coordinates, up to the fast angle l_1, which agrees with the eccentric anomaly and substitutes for the mean anomaly. Under the Keplerian action, the real bodies describe ellipses whose foci all are the moving center of mass of m_0 and m_1. In particular, the two ellipses of m_0 and m_1 are described by $\sigma_1 Q_1$ and $-\sigma_0 Q_1$. Hence they have the same excentricity

$$\sqrt{1 - \frac{G_1^2}{L_1^2}}$$

and are in opposition.

After their construction in [6], coordinates (l_1, L_1, g_1, G_1) may be extended to coordinates on some blow-up $\mathbf{T}^2 \times \mathbf{I}^2$ of $\mathbf{X}_1$ (where $\mathbf{I}$ stands for the closed interval). Just as for the Delaunay coordinates in the non-regularized Kepler problem, the blow-up consists in artificially providing circular inner ellipses with a pericenter.

The regularized analogue $(\lambda_1, \Lambda_1, \xi_1, \eta_1)$ of the Poincaré coordinates of the inner body is defined over some blow-up $\mathbf{S}^1 \times \mathbf{R}^3$ of $\mathbf{X}_1$ by

$$\begin{cases} \lambda_1 = l_1 + g_1 \\ \Lambda_1 = L_1 \\ \xi_1 + i\eta_1 = \sqrt{2(L_1 - G_1)}\, e^{ig_1}. \end{cases}$$

They are smooth coordinates on $\mathbf{X}_1$ in the neighborhood of positively oriented circular ellipses ($G_1 = L_1$), up to and without including negatively oriented circular ellipses ($G_1 = -L_1$). In turn, they have an analogue $(\tilde\lambda_1, \Lambda_1, \tilde\xi_1, \tilde\eta_1)$ which faithfully describes the neighborhood of negatively oriented circular inner ellipses ($G_1 = -L_1$) up to but not including positively oriented circular ellipses, and which is defined by

$$\begin{cases} \tilde\lambda_1 = l_1 - g_1 \\ \Lambda_1 = L_1 \\ \tilde\xi_1 + i\tilde\eta_1 = \sqrt{2(L_1 + G_1)}\, e^{-ig_1}. \end{cases}$$

3. Regularized Secular Systems

The regularized Keplerian dynamics is dynamically degenerate just as the initial Keplerian dynamics, because $\mathcal{F}_{\text{Kep}}$ depends on only half the action variables. It is the perturbing function which breaks the degeneracy down and determines the secular dynmics. We outline the construction which is made in [7] of the secular systems, in some region of the phase space where the perturbing function $\mathcal{F}_{\text{per}}$ is small in the C^∞-topology.

3.1. Perturbing region

We will *always* restrict ourselves to a small neighborhood of the hypersurface $\mathcal{F}_{\text{Kep}} = 0$, because this hypersurface is close to the perturbed hypersurface $\mathcal{F} = 0$, which is the only dynamically relevant energy level, and because a priori our estimates of the perturbing function would not hold outside such a neighborhood. More precisely, for any given energy f and masses m_0, m_1, m_2, we will assume that the Euclidean distance of $\mathbf{R}^2$ between the point $(a_1, a_2) \in \mathbf{R}^2$ and the set

$$A_f = \{(a_1', a_2'),\ \mathcal{F}_{\text{Kep}}(a_1', a_2') = 0\}$$

is smaller than $a_1/2$ (and hence than $a_2/2$); a_1 and a_2 stand for the two semi major axes.

We will also *always* assume that the eccentricity e_2 of the outer ellipse is upper bounded: $e_2 < e_2^{\max} < 1$. This symplifying hypothesis is not compulsory. In [5, 7] indeed, it is shown to which extent the outer ellipse may be close to the other two, provided that the mass of one of the two inner bodies is large. However, the singularity where the Keplerian ellipses meet one another is certainly not regularizable in the same sense as double inner collisions, since not only the perturbing function gets large in the C^k-topology, but the average system itself.

DEFINITION 3. *For $\varepsilon > 0$, the* perturbing region[11] *of order ε, which is denoted by $\mathbf{P}_\varepsilon$, is the subset of the direct product of the phase space* $\mathbf{X}$

[11] $\mathbf{P}_\varepsilon$ is a subset of what I called the perturbing region Π_ε^k in [7] ($e_2 < e_2^{\max} < 1$).

and of the space $\mathbf{M} \simeq \mathbf{R}^3$ *of the masses where* $d((a_1, a_2), A_f) < a_1/2$, $e_2 < e_2^{\max}$ *and*

$$\frac{(\mu_1 + m_2) M_2}{M_1^2} \frac{a_1}{a_2} < \varepsilon.$$

The mass ratio which occurs in the definition is small exactly if two of the real masses, including the outer mass, are small when compared to the third mass (*planetary problem*). Given that the outer eccentricity is upper bounded, the semi major axis ratio a_1/a_2 is small exactly if the outer body is far from the other two (*lunar problem*). Hence the perturbing region generalizes the planetary and the lunar regions.

The following lemma justifies the latter definition.[12]

LEMMA 2 (Appendix A of [7]). *Let* $k \geq 0$ *be an integer. There is a local* C^k*-norm* $\|\cdot\|_k$ *which depends only on the semi major axes and the masses*[13], *such that the regularized perturbing function* $\mathcal{F}_{\text{per}}$ *satisfies the estimate*

$$\|\mathcal{F}_{\text{per}}\|_k < \mathrm{Cst}_k\, \varepsilon$$

over a uniform neighborhood of the level set $\mathcal{F}_{\text{Kep}} = 0$ *in* $\mathbf{P}_\varepsilon$, *for some constant* Cst_k *which depends on* k *but not on* ε.

3.2. Averaging

We are going to try to eliminate the mean longitudes from the perturbing function over the perturbing region. This can be done at some finite order only where the Keplerian frequencies satisfy a finite number of non-resonant conditions. Since our main purpose is to eventually apply some KAM theorem, we will actually carry out the elimination only on the fixed set of diophantine Keplerian 2-tori. Let (ν_1, ν_2) be the Keplerian frequency vector:

$$\nu_1 = \frac{\partial \mathcal{F}_{\text{Kep}}}{\partial \Lambda_1} = \sqrt{\frac{2 f_1(\Lambda_2)}{\mu_1}}, \quad \nu_2 = \frac{\partial \mathcal{F}_{\text{Kep}}}{\partial \Lambda_2} = \frac{\Lambda_1}{\sqrt{2\mu_1 f_1(\Lambda_2)}} \frac{\mu_2^3 M_2^3}{\Lambda_2^3}.$$

[12] Once an adequate scaling has been made in the action variables, it suffices to prove the estimate in the C^0-norm, because estimates in the C^k-norms just change the constant in the estimate. Hence the proof is elementary.

[13] The norm $\|\cdot\|$ is the norm of Proposition 2.1 of [7] up to a normalization by a constant factor which makes this statement invariant by change of measurement units. It is parametrized by the semi major axes and the masses and it measures the size of a function and its derivatives of order less than or equal to k, with respect to some symplectic coordinates of the secular space.

If $p \geq 1$ is an integer and $\gamma > 0$ and $\tau \geq p-1$ are real numbers, let

$$\begin{cases} HD_{\gamma,\tau}(p) = \left\{\alpha \in \mathbf{R}^p \;:\; \forall k \in \mathbf{Z}^p \setminus 0,\ |k\cdot\alpha| \geq \frac{\gamma}{|k|^\tau}\right\} \\ hd_{\gamma,\tau} = \{(x,m) \in \mathbf{X}\times\mathbf{M} \;:\; (\nu_1(x,m),\nu_2(x,m)) \in HD_{\gamma,\tau}(2)\}, \end{cases}$$

where, for p-uplets k of $\mathbf{Z}^p$, $|\cdot|$ stands for the l_2-norm:

$$|k| = \sqrt{k_1^2 + \ldots + k_p^2};$$

$HD_{\gamma,\tau}(p)$ is the tranversally Cantor set of frequency vectors in $\mathbf{R}^p$ which satisfy homogeneous diophantine conditions of constants γ, τ, and $hd_{\gamma,\tau}$ is the inverse image of $HD_{\gamma,\tau}(2)$ by the Keplerian frequency map (ν_1,ν_2) in the space $\mathbf{X}\times\mathbf{M}$. In the definition of $hd_{\gamma,\tau}$, nothing prevents γ or τ to be functions on $\mathbf{X}\times\mathbf{M}$. Let

$$hd = \bigcup_{\gamma>0,\tau\geq 1} hd_{\gamma,\tau}.$$

Also, let $\overset{\vee}{\nu} = \min(\nu_1,\nu_2)$ be the smallest of the two Keplerian frequencies.

PROPOSITION 1 (Féjoz [7]). *Let $n \geq 0$ and $k \geq 0$ be integers and $\gamma > 0$ and $\tau \geq 1$ be real numbers. For every $\varepsilon > 0$ there are a C^∞-Hamiltonian $\mathcal{F}^n_{\mathrm{sec}} : \mathbf{P}_\varepsilon \to \mathbf{R}$ and a C^∞-symplectomorphism $\phi^n : \mathbf{P}_\varepsilon \to \phi^n(\mathbf{P}_\varepsilon)$ which is ε-close to the identity in the C^k-norm $\|\cdot\|_k$ and fibered[14] over the parameter space $\mathbf{M}$, and such that*

- *there exists a constant $C_{n,k} > 0$ such that, for every $\varepsilon > 0$,*

$$\|\mathcal{F}\circ\phi^n - \mathcal{F}^n_{\mathrm{sec}}\|_k \leq C_{n,k}\varepsilon^{1+n} \quad \textit{over} \quad \mathbf{P}_\varepsilon;$$

- *the restriction of the infinite jet of $\mathcal{F}^n_{\mathrm{sec}}$ to the transversally Cantor set $hd_{\gamma\overset{\vee}{\nu},\tau}$ is invariant by the Keplerian action of the two-torus and by the diagonal action of the circle[15] making the two bodies rotate simultaneously; hence $\mathcal{F}^n_{\mathrm{sec}}$ is completely integrable on $hd_{\gamma\overset{\vee}{\nu},\tau}$.*

The Hamiltonians $\mathcal{F}^n_{\mathrm{sec}}$ are built inductively. At each step, the fast angles (λ_1,λ_2) are eliminated on the transversally Cantor set $hd_{\gamma\overset{\vee}{\nu},\tau}$

[14] ϕ^n leaves the parameters unchanged.
[15] *Diagonal* means that the circle acts simultaneously on both ellipses, as if they made up a rigid solid.

up to an increasing order of smallness; then, by Whitney's extension theorem[16], the averaged infinite jet along $hd_{\gamma\check{\nu},\tau}$ may be extended into a smooth function everywhere. Eliminating the angles only over $hd_{\gamma\check{\nu},\tau}$ allows both to get a positive measure of invariant tori and to have uniform estimates of the secular Hamiltonians over $\mathbf{P}_\varepsilon$.

