Unchained polygons and the N-body problem

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October 16, 2008

To the memory of J. Moser, with admiration

Abstract

We study both theoretically and numerically the Lyapunov families which bifurcate in the vertical direction from a horizontal relative equilibrium in $\mathbb{R}^3$. As explained in [CF1], very symmetric relative equilibria thus give rise to some recently studied classes of periodic solutions. We discuss the possibility of continuing these families globally as action minimizers in a rotating frame where they become periodic solutions with particular symmetries. A first step is to give estimates on intervals of the frame rotation frequency over which the relative equilibrium is the sole absolute action minimizer: this is done by generalizing to an arbitrary relative equilibrium the method used in [BT] by V. Batutello and S. Terracini.

In the second part, we focus on the relative equilibrium of the equal-mass regular N-gon. The proof of the local existence of the vertical Lyapunov families relies on the fact that the restriction to the corresponding directions of the quadratic part of the energy is positive definite. We compute the symmetry groups $G_{\frac{r}{s}}(N,k,\eta)$ of the vertical Lyapunov families observed in appropriate rotating frames, and use them for continuing the families globally.

The paradigmatic examples are the "Eight" families for an odd number of bodies and the "Hip-Hop" families for an even number. The first ones generalize Marchal's P_{12} family for 3 bodies, which starts with the equilateral triangle and ends with the Eight [CM, Ma2, CFM, CF1, S]; the second ones generalize the Hip-Hop family for 4 bodies, which starts from the square and ends with the Hip-Hop [CV, CF1, TV].

We argue that it is precisely for these two families that global minimization may be used. In the other cases, obstructions to the method come from isomorphisms between the symmetries of different families; this is the case for the so-called "chain" choreographies (see [S]), where only a local minimization property is true (except for $N = 3$). Another interesting feature of these chains is the deciding role played by the parity, in particular through the value of the angular momentum. For the Lyapunov families bifurcating from the regular N-gon whith $N \leq 6$ we check in an appendix that locally the torsion is not zero, which justifies taking the rotation of the frame as a parameter.

Contents

List of Figures

RELATIVE EQUILIBRIA AND THEIR LYAPUNOV FAMILIES

1 Lyapunov families bifurcating normally from relative equilibria

We consider relative equilibria rotating in the horizontal plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, that is, of the form

$$e^{\mathbb{J}\omega_1 t}\, C,$$

where C is a central configuration, $\omega_1 > 0$ is the frequency and $\mathbb{J}$ is the horizontal-rotation operator acting diagonally on the horizontal components by a rotation of $\pi/2$ and trivially on the vertical ones. Along the paper, various normalizations of C and various rotating frames will be considered, and the corresponding relative equilibrium will often be denoted by $\bar{x}(t)$.

1.1 The horizontal and vertical variational equations

The variational equations associated with a solution $x_i = x_i(t)$ of the Newton's equations for N bodies in the Euclidean space $(\mathbb{R}^3, \|\cdot\|)$

$$\ddot{x}_i = \sum_{i\neq j} m_j \frac{x_j - x_i}{||x_j - x_i||^3},\ i = 1, \ldots, N$$

are

$$\ddot{\delta x_i} = \sum_{j\neq i} m_j \frac{\delta x_j - \delta x_i}{||x_j - x_i||^3} - 3\sum_{j\neq i} m_j \frac{\langle x_j - x_i, \delta x_j - \delta x_i\rangle}{||x_j - x_i||^5}(x_j - x_i). \qquad (VE)$$

Along a planar solution $x_i = x_i(t)$ (supposed to be horizontal), the Pythagoras theorem implies these equations split into horizontal and vertical parts. Namely, if $\delta x_i = h_i + z_i \in \mathbb{C} \oplus \mathbb{R}$ is the decomposition of the variations in respectively horizontal and vertical components, (VE) is equivalent to the following pair of equations: the complicated *horizontal variational equation*

$$\ddot{h}_i = \sum_{j\neq i} m_j \frac{h_j - h_i}{||x_j - x_i||^3} - 3\sum_{j\neq i} m_j \frac{\langle x_j - x_i, h_j - h_i\rangle}{||x_j - x_i||^5}(x_j - x_i), \qquad (HVE)$$

and the much simpler *vertical variational equation*

$$\ddot{z}_i = \sum_{j\neq i} m_j \frac{z_j - z_i}{||x_j - x_i||^3}. \qquad (VVE)$$

1.2 Vertical variations of a relative equilibrium

As the mutual distances $r_{ij} = ||x_j - x_i||$ stay constant along a relative equilibrium motion, the corresponding vertical variational equation has constant coefficients:

$$\begin{pmatrix} \ddot{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{z}_N \end{pmatrix} = \begin{pmatrix} -\sum_{j\neq 1} \frac{m_j}{r_{j1}^3} & \frac{m_2}{r_{21}^3} & \frac{m_3}{r_{31}^3} & \cdot & \cdot & \frac{m_N}{r_{N1}^3} \\ \frac{m_1}{r_{12}^3} & -\sum_{j\neq 2} \frac{m_j}{r_{j2}^3} & \frac{m_3}{r_{32}^3} & \cdot & \cdot & \frac{m_N}{r_{N2}^3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{m_1}{r_{1N}^3} & \frac{m_2}{r_{2N}^3} & \frac{m_3}{r_{3N}^3} & \cdot & \cdot & -\sum_{j\neq N} \frac{m_j}{r_{jN}^3} \end{pmatrix} \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_N \end{pmatrix},$$

or $\ddot{z} = \mathcal{W}z$, where $\mathcal{W}$ is, up to a factor -2, the transposed of the *Wintner-Conley matrix* (i.e., up to a factor -2 and a transposition, it represents the endomorphism A of the space of codispositions in [AC]). The sum of the elements of any line of $\mathcal{W}$ is equal to 0. This implies that it acts on the space of dispositions $\mathcal{D} = \mathbb{R}^N/(1,1,\ldots,1)\mathbb{R}$, to which z rightly belongs.

The matrix $\mathcal{W}$ is symmetric for the mass scalar product, which means that

$$z' \cdot \mathcal{W}z'' = \mathcal{W}z' \cdot z'', \quad \text{where} \quad z' \cdot z'' = \sum_{i=1}^{N} m_i z_i' z_i''.$$

Hence the eigenvalues of $\mathcal{W}$ are real; because the Newton force is attractive, they are also negative (see [Mo, Proposition 1] or [AC]). We will call their distinct values the *vertical frequencies* and denote them $-\omega_1^2, -\omega_2^2, \ldots, -\omega_\ell^2$, $\ell \leq N-1$, *where the ω_k are chosen to be positive*; note that by making $\mathcal{W}$ act on $\mathcal{D}$ we have taken away the eigenvalue 0.

Now, let $Z_1, \ldots, Z_{(N-1)}$ be a basis of $\mathcal{D}$ (which can be chosen orthogonal) consisting of eigenvectors of $\mathcal{W}$ with eigenvalues $-\omega(1)^2, \ldots, -\omega(N-1)^2$, not necessarily distinct. The general solution $Z(t)$ of (VVE) is of the form

$$Z(t) = \sum_{j=1}^{N-1} \mathrm{Re}\,(\alpha_j Z_j e^{i\omega(j)t}), \quad \alpha_j \in \mathbb{C},$$

that is

$$Z(t) = \sum_{k=1}^{\ell} \mathrm{Re}\,(W_k e^{i\omega_k t}),$$

where each W_k is a *complex* eigenvector of $\mathcal{W}$ with eigenvalue $-\omega_k^2$.

1.3 What is known about the vertical frequencies

First, one of the frequencies is that of the relative equilibrium, ω_1: it corresponds indeed to infinitesimal rotations around a horizontal axis.

Let us now compare ω_1 to the other frequencies. Let $I(x) = |x|^2 = \sum m_i ||x_i||^2$ be the moment of inertia (i.e. the square norm in the mass metric), and let $U(x) = \sum_{i<j} \frac{m_i m_j}{||x_j - x_i||}$ be the potential function. Since a central configuration

C is a critical point of the scaled potential $\tilde{U} = I^{\frac{1}{2}} U$, it is natural to write U in terms of $\tilde{U}$:

$$dU(x)\delta x = -(x \cdot \delta x) I(x)^{-\frac{3}{2}} \tilde{U}(x) + I(x)^{-\frac{1}{2}} d\tilde{U}(x)\delta x$$

and

$$d^2U(x)(\delta x, \delta x) = 3(x \cdot \delta x)^2 I(x)^{-\frac{5}{2}} \tilde{U}(x) - 2(x \cdot \delta x) I(x)^{-\frac{3}{2}} d\tilde{U}(x)\delta x$$
$$-|\delta x|^2 I(x)^{-\frac{3}{2}} \tilde{U}(x) + I(x)^{-\frac{1}{2}} d^2\tilde{U}(x)(\delta x, \delta x).$$

Let now $x = C$, the central configuration. Let us split a tangent vector δx as before, into horizontal and vertical components $h \in \mathbb{C}$ and $z \in \mathbb{R}$. Since $d\tilde{U}(C) = 0$,

$$\begin{aligned} d^2U(C)(h+z, h+z) &= 3(C \cdot h)^2 I(C)^{-\frac{5}{2}} \tilde{U}(C) - (|h|^2 + |z|^2) I^{-\frac{3}{2}} \tilde{U}(C) \\ &\quad + I(C)^{-\frac{1}{2}} d^2\tilde{U}(C)(h+z, h+z). \end{aligned}$$

In particular, for a vertical variation,

$$d^2U(C)(z,z) = -|z|^2 I(C)^{-\frac{3}{2}} \tilde{U}(C) + I(C)^{-\frac{1}{2}} d^2\tilde{U}(C)(z,z).$$

But, taking the scalar product with $\bar{x}(t) = e^{\mathbb{J}\omega_1 t} C$ of the identity

$$\nabla U(C) = \ddot{\bar{x}}(t) = -\omega_1^2 \bar{x}(t),$$

where the gradient is relative to the mass metric, one gets $I(C)^{-\frac{3}{2}} \tilde{U}(C) = \omega_1^2$. Finally, one deduces that, if $\mathcal{W}(C) Z_k = -\omega_k^2 Z_k$,

$$d^2\tilde{U}(C)(Z_k, Z_k) = (\omega_1^2 - \omega_k^2) I^{\frac{1}{2}} |Z_k|^2.$$

If we write any vertical variation $Z = \sum_{i=1}^{N-1} u_i Z_{(i)}$ in terms of the orthogonal basis introduced in section 1.2, we get

$$d^2\tilde{U}(C)(Z,Z) = \sum_{i=1}^{N-1} \left(\omega_1^2 - \omega(i)^2\right) I^{\frac{1}{2}} u_i^2 |Z_{(i)}|^2.$$

Is was proved by Pacella (in the equal-mass case) and Moeckel (in the general case) that a planar central configuration C of at least 4 bodies is never a local minimum of $\tilde{U}$: more precisely, there exists always some Z such that $d^2\tilde{U}(C)(Z,Z) < 0$. This implies the

Lemma 1 (Pacella, Moeckel) *For any central configuration C of at least 4 bodies, at least one of the normal frequencies ω_k is strictly greater than ω_1.*

This has a direct consequence on the Hessian of the Lagrangian action: let

$$\zeta(t) = Z_k(\frac{\omega_1}{\omega_k} t) = \mathrm{Re}\,(W_k e^{i\omega_1 t}).$$

Then, for the action during time $T = \frac{2\pi}{\omega_1}$, one has

$$d^2A(\bar{x}(t))(\zeta(t), \zeta(t)) = \pi |W_k|^2(\omega_1^2 - \omega_k^2),$$

which has the same sign as $d^2\tilde{U}(C)(Z_k, Z_k)$. The proof of this formula is a direct computation: on the one hand,

$$\int_0^T |\dot{\zeta}(t)|^2 dt = \pi\omega_1^2|W_k|^2,$$

on the other hand, we have seen that, for any t, $d^2U(C)(\zeta(t), \zeta(t)) = -\omega_k^2\zeta(t)$. This implies the following identity and hence the desired formula:

$$\int_0^T d^2U(\bar{x}(t))(\zeta(t), \zeta(t))dt = -\omega_k^2 \int_0^{2\pi} |\zeta(t)|^2 dt = -\omega_k^2\pi|W_k|^2.$$

It follows that, if $N \geq 4$, a relative equilibrium is never a local minimizer of the action (compare [C1]).

1.4 Lyapunov families and their lifts

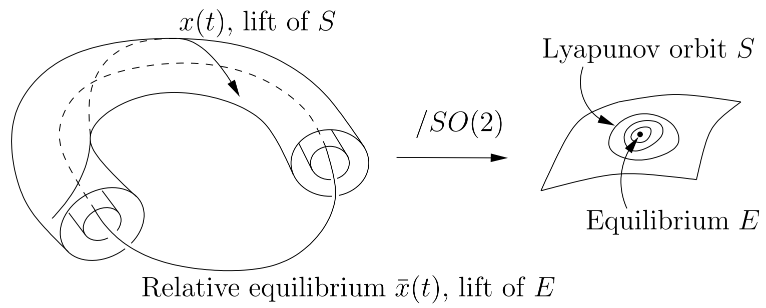


Figure 1: Lyapunov family of a relative equilibrium

The system may be reduced by fixing the angular momentum and quotienting by rotations around its axis; if the relative equilibrium $\bar{x}(t)$ is horizontal, the angular momentum vector will be vertical. Besides, by definition of a relative equilibrium, the solution $\bar{x}(t)$ projects to an equilibrium point E of the reduced system. Each pair of conjugate, purely imaginary eigenvalues of the linearization of the reduced vector field corresponds to a $2\pi/\omega_k$-periodic eigenmode. In good cases, these eigenmodes give rise to a family of solutions of the reduced system of period close to $2\pi/\omega_k$ in the neighborhood of the equilibrium, called a *Lyapunov family.*

In the simplest case where no other frequency of the equilibrium is an integer multiple of ω (this is for example the case of the Hip-Hop family of the four-body problem), the Lyapunov center theorem [Mos] proves the local existence, uniqueness and regularity of this family. In resonant cases, one can use either an argument involving a higher order normal form (such as in [CF2] for the P_{12} family), or a topological, existence argument (such as the Weinstein-Moser theorem [Mos]); for a detailed discussion on the simplest examples, see section 4. Global existence theorems also apply [AY, CMP], but they give only general information on the nature of the families.

Each Lyapunov family L lifts to the original phase space as a local one-parameter (non-reduced, Lyapunov) family of invariant two-tori foliated by quasiperiodic solutions; on a given torus, any two solutions differ only by a rotation around the vertical axis (figure 1). A non-reduced Lyapunov family associated with an eigenmode $z(t) = \operatorname{Re} Z_k \, e^{i\omega_k t}$ is tangent to the lift to the phase space of the *cylindrical* family of curves

$$t \mapsto (\bar{x}(t), Az(t-\varphi)),$$

parametrized by a phase φ and an amplitude A. When observed in a rotating frame which puts into resonance the horizontal and vertical frequencies, these curves become a cylindrical family of loops, whose period is an integer multiple of $2\pi/\omega_k$, say $s2\pi/\omega_k$, $s \in \mathbb{Z} \setminus 0$.

In a continuous family of rotating frames whose rotation frequency now depends on S (in general the rotation of the frame will vary, due to torsion, while it is fixed along the one-parameter family of solutions in the tangent cylinder), the quasiperiodic Lyapunov solutions $x(t)$ themselves become periodic solutions whose homology class in $\mathbb{T}^2$ is independant of S. More precisely, if $\mu(S) \in \mathbb{R}/\mathbb{Z}$ is their monodromy, defined by

$$x(t + 2\pi/\omega(S)) = e^{\mathbb{J}2\pi\mu(S)/\omega(S)} x(t),$$

where $\omega(S)$ is the frequency of S and $\mathbb{J}$, as before, is the rotation operator,[1] the condition is that

$$\frac{\mu(S) - \varpi}{\omega(S)} \in \mathbb{Q}.$$

The scaling invariance of Newton's equations (if $x(t)$ is a solution, so is $\lambda^{-\frac{2}{3}} x(\lambda t)$ for any $\lambda > 0$) entails that each family can be freely rescaled. Fixing the norm of the angular momentum is a way to choose a normalization. But when moving away from the relative equilibrium, this may artificially lead to singularities, at elements of the family having zero angular momentum. Fixing instead the period (in addition to the direction of the angular momentum) is then a better choice of normalization, allowing to bypass zero-angular momentum elements. (On the other hand there is no hope with our techniques to bypass singularities where the period itself tends to infinity.) This allows to assume that in the above family of rotating frames the Lyapunov solutions all have the same period $s2\pi/\omega_k$.

[1] Vertical colinear motions should be avoided since they correspond to a singularity of the reduction.

When the loops in the cylindrical family have some symmetries, the question arises whether the solutions in the the Lyapunov family, when observed in the above family of rotating frames, share the same symmetry. The uniqueness statement in the Lyapunov theorem implies that the answer is positive locally when the eigenmode is simple. However, the first eigenmode is always degenerate, which requires more analysis (see section 4).

The really interesting question is to continue the family globally: if the integrated torsion effect allows continuation up to the inertial frame, we shall have completed the primary purpose of our study –to prove the existence of symmetric solutions in the inertial frame. If the symmetry group is rich enough, this program can be achieved using minimization of the Lagrangian action. This will be illustrated in section 5 with choreographic and Hip-Hop solutions arising from the regular N-gon relative equilibrium.

2 Minimizing properties of relative equilibria

Assume here that the central configuration C is normalized so that $e^{\mathbb{J}t}C$ is a relative equilibrium.

We study the action minimizing properties of the rescaled relative equilibrium

$$\bar{x}(t) = x^{\omega}_{\varpi}(t) = e^{\mathbb{J}(\omega+\varpi)t}C^{\omega}_{\varpi},$$

where

$$C^{\omega}_{\varpi} = (\omega+\varpi)^{-\frac{2}{3}}C.$$

In a frame rotating with frequency ϖ, it becomes the $T = 2\pi/\omega$-periodic loop

$$\bar{y}(t) = y^{\omega}_{\varpi}(t) = e^{-\mathbb{J}\varpi t}x^{\omega}_{\varpi}(t).$$

Let Λ be the space of T-periodic loops in the configuration space with H^1 regularity. For each value of ϖ we define the action $\mathcal{A}_\varpi(y)$ of $y(t) \in \Lambda$ as the action of the path $x(t) = e^{\mathbb{J}\varpi t}y(t)$ in the inertial frame:

$$\mathcal{A}_\varpi(y) = \mathcal{A}(x) = \int_0^T \left(\frac{1}{2}|\dot{x}|^2 + U(x)\right)dt = \int_0^T \left(\frac{1}{2}|\dot{y} + \mathbb{J}\varpi y|^2 + U(y)\right)dt.$$

We call $\mathcal{A}_\varpi$ the action in a frame rotating with frequency ϖ.

2.1 Local minimizing properties

In a frame whose rotation frequency is ϖ, $x^{\omega}_{\varpi}(t)$ has frequency ω. The Lagrangian action during a given time T of a curve $x(t) = (x_1(t), \ldots, x_n(t))$ in the configuration space is

$$\mathcal{A}(x) = \int_0^T \left[\frac{1}{2}|\dot{x}(t)|^2 + U(x(t))\right]dt,$$

where, as before, $|\dot{x}|^2 = \sum m_i ||\dot{x}_i||^2$ and $U(x) = \sum_{i<j} \frac{m_i m_j}{||x_j - x_i||}$ is the potential function. As the kinetic and potential energies of x^ω_ϖ are independant of time, its action during time T is readily computed to be

$$\mathcal{A}(C, \omega, T, \varpi) := \mathcal{A}(x^\omega_\varpi) = (\omega + \varpi)^{\frac{2}{3}} \frac{T}{2\pi} a,$$

where a is the action of the normalized solution $x^1_0(t) = e^{\mathbb{J}t} C$.

As expected, the action tends to zero when ϖ tends to $-\omega$: at the limit, the rotation of the frame alone accounts for the motion and the bodies have only to stay still at infinity in the inertial frame.

We are interested in the local action minimizing properties under appropriate symmetry constraints of the members of this family in the following case:

$$\omega = \frac{r}{s}\omega_k, \quad T = s\frac{2\pi}{\omega_k} = r\frac{2\pi}{\omega},$$

where $\omega_1 = 1, \omega_2, \cdots, \omega_k, \cdots$ are the frequencies of the vertical variational equation associated with $x^1_0(t)$ and r, s are mutually prime integers. Note that T is the minimal period of $(x^\omega_\varpi(t), Z_k(t))$ in the rotating frame with frequency ϖ, where $Z_k(t)$ has frequency ω_k.

Let us first compute the Hessian

$$d^2\mathcal{A}(x)(\xi, \xi) = \int_0^T \left[|\dot{\xi}(t)|^2 + d^2U(x(t))(\xi(t), \xi(t)) \right] dt.$$

of the action when $x = x^\omega_\varpi$ and $\xi = (0, 0, Z_k)$ with Z_k a solution of (VVE) of the form $Z_k(t) = \mathrm{Re}\,(W_k e^{i\omega_k t})$ and W_k a complex eigenvector of $\mathcal{W}(C)$ with eigenvalue $-\omega_k^2$ (see section 1.2). As $\int_0^T \cos^2(\omega_k t) dt = \int_0^T \sin^2(\omega_k t) dt$, we get immediately that

$$\int_0^T |\dot{Z}_k(t)|^2 dt = \omega_k^2 \int_0^T |Z_k(t)|^2 dt.$$

Hence

$$d^2\mathcal{A}(x^\omega_\varpi)(Z_k, Z_k) = \int_0^T \left[\omega_k^2 |Z_k(t)|^2 + d^2U(C^\omega_\varpi)(Z_k(t), Z_k(t)) \right] dt.$$

Now $dU(x)X = X \cdot \mathcal{W}x$, where x and X belong to $\mathcal{D} \otimes \mathbb{R}^3 \equiv \mathcal{D}^3$ and $\mathcal{W}$ acts on each component. Since the mutual distances, hence $\mathcal{W}$, stay constant under a vertical variation, we get

$$d^2U(C^\omega_\varpi)(Z_k(t), Z_k(t)) = Z_k(t) \cdot \mathcal{W}(C^\omega_\varpi) Z_k(t) = -(C^\omega_\varpi / C)^{-3} \omega_k^2 |Z_k(t)|^2,$$

that is

$$d^2\mathcal{A}(x^\omega_\varpi)(Z_k, Z_k) = \left(1 - (\omega + \varpi)^2\right) \omega_k^2 \int_0^T |Z_k(t)|^2 dt.$$

In particular, $d^2\mathcal{A}(x^\omega_\varpi)(Z_k, Z_k) = 0$ iff $\varpi = \pm 1 - \omega = \pm 1 - \frac{r}{s}\omega_k$. This is easy to explain: $\mathcal{A}$ is the action in a frame which rotates in such a way that the

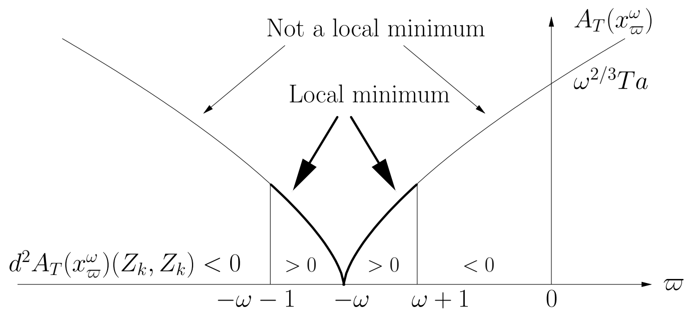


Figure 2: Action of a relative equilibrium in a frame with rotation speed ϖ

period of the relative equilibrium becomes s times the one of $Z_k(t)$. It follows that $Z_k(t)$ defines a periodic Jacobi field and this implies that the corresponding variation belongs to the kernel of the Hessian of $\mathcal{A}$.

In the rotating frame, the solution x^ω_ϖ possesses all the symmetries of the solutions of the (VVE) considered above. If the symmetry constraints that we impose are strong enough to allow only one type of solutions of (VVE) and no solution of (HVE), x^ω_ϖ will be the only local minimizer of the action under thee constraints in the whole interval $-(1+\omega)\le\varpi\le 1-\omega$.

In the case of the regular n-gon configuration of n equal masses, this will be the case for 3 or 4 bodies; but already for 5 bodies, we shall see an example where the symmetry constraint does not discriminate between two different solutions of (VVE). In the next section we study the global minimization problem.

2.2 Global minimizing properties

This section, which as well as 5.1 is taken from the unpublished manuscript [C2], develops an idea first used by Barutello and Terracini in [BT].

We shall now systematically use the shorter notation $\bar{x}(t)=e^{\mathbb{J}(\omega+\varpi)t}\bar{x}$ in place of $x^\omega_\varpi(t)=e^{\mathbb{J}(\omega+\varpi)t}C^\omega_\varpi$.

Let G be a finite subgroup of $O(2)\times\Sigma_N\times O(3)$. It has a natural action on the space Λ of T-periodic loops in the configuration space of the problem: $g=(\tau,\sigma,\rho)\in G$ acts on $y(t)=\big(y_1(t),\cdots,y_N(t)\big)\in\Lambda$ according to the diagram

$$\begin{array}{cccccc} y: & \mathbb{R}/T\mathbb{Z} & \times & \{1,...,N\} & \to & \mathbb{R}^3 \\ & \tau\downarrow & & \sigma\downarrow & & \rho\downarrow \\ gy: & \mathbb{R}/T\mathbb{Z} & \times & \{1,...,N\} & \to & \mathbb{R}^3. \end{array}$$

The transformed loop is

$$gy_j(t)=\rho y_{\sigma^{-1}(j)}(\tau^{-1}(t));$$

this convention is chosen so as to have a left action, i.e. $(gg')q=g(g'q)$.

We suppose that $\bar{y}(t) = e^{-\mathbb{J}\varpi t}\bar{x}(t) = e^{\mathbb{J}\omega t}\bar{x}$ belongs to the subset $\Lambda^G \in \Lambda$ of loops which are invariant under this action and we look for conditions on G and ϖ which insure that the sole absolute minimizer of $\mathcal{A}_\varpi$ is y.

2.2.1 Choice of a new functional

Following [BT] we are looking for an action functional $y \mapsto \bar{\mathcal{A}}_\varpi(y)$ with the following properties:

1) $\mathcal{A}_\varpi(y) \geq \bar{\mathcal{A}}_\varpi(y)$ for any $y(t) \in \Lambda^G$,

2) $\bar{\mathcal{A}}_\varpi$ attains its minimum value at $\bar{y}(t)$

3) If $y(t)$ is a solution in the rotating frame, then $\mathcal{A}_\varpi(y) = \bar{\mathcal{A}}_\varpi(y)$ if and only if $y(t) = \bar{y}(t)$.

If such a functional exists, the sole absolute minimum of $\mathcal{A}_\varpi(y)$ is $\bar{y}(t)$.

As in [BT], from which the notations are borrowed (but with different normalization), one looks for a functional of the form

$$\bar{\mathcal{A}}_\varpi(y) = \bar{\mathcal{A}}(x) = \frac{\lambda}{2}\sum_{i<j}\bar{\mu}_{ij}\xi^x_{ij} + c\left(\sum_{i<j}\bar{\mu}_{ij}\xi^x_{ij}\right)^{-\frac{1}{2}},$$

where, if $x(t) = (x_1(t), \cdots, x_N(t)) = e^{\mathbb{J}\varpi t}y(t)$,

$$\xi^x_{ij} = \int_0^T ||x_i(t) - x_j(t)||^2 dt = \int_0^T ||y_i(t) - y_j(t)||^2 dt$$

and all the coefficients $\lambda, c, \bar{\mu}_{ij}$ are positive.

2.2.2 Minoration of the potential part of the action

By Jensen inequality applied to the convex function $f(s) = s^{-\frac{1}{2}}$,

$$\frac{1}{T}\int_0^T \frac{dt}{||x_i(t) - x_j(t)||} \geq \left(\frac{1}{T}\int_0^T ||x_i(t) - x_j(t)||^2\right)^{-\frac{1}{2}} = \left(\frac{1}{T}\xi^x_{ij}\right)^{-\frac{1}{2}},$$

with equality if and only if $||x_i(t) - x_j(t)||$ is independent of t.

Hence, for any choice of positive μ_{ij}'s, there exists a maximal $\bar{c} > 0$ such that

$$\int_0^T U(x)dt \geq T^{\frac{3}{2}}\sum_{i<j} m_i m_j \left(\xi^x_{ij}\right)^{-\frac{1}{2}} \geq \bar{c}T^{\frac{3}{2}}\left(\sum_{i<j}\mu_{ij}\xi^x_{ij}\right)^{-\frac{1}{2}}.$$

Indeed, as the function

$$\left(\sum_{i<j}\mu_{ij}\xi^x_{ij}\right)^{\frac{1}{2}}\left(\sum_{i<j} m_i m_j \left(\xi^x_{ij}\right)^{-\frac{1}{2}}\right)$$

is homogeneous of degree 0, it is enough to look for a constrained minimum of $\sum_{i<j} m_i m_j \left(\xi_{ij}^x\right)^{-\frac{1}{2}}$ on $\sum_{i<j} \mu_{ij}\xi_{ij}^x = 1$, i.e. there must exist α such that

$$\forall i<j,\ \frac{1}{2}m_i m_j \left(\xi_{ij}^x\right)^{-\frac{3}{2}} = \alpha\mu_{ij}, \quad \text{i.e.} \quad \forall i<j,\ \xi_{ij}^x = \left(\frac{2\alpha\mu_{ij}}{m_i m_j}\right)^{-\frac{2}{3}}.$$

Choice of the μ_{ij}: On chooses $\mu_{ij} = \bar{\mu}_{ij}$ such that the $\bar{\xi}_{ij} = \left(\frac{\bar{\mu}_{ij}}{m_i m_j}\right)^{-\frac{2}{3}}$ be equal to the squared mutual distances $\bar{r}_{ij}^2$ of the central configuration $\bar{x}$ (with relative equilibrium motion of period T in the rotating frame). In other words, one chooses

$$\bar{\mu}_{ij} = m_i m_j \bar{r}_{ij}^{-3}.$$

One checks that $\bar{c} = \bar{U}^{\frac{3}{2}}$, where $\bar{U} = U(\bar{x})$. We have proved the

Lemma 2

$$\int_0^T U(x(t))dt \geq (\bar{U}T)^{\frac{3}{2}} \left(\sum_{i<j} \bar{\mu}_{ij}\xi_{ij}^x\right)^{-\frac{1}{2}}$$

with equality if and only if there exists $\beta > 0$ and $S(t) \in O(3)$ such that for all t, $x(t) = \beta S(t)\bar{x}$.

The only if part is because the mutual distances of the configuration $x(t)$ being independent of t and proportional to the corresponding ones of $\bar{x}$, this implies the existence of β and $S(t)$.

2.2.3 Minoration of the kinetic part of the action

Integrating $|x(t)|^2$ by parts between 0 and T does not lead to a boundary term because, if $x(t)$ itself is not T-periodic, $< x(t), \dot{x}(t) >$ is T-periodic; one gets

$$\int_0^T |\dot{x}|^2 dt = \int_0^T \sum_{i=1}^N m_i \|\dot{x}_i\|^2 dt = -\int_0^T \sum_{i=1}^N m_i < \ddot{x}_i, x_i > dt.$$

Let $\chi = \mathcal{D} \otimes \mathbb{R}^3$ be the quotient of $(\mathbb{R}^3)^N$ by the action of translations, and $\Delta : \mathcal{X} \to \mathcal{X}$ be defined by

$$(\Delta x)_i = \sum_{j,j\neq i} \frac{\bar{\mu}_{ij}}{m_i}(x_i - x_j) = \sum_{j,j\neq i} \frac{m_j}{\bar{r}_{ij}^3}(x_i - x_j) \quad \text{if} \quad x = (x_1, \cdots, x_N).$$

Then

$$\sum_i m_i \int_0^T < (\Delta x)_i(t), x_i(t) > dt = \sum_{i,j,i\neq j} \frac{m_i m_j}{\bar{r}_{ij}^3} \int_0^T < x_i - x_j, x_i > dt$$
$$= \sum_{i<j} \bar{\mu}_{ij} \int_0^T ||x_j - x_i||^2 dt.$$

Definition. Let λ_ϖ^G be the smallest (eigenvalue) λ such that there exists a solution $y(t) \in \Lambda^G$ of the equation $-\ddot{y} - 2\mathbb{J}\varpi\dot{y} + \varpi^2 y = \lambda\Delta y$, or equivalently a solution $x(t) = e^{\mathbb{J}\varpi t}y(t)$ of the equation $-\ddot{x} = \lambda\Delta x$ with $y(t) \in \Lambda^G$.