DEFINITION 4. *The Hamiltonian* $\phi^{n*}\mathcal{F}$ *can be split into the* (nth order, regularized) secular Hamiltonian $\mathcal{F}^n_{\text{sec}}$*, which is Pöschel-integrable on* $hd_{\gamma\check{\nu},\tau}$*, and the* complementary part $\mathcal{F}^n_{comp}$*, which is of* C^k*-size* $O(\varepsilon^{1+n})$*:*

$$\phi^{n*}\mathcal{F} = \mathcal{F}^n_{\text{sec}} + \mathcal{F}^n_{comp}, \qquad \left\|\mathcal{F}^n_{comp}\right\|_k \leq \mathrm{Cst}\,\varepsilon^{1+n}.$$

In turn, the secular Hamiltonian can be split into a (Liouville-) integrable part $\mathcal{F}^n_{\text{int}}$ *and a* resonant part $\mathcal{F}^n_{\text{res}}$ *whose infinite jet vanishes along* $hd_{\gamma\check{\nu},\tau}$*:*

$$\mathcal{F}^n_{\text{sec}} = \mathcal{F}^n_{\text{int}} + \mathcal{F}^n_{\text{res}}, \quad \textit{with} \quad \begin{cases} \mathcal{F}^n_{\text{int}} = \mathcal{F}_{\text{Kep}} + \langle \mathcal{F}_{\text{per}} \rangle + \ldots + \langle \mathcal{F}^{n-1}_{comp} \rangle \\ \\ j^\infty \mathcal{F}^n_{\text{res}}|_{hd_{\gamma\check{\nu},\tau}} = 0. \end{cases}$$

The Keplerian Hamiltonian $\mathcal{F}_{\text{Kep}}$ can thus be thought of as the zeroth order secular system, the perturbing function $\mathcal{F}_{\text{per}}$ as the zeroth order complementary part, and the averaged system $\mathcal{F}_{\text{Kep}} + \langle \mathcal{F}_{\text{per}} \rangle$ as the integrable part $\mathcal{F}^1_{\text{int}}$ of the first order secular system.

The purpose of this paper is to study the dynamics of the integrable Hamiltonians $\mathcal{F}^n_{\text{int}}$, whose infinite jets along $hd_{\gamma\check{\nu},\tau}$ agree with those of $\mathcal{F}^n_{\text{sec}}$. We will loosely call $\mathcal{F}^n_{\text{int}}$ the *secular Hamiltonians.*

3.3. Secular space

The secular Hamiltonians $\mathcal{F}^n_{\text{int}}$ do not depend on the fast angles (λ_1, λ_2). So they descend through the quotient

$$(\mathrm{SO}_3 \times \mathbf{R}) \times (\mathbf{S}^1 \times \mathbf{R}^3 \times \mathbf{S}^0) \xrightarrow{/\mathbf{T}^2} (\mathbf{S}^2 \times \mathbf{R}) \times (\mathbf{R}^3 \times \mathbf{S}^0)$$

by the Keplerian action[17], and the momenta (Λ_1, Λ_2) may be thought of as parameters. From the formula giving Q_1 in terms of the regularized Delaunay-like coordinates, we see that

$$\Lambda_1 = \sqrt{2\mu_1 f_1(\Lambda_2)}\, a_1;$$

[16] Cf. Appendix A of [1], or, in the context of dynamics, [16].
[17] In other words, they induce Hamiltonians on the space of pairs of Keplerian ellipses.

moreover the usual Poincaré coordinate Λ_2 equals

$$\Lambda_2 = \mu_2 \sqrt{M_2 a_2}.$$

Hence fixing these momenta is equivalent to fixing the semi major axes– this is a way to think of the first theorem of stability of Laplace –, and the symplectically reduced phase space is the space of pairs of oriented ellipses with fixed semi major axes and foci such that the two ellipses do not meet each other and the outer ellipse has an upper-bounded eccentricity. It is diffeomorphic to $\mathbf{S}^2 \times (\mathbf{R}^2 \times \mathbf{S}^0)$ and can be compactified to $\mathbf{S}^2 \times \mathbf{S}^2$, by gluing a cylinder of large-eccentricity outer ellipses at infinity.

DEFINITION 5. *The* secular space *will be the space diffeomorphic to* $\mathbf{S}^2 \times \mathbf{S}^2$ *of pairs of oriented regularized ellipses with fixed foci and semi major axes.*

The secular spaces are parametrized by the masses, the semi major axes and the regularized energy level $-f < 0$. We have compactified them for symmetry purposes, but naturally the Hamiltonians $\mathcal{F}^n_{\text{int}}$, for $n \geq 1$, are only defined on some open subset of $\mathbf{S}^2 \times \mathbf{S}^2$ where the outer eccentricity is upper bounded and where the Keplerian ellipses do not meet each other.

If we think of the 2-sphere as the configuration space of one of the two ellipses $j = 1$ or 2, we have the following non-symplectic chart: for the standard embedding $\mathbf{S}^2 \hookrightarrow \mathbf{R}^3$, as on Figure 1, the argument g_j of the pericenter of the ellipse, whenever defined, equals the longitude of the point on $\mathbf{S}^2$, and the eccentricity is the distance between the point and the vertical axes. The ellipse is positively (resp. negatively) oriented in the northern (resp. southern) hemisphere.

It will be usefull to have some additional notations at hand. Let φ_j be the colatitude on $\mathbf{S}^2$, so that the map

$$\begin{array}{llcl} \text{spher}: & \mathbf{T}^2 & \longrightarrow & \mathbf{S}^2 \hookrightarrow \mathbf{R}^3 \\ & (g_j, \varphi_j) & \longmapsto & (\cos g_j \sin \varphi_j, \sin g_j \sin \varphi_j, \cos \varphi_j) \end{array}$$

is the usual spherical-coordinate map. The *(signed) eccentricity* e_j and *centricity* ϵ_j are

$$e_j = \sin \varphi_j \quad \text{and} \quad \epsilon_j = \cos \varphi_j.$$

The northern and southern hemispheres have coordinates

$$(e_j \cos \varphi_j, e_j \sin \varphi_j),$$

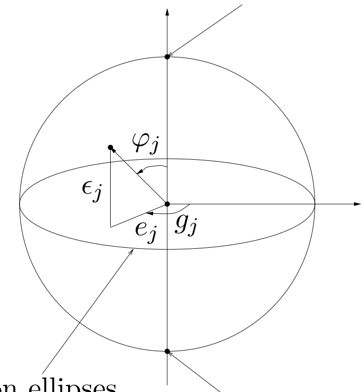

Figure 1. Sphere $\mathbf{S}^2$ of ellipses with fixed foci and semi major axes.

and the cylinder of non circular ellipses has coordinates (g_j, ϵ_j). The corresponding symplectic coordinates are the secular Poincaré variables (ξ_j, η_j) and Delaunay variables (g_j, G_j).

3.4. Expansion of the averaged system

Now, consider the averaged system

$$\mathcal{F}^1_{\text{int}} = \langle \mathcal{F} \rangle = \mathcal{F}_{\text{Kep}} + \langle \mathcal{F}_{\text{per}} \rangle.$$

The Keplerian part depends only on the masses and semi major axes. Hence it is a constant and may be omitted. After Proposition 3.1 of [6], the average $\langle F_{\text{per}} \rangle$ of the perurbing function agrees with the averaged system $\langle F_{\text{per}} \rangle$ of the non regularized problem, up to the constant factor a_1, and provided that these Hamiltonians are seen as functions on the abstract space of pairs of ellipses. Hence, after the computation of $\langle F_{\text{per}} \rangle$

in appendix C of [7],[18]

$$\langle \mathcal{F}_{\mathrm{per}} \rangle = -\mu_1 m_2 \epsilon_2 \frac{a_1}{a_2} \left[\begin{array}{l} \frac{2+3e_1^2}{8} \left(\frac{a_1}{a_2 \epsilon_2^2} \right)^2 \\ -\frac{15}{64}(\sigma_0 - \sigma_1)(4+3e_1^2) e_1 e_2 \cos g \left(\frac{a_1}{a_2 \epsilon_2^2} \right)^3 \\ +\frac{9\sigma_4}{1024} \left(\begin{array}{l} 70 e_1^2 e_2^2 (2+e_1^2)\cos(2g) \\ +45 e_1^4 e_2^2 + 30 e_1^4 \\ +120 e_1^2 e_2^2 + 80 e_1^2 \\ +24 e_2^2 + 16 \end{array} \right) \left(\frac{a_1}{a_2 \epsilon_2^2} \right)^4 \\ +O\left(\left(\frac{a_1}{a_2} \right)^5 \right) \end{array} \right],$$

where $g = g_1 - g_2$ is the difference of the arguments of the pericenters, and the σ_n's are defined, for $n \geq 2$, by

$$\sigma_n = \sigma_0^{n-1} + (-1)^n \sigma_1^{n-1}.$$

In order to study the averaged system, we will assume that the semi major axes ratio $a = a_1/a_2$ is small enough. However, once we know some given property to hold for some value of a, then we may fix a and let ε go to zero (possibly along the planetary problem) to apply KAM theorem for instance, such as in [7].

4. Reduction by the Symmetry of Rotations

4.1. Subset of aligned ellipses

The circle $\mathbf{S}^1 = \mathbf{RP}^1$ of ellipses with fixed argument g_j of pericenter plays the role of a real form of the whole space $\mathbf{S}^2 = \mathbf{CP}^1$ of ellipses.[19] Its analogue for pairs of ellipses the subset of pairs of aligned of ellipses, that is, pairs of ellipses whose difference $g = g_1 - g_2$ of arguments of pericenters satisfies $g = 0 \pmod{\pi}$. In Sections 6.2–3, the structure of secular singularities will be best understood by first restricting ourselves

[18] Lieberman [13] gives a very similar expression. But the coefficients $\sigma_j' s$ there are different, because Lieberman uses a heliocentric splitting, as opposed to the Jacobi splitting which we use.

Besides, referring to Dziobek, Lieberman parametrizes both ellipses by their true anomalies. But the computation can be simplified by rather parametrizing the inner ellipse by its eccentric anomaly. Also there are two minor mistakes in Lieberman's computation.

[19] By definition the real projective line $\mathbf{RP}^1$ is the manifold of real lines through the origin in $\mathbf{R}^2$. Similarly, the complex projective line $\mathbf{CP}^1$ is the manifold of complex lines (i.e. real planes) through the origin in $\mathbf{C}^2$.

to pairs of aligned ellipses. Dynamically, we will see in Section 4.1 that it is an invariant manifold of the Riemannian gradient of the averaged Hamiltonian.

LEMMA 3. *The subset of aligned ellipses is an embedding in the secular space* $\mathbf{S}^2 \times \mathbf{S}^2$ *of the symmetric cylinder* M_{spher} *of the spherical coordinate map* $spher : \mathbf{T}^2 \to \mathbf{S}^2$.

Here, by the *symmetric cylinder* M_{spher} we mean the topological space obtained by gluing two copies $\mathbf{S}^2 \times \mathbf{S}^0$ of the 2-sphere on the boundary of the thickening $\mathbf{T}^2 \times \mathbf{I}$, with attaching map $spher \times id_{\mathbf{S}^0}$. The upper part of Figure 2 symbolically represents this cylinder.

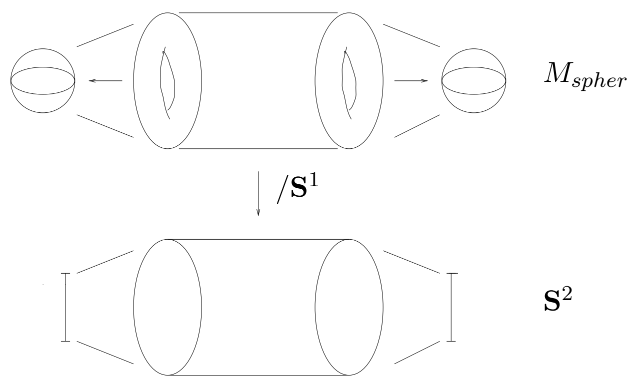

Figure 2. Cylinder M_{spher} of aligned ellipses (Lemma 3). The quotient of M_{spher} by the action of the circle is homeomorphic to $\mathbf{S}^2$ (Lemma 4).

Proof. Consider the blow-up

$$\begin{array}{rccl} cyl: & \mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I} & \longrightarrow & \mathbf{S}^2 \times \mathbf{S}^2 \hookrightarrow \mathbf{S}^2 \times \mathbf{R}^3 \\ & (x, g_2, \epsilon_2) & \longmapsto & (x, (\sqrt{1-\epsilon_2^2}\cos g_2, \sqrt{1-\epsilon_2^2}\sin g_2, \epsilon_2)) \end{array}$$

of the secular space, obtained by providing the outer ellipse with a pericenter when it is circular; $\mathbf{I}$ stands for the closed interval $[-1, 1]$. In $\mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I}$, the submanifold of aligned ellipses is

$$\left\{(x, g_2, \epsilon_2) \in \mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I};\ g = 0\ (\pi)\right\} \simeq \left\{(\varphi_1, g_2, \epsilon_2) \in \mathbf{T}^2 \times \mathbf{I}\right\}.$$

Now, let us identify what has been blown up. The blow-up is bijective on the interior of $\mathbf{S}^2 \times \mathbf{I} \times \mathbf{S}^1$. The intersection $\mathbf{T}^2 \times \mathbf{S}^0$ of the boundary $\mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{S}^0$ with the submanifold $\mathbf{T}^2 \times \mathbf{I}$ of aligned ellipses has two connected components, which both are 2-tori parametrized by (φ_1, g_2); hence the restriction of the mapping cyl to each of these tori is the spherical-coordinate map $spher$.

4.2. Quotient by the diagonal action of the circle

By Proposition 1, the secular Hamiltonians and their integrable parts $\mathcal{F}^n_{\text{int}}$ inherit from the initial Hamiltonian $\mathcal{F}$ the invariance by the lift to the phase space of the rotations in the physical plane. This lift acts on pairs of ellipses by simultaneous rotations, and on the secular space $\mathbf{S}^2 \times \mathbf{S}^2$ diagonally. Let $\pi_{\mathbf{S}^1}$ be the quotient map of the diagonal action of the circle on $\mathbf{S}^2 \times \mathbf{S}^2$.

LEMMA 4. *The quotient $\pi_{\mathbf{S}^1}(\mathbf{S}^2 \times \mathbf{S}^2)$ of the secular space by the diagonal action of the circle is homeomorphic to $\mathbf{S}^3$.*

The image by $\pi_{\mathbf{S}^1}$ of the cylinder M_{spher} of aligned ellipses is a sphere $\mathbf{S}^2$.

Proof. Intuitively, we wish we could choose a rotating frame of reference where for instance the outer ellipse had a fixed argument of pericenter.

Consider the cylindrical-coordinate map $\text{cyl} : \mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I} \to \mathbf{S}^2 \times \mathbf{S}^2$ defined in the proof of Lemma 3. The diagonal action of the circle may be lifted to $\mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I}$. This lifted action is free and its orbit space is diffeomorphic to the global section

$$\Big\{(x, g_2 = 0, \epsilon_2) \in \mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{I}\Big\} \simeq \mathbf{S}^2 \times \mathbf{I}.$$

Blowing down in $\mathbf{S}^2 \times \mathbf{I}$ what has previously been blown up yields $\mathbf{S}^3$. This identification is represented in figure 3; the thikening $\mathbf{S}^2 \times \mathbf{I}$ of $\mathbf{S}^2$ has been cut into two parts, according to the orientation of the outer ellipse, that is, to the sign of ϵ_2:

$$\mathbf{S}^2 \times [-1, 1] = \mathbf{S}^2 \times [-1, 0] \underset{\mathbf{S}^2 \times \{0\}}{\cup} \mathbf{S}^2 \times [0, 1] \ ;$$

the quotients by cyl of $\mathbf{S}^2 \times [-1, 0]$ and $\mathbf{S}^2 \times [0, 1]$ are two 3-balls, whose gluing along their boundary by the identity map is the 3-sphere indeed:

$$\mathbf{S}^3 = \mathbf{B}^3 \underset{\mathbf{S}^2}{\cup} \mathbf{B}^3.$$

By the same arguments, the set of aligned ellipses modulo simultaneous rotations is a 2-sphere obtained by gluing two copies of the 2-disc $\mathbf{D}^2$ along their boundaries. Following figure 2, the 2-sphere may also be thought of as the symmetric mapping cylinder $M_{\cos}$ of the ramified covering $\cos : \mathbf{S}^1 \to \mathbf{I}$.

The quotient map $\pi_{\mathbf{S}^1}$ fails to be a local fibration at the four points of $\mathbf{S}^2 \times \mathbf{S}^2$ where both ellipses are circular, with two possible orientations each. Indeed, these four configurations are fixed points of the diagonal action of the circle. On the other hand, the following analyticity result holds:

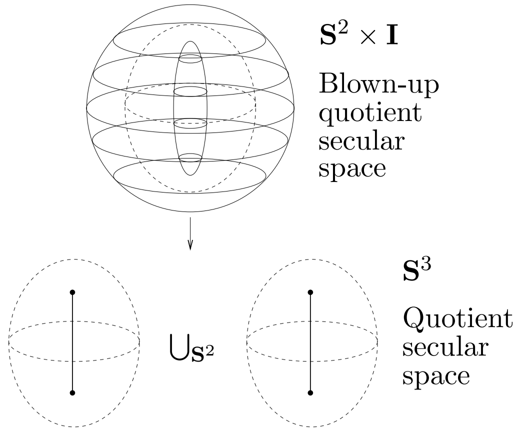

Figure 3. Blow-up of the circular outer ellipses of the quotient secular space.

LEMMA 5. *The quotient map* $\pi_{\mathbf{S}^1} : \mathbf{S}^2 \times \mathbf{S}^2 \to \mathbf{S}^3$ *is analytic.*

Proof. The result is clear in the neighborhood of points where the quotient map is a local fibration.

So consider one of the four singularities, for instance the one where both ellipses are circular and positiveley oriented. Let U be the neighborhood of this singularity defined as the direct product of the two open hemispheres where both ellipses are positively oriented. The map $(e_1 e^{ig_1}, e_2 e^{ig_2})$ defines a diffeomorphism from U to $\mathbf{C}^2$, or, equivalently, to the set $\mathbf{H} = \mathbf{R} \oplus i\mathbf{R} \oplus j\mathbf{R} \oplus k\mathbf{R} = \mathbf{C} \oplus j\mathbf{C}$ of quaternions. The induced diagonal action of the circle on $\mathbf{H}$ is given by

$$e^{i\theta} \cdot (a + bj) = e^{i\theta} a + e^{i\theta} bj$$

for $\theta \in \mathbf{R}/2\pi\mathbf{Z} \simeq \mathbf{S}^1$ and $a, b \in \mathbf{C}$.

Now consider the Kustaanheimo–Stiefel map

$$\begin{array}{rcll} \text{K.S.}: \mathbf{H} & \longrightarrow & \Im\mathbf{H} = \{x \in \mathbf{H} \mid \Re(x) = 0\} \hookrightarrow \mathbf{H} \\ x & \longmapsto & \bar{x} i x, \end{array}$$

where $\Re(x)$ and $\bar{x}$ stand for the real part and conjugate of x, and $\Im\mathbf{H}$ is the subspace of purely imaginary quaternions. If $x = a + bj$ is a quaternion with $a, b \in \mathbf{C}$, a straightforward computation shows that

$$\text{K.S.}(x) = \alpha i + Ak, \qquad \text{with} \quad \alpha = |a|^2 - |b|^2 \quad \text{and} \quad A = 2\bar{a}b.$$

Obviously, the images by K.S. of two quaternions x and $e^{i\theta} \cdot x$ belonging to the same $\mathbf{S}^1$-orbit agree. Conversely, if x and y are two quaternions such that $\text{K.S.}(x) = \text{K.S.}(y)$, setting $z = xy^{-1}$ yields $\text{K.S.}(z) = i$ or, $z \in \mathbf{C}$ and $|z| = 1$; hence x and y belong to the same $\mathbf{S}^1$-orbit. In other words, the quotient map $\pi_{\mathbf{S}^1}$ is locally realized by the analytic map K.S.

The "real form" of the map $\pi_{\mathbf{S}^1} : \mathbf{S}^2 \times \mathbf{S}^2 \to \mathbf{S}^3$ is a ramified two-sheeted covering from $\mathbf{T}^2$ to $\mathbf{S}^2$. Hence, $\pi_{\mathbf{S}^1}$ may be thought of as a quaternionic generalization of Jacobi's elliptic fonction. Also, in the neighborhood of each ramified point, the real form of $\pi_{\mathbf{S}^1}$[20] can be identified to the base part

$$\underline{\text{L.C.}} : \begin{array}{l} \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \\ z \longmapsto z^2 \end{array}$$

of the Levi-Civita map L.C. (cf. Section 1.2).

The following commutative diagrams summarize the quotient maps. The first one is local in the neighborhood of one of the four singularities, while the second one is global:

$$\begin{array}{ccccccc} \mathbf{S}^1_r & \hookrightarrow & \mathbf{R}^2 & \hookrightarrow & \mathbf{R}^4 & \hookleftarrow & \mathbf{S}^3_r \\ \downarrow \theta \mapsto 2\theta & & \downarrow \underline{\text{L.C.}} & & \downarrow \text{K.S.} & & \downarrow \text{Hopf} \\ \mathbf{S}^1_{r^2} & \hookrightarrow & \mathbf{R}^2 & \hookrightarrow & \mathbf{R}^3 & \hookleftarrow & \mathbf{S}^2_{r^2}, \end{array} \qquad \begin{array}{ccc} \mathbf{S}^1 \times \mathbf{S}^1 & \hookrightarrow & \mathbf{S}^2 \times \mathbf{S}^2 \\ \downarrow & & \downarrow \pi_{\mathbf{S}^1} \\ \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^3. \end{array}$$

The K.S. map sends each sphere $\mathbf{S}^3_r$ centered at the origin and of radius r in $\mathbf{R}^4$ onto a sphere $\mathbf{S}^2_{r^2}$ of radius r^2 in $\mathbf{R}^3$, through a Hopf fibration;[21] hence it is a cone over the Hopf fibration. As a consequence, it does not have any continuous section. Hence there is no reason a priori for the Hamiltonian induced by $\mathcal{F}^n_{\text{int}}$ on $\mathbf{S}^3$ to be differentiable, although for instance $\mathcal{F}^1_{\text{int}}$ is analytic outside the branch points of $\pi_{\mathbf{S}^1}$.

LEMMA 6. *If the semi major axes ratio a_1/a_2 and the order ε of the perturbing region $\mathbf{P}_\varepsilon$ are small, the Hamiltonians induced by the secular Hamiltonians $\mathcal{F}^n_{\text{int}}$ on the quotient secular space $\mathbf{S}^3$ are not differentiable at the branch points of the map $\pi_{\mathbf{S}^1}$.*

However, the restrictions of these induced Hamiltonians to a regular level sphere $\mathbf{S}^2$ of the angular momentum are of class C^∞.

Proof. The critical level sets of the angular momentum agree with the ramification points of the quotient map $\pi_{\mathbf{S}^1}$. So the restriction of $\pi_{\mathbf{S}^1}$ to a regular sphere of constant angular momentum is a local fibration. Hence, for regular values of the angular momentum, the reduced secular Hamiltonians are smooth.