Following [BT], the Poincaré inequality of the Kepler case is replaced by

Lemma 3 *For any $y(t) \in \Lambda^G$, one has*

$$\int_0^T |\dot{y}(t) + i\mathbb{J}\varpi y(t)|^2 dt = \int_0^T |\dot{x}(t)|^2 dt \geq \lambda_\varpi^G \sum_{i<j} \bar{\mu}_{ij}\xi_{ij}^x,$$

with equality if and on if $-\ddot{x} = \lambda_\varpi^G \Delta x$.

Proof. Define λ_0 as the minimum of the positive functional

$$J(x) = \frac{\sum_{i=1}^N m_i \int_0^T ||\dot{x}_i||^2 dt}{\sum_{i<j} \frac{m_i m_j}{\bar{r}_{ij}^3} \int_0^T ||x_i(t) - x_j(t)||^2 dt}$$

defined on the set of $x(t) = e^{\mathbb{J}\varpi t}y(t)$ in H^1 with $y(t) \in \Lambda^G$. The existence of a minimizer $x(t)$ is insured as soon as the class of loops Λ^G implies coercivity. Writing the Euler equations, one gets that for such a minimizer,

$$-\ddot{x} = J(x)\Delta x.$$

Lemma 4 $\lambda_0 = \lambda_\varpi^G$.

Proof. We have seen that if $x(t) = e^{\mathbb{J}\varpi t}y(t)$ with $y(t) \in \Lambda^G$ minimizes $J(x)$, it satisfies the equation $-\ddot{x} = \lambda\Delta x$ with $\lambda = J(x)$. The converse comes from the identity

$$J(x) = \frac{\sum_{i=1}^N m_i \int_0^T < -\ddot{x}_i, x_i(t) > dt}{\sum_{i=1}^N m_i \int_0^T < (\Delta x_i)(t), x_i(t) > dt}.$$

Being the minimum of a positive functional, λ_ϖ^G is necessarily non negative; this allows to write its definition as follows:

Lemma 5 λ_ϖ^G *is the smallest $\lambda \geq 0$ such that there exists a solution $\xi(t)$ of the equation $-\ddot{\xi} = \Delta\xi$ such that $y(t) = e^{-\mathbb{J}\varpi t}\xi(\sqrt{\lambda}t) \in \Lambda^G$ (and in particular is T-periodic).*

Remark If Λ is the space of all H^1 loops of period T, $\lambda_{min} = 0$ because of the lack of coercivity: the min is attained by motionless bodies at infinity.

Interpretations of Δ: up to a harmless factor 2, a transposition changes Δ into the Wintner-Conley matrix associated with the central configuration $\bar{x}$ (cf. 1.2). In other words, one checks that if $X = (X_1, \cdots, X_N)$ is a tangent vector at $\bar{x}$ to the configuration space $(\mathbb{R}^3)^N$ (or to its quotient by translations),

$$dU(\bar{x})X = \langle X, -\Delta\bar{x}\rangle.$$

Also, if $Z = (Z_1, \cdots, Z_N)$ is a vertical variation,

$$d^2U(\bar{x})(Z, Z) = \langle Z, -\Delta Z \rangle ,$$

which explains that Δ is symmetric with respect to the mass scalar product. Considered as living in the space $\mathbb{R}^N$ of vertical variations (resp. in its quotient by translations), the equation

$$\ddot{Z} = -\Delta Z$$

is the vertical variational equation (VVE) at $\bar{x}(t)$.

2.2.4 Minima of $\bar{\mathcal{A}}_\varpi$

Let g be the real function defined on $\mathbb{R}_+$ by

$$g(s) = \frac{\lambda_\varpi^G}{2} s + (\bar{U}T)^{\frac{3}{2}} s^{-\frac{1}{2}}.$$

The functional $\bar{\mathcal{A}}_\varpi$ is defined by

$$\bar{\mathcal{A}}_\varpi(y) = \bar{\mathcal{A}}(x) = g\left(\sum_{i<j} \bar{\mu}_{ij} \xi_{ij}^x\right).$$

The function g is strictly convex and its unique minimum is

$$s_{min} = \frac{\bar{U}T}{(\lambda_\varpi^G)^{\frac{2}{3}}}.$$

The minimum of $\bar{\mathcal{A}}_\varpi(y) = \bar{\mathcal{A}}(x)$ is attained when $\sum_{i<j} \bar{\mu}_{ij} \xi_{ij}^x = s_{min}$, that is

$$\sum_{i<j} \frac{m_i m_j}{\bar{r}_{ij}^3} \int_0^T ||x_i(t) - x_j(t)||^2 dt = \frac{\bar{U}T}{(\lambda_\varpi^G)^{\frac{2}{3}}}.$$

This implies the

Lemma 6 *A necessary and sufficient condition for the relative equilibrium $\bar{y}(t)$ to be a minimizer of the action $\bar{\mathcal{A}}_\varpi$ in Λ^G, is that $\lambda_\varpi^G = 1$.*

Proof. If $\bar{y}(t)$ is a minimizer, $\sum_{i<j} \frac{m_i m_j}{\bar{r}_{ij}^3} \int_0^T ||x_i(t) - x_j(t)||^2 dt = \bar{U}T$ and the above identity becomes $\lambda_\varpi^G = 1$.

Conversely, if $\lambda_\varpi^G = 1$, the unique minimum of the convex function $g(s) = \frac{1}{2}s + (\bar{U}T)^{\frac{3}{2}} s^{-\frac{1}{2}}$ is attained at $s_{min} = \bar{U}T$. But $\bar{y}(t)$ verifies $\forall i, j,\ \xi_{ij}^x = \bar{r}_{ij}^2 T$. Hence $\sum_{i<j} \bar{\mu}_{ij} \xi_{ij}^x = \bar{U}T$, which proves the assertion.

Lemma 7 *Let $y(t) \in \Lambda^G$ be a solution in the rotating frame. Then, the equality $\mathcal{A}_\varpi(y) = \bar{\mathcal{A}}_\varpi(y)$ occurs if and only if $\lambda_\varpi^G = 1$ and $y(t) = \bar{y}(t)$.*

Proof. $\mathcal{A}_\varpi(y) = \bar{\mathcal{A}}_\varpi(y)$ if and only if the equality holds for both the kinetic and the potential part of the action. Equality in the potential part implies that $x(t) = e^{\mathbb{J}\varpi t}y(t) = \beta S(t)\bar{x}$ is a rigid motion. As it is a solution, it follows from [AC] that it is a relative equilibrium and hence of the form

$$x(t) = \beta S e^{i\alpha t}\bar{x},$$

where S is a rotation. Now, equality in the kinetic part of the action implies that $x(t)$ is a solution of the differential equation $-\ddot{x} = \lambda\Delta x$, where $\lambda = \lambda^G_\varpi$. Hence

$$\beta\alpha^2 S e^{i\alpha t}\bar{x} = \lambda\Delta(\beta S e^{i\alpha t}\bar{x}).$$

But $\beta S e^{i\alpha t}(x_1, \cdots, x_N) = (\beta S e^{i\alpha t}x_1, \cdots, \beta S e^{i\alpha t}x_N)$ and hence

$$\Delta(\beta S e^{i\alpha t}\bar{x})_i = \sum_{j, j\neq i}\frac{m_j}{\bar{r}_{ij}^3}(\beta S e^{i\alpha t}x_i - \beta S e^{i\alpha t}x_j) = \beta S e^{i\alpha t}(\Delta\bar{x})_i.$$

Finally,

$$\beta\alpha^2 S e^{i\alpha t}\bar{x} = \beta S e^{i\alpha t}\lambda\Delta\bar{x} = \beta(\omega+\varpi)^2\lambda S e^{i\alpha t}\bar{x},$$

and

$$\alpha^2 = (\omega+\varpi)^2\lambda.$$

So, $x(t) = \beta S e^{i(\omega+\varpi)\sqrt{\lambda}t}\bar{x}$, and $y(t) = e^{-i\varpi t}\beta S e^{i(\omega+\varpi)\sqrt{\lambda}t}\bar{x}$. But $y(t)$ is supposed to belong to Λ^G and, in particular, it must be ($T = \frac{2\pi}{\omega}$)-periodic, that is for all t,

$$e^{-i2\pi\frac{\varpi}{\omega}}e^{-i\varpi t}\beta S e^{i(\omega+\varpi)\sqrt{\lambda}t}e^{i2\pi\frac{\omega+\varpi}{\omega}\sqrt{\lambda}}\bar{x} = e^{-i\varpi t}\beta S e^{i(\omega+\varpi)\sqrt{\lambda}t}\bar{x}.$$

In particular, taking $t = 0$, this implies

$$e^{i2\pi(\frac{\omega+\varpi}{\omega}\sqrt{\lambda}-\frac{\varpi}{\omega})} = 1, \quad \text{i.e.} \quad \lambda = 1,$$

provided $\varpi \neq -\omega$. This last condition is necessarily satisfied if y is a true solution, not lying still at infinity. From $\lambda = 1$ and the fact that y and hence also x is a solution, one deduces that $\beta = 1$ and hence $y(t) = \bar{y}(t)$.

We conclude that

Proposition 8 *As long as λ^G_ϖ=1, $\bar{y}(t) = \bar{x}e^{\mathbb{J}\omega t}$ is the sole absolute minimizer of $\mathcal{A}_\varpi$ in Λ^G.*

Remarks 1) An elementary computation shows that

$$\begin{aligned}\mathcal{A}_\varpi(\bar{y}(t)) = \frac{1}{2M}(\omega+\varpi)^2\sum_{i<j}m_i m_j\bar{r}_{ij}^2 T + \bar{U}T &= \frac{\sum_{i<j}m_i m_j\bar{r}_{ij}^2}{2MI}\bar{U}T + \bar{U}T \\ &= \frac{3}{2}\bar{U}T,\end{aligned}$$

while

$$\bar{\mathcal{A}}_\varpi(\bar{y}(t)) = (\frac{\lambda^G_\varpi}{2}+1)\bar{U}T$$

and

$$\min \bar{\mathcal{A}}_\varpi = (\lambda^G_\varpi)^{\frac{1}{3}}(\frac{\lambda^G_\varpi}{2}+1)\bar{U}T = (\lambda^G_\varpi)^{\frac{1}{3}}\bar{\mathcal{A}}_\varpi(\bar{y}(t)).$$

2) In the case of the *choreography symmetry*, that is of $G = \mathbb{Z}/N\mathbb{Z}$ acting by circularly permuting N equal masses after one Nth of the period, this proposition is a rewording of [BT]. It is proved there that it implies that, for any N, the regular N-gon is the sole absolute minimizer of the action among choreographies.

2.2.5 Not saying anything on the Italian symmetry

As soon as relative equilibria with different central configurations are allowed by the group action, the method cannot work: a criterium which is essentially local cannot detect the difference in action between relative equilibria whose configurations are unrelated to each other. This is coherent with the fact that no conclusion can be drawn from the property that the criterion for absolute minimization does not apply.

In order to illustrate this remark we look at the Italian symmetry

$$x(t+\frac{T}{2}) = -x(t).$$

It is the simplest invariance requirement which implies coercivity and it possesses two properties needed for our purpose:
i) any relative equilibrium satisfies the Italian symmetry;
ii) any solution of the (VVE) of a relative equilibrium solution satisfies the Italian symmetry.

In order to compare easily with the formulae in the next sections, we consider loops in the configuration space which become 1-periodic in a frame rotating with frequency ϖ. Moreover, when $G = \mathbb{Z}/2\mathbb{Z}$ with the Italian action, we shall use the notation λ^{It}_ϖ instead of λ^G_ϖ.

Let $x(t) = e^{\mathbb{J}(2\pi+\varpi)t}\bar{x}$ be some relative equilibrium ($\bar{x}$ can be any central configuration). Let $\omega_1, \omega_2, \cdots$ be the *vertical frequencies* (i.e. the ones of the Vertical Variational Equation $-\ddot{z} = \Delta z$) of a relative equilibrium with similar configuration and frequency ω_1 (see [CF1, Mo]).

Lemma 9 *Whatever the central configuration $\bar{x}$, and whatever the dimension (two or three) of the ambiant space, whatever the value of ϖ,*

$$\sqrt{\lambda^{It}_\varpi} \leq \inf_k \frac{\omega_1}{\omega_k}.$$

Corollary 10 *The criterion for being an absolute minimizer does not apply to the Italian symmetry as soon as $N \geq 4$: λ^{It}_ϖ is always strictly smaller than 1.*

Proof of the lemma. We shall denote by $\hat{\omega}_1 = 2\pi + \varpi$, $\hat{\omega}_2, \cdots$ the vertical frequencies associated with the relative equilibrium $\bar{x}(t) = \bar{x}e^{i(2\pi+\varpi)t}$, i.e.

$$\hat{\omega}_k = \frac{\omega_k}{\omega_1}(2\pi + \varpi).$$

Let us consider a horizontal solution $\xi(t)$ of the equation $-\ddot{\xi} = \Delta\xi$, of the form (we identify the horizontal plane to the complex plane) $\xi(t) = W_k e^{i\hat{\omega}_k t}$. The periodicity condition of $y(t) = e^{i\varpi t}\xi(\sqrt{\lambda}t)$ and the Italian symmetry are satisfied if and only if it is of the form $y(t) = W_k e^{m2\pi it}$ with m odd, i.e. if $\hat{\omega}_k\sqrt{\lambda} - \varpi = m2\pi$, that is $\frac{\omega_k}{\omega_1}(2\pi + \varpi) = m2\pi$, that is

$$\sqrt{\lambda} = \frac{\omega_1}{\omega_k}\left(\frac{m2\pi + \varpi}{2\pi + \varpi}\right).$$

One concludes by choosing $m = 1$.

Proof of the corollary. Let $Z(t) = \mathrm{Re}\,(W_k e^{i\omega_k t})$ be a solution of the (VVE) equation associated with the relative equilibrium $x(t) = A\bar{x}e^{i\omega_1 t}$ with configuration similar to $\bar{x}$ and frequency ω_1. Here, W_k is a complex eigenvector with eigenvalue ω_k^2 of the associated Δ operator. Finally, let

$$\zeta(t) = Z(\frac{\omega_1}{\omega_k}t) = \mathrm{Re}\,(W_k e^{i\omega_1 t}).$$

Then, for the action during time $T = \frac{2\pi}{\omega_1}$, we have proved in 1.3 that

$$d^2\mathcal{A}(x(t))(\zeta(t),\zeta(t)) = \frac{T}{2}|W_k|^2(\omega_1^2 - \omega_k^2).$$

The corollary follows from the Pacella-Moeckel lemma recalled in 1.3.

Remark 1 Not knowing the result of Pacella-Moeckel, one could have deduced that, in order that the relative equilibrium solution $x(t) = x_0 e^{\mathbb{J}\omega_1 t}$ be the sole absolute minimizer of the action in Λ^{It}, it is enough that it be a local minimizer with respect to the vertical variations, which would have been a mirific result indeed if it had not been an empty one, except in the case of the equilateral relative equilibrium of 3 bodies.

Remark 2 In the following sections the choice of the groups $G_{r/s}(N,k,\eta)$ will eliminate the non trivial horizontal variations, since the only horizontal loops which are invariant are relative equilibria of the regular N-gon, possibly with a different frequency and a reordering of the bodies. The same was true in the case of choreographies, studied in [BT]: the regular N-gon is the sole central configuration whose relative equilibrium motions are choreographies.

The main difference with the study below of the groups $G_{r/s}(N,k,\eta)$ lies in the fact that the two integers m and k will not be any more independant!