Now consider the neighborhood of the ramification points of $\pi_{\mathbf{S}^1}$. Since $\mathcal{F}^n_{\text{int}}$ is ε-close to the averaged system $\langle \mathcal{F}_{\text{per}} \rangle$ in the C^1-topology,

[20] that is, the restriction of $\pi_{\mathbf{S}^1}$ to aligned ellipses.

[21] By definition, the Hopf fibration $\mathbf{S}^3 \to \mathbf{S}^2$ maps $(x, y) \in \mathbf{S}^3 \hookrightarrow \mathbf{C}^2$, $|x|^2 + |y|^2 = 1$, to its orbit $\{(e^{i\alpha}x, e^{i\alpha}y), \alpha \in \mathbf{R}/2\pi\mathbf{Z}\}$ under the diagonal action of the circle.

it suffices to prove that $\langle \mathcal{F}_{\mathrm{per}} \rangle$ is not differentiable on $\mathbf{S}^3$. After the expansion of $\langle \mathcal{F}_{\mathrm{per}} \rangle$ given in § 2.4, it even suffices to prove that the function e_1^2 is not differentiable on $\mathbf{S}^3$.

If $x = a + bj$ is a quaternion with $a, b \in \mathbf{C}$, recall from Lemma 5 that

$$\mathrm{K.S.}(x) = \alpha i + Ak, \qquad \text{with} \quad \alpha = |a|^2 - |b|^2 \quad \text{and} \quad A = 2\bar{a}b.$$

The function $e_1^2 = |a|^2$ on $U \simeq \mathbf{H}$ (cf. the proof of lemma 5) is invariant by the diagonal action of the circle, and can thus be factorized by the K.S. map; indeed, we have

$$e_1^2 = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 2|A|^2}\right) \circ \mathrm{K.S.}$$

The function $\sqrt{\alpha^2 + 2|A|^2}$ of $\alpha i + Aj \in \Im\mathbf{H}$ is not differentiable at 0, and hence neither are the Hamiltonians induced by the averaged Hamiltonian or higher order secular Hamiltonians.

In the quotient secular space $\mathbf{S}^3$, outside the four singularities of $\pi_{\mathbf{S}^1}$, we have the following local coordinates:

- $(\varphi_1, g = g_1 - g_2, \varphi_2)$, and $(\epsilon_1, g, \epsilon_2)$, if $e_1 \neq 0$ and $e_2 \neq 0$, because then $g_1 + g_2$ is a submersion and its level sets are sections of the $\mathbf{S}^1$-orbits of $\mathbf{S}^2 \times \mathbf{S}^2$;

- $(e_1 \cos g, e_1 \sin g, \varphi_2)$ if $e_1 \simeq 0$, because then $e_2 \neq 0$, so locally g_2 is a submersion and its level sets are sections of the $\mathbf{S}^1$-orbits of $\mathbf{S}^2 \times \mathbf{S}^2$;

- $(\varphi_1, e_2 \cos g, e_2 \sin g)$ if $e_2 \simeq 0$, by the same kind of argument.

At the four singularities, where both ellipses are circular, coordinates mix eccentricities of the two ellipses. So it is more pleasant to consider the blow-up

$$\begin{array}{ccc} \mathbf{S}^2 \times \mathbf{S}^2 & \xleftarrow{id_{\mathbf{S}^2} \times spher} & \mathbf{S}^2 \times \mathbf{T}^2 \\ \big\downarrow /\mathbf{S}^1 & & \big\downarrow /\mathbf{S}^1 \\ \mathbf{S}^3 & \longleftarrow & \mathbf{S}^2 \times \mathbf{S}^1, \end{array}$$

where, in $\mathbf{S}^2 \times \mathbf{S}^1$, $\mathbf{S}^2$ is the configuration space of the inner ellipse and $\mathbf{S}^1$ is parametrized by φ_2.

In the proof of Theorem 1, we will also consider the blow-up spher $\times$ spher. In particular, the quotient of the blow-up of the subset of aligned ellipses is a 2-torus $\mathbf{T}^2$ parametrized by (φ_1, φ_2):

$$\begin{array}{ccc} M_{\text{spher}} & \xleftarrow{\text{spher}\times\text{spher}} & \mathbf{T}^3 \times \mathbf{S}^0 \\ \big\downarrow /\mathbf{S}^1 & & \big\downarrow /\mathbf{S}^1 \\ \mathbf{S}^2 & \xleftarrow[(\varphi_1,\varphi_2)\sim(-\varphi_1,-\varphi_2)]{\text{antipodal}} & \mathbf{T}^2. \end{array}$$

Furthermore, as we have already noticed it, $\mathbf{T}^2$ embeds in $\mathbf{S}^2 \times \mathbf{S}^2$, onto the set of pairs of ellipses with some fixed argument of pericenter modulo π. Hence $\mathbf{T}^2$ may be thought of as a real form of $\mathbf{S}^2 \times \mathbf{S}^2$.

4.3. Foliation by the level sets of the angular momentum

Since the diagonal action of the circle is the Hamiltonian flow of the angular momentum

$$C = G_1 + G_2 = \pm\left(\Lambda_1 - \frac{\xi_1^2 + \eta_1^2}{2}\right) \pm \left(\Lambda_2 - \frac{\xi_2^2 + \eta_2^2}{2}\right),$$

the following result holds.

LEMMA 7 (Singularities of the angular momentum on $\mathbf{S}^2 \times \mathbf{S}^2$). *The critical points of the angular momentum are the four ramification points of the map $\pi_{\mathbf{S}^1}$. They are non- degenerate: elliptic when the two ellipses have the same orientation, and hyperbolic of index* 2 *otherwise.*

When the parameter Λ_1/Λ_2 goes through the value 1, the angular momentum undergoes a heteroclinic bifurcation: the two hyperbolic level sets

$$C = \pm|\Lambda_1 - \Lambda_2|$$

agree and the foliation of the secular space $\mathbf{S}^2 \times \mathbf{S}^2$ by the level sets of the angular momentum is symmetric with respect to the diagonal of pairss of ellipses with equal eccentricies. Also, since the flows of C and of the secular Hamiltonians commute, the critical points of C, which are isolated, automatically are fixed points of the secular systems, at any order of averaging.

LEMMA 8 (Foliation by the level sets of the angular momentum). *In the secular space $\mathbf{S}^2 \times \mathbf{S}^2$, the level sets of the angular momentum C are diffeomorphic to $\mathbf{S}^3$ within the two bounded outer intervals of regular values of C, and diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$ between the two hyperbolic critical values of C.*

In the quotient secular space $\mathbf{S}^3$, the regular level sets of C are diffeomorphic to $\mathbf{S}^2$.

Proof. Assume that $\Lambda_1 < \Lambda_2$. In $\mathbf{S}^2 \times \mathbf{S}^2$, the local expression of the angular momentum in the Poincaré coordinates shows that, in the neighborhood of its extrema, the level sets of C are standard round spheres $\mathbf{S}^3$. The quotient map by rotations is the Hopf fibration, whose image is a family of standard round spheres $\mathbf{S}^2$. Within the two whole outer bounded intervals

$$]-\Lambda_1 - \Lambda_2, \Lambda_1 - \Lambda_2[\quad \text{and} \quad]\Lambda_2 - \Lambda_1, \Lambda_1 + \Lambda_2[,$$

C is a fibration and hence its level sets still are 3- and 2-spheres before and after quotient by the rotations.

Within the middle interval

$$]\Lambda_1 - \Lambda_2, \Lambda_2 - \Lambda_1[$$

of regular values of C, the expression

$$C = \Lambda_1 \cos\varphi_1 + \Lambda_2 \cos\varphi_2$$

shows that the conservation of the angular momentum does not prevent the inner ellipse ($\Lambda_1 < \Lambda_2$) to be a collision ellipse ($e_1 = 1$), but prevents the outer ellipse to be circular. In other words, the level sets of C are diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$, where $\mathbf{S}^2$ is the configuration sphere of the inner ellipse and $\mathbf{S}^1$ is parametrized by the argument g_2 of the pericenter of the outer ellipse. The quotient map is the trivial fibration $\mathbf{S}^2 \times \mathbf{S}^1 \to \mathbf{S}^2$.

If $\Lambda_1 > \Lambda_2$, it suffices to switch the roles played by the two ellipses. If eventually $\Lambda_1 = \Lambda_2$, the two hyperbolic level sets agree.

The angular momentum does not depend on the arguments of the pericenters of the ellipses. Hence there is no loss of information on Figure 4, which represents the level sets of the restriction of C to (a fundamental domain of) the real form $\mathbf{T}^2$ of the secular space. The circles corresponding to higher or lower regular levels are homotopic to 0; those corresponding to the levels which are close to 0 are homotopic to one of the two generators of the homology, depending on the sign of $\Lambda_1 - \Lambda_2$.

5. Secular Symmetries and Singularities

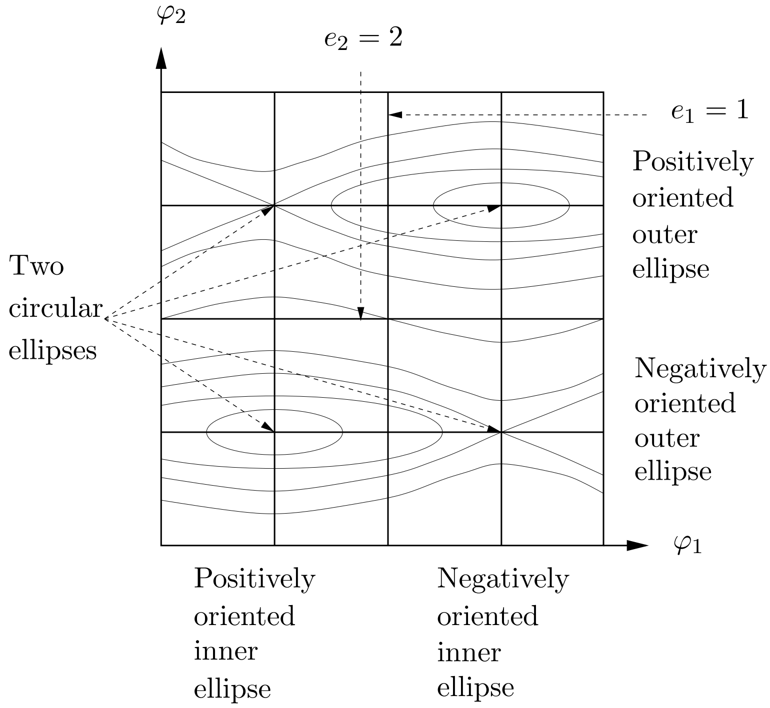

Figure 4. Level sets of the angular momentum in the real form $\mathbf{T}^2$ of the secular space, in the case where $\Lambda_1 < \Lambda_2$. The fundamental domain of $\mathbf{T}^2$ which is drawn is $(\varphi_1, \varphi_2) \in [-\pi/2, 3\pi/2]^2$.

5.1. Symmetries of the secular systems

Up to the additive constant $\mathcal{F}_{\text{Kep}}$, the averaged system $\langle \mathcal{F} \rangle$ equals

$$\langle \mathcal{F}_{\text{per}} \rangle = -\frac{a_1}{4\pi^2} \int_{\mathbf{T}^2} \left(\frac{m_0 m_2}{|Q_2 + \sigma_1 Q_1|} + \frac{m_1 m_2}{|Q_2 - \sigma_0 Q_1|} - \frac{M_1 m_2}{|Q_2|} \right) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2.$$

The coming three symmetries will let us forsee two new families of singularities which are not singularities of the angular momentum.

Since the angular momentum commutes with the averaged system, each family of singularities necessarily is an $\mathbf{S}^1$-orbit. Consider the Hamiltonian induced by $\langle \mathcal{F}_{\text{per}} \rangle$ over the quotient secular space $\mathbf{S}^3$; we still denote this Hamiltonian by $\langle \mathcal{F}_{\text{per}} \rangle$:

LEMMA 9 (Symmetries of the averaged system on $\mathbf{S}^3$).

1. *$\langle \mathcal{F}_{\text{per}} \rangle$ can be factorized through the* 4*-sheeted ramified covering $s_1 : \mathbf{S}^3 \to \mathbf{B}^3$ which sends a pair of oriented ellipses (modulo*

rotations) to the corresponding pair of non oriented ellipses.[22] *The ramification set of* s_1 *is the set* $\overset{4}{\underset{\mathbf{S}^1}{\cup}} \mathbf{D}^2$ *obtained by gluing the four discs where one of the ellipses is degenerate* ($\epsilon_1\epsilon_2 = 0$) *along their boundaries.*[23]

2. $\langle \mathcal{F}_{\mathrm{per}} \rangle$ *can be factorized through the ramified covering* $s_2 : \mathbf{S}^3 \to \mathbf{B}^3$ *which identifies two pairs of oriented ellipses (modulo rotations) which differ only by the sign of* g*. The ramification set of* s_2 *is the sphere* $\mathbf{S}^2$ *of aligned ellipses* ($g = 0\ (\pi)$)*.*

3. *When* $m_0 = m_1$, $\langle \mathcal{F}_{\mathrm{per}} \rangle$ *can be factorized through the ramified covering* $s_3 : \mathbf{S}^3 \to \mathbf{B}^3$ *which identifies two pairs of oriented ellipses (modulo rotations) which differ only by their aguments of pericenters* g *and* $\pi - g$*. The ramification set of* s_3 *is the sphere* $\mathbf{S}^2$ *of ellipses with orthogonal major axes.*

Proof.

1. The change of variable formula for integrals shows that $\langle \mathcal{F}_{\mathrm{per}} \rangle$ does not depend on the orientation of ellipses. Hence $\langle \mathcal{F}_{\mathrm{per}} \rangle$ descends through the map s_1. The ramification set of s_1 is the inverse image of the boundary $\mathbf{S}^2$ of the space $\mathbf{B}^3$ of pairs of non oriented ellipses.

2. Thanks to the latter symmetry and to the invariance of the perturbing function by the change of orientation of the physical plane, the Hamiltonian $\langle \mathcal{F}_{\mathrm{per}} \rangle$ is invariant by the change of $g = g_1 - g_2$ into $-g$, the only angle it depends on. Hence $\langle \mathcal{F}_{\mathrm{per}} \rangle$ descends through the map s_2.

3. The perturbing function is invariant by

$$(Q_1, m_0, m_1) \mapsto (-Q_1, m_1, m_0),$$

and $\langle \mathcal{F}_{\mathrm{per}} \rangle$ by $(g, m_0, m_1) \mapsto (g + \pi, m_1, m_0)$. Hence, if $m_0 = m_1$, the perturbing function is symmetric with respect to the bodies m_0 and m_1, and $\langle \mathcal{F}_{\mathrm{per}} \rangle$ descends through the map s_3.

These three symmetries can easily be visualized in the following interpretation on $\langle \mathcal{F}_{\mathrm{per}} \rangle$, which does not have a simple analogue for higher order secular systems (cf. Figure 5):

[22] In other words, there is a function $\overline{\langle \mathcal{F}_{\mathrm{per}} \rangle}$ such that $\langle \mathcal{F}_{\mathrm{per}} \rangle = \overline{\langle \mathcal{F}_{\mathrm{per}} \rangle} \circ s_1$.

[23] By definition, the *ramification set* of s_1 is the set of points in $\mathbf{S}^3$ where s_1 fails to be a local fibration. Namely, the preimage of a generic point by s_1 consists of four points, whereas the preimage of a point in the image of the ramification set consists of fewer points.

- Let E_0 and E_1 be the homothetic ellipses of masses m_0 and m_1, which under the Keplerian flow are described by the position vectors $\sigma_0 Q_1$ and $-\sigma_1 Q_1$. Hence these two ellipses are rigidly attached in opposition, and they have semi major axes $\sigma_1 a_1$ and $\sigma_0 a_1$, same eccentricity e_1 and arguments of pericenters $g_1 + \pi$ and g_1.
- Let E_2 be the ellipse of mass m_2 which under the Keplerian flow is described by Q_2. Thus its semi major axis, eccentriciy and argument of pericenter are a_2, e_2 and g_2.
- Let $E_{1/2}$ be the repulsive center at the origin, with negative mass $-M_1$.

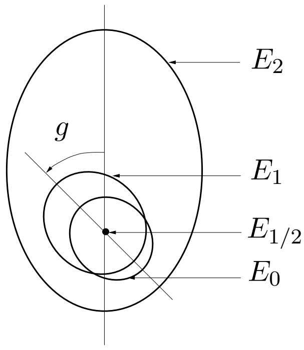

Figure 5. System of ellipses for which $\langle \mathcal{F}_{\mathrm{per}} \rangle$ is the Newtonian potential (case where $m_0 > m_1$).

Then the averaged Hamiltonian is the potential energy of the ellipse E_2 undergoing the quadrupolar potential created by E_0, $E_{1/2}$ and E_1. In particular, when $m_0 = m_1$, the two ellipses E_0 and E_1 are symmetric with respect to the focus.

Now, consider the function $\langle \mathcal{F}_{\mathrm{per}} \rangle$ along the circle of $\mathbf{S}^3$ which is parametrized by φ_2, where the ellipses are aligned ($g = 0 \pmod{\pi}$) and where the inner ellipse is degenerate ($e_1 = 1$). The intersection of this circle with the domain of definition of $\langle \mathcal{F}_{\mathrm{per}} \rangle$ is the disjoint union of the two line segments where the real ellipses do not meet each other. When $a = a_1/a_2$ is small enough, computations in Section 4.2 will show that $\langle \mathcal{F}_{\mathrm{per}} \rangle$ reaches a maximum on each one of these line segments. After the first two symmetries of Lemma 9, these two points are critical points of $\langle \mathcal{F}_{\mathrm{per}} \rangle$.

Figure 6 represents the level sets of the restriction of the Hamiltonian $\langle \mathcal{F}_{\mathrm{per}} \rangle$ to the real form $\mathbf{T}^2$ of aligned ellipses. The domain of definition of this restriction consists of two connected components ($e_2 < \mathrm{Cst}$) which both are diffeomorphic to the cylinder $\mathbf{S}^1 \times \mathbf{R}$. Only one of these components is represented, but thanks to the first symmetry of Lemma 9 the foliations on both components are diffeomorphic.

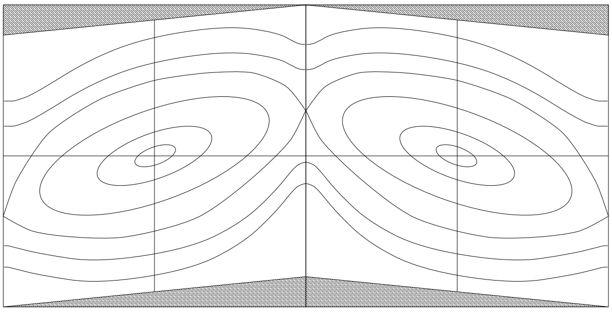

Figure 6. Level curves of the averaged system $\langle F_{\mathrm{per}} \rangle$ over the cylinder of aligned ellipses, parametrized by $(\varphi_1, \varphi_2) \in [-\pi/2, 3\pi/2] \times [-\pi/2, \pi/2]$.

The symmetry s_1 of Lemma 9 comes from the noteworthy fact that the chosen perturbing function depends only on the positions; hence higher order secular systems have no reason a priori for factorizing through s_1. Hence, although the hyperbolic singularities of the averaged system necessarily persist for $\mathcal{F}^n_{\mathrm{int}}$ if ε is small enough, they may move away from the sphere of degenerate inner ellipses. However, the two last symmetries of lemma 9 hold, provided that they are expressed without referring to s_1.

COROLLARY 1 (Symmetries of the secular systems on $\mathbf{S}^3$).

1. *$\mathcal{F}^n_{\mathrm{int}}$ can be factorized through the ramified covering $s'_2 : \mathbf{S}^3 \to \mathbf{B}^3$ whose fibers consist of pairs (E_1, E_2), (E'_1, E'_2) of pairs of ellipses such that E_i and E'_i have opposite orientations ($i = 1, 2$) and the difference g (resp. g') of arguments of the pericenters of E_1 and E_2 (resp. E'_1 and E'_2) satisfy $g = -g' \pmod{2\pi}$. The ramification set of s'_2 consists of the two points where the two ellipses are aligned and both degenerate.*

2. *When $m_0 = m_1$, $\mathcal{F}^n_{\mathrm{int}}$ can be factorized through the ramified covering $s'_3 : \mathbf{S}^3 \to \mathbf{B}^3$ which identifies two pairs of oriented ellipses which differ only by their aguments of pericenters g and $g + \pi$. The ramification set of s'_3 is the circle $e_1 e_2 = 0$ where at least one of the two ellipses is cicular.*

The proof is similar to that of Lemma 9 and is left to the reader. The two symmetries s'_2 and s'_3 do not have the same consequences as those of the averaged system in terms of existence of singularities because their ramification sets are too small. However, non-degenerate singularities of the averaged system persist for higher order secular systems, provided the order ε of the perturbing region $\mathbf{P}_\varepsilon$ is small enough.

5.2. Singularities

We now prove the heuristic description that we made in section 4.1 of the critical points of $\langle \mathcal{F}_{\text{per}} \rangle$, in the case when the semi major axes ratio $a = a_1/a_2$ is small. Since $\langle \mathcal{F}_{\text{per}} \rangle$ descends through the quotient by the diagonal action of the circle, we get rid of this invariance by listing the singularities of the induced Hamiltonian on $\mathbf{S}^3$, which we still denote by $\langle \mathcal{F}_{\text{per}} \rangle$. Actually, this Hamiltonian is only defined on the region of $\mathbf{S}^3$ where the ellipses do not meet each other and where the outer eccentricity is upper bounded; this region is diffeomorphic to the disjoint union of two 3-balls, which correspond to the two possible orientations of the outer ellipse. Note that by Lemma 6 the Hamiltonian $\langle \mathcal{F}_{\text{per}} \rangle$ induced on $\mathbf{B}^3 \cup \mathbf{B}^3 \hookrightarrow \mathbf{S}^3$ is not differentiable at the four branch points of the map $\pi_{\mathbf{S}^1}$. Hence in the following statement "elliptic singularity" stands for "local extremum", or, more precisely, for "image by $\pi_{\mathbf{S}^1}$ of an elliptic singularity in the usual sense in $\mathbf{S}^2 \times \mathbf{S}^2$".