THE CASE OF THE REGULAR N-GON

CAUTION: It will be convenient from now on to number the bodies from 0 to $N-1$ *considered as elements of* $\mathbb{Z}/n\mathbb{Z}$*. Moreover, we suppose that the mass of each body is equal to 1.*

3 Infinitesimal continuation

3.1 Vertical variations

We focus on the central configuration $C = (1, \zeta, ..., \zeta^{N-1})$, with $N \geq 3$ and $\zeta = e^{i2\pi/N}$, in the horizontal plane identified with $\mathbb{C}$, and on the associated relative equilibrium

$$\bar{x}(t) = (\bar{x}_0(t), \bar{x}_1(t), \ldots, \bar{x}_{N-1}(t)) = e^{\mathbb{J}\omega_1 t}\, C$$

of the equal mass N-body problem.

Let

$$\rho_k = r_{i,i+k} = |\zeta^k - 1|,\; k = 1, \ldots, N-1.$$

In particular, $\rho_{N-k} = \rho_k$ if $k \leq [N/2]$. The vertical variational equation (VVE) reads

$$\ddot{z}_i = \sum_{j \neq i} \frac{z_j - z_i}{\rho^3_{|j-i|}}$$

or,

$$\begin{pmatrix} \ddot{z}_0 \\ \ddot{z}_1 \\ \vdots \\ \ddot{z}_{N-1} \end{pmatrix} = \begin{pmatrix} -\sum \frac{1}{\rho_k^3} & \frac{1}{\rho_1^3} & \frac{1}{\rho_2^3} & \cdots & \frac{1}{\rho_{N-1}^3} \\ \frac{1}{\rho_{N-1}^3} & -\sum \frac{1}{\rho_k^3} & \frac{1}{\rho_1^3} & \cdots & \frac{1}{\rho_{N-2}^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\rho_1^3} & \frac{1}{\rho_2^3} & \frac{1}{\rho_3^3} & \cdots & -\sum \frac{1}{\rho_k^3} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{pmatrix}.$$

Such a circulant matrice has an explicit basis of eigenvectors: a basis of complex eigenvectors is formed by the

$$X_k = (\zeta^k, \zeta^{2k}, \ldots, \zeta^{Nk} = 1), \quad k = 0, \ldots, N-1,$$

with eigenvalues

$$\lambda_k = -\sum \frac{1}{\rho_j^3} + \frac{1}{\rho_1^3}\zeta^k + \frac{1}{\rho_2^3}\zeta^{2k} + \ldots + \frac{1}{\rho_{N-1}^3}\zeta^{(N-1)k} = -\sum_{1 \leq j \leq N-1} \frac{1 - \zeta^{jk}}{\rho_j^3}.$$

In particular, $\lambda_1 = -\omega_1^2$ corresponds to the frequency of the relative equilibrium.

The cases N odd and N even behave slightly differently from one another.

(i) Case $N = 2n+1$ *odd:* the $4n$-dimensional phase space $\mathcal{D}^2$ of the variational equation splits into the direct sum of $2n$ invariant eigenplanes. Indeed, there are exactly n distinct mutual distances

$$\rho_1 = \rho_{N-1}, \rho_2 = \rho_{N-2}, \ldots, \rho_n = \rho_{N-n} = \rho_{n+1}$$

(in increasing order: see lemma 11 below) and $\bar\zeta^k = \zeta^{(N-k)}$. Hence, for $k = 1, \ldots, n$,

$$\lambda_k = \lambda_{N-k} = \sum_{j=1}^{n} \frac{1}{\rho_j^3}(\zeta^{jk} + \bar\zeta^{jk} - 2) = -2\sum_{j=1}^{n} \frac{1}{\rho_j^3}(1 - \cos\frac{2\pi jk}{N}).$$

(the eigenvalue $\lambda_0 = 0$, with eigenvector $X_0 = (1, 1, \ldots, 1)$, disappears after the quotient by vertical translations).

To each eigenvalue λ_k $(1 \le k \le n)$ of the circulant matrix, corresponds the 4-dimensional space of solutions of the (VVE) consisting in solutions of the following form, where the amplitude A and the phase t_0 are parameters and $\omega_k = \sqrt{-\lambda_k}$:

$$z(t) = \Big(A\,\mathrm{Re}\,(e^{i\omega_k(t-t_0)}), A\,\mathrm{Re}\,(\zeta^k e^{i\omega_k(t-t_0)}), \ldots, A\,\mathrm{Re}\,(\zeta^{k(N-1)} e^{i\omega_k(t-t_0)})\Big)$$

and

$$z(t) = \Big(A\,\mathrm{Re}\,(e^{i\omega_k(t-t_0)}), A\,\mathrm{Re}\,(\bar\zeta^k e^{i\omega_k(t-t_0)}), \ldots, A\,\mathrm{Re}\,(\bar\zeta^{k(N-1)} e^{i\omega_k(t-t_0)})\Big).$$

(ii) Case $N = 2n$ even: the $4n-2$-dimensional phase space splits into the sum of a $4(n-1)$-dimensional invariant space in which the solutions are of the same form as in the odd case and an invariant plane which corresponds to the eigenvalue λ_n of the circulant matrix. The eigenvalue λ_n plays a special role for the sole reason that $\zeta^n = -1$ is real and hence corresponds to a 2-dimensional space of solutions and not a 4-dimensional one.

3.2 The vertical eigenvalues

Partly following the method of [PW, Mo], we prove the following fact, which will be used in sections 4 and 5.

Lemma 11 *The half sequence $(\lambda_k)_{1\le k\le N/2}$ is negative and decreasing.*

Case of an odd number of bodies: $N = 2n + 1$. Using the fact that $\zeta^{N-j} = \bar\zeta^j$ and $\rho_{N-j} = \rho_j$, and introducing $\theta = 2\pi/N$, we get

$$\lambda_k = -\frac{1}{4}\sum_{j=1}^{n} \frac{1 - \cos jk\theta}{\sin^3 \frac{j\theta}{2}} \quad (1 \le k \le n).$$

Three successive discrete derivations yield

$$\begin{aligned}
\delta\lambda_k &= \lambda_{k+1} - \lambda_k = -\tfrac{1}{2}\textstyle\sum_{j=1}^{n} \frac{\sin j(2k+1)\frac{\theta}{2}}{\sin^2 j\frac{\theta}{2}} && (1 \le k \le n-1)\\
\delta^2\lambda_k &= \delta\lambda_{k+1} - \delta\lambda_k = -\textstyle\sum_{j=1}^{n} \frac{\cos j(k+1)\theta}{\sin j\frac{\theta}{2}} && (1 \le k \le n-2)\\
\delta^3\lambda_k &= \delta^2\lambda_{k+1} - \delta^2\lambda_k = 2\textstyle\sum_{j=1}^{n} \sin j(2k+3)\frac{\theta}{2} && (1 \le k \le n-3)\\
&= 2\frac{\sin\big((2k+3)(n+1)\frac{\theta}{4}\big)\sin\big((2k+3)n\frac{\theta}{4}\big)}{\sin(2k+3)\frac{\theta}{4}}.
\end{aligned}$$

We want to show that λ_k decreases with k, or that $\delta\lambda_k$ is negative. The reason for derivating three times is that $\delta^3\lambda_k$ is a trigonometric polynomial and that its sign is related to the convexity of $\delta\lambda_k$.

Notice that these sequences still make sense for larger values of k, provided that the denominators do not vanish. Besides, the property that $\delta\lambda_k$ is ≤ 0 for extremal values of k is easier to check outside the initial interval of definition:

$$\delta\lambda_0 = -\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\sin\left(j\frac{\theta}{2}\right)} < 0$$

(because $j\frac{\theta}{2} = j\frac{\pi}{2n+1} < \pi$) and

$$\delta\lambda_n = -\frac{1}{2}\sum_{j=1}^{n}\frac{\sin j\pi}{\sin^2 j\frac{\theta}{2}} = 0.$$

Hence it suffices to show that $\delta\lambda_k$ is convex with respect to k, i.e. $\delta^3\lambda_k \geq 0$, over $\{0, ..., n-2\}$ (see figure 3).

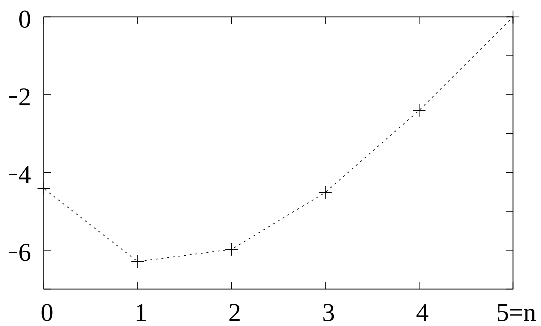


Figure 3: Graph of $\delta\lambda_k$ for $n = 5$ ($2n+1 = 11$ bodies)

In the above expression of $\delta^3\lambda_k$, the denominator is the sine of

$$0 < (2k+3)\frac{\theta}{4} \leq \frac{2n-1}{2n+1}\cdot\frac{\pi}{2} < \frac{\pi}{2} \quad \text{if} \quad 0 \leq k \leq n-2,$$

and thus is > 0. It remains to check that the numerator

$$\nu_k = \sin\left((2k+3)(n+1)\frac{\theta}{4}\right)\sin\left((2k+3)n\frac{\theta}{4}\right)$$

itself is ≥ 0, which follows from linearizing:

$$\nu_k = \frac{1}{2}\cos\frac{k+\frac{3}{2}}{n+\frac{1}{2}}\frac{\pi}{2} \geq 0 \quad \text{if} \quad 0 \leq k \leq n-2.$$

Case of an even number of bodies: $N = 2n$. In the sum defining λ_k, all terms come pairwise (indexes j and $N-j$), except that of the n-th body. Let $\epsilon_j = 1/2$ if $j = n$ and $\epsilon_j = 1$ otherwise. Then

$$\lambda_k = -\frac{1}{4}\sum_{j=1}^{n}\epsilon_j \frac{1-\cos jk\theta}{\sin^3 \frac{j\theta}{2}} \quad (1 \le k \le n).$$

It follows that

$$\begin{array}{rcll}
\delta\lambda_k & = & -\frac{1}{2}\sum_{j=1}^{n}\epsilon_j \frac{\sin j(2k+1)\frac{\theta}{2}}{\sin^2 j\frac{\theta}{2}} & (1 \le k \le n-1)\\
\delta^2\lambda_k & = & -\sum_{j=1}^{n}\epsilon_j \frac{\cos j(k+1)\theta}{\sin j\frac{\theta}{2}} & (1 \le k \le n-2)\\
\delta^3\lambda_k & = & 2\sum_{j=1}^{n}\epsilon_j \sin j(2k+3)\frac{\theta}{2} & (1 \le k \le n-3)\\
 & = & 2\frac{\sin\left((2k+3)(n+1)\frac{\theta}{4}\right)\sin\left((2k+3)n\frac{\theta}{4}\right)}{\sin(2k+3)\frac{\theta}{4}} + (-1)^k. &
\end{array}$$

Again, we will check that $\delta\lambda_k$ is < 0 at the boundary of some interval of integers containing $\{1, ..., n-1\}$, and that $\delta\lambda_k$ is convex inside this interval.

Similarly to the odd case, obviously we have $\delta\lambda_0 < 0$ and $\delta\lambda_{n-1/2} = 0$. Unfortunately, the latter equality is not of any direct use since the index is not an integer. Let us rather show that

$$\delta\lambda_{n-1} = \frac{1}{2}\left(\sum_{j=1}^{n}\frac{(-1)^j\epsilon_j}{\sin^2 j\frac{\pi}{2n}}\right)\sin\frac{\pi}{2n}$$

is < 0. It suffices to see that

$$a_n = \sum_{j=1}^{n}\frac{(-1)^j\epsilon_j}{\sin^2 j\frac{\pi}{2n}} = \sum_{j=1}^{n-1}\frac{(-1)^j}{\sin^2 j\frac{\pi}{2n}} + \frac{(-1)^n}{2} = \sum_{j=1}^{n}\frac{(-1)^j}{\sin^2 j\frac{\pi}{2n}} + \frac{(-1)^{n+1}}{2}$$

is < 0. Note that in these sums if $j \le n-1$ is odd the pair of terms of indices j and $j+1$ have a negative contribution. Hence, if n is odd, the last but one given expression of a_n shows that $a_n \le -1/2$; and, if n is even, the same estimate follows from the last expression.

It remains to show that

$$\delta^3\lambda_k = \frac{2\sin\left((2k+3)(n+1)\frac{\theta}{4}\right)\sin\left((2k+3)n\frac{\theta}{4}\right) + (-1)^k\sin(2k+3)\frac{\theta}{4}}{\sin(2k+3)\frac{\theta}{4}}$$

is positive if $1 \le k \le n-2$. The denominator being > 0, focus on the numerator ν_k. Setting $\alpha = (k+\frac{3}{2})\frac{\pi}{2n}$, we get

$$\nu_k = 2\sin(n+1)\alpha\sin n\alpha + (-1)^k\sin\alpha;$$

splitting $(n+1)\alpha$ into $n\alpha + \alpha$ and partially linearizing yields

$$\underbrace{\cos\alpha}_{>0}\left(\underbrace{(1+\cos 2n\alpha)}_{\ge 0} + \tan\alpha\,\underbrace{\left(\sin 2n\alpha + (-1)^k\right)}_{=0}\right) \ge 0,$$

which proves the result.

3.3 The symmetry group $G_{r/s}(N,k,\eta)$

We will now analyze the symmetries of vertical eigenmodes in the full phase space. Recall that most of the time a vertical eigenfrequency is degenerate and has vertical multiplicity 4. We will let the amplitude and the phase be respectively $A=1$ and $t_0=0$, and consider the most symmetric generators of its eigenspace i.e. 2-frequency motions of the form

$$x_j(t) = \left(\zeta^j e^{i\omega_1 t}, \mathrm{Re}\,(\zeta^{\eta kj}e^{i\omega_k t})\right) \in \mathbb{C}\times\mathbb{R}, \quad (j=0,\cdots,N-1), \qquad S(N,k,\eta)$$

where $\eta=\pm 1$ and, like in the former section, $\zeta = e^{i\frac{2\pi}{N}}$ and $k\in\{1,...n\}$. When observed in a frame rotating around the vertical axis with angular velocity ϖ such that

$$\frac{\omega_1-\varpi}{\omega_k} = \frac{r}{s} \in \mathbb{Q},$$

the horizontal and vertical frequencies are set into resonance and the motion becomes periodic of period $T=\frac{2\pi s}{\omega_k}$:

$$x_j(t) = \left(\zeta^j e^{i\frac{r}{s}\omega_k t}, \mathrm{Re}\,(\zeta^{\eta kj}e^{i\omega_k t})\right) \quad (j=0,...,N-1) \qquad S_{r/s}(N,k,\eta).$$

The discrete symmetry group of such a motion is seeked as before as a subgroup of

$$G_0 = O(\mathbb{R}/T\mathbb{Z})\times\Sigma(N)\times O(\mathbb{R}^3),$$

where $g=(\tau,\sigma,\rho)\in G_0$ acts naturally on the space of T-periodic loops: $gx_j(t) = \rho x_{\sigma^{-1}(j)}(\tau^{-1}(t))$.

Let $G_{r/s}(N,k,\eta)$ be the stabilizer of $S_{r/s}(N,k,\eta)$. The group structure of $G_{r/s}(N,k,\eta)$ does not depend on r and it depends on s only through the fact that we are looking to the solution during a time interval s times longer than the minimal period of the relative equilibrium in the inertial frame. We let (s,k) be the gcd of s and k, and $s=(s,k)s',\ k=(s,k)k'$.

Lemma 12 *$G_{r/s}(N,k,\eta)$ is a semi-direct product of an Abelian group H of order $2Ns$ by $\mathbb{Z}/2\mathbb{Z}$. The group H is an extension by $\mathbb{Z}/2\mathbb{Z}$ of a group K which is itself an extension of $\mathbb{Z}/Ns'\mathbb{Z}$ by $\mathbb{Z}/(k,s)\mathbb{Z}$.*

Proof.