PROPOSITION 2 (Singularities of the averaged system on $\mathbf{S}^3$). *If the semi major axes ratio $a = a_1/a_2$ is small, the averaged Hamiltonian $\langle \mathcal{F}_{\text{per}} \rangle$ possesses exactly six singularities:*

– *four elliptic singularities at the four branch points of $\pi_{\mathbf{S}^1}$ ($e_1 = e_2 = 0$)*

– *and two hyperbolic singularities on the circle of aligned ellipses ($g = 0\ (\pi)$) and degenerate inner ellipses ($e_1 = 1$).*

On both hyperbolic singularities, the outer eccentricity goes to zero when a or $m_0 - m_1$ go to 0.

Proof. First consider a small neighborhood of the four branch points $e_1 = e_2 = 0$. The induced averaged system on $\mathbf{S}^3$ is not differentiable at $e_1 = e_2 = 0$ (cf. Lemma 6). However, $\langle \mathcal{F}_{\text{per}} \rangle$ is analytic on $\mathbf{S}^2 \times \mathbf{S}^2$ and so is (the pull-back of) $\langle \mathcal{F}_{\text{per}} \rangle$ on the ramified covering

$$\mathbf{S}^2 \times \mathbf{T}^2 = (id_{\mathbf{S}^2} \times \text{spher})^{-1}(\mathbf{S}^2 \times \mathbf{S}^2).$$

Hence it induces an analytic Hamiltonian on the quotient $\mathbf{S}^2 \times \mathbf{S}^1$ of the blow-up by the diagonal action of the circle. Now, in the neighborhood of $e_1 = e_2 = 0$ in $\mathbf{S}^2 \times \mathbf{S}^1$, the variables $(x = e_1 \cos g, y = e_1 \sin g, e_2)$

are analytic local coordinates. The first term of the expansion of $\langle \mathcal{F}_{\mathrm{per}} \rangle$ (cf. § 2.4) is positively proportional to

$$f_0 = -\frac{2 + 3(x^2 + y^2)}{(1 - e_2^2)^{3/2}}.$$

The branch points $x = y = e_2 = 0$ are elliptical singularities of f_0, and by the implicit function theorem, they persist as such for $\langle \mathcal{F}_{\mathrm{per}} \rangle$ provided that a is small enough. Moreover, by local unicity and symmetry, the perturbed singularities necessarily agree with the very branch points themselves.

Second, assume that $e_2 \neq 0$. The variables $(e_1 \cos g, e_1 \sin g, \varphi_2)$ are local coordinates in $\mathbf{S}^3$. Using these variables shows that f_0 does not have any critical point in this open set of $\mathbf{S}^3$. Since $\langle \mathcal{F}_{\mathrm{per}} \rangle$ is uniformly C^k-close to f_0 when a is small, given any compact subset K of $\{e_2 \neq 0\} \subset \mathbf{S}^3$, there is a small value of a below which $\langle \mathcal{F}_{\mathrm{per}} \rangle$ does not have any critical point in K.

Lastly, assume that $e_1 \neq 0$. The variables $(\varphi_1, x = e_2 \cos g, y = e_2 \sin g)$ are local coordinates, in terms of which the function f_0 equals

$$f_0 = \frac{2 + 3 \sin^2 \varphi_1}{(1 - x^2 - y^2)^{3/2}}.$$

Since $\varphi_1 \neq 0$ (π), f_0 has two unique critical points ($\varphi_1 = \pi/2$ (π), $e_2 = 0$), which are hyperbolic. By symmetry and local unicity, they persist for $\langle \mathcal{F}_{\mathrm{per}} \rangle$ and their perturbations belong to the sphere $\mathbf{S}^2$ of aligned ellipses. Furthermore, when $m_0 = m_1$, the third symmetry of Lemma 9 prevents these hyperbolic singularities from moving away from $\{e_2 = 0\}$.

Figure 7 represents the foliation by the level sets of $\mathcal{F}_{\mathrm{int}}^n$ of one of the two connected components $\mathbf{B}^3$ of the domain of definition of $\mathcal{F}_{\mathrm{int}}^n$ on $\mathbf{S}^3$.

Higher order secular systems $\mathcal{F}_{\mathrm{int}}^n$ are ε-close to $\mathcal{F}_{\mathrm{int}} = \langle \mathcal{F}_{\mathrm{per}} \rangle$ in $\mathbf{P}_\varepsilon$. So, if both a and ε are small, $\mathcal{F}_{\mathrm{int}}^n$ is close to f_0 (cf. the latter proof). Since f_0 is a Morse function on $\mathbf{S}^2 \times \mathbf{S}^1$, a similar result as Proposition 2 holds for $\mathcal{F}_{\mathrm{int}}^n$. The only difference is that $\mathcal{F}_{\mathrm{int}}^n$ depends a priori on the orientation of the ellipses. Hence the hyperbolic singularities have no obvious reason for remaining located on the circle where the ellipses are aligned and the inner ellipse is degenerate.

COROLLARY 2 (Singularities of secular systems on $\mathbf{S}^3$). *If a and ε are small, the n-th order secular Hamiltonian $\mathcal{F}_{\mathrm{int}}^n$ possesses exactly six singularities:*

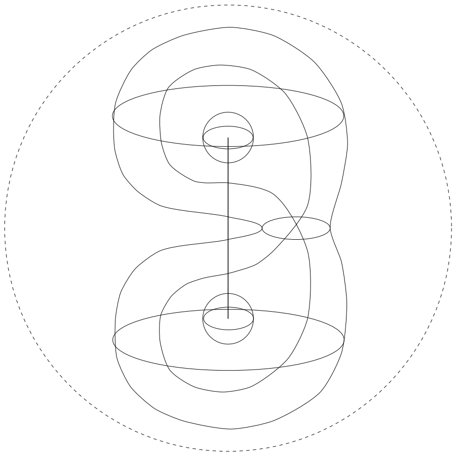

Figure 7. Foliation by the level surfaces of $\langle F_{\rm per}\rangle$ of the ball $\mathbf{B}^3 \hookrightarrow \mathbf{S}^3$ where the outer ellipse is positively oriented. The dashed line represents the boundary of $\mathbf{B}^3$. For high values of $\langle F_{\rm per}\rangle$, the level sets are pairs of topological 2-spheres around the bi-circular points. At the hyperbolic critical value, the two spheres meet at one point. For lower levels of $\langle F_{\rm per}\rangle$, the level sets are connected and consist of one topological 2-sphere each.

– *four elliptic singularities at the four branch points of* $\pi_{\mathbf{S}^1}$ *(*$e_1 = e_2 = 0$*),*

– *two hyperbolic singularities, which are* ε*-close to the circle of aligned ellipses (*$g = 0\ (\pi)$*) and degenerate inner ellipses (*$e_1 = 1$*).*

Singularities of Proposition 2 all arise from the symmetries listed in Lemma 9, and there are no other obvious symmetries. This pleads for the following conjecture.

CONJECTURE 1. *Corollary 2 holds globally in the perturbing region* $\mathbf{P}_\varepsilon$ *without assuming that a is small.*

6. Bifurcation Diagram of the Reduced Secular systems

In this section, we study the secular systems $\mathcal{F}^n_{\rm int}$ after the symplectic reduction by the symmetry of rotations, i.e. restricted to a sphere $\mathbf{S}^2$ of constant angular momentum (cf. Lemma 8) in $\mathbf{S}^3$. In particular, singularities of the reduced system are secular fixed points in some adequate rotating frame of reference.

6.1. Parameter space

The dynamics is invariant by changes of length and mass units. Hence without loss of generality we may parametrize the two lengths a_1, a_2, the three masses m_0, m_1, m_2, the angular momentum C and the energy f with only five parameters, by fixing $M_2 = a_1 = 1$ for instance.

Let a, b, c and d be the adimensional parameters defined by

$$a = \frac{a_1}{a_2}, \quad b = \frac{\Lambda_1}{\Lambda_2}, \quad c = b\epsilon_1 + \epsilon_2, \quad d = \sigma_0 - \sigma_1;$$

a is the semi major axis ratio, b is the ratio of the 'circular linear momenta', c is the normalized angular momentum and d measures the difference between the two inner masses. The careful reader will check that the map

$$(a_1, a_2, m_0, m_1, m_2, C, f) \longmapsto (a_1, M_2, a, b, c, d, f)$$

is a homeomorphism between the two open sets of $\mathbf{R}^7$ defined by the inequalities

$$\begin{cases} a_1, a_2, m_0, m_1, m_2, f \in]0, +\infty[\\ |C| < \Lambda_1 + \Lambda_2 \\ d((a_1, a_2), A_f) < a_1/2 \quad (cf. \S\ 2.1) \end{cases}$$

in the range, and

$$\begin{cases} a_1, M_2, a, b, f \in]0, +\infty[\\ |c| < 1 + b \\ |d| < 1 \\ d((a_1, a_1/a), A_f) < a_1/2 \end{cases}$$

in the image. For computational reasons we will describe the bifurcation diagram of the secular systems in terms of the coordinates (a, b, c, d, f) of the parameter space.

6.2. Averaged system

Recall from Lemma 8 that regular level surfaces of the angular momentum C (or c) in the quotient secular space $\mathbf{S}^3$ are differomorphic to $\mathbf{S}^2$. In particular, since the reduced secular space is 2-dimensional, singularities of the Hamiltonian vector field on $\mathbf{S}^2$ agree with and have the same index as critical points of the Hamiltonian itself.

For the sake of simplicity, we assume that the angular momentum C is large enough so that the secular systems are defined over all the level sphere $\mathbf{S}^2$ of constant angular momentum: $c - b > \epsilon_2^{\min}$, where $\epsilon_2^{\min} = \sqrt{1 - e_2^{\max}}$ (cf. the beginning of § 2.1). Dropping this assumption would only allow singularities for which the outer eccentricity is large, to drift out of the perturbing region $\mathbf{P}_\varepsilon$.

We also assume that the angular momentum is larger, in absolute value, than its hyperperbolic critical values: $|c| > |1-b|$. When $|c|$ decreases through $|1-b|$, $\langle \mathcal{F}_{\text{per}} \rangle$ undergoes a singular saddle-node bifurcation and one additional singularity appears, where the inner ellipse is almost circular and the two ellipses have opposite orientations; this singularity is the analogue of Lieberman's singularity [13], but for ellipses of opposit orientations.

THEOREM 1 (Bifurcation diagram of the reduced averaged system). *There are an open set $\mathcal{W}$ of the parameter space, open sets $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ of $\mathcal{W}$, functions $c_1(a,b,d)$, $c_2(a,b,d)$, $d_1(a,b,c)$ and $d_2(a,b,c)$ and a constant* Cst *such that:*

- *$\mathcal{W}$ is defined by $a < \text{Cst}$, $|c| > |1-b|$, and $c - b > \epsilon_2^{\min}$;*
- *the $\mathcal{W}_i$'s are subsets of $\mathcal{W}$ defined by*

$$\begin{cases} \mathcal{W}_1 \ : \ b \notin [0,1] \quad \textit{or} \quad |c| \notin [c_1, c_2], \\ \mathcal{W}_2 \ : \ 0 < b < 1, \quad c_1 < |c| < c_2, \quad d \notin [d_1, d_2], \\ \mathcal{W}_3 \ : \ 0 < b < 1, \quad c_1 < |c| < c_2, \quad d_1 < d < d_2, \end{cases}$$

so that $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \pmod 0$;

- *in $\mathcal{W}_1$, $\langle F_{\text{per}} \rangle$ has exactly two elliptic singularities, such that $g = 0$ (π) and $e_1 e_2 = O(a)$, that is these singularities belong to the circle of aligned ellipses and are a-close to the two points of $\mathbf{S}^2$ where one of the ellipses is circular;*
- *in $\mathcal{W}_2$, $\langle F_{\text{per}} \rangle$ has exactly two additional singularities: a hyperbolic singularity such that $g = O(a)$ (2π) (almost conjunction), and an elliptic singularity such that $g = \pi + O(a)$ (2π) (almost opposition);*
- *in $\mathcal{W}_3$, $\langle F_{\text{per}} \rangle$ has exactly four more singularities than in $\mathcal{W}_1$: two elliptic singularities such that $g = O(a)$ (π), and two hyperbolic singularities for which $g = \pi/2 + O(a) + O(d/a)$ (π).*

Figure 8 represents a section of $\mathcal{W}$ by the codimension-2 space of equations $a = \text{Cst}$ and $f = \text{Cst}'$, where Cst is the same constant as in the theorem, and Cst′ is any positive real number. The graphs of the functions c_1 and c_2 are saddle-node bifurcation surfaces; the graphs of d_1 and d_2 are $\mathbf{Z}_2$-symmetric saddle-node bifurcation surfaces; and the hyperplane $d = 0$ is a heteroclinic bifurcation surface.

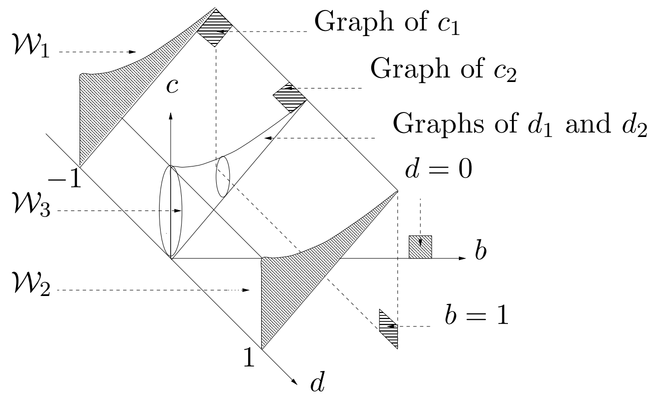

Figure 8. Section of the bifurcation diagram of $\langle F_{\rm per}\rangle$ by the codimension-2 space $a = \mathrm{Cst}$, $f = \mathrm{Cst}'$, where Cst is small enough.

Proof. Singularities of the reduced averaged system are more easily described when unfolded in the quotient secular space $\mathbf{S}^3$, without fixing the angular momentum. The set of these singularities consists of the points where the level surfaces of $\langle \mathcal{F}_{\rm per}\rangle$ and C are in contact in $\mathbf{S}^3$. Hence they are the solutions of the equation

$$d\langle \mathcal{F}_{\rm per}\rangle \wedge dc = 0.$$

Here the computations in local coordinates are similar to those of the proof of Proposition 2. Hence some details will be left to the reader.

First consider the pull-back of this equation by

$$\mathbf{T}^2 \xrightarrow{\alpha} \mathbf{S}^2 \overset{j}{\hookrightarrow} \mathbf{S}^3,$$

where α is the ramified antipodal covering (cf. end of § 3.2) and j is the inclusion of aligned ellipses in the quotient secular space. The 2-torus has coordinates (φ_1, φ_2), in terms of which the equation becomes

$$\sin\varphi_1 \sin\varphi_2 (2\cos\varphi_1 \cos\varphi_2 - b(5 - 3\cos^2\varphi_1)) = O(a).$$

At the limit $a = 0$, this equation makes sense over $\mathbf{T}^2$ (even where the outer eccentricity is large) and the set of its solutions consists of

- four circles $e_1 e_2 = 0$, whose images by α in $\mathbf{S}^2$, after the proof of lemma 4, consist of a unique circle;
- if $0 < b < 1$, the two circles $\epsilon_1\epsilon_2 = b(5 - 3\epsilon_1^2)/2$, whose images by α in $\mathbf{S}^2$ consist of two circles which entirely lie in the hemisphere $\epsilon_1\epsilon_2 > 0$ of pairs of ellipses whose orientations agree;

– if $b = 0$, the two circles $\epsilon_1\epsilon_2 = 0$, which are the standard generators of the homology of $\mathbf{T}^2$, and whose images by α are consist of two circles.

These solutions are represented by dashed lines in figure 9 when $0 < b < 1$. Most of these points are non degenerate solutions of the equation $\alpha^* j^*(d\langle\mathcal{F}_{\text{per}}\rangle \wedge dc)|_{a=0} = 0$ and survive when $a > 0$ provided a is small enough. The perturbed solutions are represented by solid lines on the same figure. The degenerate unperturbed solutions are:

– the four points $e_1 = e_2 = 0$ which, by symmetry, are critical points of both $\langle\mathcal{F}_{\text{per}}\rangle$ and c, and hence which are singularities of the reduced averaged system for $a > 0$;

– the eight singular points which belong to both a circle $e_1e_2 = 0$ and a circle $\epsilon_1\epsilon_2 = b(5-3\epsilon_1^2)/2$, and which, by perturbation, disappear as the point $(0,0)$ in the xy-plane would, for the local model $xy = a$.

The traces of the solutions of $j^*(d\langle\mathcal{F}_{\text{per}}\rangle \wedge dc) = 0$ on a level circle of the angular momentum c in the sphere $\mathbf{S}^2$ of aligned ellipses consist of either two or four points. Indeed, the curves $e_1e_2 = O(a)$ meet all the level curves of the angular momentum $c = b\cos\varphi_1 + \cos\varphi_2$, which yields two solutions. On the other hand, the curves

$$\epsilon_1\epsilon_2 = b(5 - 3\epsilon_1^2)/2 + O(a)$$

meet only those level curves of c for which c lies within two limiting values satisfying

$$c_1 = 2b + O(a) \quad \text{and} \quad c_2 = \frac{1}{3}(2 + \sqrt{(1 + 15b^2)}) + O(a).$$

The values c_1 and c_2 give the boundary of the region $\mathcal{W}_1$ and are saddle-node bifurcations (cf. Figure 10 when $0 < b < 1$).

Consider the full quotient secular space $\mathbf{S}^3$ again. Using local coordinates, it is straightforward to check that the non degenerate singularities of $j^*\langle\mathcal{F}_{\text{per}}\rangle$ yield the singularities of the reduced averaged system:

– The critical points of $j^*\langle\mathcal{F}_{\text{per}}\rangle$ such that $e_1e_2 = O(a)$ give rise to two elliptic singularities for the reduced system. A normal form of the reduced averaged Hamiltonian the neighborhood of these singularities is computed is the last section of [7].

– The points such that $\epsilon_1\epsilon_2 = b(5 - 3\epsilon_1^2)/3)$ are degenerate singularities of the first term of the expansion of $\langle\mathcal{F}_{\text{per}}\rangle$. It is the second

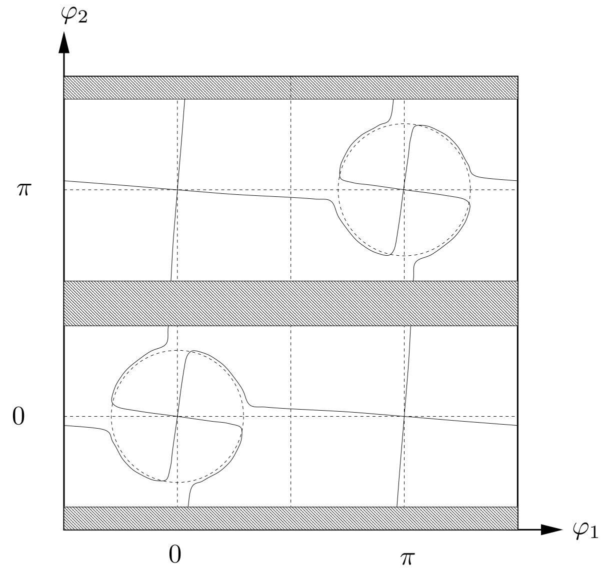

Figure 9. Contact curves of the foliations by the level curves of $\langle F_{\mathrm{per}}\rangle$ and c in the ramified covering $\mathbf{T}^2$ of the sphere $\mathbf{S}^2$ of aligned ellipses. It is assumed that $0 < b < 1$. The dashed lines represents the limit of these contact curves when $a \to 0$, that is, the contact curves of f_0 and c.

term, in $\cos g$, or the third one, in $\cos(2g)$, which breaks up the degeneracy, according to the relative size of a and d. We have

$$\frac{\partial}{\partial g}\langle \mathcal{F}_{\mathrm{per}}\rangle = f_1.\left(A\sin g + \sin(2g) + O(a)\right),$$

where f_1 is a non vanishing function and

$$A = \frac{4}{21}\frac{1}{\sigma_4 e_1 e_2}\frac{4+3e_1^2}{2+e_1^2}\frac{d}{a}.$$

The value $g = 0 \pmod{\pi}$ is always a zero of the function $g \mapsto A\sin g + \sin(2g)$. This function has an additional zero g in $]0,\pi[\pmod{2\pi}$ if and only if $-2 < A < 2$. These inequalities yield a criterion for determining the boundary between $\mathcal{W}_2$ and $\mathcal{W}_3$, as defined in the statement of the theorem:

$$-d_1 = d_2 = \frac{6}{5}\sigma_4\frac{2+e_1^{*2}}{4+3e_1^{*2}}e_1^* e_2^* a \mod O(a^2),$$

where $e_j^* = \sqrt{1-\epsilon_j^{*2}}$, and $(\epsilon_1^*, \epsilon_2^*)$ is either pair of solutions of the equation $\epsilon_1\epsilon_2 = b(5-3\epsilon_1^2)/3$, with $c = b\epsilon_1 + \epsilon_2$. Hence, in addition to the two already mentionned elliptic singularities, the reduced averaged Hamiltonian, in $\mathcal{W}_2$, has two additional singularities: one hyperbolic singularity with $g = 0 \pmod{2\pi}$ (ellipses in conjunction) and one elliptic singularity with $g = \pi \pmod{2\pi}$ (ellipses in opposition). In $\mathcal{W}_3$, the reduced averaged Hamiltonian has four additionnal singularities rather than two: two elliptic singularities with $g = 0 \pmod{\pi}$ (aligned ellipses), and two hyperbolic singularities having non aligned ellipses.

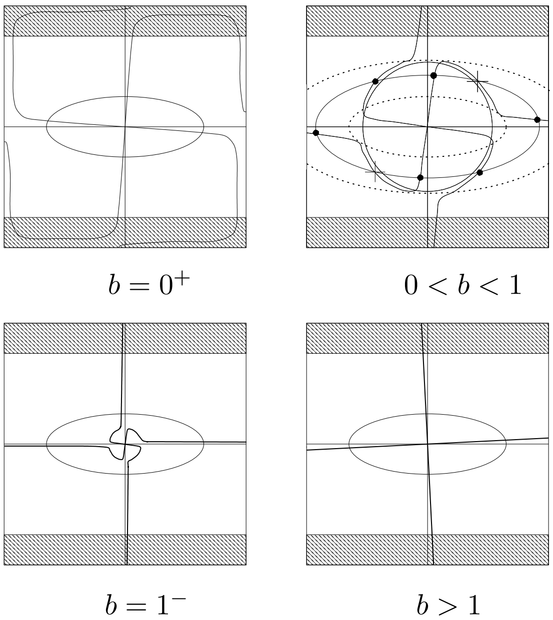

Figure 10. Contact curves of the foliations by the level curves of $\langle F_{\text{per}} \rangle$ and c in the disc of positively oriented and aligned ellipses, for different values of b.