1^o *Restriction to a subgroup G_1 of G_0.* Certainly elements of $G_{r/s}(N,k,\eta)$ stabilize the regular N-gon relative equilibrium (as the horizontal component of any infinitesimal vertical variations), as well as the cylinder of infinitesimal vertical variations. Hence $G_{r/s}(N,k,\eta)$ is contained in the subgroup G_1 consisting of elements $g=(\tau,\sigma,\rho)\in G_0$ satisfying the following conditions:

— The isometry $\rho\in O(\mathbb{R}^3)$ is of the form $\rho=(\rho_{hor},\rho_{ver}):\mathbb{C}\times\mathbb{R}\mapsto\mathbb{C}\times\mathbb{R}$ where

$$\rho_{hor}(h) = e^{i2\pi\alpha}h \quad \text{or} \quad e^{i2\pi\alpha}\bar{h} \quad \text{and} \quad \rho_{ver}(v) = e^{i\pi\beta}v$$

with $\alpha\in\mathbb{R}/\mathbb{Z}$ and $\beta\in\mathbb{Z}/2\mathbb{Z}$.

— If we set $\xi = \pm 1$ according to whether $\rho_{hor}(h) = e^{i2\pi\alpha}h$ or $e^{i2\pi\alpha}\bar{h}$, $\tau^{-1}(t) = \xi(t-\theta)$ with $\theta \in \mathbb{R}/T\mathbb{Z}$.

Hence an element $g \in G_1$ can be identified with a quintuple

$$(\theta, \sigma, \alpha, \beta, \xi) \in \mathbb{R}/T\mathbb{Z} \times \Sigma(N) \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_2,$$

the multiplication law being given by $\tilde{g} = g'g$ with

$$\tilde{\theta} = \theta' + \xi'\theta,\ \tilde{\sigma} = \sigma'\sigma,\ \tilde{\alpha} = \alpha' + \xi'\alpha,\ \tilde{\beta} = \beta' + \beta,\ \tilde{\xi} = \xi'\xi.$$

In the rest of this proof, let us set the time unit so that $\omega_k = 2\pi$, i.e. $T = s$: if $x \in S_{r/s}(N, k, \eta)$,

$$gx_j = \begin{pmatrix} \exp\left(2\pi i\left[\alpha + \frac{1}{N}\xi\sigma^{-1}(j) + \frac{r}{s}(t-\theta)\right]\right) \\ \cos\left(2\pi\left[\frac{\beta}{2} + \frac{\eta}{N}k\sigma^{-1}(j) + \xi(t-\theta)\right]\right) \end{pmatrix}.$$

2 ^{o} *Restriction to a subgroup G_2 of G_1.* Setting $\delta = \xi\sigma^{-1}(j) - j \in \mathbb{Z}/N\mathbb{Z}$, the symmetry equation $gx = x$ reduces to

$$\begin{cases} \alpha + \frac{\delta}{N} - \frac{r}{s}\theta \equiv 0 \pmod 1 \\ \xi\frac{\beta}{2} + k\eta\frac{\delta}{N} - \theta \equiv 0 \pmod 1. \end{cases}$$

The first equation shows that δ is independent of j, i.e. $\xi\sigma$ is a circular permutation. Hence, using the fact that $\xi\beta = \beta \pmod 1$,

$$\begin{cases} \alpha \equiv \frac{r}{s}\theta - \frac{\delta}{N} \pmod 1 \\ \theta \equiv \frac{\beta}{2} + k\eta\frac{\delta}{N} \pmod 1 \end{cases}$$

These equations completely determine $\alpha \in \mathbb{R}/\mathbb{Z}$ as a function of δ and θ but $\theta \in \mathbb{R}/s\mathbb{Z}$ is only determined mod 1, as a function of $(\delta, \beta, \xi) \in \mathbb{Z}_N \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_2$. Let

$$G_2 = (\mathbb{R}/s\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{F}_2 = \{(\theta, \delta, \beta, \xi)\},$$

where the semi-direct product is defined by the law $\tilde{g} = g'g$, with

$$\tilde{\theta} = \theta' + \xi'\theta,\ \tilde{\delta} = \delta' + \xi'\delta,\ \tilde{\beta} = \beta' + \beta,\ \tilde{\xi} = \xi'\xi.$$

Then $G_{r/s}(N, k, \eta)$ identifies with the subgroup of G_2 defined by the equation

$$\theta \equiv \frac{\beta}{2} + k\eta\frac{\delta}{N} \pmod 1.$$

3 ^{o} *Group structure of $G_{r/s}(N, k, \eta)$.*

*(*i) The semi-direct product structure of G_2 goes down to $G_{r/s}(N, k, \eta)$. Indeed, the section $\xi \mapsto (0, 0, 0, \xi)$ of the projection $G_2 \to \mathbb{F}_2$ takes its values in the subgroup $G_{r/s}(N, k, \eta)$. Hence $G_{r/s}(N, k, \eta) = H \rtimes \mathbb{Z}/2\mathbb{Z}$, where H is the

subgroup of the abelian group $\mathbb{R}/s\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ defined as the set of triples (θ,δ,β) which satisfy

$$\theta \equiv \frac{\beta}{2} + k\eta\frac{\delta}{N} \pmod 1.$$

*(*ii) Let K be the kernel of the group homomorphism $(\theta,\delta,\beta)\mapsto\beta$ from H to $\mathbb{Z}/2\mathbb{Z}$. It can be identified with the subgroup of $\mathbb{R}/s\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}$ defined as the set of pairs (θ,δ) which satisfy

$$\theta \equiv k\eta\frac{\delta}{N} \pmod 1.$$

*(*iii) One has the exact sequence

$$0\to\mathbb{Z}/Ns'\mathbb{Z}\to K\to\mathbb{Z}/(k,s)\mathbb{Z}\to 0,$$

where the first arrow sends 1 to $(k\eta/N,1)$ and the second one sends (θ,δ) to the class of $\theta-k\eta\delta/N$. This ends the proof of the lemma.

The precise structure of $G_{r/s}(N,k,\eta)$ depends on s,N,k,η. We study it in some cases.

Lemma 13 *If one of the following conditions is satisfied, H is isomorphic to $K\times\mathbb{Z}/2Z$:*
1) s is odd
2) s and N are even, k is odd.

Proof.
1) If s is odd, the mapping $\beta\mapsto(\frac{s\beta}{2},0,\beta)$ defines a section of the projection from H to $\mathbb{Z}/2\mathbb{Z}$. Hence the mapping $(\theta,\delta,\beta)\mapsto\big((\theta-\frac{s\beta}{2},\delta),\beta\big)$ is an isomorphism from H to $K\times\mathbb{Z}/2\mathbb{Z}$.
2) If $s=2\sigma$ is even, a section must send 1 onto an element of order 2 in H not belonging to K, i.e. an element of the form $(\theta,\delta,1)$ such that there exist $a,b\in\mathbb{Z}$ with

$$2\delta = bN, \qquad 1+2k\eta\frac{\delta}{N} = 1+bk\eta = as = 2a\sigma.$$

From the second equation, it follows that b and k must be odd. The first one then implies that N is even. If all these conditions are satisfied, $\beta\mapsto(\beta\frac{s}{2},\beta\frac{N}{2},\beta)$ defines a section and hence the mapping $(\theta,\delta,\beta)\mapsto\big((\theta-\beta\frac{s}{2},\delta-\beta\frac{N}{2}),\beta\big)$ is an isomorphism from H to $K\times\mathbb{Z}/2\mathbb{Z}$.

Lemma 14 *If $(k,s')=1$, K is isomorphic to $\mathbb{Z}/Ns'\mathbb{Z}\times\mathbb{Z}/(k,s)\mathbb{Z}$ (and hence to $\mathbb{Z}/Ns\mathbb{Z}$ if $(Ns',(k,s))=1$); if $(Nk',(k,s))=1$, K is isomorphic to $\mathbb{Z}/Ns\mathbb{Z}$.*

1) If $(k,s')=1$, the mapping which sends $1\in\mathbb{Z}/(k,s)\mathbb{Z}$ to $(s',0)\in K$ is a section.

2) If $(Nk',(k,s))=1$, this implies $(N,(k,s))=1$, and hence the existence of integers l' and δ such that $Nl'+\eta\delta(k,s)=1$. Let us set $l=l'k'$ and $\theta=k\eta\delta/N+l$. We assert that the element (θ,δ) is of order Ns. Indeed, $p(\theta,\delta)=0\in K$ if and only if there exists integers a,b such that

$$pk\eta\delta/N+pl=k\eta b+pl=as,\quad p\delta=bN.$$

From its definition δ satisfies $(N,\delta)=1$. Then, the second equation above implies the existence of an integer p' such that $p=p'N$, hence $b=p'\delta$ and the first equation above becomes $p'\big((k,s)\eta\delta+Nl'\big)k'=as$, that is $p'k'=as$. As $(Nk',(k,s))=1$ implies $(k',(k,s))=1$ and hence $(k',s)=1$, there exists an integer p'' such that $p'=p''s$, and hence that $p=p''Ns$.

Corollary 15 *When $s=1$, $G_{r/s}(N,k,\eta)$ is isomorphic to the direct product $D_N\times\mathbb{Z}/2\mathbb{Z}$, where D_N, of order $2N$, is the dihedral group. In particular, when N is odd, it is isomorphic to D_{2N}.*

3.4 Invariant loops

An s-periodic loop of configurations $x(t)=\big(x_1(t),\cdots,x_N(t)\big)$ is invariant under the action of $G_{r/s}(N,k,\eta)$ if and only if, for every $(\theta,\delta,\beta,\xi)\in G_2$ representing an element of $G_{r/s}(N,k,\eta)$, i.e. such that $\theta-\frac{\beta}{2}-k\eta\frac{\delta}{N}=l\in\mathbb{Z}$, one has

$$\forall j\in\mathbb{Z}/N\mathbb{Z},\ x_j(t)=\rho x_{\xi(j+\delta)}\big(\xi(t-\theta)\big),$$

where the action of ρ on $\mathbb{R}^3=\mathbb{C}\times\mathbb{R}$ is defined by

$$\rho(h,z)=(e^{i2\pi\alpha}\bar{h}^{\xi},e^{i\pi\beta}z)\quad\text{with}\quad\alpha=\frac{r}{s}\theta-\frac{\delta}{N}\pmod 1,$$

where $\bar{h}^{\xi}=h$ if $\xi=+1$ and $\bar{h}^{\xi}=\bar{h}$ if $\xi=-1$.

Before looking at remarkable classes of invariant loops, let us make some general comments:

1. Taking $(\theta=\frac{1}{2}+l,\delta=0,\beta=1,\xi=1)$ and setting $x_j(t)=(h_j(t),z_j(t))$, we get

$$h_j(t)=e^{i2\pi\frac{r}{s}(\frac{1}{2}+l)}h_j(t-\frac{1}{2}+l),\quad z_j(t)=-z_j(t-\frac{1}{2}+l).$$

In particular, if $s=2l+1$ and r are odd, an invariant loop possesses the Italian symmetry. This is the case of the Hip-Hops and of chains with an odd number of lobes (see below).

2. Taking $(\theta=0,\delta=0,\beta=0,\xi=-1)$, we get

$$h_j(t)=\bar{h}_{-j}(-t),\quad z_j(t)=z_{-j}(-t).$$

In particular, when $t=0$, the configuration is always symmetric with respect to the vertical plane containing the first coordinate axis.

3. Taking $(\theta = \frac{k\eta}{N}, \delta = 1, \beta = 0, \xi = 1)$, we get

$$h_j(t) = e^{i2\pi\frac{A}{N}} h_{j+1}(t - \frac{k\eta}{N}), \quad z_j(t) = z_{j+1}(t - \frac{k\eta}{N}).$$

where $A = \frac{r}{s}k\eta - 1$.

More generally, the smallest non zero value of $\theta = \frac{\beta}{2} + k\eta\frac{\delta}{N} + l = \frac{N(\beta+2l)+2k\eta\delta}{2N}$ is $\theta_0 = \frac{(N,2k)}{2N}$ because $\beta + 2l$ and δ are arbitrary integers.

One can distinguish two cases in the action of $G_{r/s}(N, k, \eta)$ on the space of loops of N-body configurations. In the first one, exemplified by the P_{12} family for 3 bodies, no a priori spatial symmetry of the configuration exists for all times, and even $\alpha = 0$ for any element of the group. In this case, the group acts on the full space of similitude classes of N-body configurations. The condition is that for any $l \in \mathbb{Z}$, $\delta \in \mathbb{Z}/\mathbb{N}\mathbb{Z}$, $\beta \in \mathbb{Z}$, $\frac{r}{s}\left[\frac{\beta}{2} + k\eta\frac{\delta}{N} + l\right] - \frac{\delta}{N} = 0 \pmod 1$. This is equivalent to $s = 1$, $r = 2r'$, $2r'k\eta - 1 = 0 \pmod N$ (which implies that N is odd). The case $\frac{r}{s} = 2r' = N - 1, k = 1, \eta = -1$ corresponds to the "unchained polygons" with $N - 1$ lobes.

In the second one, exemplified by the Hip-Hop family for 4 bodies, such a symmetry does exist: there is a group element such that $\theta = 0$, $\delta = -1$, $\alpha = 1/N$. In this case, the group acts on a subspace of the space of similitude classes of N-body configurations. The condition is that $\frac{\beta}{2} - \frac{k\eta}{N} = 0 \pmod 1$ which, because $k \leq \frac{N}{2}$) implies $\beta = 1$ and $k\eta = \frac{N}{2}$ (the choice of $\eta = \pm 1$ is then immaterial because $e^{k\frac{2\pi}{N}} = e^{-k\frac{2\pi}{N}}$, so we set $\eta = 1$). The $N = 2N'$-body Hip-Hops studied in [TV, BCPS] fall into this category; the group element is $(\theta = 0, \delta = -1, \xi = 1, \alpha = \frac{1}{N})$, which sends one body onto the next one by a rotation of $\frac{2\pi}{N}$ followed by a change of sign in the vertical component. We now study some types of symmetric loops in the two cases.

The choreographic symmetry (bodies gather by pairs, at least, following the same curve in space): this corresponds to a group element g with $\xi = 1, \delta \neq 0, \alpha = 0, \beta = 0$. The equations defining the symmetry group become

$$\begin{cases} \frac{\delta}{N} - \frac{r}{s}\theta \equiv 0 \pmod 1 \\ k\eta\frac{\delta}{N} - \theta \equiv 0 \pmod 1. \end{cases}$$

Choreographies can be *simple* or *multiple*. They are *simple* if all the bodies lie on the same curve, i.e. if in addition σ has a unique cycle:

$$(\delta, N) = 1.$$

Lemma 16 *$S_{r/s}(N, k, \eta)$ is a simple choreography if and only if*

$$s - k\eta r = 0 \pmod N.$$

Proof. We rewrite more explicitely the conditions for being a choreography: there exists integers l, m such that

$$\begin{cases} s\delta - Nr\theta = lsN \\ k\eta\delta - N\theta = mN. \end{cases}$$

This is equivalent to

$$\begin{cases} (s - k\eta r)\delta = (ls - mr)N \\ k\eta\delta - N\theta = mN. \end{cases}$$

Due to the condition of simple choreography $(\delta, N) = 1$, this is equivalent to the existence of an integer a such that

$$\begin{cases} ls - mr = a\delta \\ s - k\eta r = aN \\ k\eta\delta - N\theta = mN. \end{cases}$$

This proves the "only if" part of the lemma.

For the "if" part, let us suppose that there exists an integer a such that $s - k\eta r = aN$. As $(r, s) = 1$, we can choose l, m such that $ls + mr = a$. We then define

$$g = \left(\theta = \frac{k\eta}{N} - m, \delta = 1, \xi = 1, \alpha = 0, \beta = 0\right) \in G_{r/s}(N, k, \eta),$$

which completes the proof.

Remark The condition of simple choreography could of course have been obtained directly from the formulae for invariant loops or by writing that for all j, one has $\left(\zeta^{\eta k j} e^{i\omega_k t}\right)^{\frac{r}{s}} = \zeta^j e^{i\frac{r}{s}\omega_k t}$, which expresses that all the bodies belong to the curve described by the first.