The phase portraits of the reduced averaged system are represented on figure 11 in three particuliar cases of Theorem 1. They can all be factorized by a fold along the circle of aligned ellipses. The thrid one, where $d = 0$, can also be factorized by a fold along the circle where the inner and outer ellipses are perpendicular to one another.

6.3. Consequences

By Theorem 1, in the parameter region $\mathcal{W}_3$ the reduced averaged system has two hyperbolic singularities in the subset of non-aligned ellipses. One singularity is such that $0 < g < \pi \pmod{2\pi}$, and the other one is such that $-\pi < g < 0 \pmod{2\pi}$. Let $\mathbf{B}^3_0 \hookrightarrow \mathbf{S}^3$ be

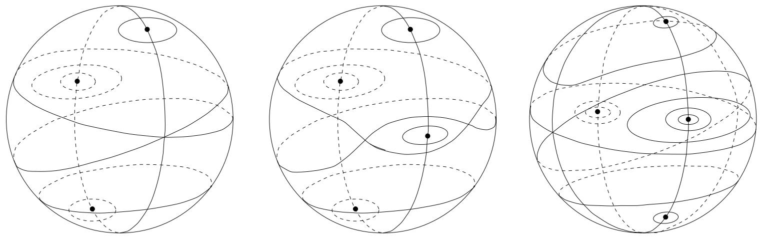

Figure 11. Phase portrait of the reduced averaged system, respectively in $\mathcal{W}_2$, $\mathcal{W}_3 \cap \{d \neq 0\}$ and $\mathcal{W}_3 \cap \{d = 0\}$.

the ball consisting of pairs of positively oriented, non circular ellipses, with a difference g of arguments of pericenters such that $0 < g < \pi \pmod{2\pi}$.

COROLLARY 3 (Surjectivity of singularities of the averaged system). *The function*

$$\begin{array}{lcl} \mathcal{W}_3 & \longrightarrow & \mathbf{B}_0^3 \\ (a,b,c,d,f) & \mapsto & \textit{Hyperbolic singularity of } \langle \mathcal{F}_{\mathrm{per}} \rangle \textit{ in } \mathbf{B}_0^3 \end{array}$$

is onto.

Proof. Consider a pair of ellipses in $\mathbf{B}^3$, with eccentricities e_1 and e_2 and difference of arguments of pericenters $g \in]0, \pi[$. From the proof of Theorem 1, if $f > 0$ is given and if a is small, there are unique values of b and c such that (e_1, e_2) are solutions of the equation

$$\alpha^*(d\langle \mathcal{F}_{\mathrm{per}} \rangle \wedge dc) = 0;$$

and there is a unique value of d such that the function $g \mapsto A \sin g + \sin(2g)$ has a zero in $]0, \pi[$, with

$$A = \frac{4}{21} \frac{1}{\sigma_4 e_1 e_2} \frac{4 + 3e_1^2}{2 + e_1^2} \frac{d}{a} + O(a).$$

For those values of the parameters, the point $(e_1, e_2, g) \in \mathbf{B}_0^3$ is a singularity of the averaged system with $g \neq 0 \pmod{\pi}$. Hence the point (a, b, c, d, f) is in $\mathcal{W}_3$, and fits the bill.

COROLLARY 4 (Reduced secular Hamiltonians). *If ε is small, there are open sets $\mathcal{W}_1^n$, $\mathcal{W}_2^n$ and $\mathcal{W}_3^n$ of $\mathcal{W} \cap \mathbf{P}_\varepsilon$ such that*

– *$\mathcal{W}_1^n$, $\mathcal{W}_2^n$ and $\mathcal{W}_3^n$ are ε-close to the subsets $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ defined in Theorem 1, and*

$$\mathcal{W} \cap \mathbf{P}_\varepsilon = \mathcal{W}_1^n \cup \mathcal{W}_2^n \cup \mathcal{W}_3^n \pmod{0};$$

– *$\mathcal{F}^n_{\rm int}$ has the same number of singularities in $\mathcal{W}^n_1$ and $\mathcal{W}^n_2$ as $\langle\mathcal{F}_{\rm per}\rangle$ in $\mathcal{W}_1$ and $\mathcal{W}_2$, and is C^k-orbitally conjugate and C^k-close to $\langle\mathcal{F}_{\rm per}\rangle$ in these respective regions of the parameter space;*

– *$\mathcal{F}^n_{\rm int}$ has at least two elliptic singularities (and possibly more) $\mathcal{W}^n_3$.*

Proof. Consider the expansions of $\langle\mathcal{F}_{\rm per}\rangle$ and $\mathcal{F}^n_{\rm int} - \langle\mathcal{F}_{\rm per}\rangle$ with respect to the powers of a. The first non-constant term of $\mathcal{F}^n_{\rm int} - \langle\mathcal{F}_{\rm per}\rangle$ certainly is smaller than the first term of $\langle\mathcal{F}_{\rm per}\rangle$, but maybe not than the second or third terms of $\langle\mathcal{F}_{\rm per}\rangle$.

Now, as the proof of theorem 1 shows, when a is small the elliptic singularities of $\langle\mathcal{F}_{\rm per}\rangle$ for which $e_1e_2 = O(a)$ are perturbations of non degenerates singularities of the first term

$$-\mu_1 m_2 \frac{2+3e_1^2}{8\epsilon_2^3} a^3$$

of $\langle\mathcal{F}_{\rm per}\rangle$; hence they survive an additional small perturbation and yield similar singularities for $\mathcal{F}^n_{\rm int} = \langle\mathcal{F}_{\rm per}\rangle + \langle\mathcal{F}^1_{comp}\rangle$ if ε is small enough.

In $\mathcal{W}_2$, the other non degenerate singularities of $\langle\mathcal{F}_{\rm per}\rangle$ with aligned ellipses are perturbations of non degenerate singularities of the first two terms

$$-\mu_1 m_2 \left(\frac{2+3e_1^2}{8\epsilon_2^3} a^3 - \frac{15}{64}(\sigma_0 - \sigma_1) \frac{(4+3e_1^2)}{\epsilon_2^5} e_1 e_2 \cos g\, a^4 \right)$$

of $\langle\mathcal{F}_{\rm per}\rangle$. In order to prove that these singularities persist for $\mathcal{F}^n_{\rm int}$, we need to show that the first term of $\mathcal{F}^n_{\rm int} - \langle\mathcal{F}_{\rm per}\rangle$ is smaller than the second term of $\langle\mathcal{F}_{\rm per}\rangle$, in the norm $\|\cdot\|_k$ of proposition 1. After appendix A of [7], it actually suffices to look to the C^0-estimates. The norm $\|\cdot\|_0$ is defined in [7] by

$$\|\mathcal{F}\|_0 = \sup_{\mathbf{T}^4} \frac{|\mathcal{F}|}{\check{\Lambda}\check{\nu}} = \sup_{\mathbf{T}^4} \frac{|\mathcal{F}|}{\check{\Lambda}\left(\frac{\partial \mathcal{F}_{\rm Kep}}{\partial \Lambda}\right)^{\vee}},$$

where $\mathbf{T}^4$ is the torus of the fast angles (λ_1, λ_2) and the arguments of the pericenters (g_1, g_2), $\check{\nu} = \min(\nu_1, \nu_2)$ is the smallest of the Keplerian frequencies and $\check{\Lambda} = \min(\Lambda_1, \Lambda_2)$ is the smallest of the Keplerian momenta; one of the reasons for this norm to be natural is that it is invariant by change of units. In this norm, the size of the first term of $\mathcal{F}^n_{\rm int} - \langle\mathcal{F}_{\rm per}\rangle$ is

$$\left(\frac{\mu_1 m_2 a^3}{\check{\Lambda}\check{\nu}}\right)^2,$$

and the size of the second term of the averaged system is

$$\frac{\mu_1 m_2 a^4}{\overset{\vee}{\Lambda}\overset{\vee}{\nu}}.$$

We want this ratio to go to zero when ε goes to zero. Since the relevant singularities exist only in $\mathcal{W}_2$, we may assume $b < 1$, that is $\overset{\vee}{\Lambda} = \Lambda_1$. Then the ratio is

$$\frac{m_2\sqrt{a}}{\sqrt{M_1 M_2}} \max\left(\sqrt{\frac{M_2}{M_1}} a^{3/2}, 1\right).$$

By using the inequality

$$\frac{(\mu_1 + m_2)M_2}{M_1^2} a < \varepsilon,$$

which holds in $\mathbf{P}_\varepsilon$ (cf. Definition 3), it is elementary to check that this ratio goes to zero with ε.

In $\mathcal{W}_1$ and $\mathcal{W}_2$, there are no other singularities. Therefore the statement of the theorem concerning $\mathcal{W}_1^n$ and $\mathcal{W}_2^n$ holds.

In $\mathcal{W}_3$, the non circular singularities of the averaged system are determined by the first three terms of the averaged Hamiltonian, and a similar computation shows that in general the third term is no larger than the first term in $\mathcal{F}_{\text{int}}^n$ which comes from the second order averaging.

The first non-vanishing term of the expansion of the perturbing function $\mathcal{F}_{\text{per}}$ depends on g only through $\cos(2g)$–this is a consequence of the fact that the second Legendre polynomial is even. Hence the first term of $\mathcal{F}_{\text{int}}^n - \langle\mathcal{F}_{\text{per}}\rangle$ too depends on g only through $\cos(2g)$. Hence, there is a region $\mathcal{W}_3^n$ close to $\mathcal{W}_3$ such that, up to this term, higher order secular systems have at least as many singularities as the averaged system in $\mathcal{W}_3$. However, proving that these singularities are non degenerate and unique requires to actually compute this term coming from the second-order averaging.

Figure 11 and Corollary 4 show that trajectories of the secular systems fall into several categories:

– regular trajectories where ellipses rotate with respect to one another (such a trajectory generates the first homotopy group of the sphere of constant angular momentum minus the two poles $e_1 e_2 = 0$,[24]

[24] These two poles only make sense asymptotically when ε goes to 0. Indeed, e_1 and e_2 do not denote the physical eccentricities, since the coordinates have been deformed in order to compute the normal forms in Section 2.2.

– regular trajectories where ellipses oscillates with respect to one another (they are homotopic to zero),

– two regular trajectories of undetermined kind, along which one of the two ellipses goes through its circular configuration,

– elliptic or hyperbolic singularities,

– degenerate singularities,

– heteroclinic or homoclinic trajectories.

In [7] it is shown how to apply a sophisticated version of KAM theorem à la Herman [10] in order to prove the existence of a positive measure of various types of quasiperiodic motions in the initial planar three-body problem:

– regular secular orbits which do not meet the set of degenerate inner ellipses give rise to some invariant quasiperiodic Lagrangian four-tori,

– non degenerate secular singularities give rise to some quasiperiodic isotropic invariant three-tori,

– and regular secular orbits which are transverse to the set of physical collisions give rise to some invariant quasiperiodic Lagrangian punctured four-tori.

Also, according to Hanssmann [9], under appropriate transversality conditions parabolic tori persist and furthermore whole saddle-node bifurcations persist, with all lower dimensional invariant tori parametrized by pertinent transversally Cantor sets.

Additionally, non-degenerate secular singularities may be used to prove the existence of short periodic orbits [7] which generalize Poincaré's periodic orbits of the second kind [14]. These orbits actually have two frequencies and are periodic proper in some adequate rotating frame of reference.

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