Corollary 17 *For any N, k, η, the set of values of r/s for which $S_{r/s}(N, k, \eta)$ is a simple choreography is dense in $\mathbb{R}$.*

Proof. The integers N, k, η being given, r, s, a must be such that

$$(r, s) = 1 \quad \text{and} \quad s - k\eta r = aN.$$

The primality condition is equivalent to the existence of integers l, m such that $ls + mr = 1$, that is $(m + lk\eta)r + l(aN) = 1$. As $(m + lk\eta, l)$ can be an arbitrary pair of integers, this is equivalent to $(r, aN) = 1$. But the density of the r/s is equivalent to the density of $s/r = k\eta + aN/r$ and hence to the density of the aN/r or equivalently of the r/aN, with the unique condition that $(r, aN) = 1$. As the irreducible fractions of the form r/N^i are dense, the corollary follows.

Eights and maximal chains symmetries The two choreographic cases with extreme values of k are $G_2(N = 2n + 1, n, -1)$ (which we call the *Eight* symmetry) and $G_{N-1}(N, 1, -1)$ (resp. the *maximal chain* symmetry). In the first case, the group element $(\theta = \frac{1}{2N}, \delta = 1, \beta = 1, \xi = 1)$ leads to

$$x_j(t) = x_{j+1}(t - \frac{1}{2N}),\ z_j(t) = -z_{j+1}(t - \frac{1}{2N});$$

in the second case, the group element ($\theta = \frac{1}{2N}, \delta = n$, $\beta = 1$, $\xi = 1$) leads to

$$x_j(t) = x_{j+n}(t - \frac{1}{2N}),\ z_j(t) = -z_{j+n}(t - \frac{1}{2N}).$$

Solutions invariant under such a symmetry are called *unchained polygons* because the horizontal rotation of the regular N-gon is unfolded in the vertical direction, into a choreographic chain with 2 and $N-1$ lobes respectively. Examples are shown on figures 4.2 and 4.1.

The Hip-Hop symmetry Writing $k = (N,k)\tilde{k}$, $N = (N,k)\tilde{N}$, we notice that

$$\zeta^{\eta k j_1} = \zeta^{\eta k j_2} \quad \text{if and only if} \quad j_2 - j_1 \quad \text{is a multiple of} \ \tilde{N}.$$

Hence the bodies are divided into $\tilde{N}$ regular (N,k)-gons which remain horizontal at each instant. Independantly of the chosen rotating frame, the motion is invariant under a group element g of the form

$$g = \left(\theta = 0,\, \delta = \tilde{N},\, \alpha = -\frac{1}{(N,k)},\, \beta = 0\right) \in G_{1,1}(N,k,\eta).$$

If we make the simplest possible choice $r/s = 1$, we get a *Hip-Hop motion.* The name is a reference to the original Hip-Hop solution where the two diagonals of a square stay horizontal while undergoing vertical oscillations (see figure 4.6). Note that this symmetry can be combined with a choreographic symmetry.

A few examples of Lyapunov cylinders are shown on figure 4. They are respectively tangent at the relative equilibrium to the following Lyapunov families (the terminology will be explained in section 5):

1. $S_4(5,1,-1)$ Maximal unchained polygon ($\ell = 4$)
2. $S_2(5,2,-1)$ Five-body Eight
3. $S_3(4,1,-1)$ Unchained polygon (Gerver family)
4. $S_{3/2}(4,2,1)$ Terracini-Venturelli choreography (Hip-Hop family)
5. $S_1(6,1,-1)$ Six-body Hip-Hop
6. $S_2(6,2,1)$ Yet another six-body Hip-Hop.

3.5 Isomorphic symmetries

In order to better understand minimizers of the Lagrangian action among loops which in the rotating frame are invariant under the $G_{r/s}(N,k,\eta)$ action described above, we study the cases when the actions of two groups $G_{r/s}(N,k,\eta)$ and $G_{r'/s'}(N,k',\eta')$ coincide up to a relabelling $j \mapsto j' = \Sigma(j)$ of the bodies. A first condition is that the periods coincide, i.e. $s' = s$.

We will stick for simplicity to the case $s = s' = 1$.

Let as usual $\zeta = e^{2\pi i/N}$; the horizontal relative equilibrium whose j'th body's motion is $h'_{j'}(t) = \zeta^{j'} e^{2\pi i r' t}$ is $G_{r'}(N,k',\eta')$-symmetric. Hence a necessary

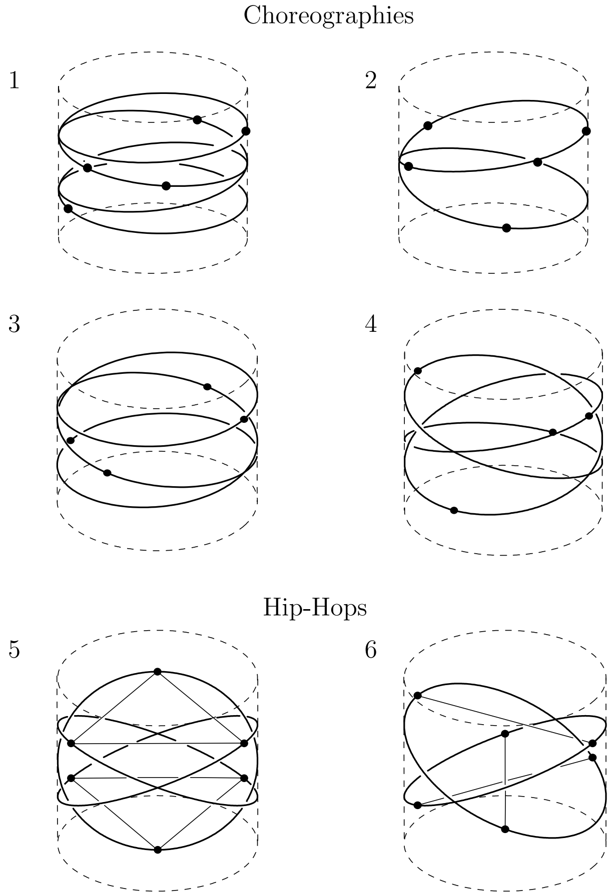


Figure 4: Some examples of first order solutions having special symmetries in a resonant rotating frame.

condition for the two actions to coincide up to the relabelling Σ is that the horizontal relative equilibrium whose jth body's motion is

$$h_j(t) = h'_{\Sigma(j)}(t) = \zeta^{\Sigma(j)} e^{2\pi i r' t}$$

be $G_r(N,k,\eta)$-symmetric. This means that for any $(\delta,\beta,\xi) \in G_r(N,k,\eta)$,

$$e^{2\pi i\alpha} \zeta^{\xi\Sigma(\xi(j+\delta))} e^{2\pi i r' \xi^2 (t-\theta)} = \zeta^{\Sigma(j)} e^{2\pi i r' t}.$$

Using the identities $\theta = \frac{\beta}{2} + k\eta\frac{\delta}{N} \pmod 1$ and $\alpha = \frac{r}{s}\theta - \frac{\delta}{N} \pmod 1$, this becomes

$$(r-r')\big[\frac{\beta}{2} + k\eta\frac{\delta}{N}\big] - \frac{\delta}{N} + \frac{1}{N}\big[\xi\Sigma(\xi(j+\delta)) - \Sigma(j)\big] = 0 \pmod 1.$$

$\delta = 0, \beta = 1, \xi = 1$, we get a first necessary condition

$$1) \quad r - r' = 2p \in 2\mathbb{Z}.$$

Taking then $\delta = 0, \beta = 0$, we get $\xi\Sigma(\xi j) = \Sigma(j) \pmod N$. In particular, if $\xi = -1$ and $j = 0$ we get $\Sigma(0) = 0 \pmod N$. Finally, taking $\delta = 1, \beta = 0$, we get $\Sigma(j+1) - \Sigma(j) = 1 - 2pk\eta \pmod N$, hence $\Sigma(j) = (1 - 2pk\eta)j \pmod N$. Interchanging the roles of the two groups we see that necessarily $\Sigma^{-1}(j') = (1 + 2pk'\eta')j' \pmod N$. This yields a second necessary condition

$$2) \quad (1 - 2pk\eta)(1 + 2pk\eta') = 1 \pmod N.$$

It turns out that these necessary conditions are essentially sufficient. Indeed, it is enough to replace the condition 2), which can be written $-2p[-k\eta + k'\eta' - 2pk\eta k'\eta'] = 0 \pmod N$ by the slightly stronger condition 2') below:

Proposition 18 *A necessary and sufficient condition for the existence of a permutation $\mathfrak{S}$ of $\mathbb{Z}/N\mathbb{Z}$ such that for any $G_{r'}(N, k', \eta')$-invariant loop $x'(t)$ whose j'th body motion is $(h'_{j'}(t), z'_{j'}(t))$, the loop $x(t)$ whose jth body motion is $(h_j(t), z_j(t)) := \big(h'_{\mathfrak{S}(j)}(t), z'_{\mathfrak{S}(j)}(t)\big)$ be $G_r(N, k, \eta)$-invariant is that the following conditions be satisfied:*
1) $r - r' = 2p \in 2\mathbb{Z}$,
2') $-k\eta + k'\eta' - 2pk\eta k'\eta' = 0 \pmod N$.
The permutation $\mathfrak{S}$ and its inverse are then respectively $j \mapsto j' = (1-2pk\eta)j$ and $j' \mapsto j = (1 + 2pk'\eta')j'$ in $\mathbb{Z}/N\mathbb{Z}$.

Proof. We already know that the asserted form of the permutation $\mathfrak{S}$ is necessary. What remains is a direct computation: in order to show the invariance of the loop $x(t)$ under the element of $G_r(N, k, \eta)$ represented by (δ, β, ξ), one must find an element of $G_{r'}(N, k', \eta')$ repesented by (δ', β', ξ') such that

$$e^{2\pi i(\alpha - \alpha')}\overline{h'_{(1-2pk\eta)\xi(j+\delta)}(\xi(t-\theta))}^{\xi} = \overline{h'_{(1-2pk\eta)\xi' j + \xi'\delta'}(\xi'(t-\theta'))}^{\xi'}$$

and

$$e^{\pi i(\beta - \beta')} z'_{(1-2pk\eta)\xi(j+\delta)}(\xi(t-\theta)) = z'_{(1-2pk\eta)\xi' j + \xi'\delta'}(\xi'(t-\theta')).$$

Straightforward computations show that this reduces to condition 2'), and that the unique solution is

$$\delta' = (1 - 2pk\eta)\delta,\ \beta' = \beta,\ \xi' = \xi,$$

Examples We note $G \equiv G'$ for two groups which satisfy the conditions of the proposition. We have

$G_{N-1}(N, 1, -1) \equiv G_{-(N-1)}(N, 1, 1)$, with $\mathfrak{S}(j) = -j \pmod N$.

$G_2(N, n, -1) \equiv G_{-2}(N, n, 1)$, with $\mathfrak{S}(j) = -j \pmod{N = 2n + 1}$.

$G_{N-1}(N, 1, -1) \equiv G_2(N, n, -1)$ for $N = 2n + 1$. This is the isomorphism between the symmetries of the maximal chains (see 5.3.1) and the symmetries of the Eight with an odd number of bodies. Here $k\eta = -1$, $k\eta' = -n$, $p = n - 1$, $\mathfrak{S}(j) = (2n-1)j = -2j \pmod N$.

$G_3(4, 1, -1) \equiv G_1(4, 1, 1)$. This is the isomorphism between the symmetries of the Gerver solution (see 5.3) and the symmetries of the relative equilibrium of the square. Here $k\eta = -1$, $k\eta' = 1$, $p = 1$, $\mathfrak{S}(j) = 3j \pmod 4 = -j \pmod 4$.

$G_3(5, 2, 1) \equiv G_1(5, 1, 1)$. This is the isomorphism between the symmetries of the 3 lobes chain for 5 bodies (see ....) and the symmetries of the relative equilibrium of the pentagon. Here $k\eta = 2$, $k\eta' = 1$, $p = 1$, $\mathfrak{S}(j) = -3j \pmod 5 = 2j \pmod 5$.

Remark The difference between conditions 2) and 2') is most easily understood on the example of the groups $G_1(4, 1, 1)$ and $G_1(4, 2, \pm 1)$. The horizontal relative equilibrium of the square possesses both symmetries but the two actions are not isomorphic: indeed, relative equilibria in an inclined plane are symmetric under $G_1(4, 1, 1)$ but not under the symmetry group of the Hip-Hop $G_1(4, 2, \pm 1)$. This comes from the fact that condition 2) is trivially satisfied because $p = 0$ while condition 2') is not.

4 Local continuation

After reduction by translations and rotations, the $(6N-10)$-dimensional reduced phase space is the sum of a $(4N-6)$-dimensional "horizontal" subspace and a $(2N-4)$-dimensional "vertical" subspace, both invariant under the linearized flow. The eigenvalues of this flow are that of the variational equation, the only difference being that the generalized eigenspace corresponding to $\pm i\omega_1$ has lost four dimensions, two horizontally (corresponding to fixing the value of the angular momentum and to rotations around it), and two vertically (corresponding to rotations around a horizontal axis).

We have already studied the vertical part of the spectrum. A thourough study of the horizontal part appears in [Mo]. See figures 5 and 6. Note that purely imaginary horizontal eigenvalues exist only for $N \geq 6$ bodies.

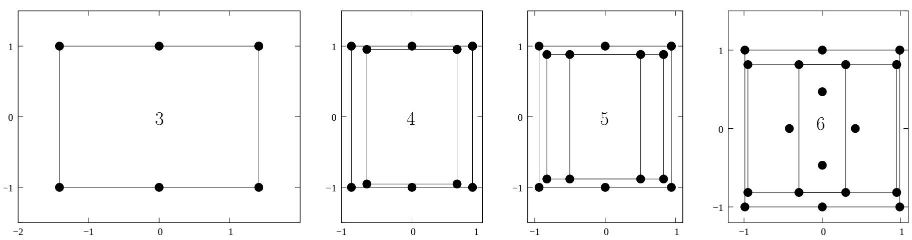


Figure 5: Horizontal spectrum for three to six bodies

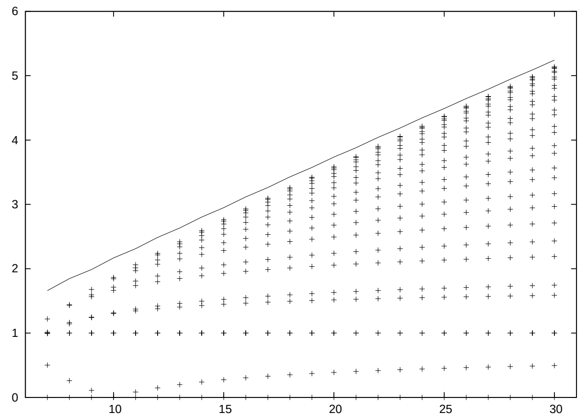


Figure 6: Imaginary parts of purely imaginary horizontal eigenvalues of the regular N-gon, as functions of $N \in \{7, ..., 30\}$ are upper bounded by ω_n (solid line).

Recall that for each vertical frequency ω_k the vertical eigenspace, most of the time, has dimension 4. The only exception is when $N = 2n$ is even and $k = n$; if moreover no other vertical or horizontal frequency is an integer multiple of ω_n, Lyapunov's classical theorem implies the local existence and uniqueness of the Lyapunov family. In the other cases, one can try to get existence from the following theorem of Weinstein, here in the form given by Moser, of whom we have kept the notations.

Theorem 19 ([Mos, p. 743]) *Let H be a Hamiltonian on R^{2n} with an equilibrium point at the origin, and C be the linearization of the Hamiltonian vector field at the origin. Assume that $\mathbb{R}^{2n} = E \oplus F$, where E and F are invariant subspaces of C such that all solutions of C in E have the same period $T > 0$ while no nontrivial solution in F has this period. Moreover, assume that the Hessian $D^2H(0)$ restricted to E is positive definite. Then, for sufficiently small ε, on each energy surface $H(z) = H(0) + \varepsilon^2$ the number of periodic orbits of H is at least $\frac{1}{2} dim E$.*

We will see in the next section that the theorem applies to the vertical frequencies provided that no unexpected resonance occurs. The cases $N = 3$ and $N = 2n$ with Hip-Hop symmetry are respectively studied in [CF2] and [BCPS] (see also [MS] for the case of the regular n-gon with a central mass at its center).

4.1 Partial convexity of the energy

In order to apply the Weinstein-Moser theorem to some vertical frequency ω_ℓ, one must check that the energy level sets, restricted to the space of $2\pi/\omega_\ell$-

periodic solutions of the linearized vector field, are compact in the neighborhood of the relative equilibrium.

Let $\mathcal{V}_\ell$ be the vertical eigenspace of the frequency ω_ℓ ($1 \leq \ell \leq N/2$), and $\mathcal{H}_1$ be the plane tangent to the homographic motions. (Recall that the total eigenspace of ω_1 contains $\mathcal{V}_1 \oplus \mathcal{H}_1$.) If no other frequency, horizontal or vertical, is an integer multiple of (possibly equal to) ω_ℓ, in order to apply the Weinstein-Moser theorem it suffices to prove that the energy is convex on $\mathcal{V}_1 \oplus \mathcal{H}_1$ for $\ell = 1$ and on $\mathcal{V}_\ell$ for $2 \leq \ell \leq N/2$. Below we will prove the fact that the quadratic part H is positive definite on the whole vector space

$$\mathcal{F} = \mathcal{H}_1 \bigoplus \oplus_{1 \leq \ell \leq N/2} \mathcal{V}_\ell.$$

The first cases, studied in the following section, all satisfy the required non resonance condition. Furthermore, numerical experiment suggests that the purely imaginary horizontal eigenvalues, in general, cannot resonate with the vertical eigenvalue $i\omega_n$, at least, for they are smaller in module (see figure 6, where the horizontal frequencies have been computed using the factorization of the stability polynomial which is described in [Mo]).[2] When this is true, the proposition below and the Weinstein-Moser theorem show in a weak sense the local existence of Lyapunov families associated with ω_n, in particular.

Proposition 20 *After reduction by rotations, the restriction of the quadratic part of the energy to $\mathcal{F}$ is definite positive.*

We first prove two lemmas. The frequency ω_1 plays a special role, for its eigenspace always includes a horizontal plane, namely the plane tangent to the homographic motions.

Lemma 21 *After reduction by rotations, the restriction of the energy to $\mathcal{V}_1 \oplus \mathcal{H}_1$ is definite positive.*

This lemma allows to apply the Weinstein-Moser theorem [Mos, p. 743] to the frequency ω_1 provided that the reduced linearized equation have no solution of period $2\pi/\omega_1$ outside $\mathcal{V}_1 \oplus \mathcal{H}_1$. It is a straightforward generalization of [CF2, lemma 2.1].

Proof. For the same reason as in [CF2, lemma 2.1], the lift of $\mathcal{V}_1 \oplus \mathcal{H}_1$ to the non-reduced phase space is tangent to the submanifold

$$\mathcal{E}_1 = \{(x, \dot{x}) \in (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N,\ \exists \rho, \sigma > 0\ \exists R, S \in SO_3,\ x = \rho R C,\ \dot{x} = \sigma S \mathbb{J} C\}$$

where, as before, C is the central configuration $(1, \zeta, ..., \zeta^{N-1})$ and $\mathbb{J}$ is the horizontal-rotation operator. It suffices to prove that the restriction of H to $\mathcal{E}_1$ is definite. For the sake of completeness, we shortly explain the computation.

[2] Note however that the largest horizontal frequency seems asymptotic to ω_n, when N tends to infinity.

Before reduction, $(\rho, \sigma, R, S) \in (\mathbb{R}_+)^2 \times SO(3) \times SO(3)$ are (generalized) coordinates on $\mathcal{E}_1$ and the restriction of H is

$$H|_{\mathcal{E}_1} = \frac{\sigma^2}{2} \sum_{0 \le j \le N-1} ||\omega_1 \mathbb{J} \zeta^j||^2 - \frac{1}{\rho} \sum_{0 \le j < k \le N-1} \frac{1}{||\zeta^j - \zeta^k||}.$$

As in 1.3, the mass dot product by C of Newton's equation shows that

$$N\omega_1^2 = \sum_{0 \le j < k \le N-1} \frac{1}{||\zeta^j - \zeta^k||}.$$

Hence

$$H|_{\mathcal{E}_1} = N\omega_1^2 \left(\frac{\sigma^2}{2} - \frac{1}{\rho} \right).$$

We compute the reduced system by first quotienting by the full group $SO(3)$ and then fixing the length of the angular momentum L. This amounts to replacing (ρ, σ, R, S) by $(\rho, \sigma, R^{-1}S)$ and imposing the relation

$$\rho\sigma || \sum_{i=0}^{N-1} (\zeta^j, 0) \wedge R^{-1} S (i\omega_1 \zeta^j, 0) || := ||L||.$$

Any element of a neighborhood of the Identity in $SO(3)$ can be uniquely written as $\exp A$, where A is an antisymmetric 3×3 matrix. In particular,

$$R^{-1}S = \exp \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

$$= Id + \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -(b^2+c^2) & ab & ac \\ ab & -(c^2+a^2) & bc \\ ac & bc & -(a^2+b^2) \end{pmatrix} + \cdots,$$

where the dots represent terms of order higher than or equal to 3 in a, b, c. Locally, the angular momentum equals

$$L = \rho\sigma\omega_1 \left(\begin{pmatrix} 0 \\ 0 \\ N \end{pmatrix} + \frac{N}{2} \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} - \frac{N}{4} \begin{pmatrix} ac \\ bc \\ a^2+b^2+2c^2 \end{pmatrix} + \cdots \right),$$

and its norm

$$\|L\| = \rho\sigma\omega_1 N \left[1 - \frac{1}{2} \left(a^2 + b^2 + 4c^2 \right) + \cdots \right].$$

In the coordinates $(a, b, c, d = \sigma - 1)$ on the space $\mathcal{E}_1$ reduced by the rotations, the reduced Hamiltonian is

$$\frac{N\omega_1^2}{2} \left(-1 + a^2 + b^2 + 4c^2 + d^2 \right) + \cdots,$$

which proves the lemma.

We will now study the other eigenspaces $\mathcal{V}_\ell$. Usually $\mathcal{V}_\ell$ is 4-dimensional, generated by the sum of the Lyapunov "cylinders" $S(N,\ell,\pm 1)$. As we will see in examples, the first interesting case is that of $N=5$, $\ell=2$. The case where $N=2n$ is even and $\ell=n$ is special: $\mathcal{V}_\ell=\mathcal{V}_n$ is then 2-dimensional (and the Lyapunov center theorem applies with no need of convexity property, provided that no other frequency is an integer multiple of ω_n).

Lemma 22 *If $2\leq \ell \leq N/2$, $\mathcal{V}_\ell$ lies in a submanifold of fixed vertical angular momentum and in the reduced space the restriction of the energy to $\mathcal{V}_\ell$ is definite positive.*

Proof. A point on $\mathcal{V}_\ell$ can be uniquely represented by $(x,\dot{x})$ with

$$\begin{aligned} x_j &= (\cos j\theta, \sin j\theta, a\cos j\ell\theta + b\sin j\ell\theta) \\ \dot{x}_j &= (-\omega_1 \sin j\theta, \omega_1\cos j\theta, c\omega_\ell \sin j\ell\theta + d\omega_\ell\cos j\ell\theta), \end{aligned}$$

with $\theta = 2\pi/N$, as before, and where $a,b,c,d\in\mathbb{R}$; (a,b,c,d) is a coordinate system on $\mathcal{V}_\ell$, except in the case $N=2n$ and $\ell=n$, where (a,c) alone is a coordinate system (with $b=d=0$).

The statement on the angular momentum comes from the fact that the angular momentum of the j-th body is

$$L_j = \begin{pmatrix} \cos j\theta \\ \sin j\theta \\ a\cos j\ell\theta + b\sin j\ell\theta \end{pmatrix} \wedge \begin{pmatrix} -\omega_1\sin j\theta \\ \omega_1\cos j\theta \\ \omega_\ell c\cos j\ell\theta + \omega_\ell d\sin j\ell\theta \end{pmatrix};$$

its vertical component depends only on the radius of the horizontal circle, and its two horizontal components boil down to degree-one trigonometric polynomials in $j(\ell+1)\theta$ and $j(\ell-1)\theta$, whose sums with respect to $j\in\{0,...,N-1\}$ are zero if $\ell>1$.

We now need to compute the quadratic part of H with respect to a,b,c,d. A straightforward computation shows that the kinetic energy is

$$\frac{K}{2} = \frac{1}{2}\sum_{j=0}^{N-1}\|\dot{x}_j\|^2 = \frac{N}{2}\left(\omega_1^2 + \omega_\ell^2(c^2+d^2)\right),$$

hence positive definite with respect to c and d. On the other hand, the potential part depends only on a and b and, at the second order,

$$\begin{aligned} -U &= -\sum_{0\leq j<k\leq N-1}\frac{1}{\|x_k-x_j\|} \\ &= -\sum_{j<k}\frac{1}{4|\sin(j-k)\frac{\ell\theta}{2}|}\begin{pmatrix} 1 - a^2\sin^2(j+k)\frac{\ell\theta}{2} - b^2\cos^2(j+k)\frac{\ell\theta}{2} + \\ 2ab\sin(j+k)\frac{\ell\theta}{2}\cos(j+k)\frac{\ell\theta}{2} + \cdots \end{pmatrix} \\ &= -N\omega_1^2 + \sum_{j<k}\frac{\left(a\sin(j+k)\frac{\ell\theta}{2} - b\cos(j+k)\frac{\ell\theta}{2}\right)^2}{4|\sin(j-k)\frac{\ell\theta}{2}|} + \cdots, \end{aligned}$$

which is positive, definite if $\ell \neq N/2$ and null if $\ell = N/2$.

We now complete the proof of the proposition.

Proof. We want to show that in the neighborhood of the Lagrange relative equilibrium the quadratic part of the energy H is positive definite on the space

$$\mathcal{F} = \mathcal{H}_1 \bigoplus \oplus_{1\leq \ell \leq N/2} \mathcal{V}_\ell.$$

The two lemmas above show that the quadratic part of the energy H is positive definite in restriction to $\mathcal{H}_1 \oplus \mathcal{V}_1$ and to each $\mathcal{V}_\ell$, $\ell \geq 2$. As eigenspaces of a symplectic linear operator (namely, the linearization of the reduced vector field) are orthogonal, it follows that the aforementioned spaces are pairwise symplectically orthogonal. Since additionally they are invariant by the flow of the quadratic part Q of $H|_\mathcal{F}$, Q does not have off-diagonal terms. Hence it is positive definite.

4.2 The first cases

We will now describe the simplest families. It is interesting that new qualitative features appear for each successive value of N from 3 to 6.

A natural name for each Lyapunov family we study would be the label $S(N, k, \eta)$ of the tangent Lyapunov cylinder in the inertial frame. On the other hand, for the sake of concreteness we like to look at a given family in rotating frames where some fixed symmetry $G_{r/s}(N, k, \eta)$ holds. This amounts to considering the Lyapunov family as a deformation of the Lyapunov cylinder $S_{r/s}(N, k, \eta)$. Among the infinitely many possible choices, we have selected the symmetry of the most remarkable periodic orbit of the (global) family in the inertial frame. Of course, this is a subjective choice and a family may bear several names.

Three bodies The case of $N = 3$ bodies is fully analyzed in [CF2], which we now summarize. The eigenvalues of the linearized reduced vector field (which can be computed following [Mo]) are:

Horizontally: $\pm i\omega_1, \left(\pm\sqrt{2} \pm i\right)\, \omega_1$

Vertically: $\pm i\omega_1$

The eigenvalue $\pm i\omega_1$ has multiplicity 2 (1 horizontal and 1 vertical), corresponding to two Lyapunov families:

– The horizontal family of homographic motions; as a curiosity, see figure 7 for a view of a member of the family in a choreographic rotating frame. The so-called proper degeneracy of the Newtonian potential shows up here in that the family has no torsion.

– The vertical P_{12}-family, tangent to the Lyapunov cylinder $S(3, 1, -1)$.

The P_{12}-family ($S(3, 1, -1)$) Because of the $1 - 1$ resonance between the vertical and horizontal eigenvalues $\pm i\omega_1$, one has to understand the third order normal form of the vector field in order to prove that there exists a unique Lyapunov family tangent to a non-trivial solution of the (VVE); in an adequately

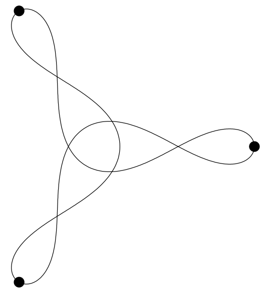


Figure 7: Homographic solution in a choreographic frame, rotating twice per period

rotating frame, this unique family possesses a $G_2(3,1,-1) = D_6$ symmetry, which is the Γ_1-symmetry described in [CFM]. See figure 8. As elsewhere, the solutions have been computed using

– the Dopri-8 algorithm [HNW], as implemented in the C language by F. Joutel and M. Gastineau (Astronomie et Systèmes dynamiques, IMCCE), for integrating Newton's equations

– a hybrid Newton algorithm, implemented in the Gnu Scientific Library [GSL], for solving symmetry equations.

The germ of the P_{12}-family was described in [Ma1] as the family of relatively periodic solutions of the three-body problem with the highest possible symmetry; it is immediately after he learnt about the existence of the Eight that Christian Marchal noticed [Ma2] that these solutions turned into choreographies in a rotating frame. This is the origin of [CF1], where we give its full scope to this remark, and prove in particular that:

– the flow of the reduced three-body problem in the neighborhood of the Lagragange relative equilibrium has a unique 4-dimensional center manifold $\mathcal{N}$,

– the energy levels within $\mathcal{N}$ are 3-spheres,

– the restriction of the flow to an energy level has a Poincaré section diffeomorphic to an annulus, whose return map is a monotone twist map which is the identity on the boundary corresponding to the homographic motions.

Four bodies

Horizontal eigenvalues

$\pm i\omega_1$

$\left(\pm\frac{1}{7}\sqrt{56-14\sqrt{2}}\pm i\right)\omega_1 \qquad \simeq (\pm 0.8595325038 \pm i)\,\omega_1$

$\Big(\pm\sqrt{42\sqrt{-16+18\sqrt{2}}-49} \qquad \simeq \Big(\pm 0.6394812009$

$\quad\pm i\sqrt{42\sqrt{-16+18\sqrt{2}}+49}\Big)\frac{\omega_1}{14} \qquad \pm 0.9533814590\,i\Big)\,\omega_1$

Vertical eigenvalues

$\pm i\omega_1 = \pm\frac{i}{2}\sqrt{2\sqrt{2}+1}$

$\pm i\omega_2 = \pm i\sqrt[4]{2}\omega_1 \qquad \simeq \pm 1.2155625241\,i\,\omega_1$

There are now two non-trivial vertical Lyapunov families.

The 4-body, 3-lobe chain family ($S(4,1,-1)$) The Weinstein-Moser theorem applied to the reduced vector field and to the eigenvalue $\pm i\omega_1$ (using lemma 21 and the fact that $\omega_1/\omega_2 = 2^{-1/4} \notin \mathbb{N}$) implies the existence of at least one more Lyapunov family corresponding to this frequency in addition to the horizontal homographic family.

Numerical computations show that the Lyapunov cylinder $S(4,1,-1)$ is tangent to a family which is choreographic in a rotating frame which starts making two full turns per period in the negative direction (see the top part of figure 13). This does not follow from the Lyapunov theorem, for the frequency ω_1 is also the frequency of the horizontal homographic family.

Yet without uniqueness it is theoretically not obvious that the family shares the $G_2(4,1,-1)$-symmetry with $S(4,1,-1)$, in the rotating frame making one turn per period, at the limit of the regular pentagon. The proof of the local existence and uniqueness of this family would require similar computations (of a normal form of the third order) and arguments as for the P_{12}-family described above.

The 4-body Hip-Hop family ($S(4,2,\pm 1)$) Since ω_2 has multiplicity one and since the only other frequency, ω_1, is not an integer multiple of ω_2, it follows from Lyapunov's theorem that there exists a unique family of relatively periodic orbits bifurcating from the square with vertical frequency close to ω_2. The eigenmode $S_1(4,2,\pm 1)$ is tangent to the family and, by uniqueness, the family shares its symmetry. Hence the family is the Hip-Hop family, with symmetry group $G_1(4,2,1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (see figure 9). It is studied in [BCPS].

Five bodies

Horizontal eigenvalues

$\pm i\,\omega_1$

... $\simeq (\pm 0.9391304549 \pm i)\,\omega_1$

... $\simeq (\pm 0.8281366700 \pm 0.8822431635\,i)\,\omega_1$

... $\simeq (\pm 0.5028535236 \pm 0.8822431635\,i)\,\omega_1$

Vertical eigenvalues

$$\pm i\omega_1 = \pm \frac{i\sqrt{2}}{4}\sqrt{\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\left(\sqrt{5}+2\right)}{1/4\,\sqrt{5}-3/4+2\left(1/4\,\sqrt{5}+1/4\right)^2}}$$

$$\pm i\omega_2 = \pm i\omega_1\sqrt{2}\sqrt{\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\left(11/4+3/4\,\sqrt{5}\right)}{\sqrt{2}\sqrt{5-\sqrt{5}}\left(\sqrt{5}+2\right)}} \simeq \pm 1.3281310261\,i\,\omega_1$$

The 5-body, 4-lobe chain family ($S(5,1,-1)$) Figure 11 depicts it in rotating frames which start by making three full turns in the negative direction (symmetry $G_4(5,1,-1)$).

Contrary to its analogue with four bodies, the maximal vertical frequency ω_2 now has multiplicity two. Using lemma 22 and the above expression of $\omega_1/\omega_2 \notin \mathbb{N}$, the Weinstein-Moser theorem proves the existence of at least two Lyapunov families. Numerically, there are exactly two families, which thus display the same symmetries as their corresponding Lyapunov cylinders:

The 5-body Eight family ($S(5,2,-1)$) It owes its name to the $G_2(5,2,-1)$-symmetry it acquires in a frame with $r/s = 2$ (figure 12).

The 5-body, 3-lobe chain family ($S(5,2,1)$) It owes its name to the non-maximal chain $G_3(5,2,1)$-symmetry in a frame with $r/s = 3$ (see figure 14).

Six bodies

Horizontal eigenvalues

$\pm\, i\,\omega_1$

... $\simeq \pm 0.4687282051\,i\,\omega_1$

... $\simeq \pm 0.4211614102\,\omega_1$

... $\simeq (\pm 0.9499800584 \pm 0.8151022048\,i)\;\omega_1$

... $\simeq (\pm 0.2986755303 \pm 0.8151022048\,i)\;\omega_1$

... $\simeq (\pm 0.9893611078 \pm\, i)\;\omega_1$

Vertical eigenvalues

$\pm i\omega_1 = \pm i\sqrt{\frac{5}{4}+\frac{\sqrt{3}}{3}}$

$\pm i\omega_2 = \pm i\sqrt{3+\frac{\sqrt{3}}{3}} \quad \simeq \pm 1.3991678967\,i\,\omega_1$

$\pm i\omega_3 = \pm \frac{i}{2}\sqrt{17} \quad \simeq \pm 1.5250481798\,i\,\omega_1$

From six bodies on, some horizontal eigenvalues are purely imaginary, which could give rise to horizontal Lyapunov families. For example, could the choreography shown in figure 3 (4th of first column) of [S] belong to such a family? Moreover the purely imaginary horizontal eigenvalues can resonate with the vertical frequencies, on which we focus (see the remark before lemma 22 however).

The 6-body, 5-loop chain ($S(6,1,-1)$) It is the 6-body instance of maximal chains. Its proof of existence has the same status as with 4 and 5 bodies.

Some 6-body Hip-Hops Since 6 is not prime, the 6-body problem has a richer set of Hip-Hop families than problems with fewer bodies.

The Weinstein-Moser theorem and numerical experiments indicate the existence of a Hip-Hop family sharing the Hip-Hop symmetry of and tangent to the Lyapunov cylinder $S(6,1,-1)$; it belongs to an invariant problem having as few dimensions as the two-body problem, where bodies are symmetric with respect to each other within two groups of three (see part 5 of figure 4).

The second frequency gives rise to two Lyapunov families $S(6,2,1)$ and $S(6,2,-1)$, each belonging to an invariant problem having the dimensions of the three-body problem, where bodies are symmetric with respect to each other within three groups of two (see part 6 of figure 4 for $S_1(6,2,-1)$).

Since all other frequencies (vertical or horizontal) are smaller than ω_3, by the Lyapunov theorem the third vertical frequency gives rise to a unique Hip-Hop family sharing the symmetries of $S(6,3,\pm 1)$. In a frame rotating once per period (at the limit of the hexagon), this family is both a Hip-Hop (bodies are symmetric with respect to each other within two groups of three) and a partial choreography (bodies chase each other by pairs, along three distinct closed curves).

5 Global continuation

We are interested in the following two questions:

– Existence: Does the range of frequency rotation ϖ of the frame over which the family exists contain 0?

– Uniqueness: Can one take the frequency ϖ as a monotonous continuous parameter over the whole family, i.e. has the torsion constant sign?

Minimization under the $G_{r/s}(N,k,\eta)$-symmetry is a natural tool for the existence question. It will turn out that, because of isomorphisms between the actions of different groups $G_{r/s}(N,k,\eta)$ (cf. section 3.5), its use is essentially restricted to the case $k = n$, i.e. to the largest vertical frequency.

As a preliminary study we apply the results of section 2.2 to estimate intervals of the rotation frequency of the frame over which the relative equilibrium family itself is the sole absolute minimizer under the $G_{r/s}(N,k,\eta)$-symmetry.

5.1 Minimization properties of the N-gon family under $G_{r/s}(N,k,\eta)$-symmetry

Consider solutions $x(t) = e^{\mathbb{J}\varpi t}y(t)$ of Newton's equations which, in a frame rotating with frequency ϖ, become s-periodic loops $y(t) \in \Lambda^G$, where $G = G_{r/s}(N,k,\eta)$. A relative equilibrium solution with the required symmetry accomplishes one turn in time $\frac{s}{r}$ in the rotating frame. Hence it has frequency $\hat{\omega}_1 = 2\pi\frac{r}{s} + \varpi$ in the inertial frame.

Referring to proposition 8, we now study conditions under which the inf of the positive λ's for which there is a solution $x(t) = e^{\mathbb{J}\varpi t}y(t)$ with $y(t) \in \Lambda^G$ of the equation

$$-\ddot{x} = \lambda\Delta x$$

is equal to 1, where Δ is computed with the mutual distances $\bar{r}_{ij}$ of the relative equilibrium of frequency $2\pi\frac{r}{s} + \varpi$ in the inertial frame.

Let $\xi(t)$ be defined by $x(t) = \xi(\sqrt{\lambda}t)$. As $\xi(t)$ is a solution of $-\ddot{\xi} = \Delta\xi$, each component of $\xi(t)$ is of the form $\sum_k a_k e^{\mathbb{J}\hat{\omega}_k t}$, where $\omega_1, \omega_2, \cdots, \omega_k, \cdots$ are the frequencies of the (VVE) and

$$\hat{\omega}_k = \frac{\omega_k}{\omega_1}(2\pi\frac{r}{s} + \varpi).$$

The linearity of the differential equation and the structure of the action of $G_{r/s}(N,k,\eta)$ allows us to study separately solutions lying in the horizontal plane and solutions lying on the vertical axis: if $y(t)$ is invariant, so are the loops $h(t)$ and $v(t)$ of N-body configurations in $\mathbb{R}^3$ obtained from $y(t)$ by projecting each body on the horizontal plane and the vertical axis respectively.

1) VERTICAL SOLUTIONS. A solution $x(t)$ of the (VVE) of the form

$$(x_0(t), \cdots, x_{N-1}(t)) \text{ with } \forall j \in \mathbb{Z}/N\mathbb{Z}, x_j(t) = \big(0, 0, x_j^z(t)\big),$$

can be written

$$x_j^z(t) = \sum_l \mathrm{Re}\left[(u_l\zeta^{jl} + v_l\bar{\zeta}^{jl})e^{i\hat{\omega}_l\sqrt{\lambda}t}\right],\ j \in \mathbb{Z}/N\mathbb{Z}.$$

The symmetry conditions impose not only s-periodicity but even 1-periodicity. Indeed, if we choose $\beta = 0, \delta = 0, q = 1$, we get $\theta = 1 \in \mathbb{R}/s\mathbb{Z}$, hence

$$x_j^z(t) = y_j^z(t) = y_j^z(t-1) = x_j^z(t-1),$$

that is

$$\sum_l \mathrm{Re}\left[(u_l\zeta^{jl} + v_l\bar{\zeta}^{jl})(1 - e^{-i\hat{\omega}_l\sqrt{\lambda}})e^{i\hat{\omega}_l\sqrt{\lambda}t}\right] \equiv 0.$$

It follows that for any l, $(u_l\zeta^{jl} + v_l\bar{\zeta}^{jl})(1 - e^{-i\hat{\omega}_l\sqrt{\lambda}}) = 0$, that is

$$\text{either} \quad u_l = v_l = 0 \quad \text{or} \quad \exists m \in \mathbb{Z},\ \hat{\omega}_l\sqrt{\lambda} = 2\pi m.$$

The second possibility occurs for a single index l provided no relation of the form $m\hat{\omega}_{l'} = m'\hat{\omega}_l$ exists with $m, m' \in \mathbb{Z}$. In any case, as we are only interested in the values that λ can take, we can restrict the attention to solutions corresponding to a single index l such that $\hat{\omega}_l\sqrt{\lambda} = 2\pi m$.

Now, the invariance of $x(t)$ under the action of $G_{\frac{r}{s}}(N, k, \eta)$ means that for every $(\theta, \delta, \beta, \xi) \in \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \{-1, +1\}$, such that $\theta - \frac{\beta}{2} - k\eta\frac{\delta}{N} = q \in \mathbb{Z}$,

$$\mathrm{Re}\left[(u_l\zeta^{jl} + v_l\bar{\zeta}^{jl})e^{i\hat{\omega}_l\sqrt{\lambda}t} - (u_l\zeta^{\xi(j+\delta)l} + v_l\bar{\zeta}^{\xi(j+\delta)l})e^{i[\pi\beta+\hat{\omega}_l\sqrt{\lambda}\xi(t-\theta)]}\right] \equiv 0.$$

Choosing $\delta = 0, \beta = 1, \xi = 1$, hence $\theta = 1/2$, we get that $m = 1 + 2p$ is odd.
Choosing $\delta = 0, \beta = 0, \xi = -1$, hence $\theta = 0$, we get that for all $j \in \mathbb{Z}/N\mathbb{Z}$, $(u_l - \bar{u}_l)\zeta^{jl} + (v_l - \bar{v}_l)\bar{\zeta}^{jl} = 0$, which implies that u_l and v_l must be real.
Finally, for $\xi = 1$ the condition is that for every j, δ,

$$u_l\zeta^{jl}\left(1 - e^{2\pi i\left(-\frac{mk\eta\delta+\delta l}{N}\right)}\right) + v_l\bar{\zeta}^{jl}\left(1 - e^{2\pi i\left(-\frac{mk\eta\delta-\delta l}{N}\right)}\right) = 0,$$

which implies that one of the two coefficients u_l and v_l must vanish and that, according to which one does vanish,

$$l = \pm mk\eta \mod N = \pm(1 + 2p)k\eta \mod N.$$

Finally, the inf of the corresponding $\sqrt{\lambda}$'s is the inf of the $\left|\frac{(1+2p)2\pi}{\hat{\omega}_l}\right|$, that is

$$\inf_p \frac{\omega_1}{\omega_{(1+2p)k\eta}}\left|\frac{(1+2p)2\pi}{2\pi\frac{r}{s} + \varpi}\right|,$$

where the frequencies $\omega_{(1+2p)k\eta} = \omega_{-(1+2p)k\eta}$ involved are the ones of bifurcating vertical solutions with the required symmetry (in particular, $\omega_{k\eta} = \omega_k$ is always among them). One deduces immediately that the condition $\inf \lambda \geq 1$ reads

$$\left|\varpi + 2\pi\frac{r}{s}\right| \leq V = \inf_{p>0} \frac{\omega_1}{\omega_{(1+2p)k\eta}}|(1 + 2p)2\pi|,$$

where the retriction to the $p \geq 0$ comes from the fact that if $p' = -(1 + p)$, one has $|1 + 2p'| = |1 + 2p|$, hence $\omega_{1+2p'k\eta} = \omega_{1+2pk\eta}$.

2) HORIZONTAL SOLUTIONS. The general "horizontal" solution

$$\xi = (\xi_0, \cdots, \xi_{N-1}), \text{ with } \xi_j(t) = (\xi_j^h(t), 0),$$

of $-\ddot{\xi} = \Delta\xi$ is such that, identifying $\mathbb{R}^2$ with $\mathbb{C}$, there exists $k_0, a_l, b_l, c_l, d_l \in \mathbb{C}$ such that

$$\forall j,\ \xi_j^h(t) = k_0 + \sum_{l\neq 0}\left[\left(a_le^{i\hat{\omega}_lt} + b_le^{-i\hat{\omega}_lt}\right)\zeta^{jl} + \left(c_le^{i\hat{\omega}_lt} + d_le^{-i\hat{\omega}_lt}\right)\bar{\zeta}^{jl}\right].$$

If $N = 2n + 1$ is odd, to each of the n pairs $\lambda_l = \lambda_{N-l} = -\hat{\omega}_l^2$ of non-zero eigenvalues of the matrix which defines Δ, corresponds an eight dimensional space of solutions parametrized by the four complex constants a_l, b_l, c_l, d_l. This gives the dimension $4n$ of the phase space after quotienting by the translations.

If $N = 2n$ is even, to the eigenvalue λ_n corresponds only a four dimensional space of solutions because $\zeta^n = \bar{\zeta}^n = -1$. In this case the two complex parameters are $a_l + c_l$ and $b_l + d_l$.

We shall now study the conditions imposed on horizontal solutions by the invariance under $G_{\frac{r}{s}}(N, k, \eta)$.

1) The condition of s-periodicity on $y(t)$ imposes that, in addition to $\hat{\omega}_0 = 0$, only one frequency $\hat{\omega}_l$ is present: indeed, for all j and all t,

$$0 = \sum_{l \neq 0} \big[(e^{i\alpha_l s} - 1)(a_l \zeta^{jl} + c_l \bar{\zeta}^{jl}) e^{i\alpha_l t} + (e^{i\beta_l s} - 1)(b_l \zeta^{jl} + d_l \bar{\zeta}^{jl}) e^{i\beta_l t}\big],$$

where

$$\alpha_l = \hat{\omega}_l \sqrt{\lambda} - \varpi, \ \beta_l = -\hat{\omega}_l \sqrt{\lambda} - \varpi.$$

Hence, as $\omega_l \pm \omega_{l'} \neq 0$ if $l \neq l'$,

$$\forall j, l, \ (e^{i\alpha_l s} - 1)(a_l \zeta^{jl} + c_l \zeta^{jl}) = (e^{i\beta_l s} - 1)(b_l \zeta^{jl} + d_l \bar{\zeta}^{jl}) = 0,$$

As $a_l \zeta^{jl} + c_l \bar{\zeta}^{jl}$ (resp. $b_l \zeta^{jl} + d_l \bar{\zeta}^{jl}$) cannot be equal to 0 for all $j = 0, \cdots, N-1$ except if $a_l = c_l = 0$ (resp. $b_l = d_l = 0$), one deduces that there exists a (necessarily unique) $l \neq 0$ such that $e^{i\alpha_l s} = 1$ or $e^{i\beta_l s} = 1$, that is

$$\alpha_l s = (\hat{\omega}_l \sqrt{\lambda} - \varpi)s = m2\pi, \quad \text{or} \quad \beta_l s = (-\hat{\omega}_l \sqrt{\lambda} - \varpi)s = m2\pi,$$

i.e.

$$y_j(t) = (k_0 + (a_l \zeta^{jl} + c_l \bar{\zeta}^{jl}) e^{i2\pi \frac{m}{s} t}, 0) \quad \text{or} \quad y_j(t) = (k_0 + (b_l \zeta^{jl} + d_l \bar{\zeta}^{jl}) e^{i2\pi \frac{m}{s} t}, 0).$$

Both cases are identical hence we suppose that the first identity holds. Hence we are looking for the invariance of

$$y_j(t) = \left(k_0 + (u_l \zeta^{jl} + v_l \bar{\zeta}^{jl}) e^{i2\pi \frac{m}{s} t}, 0\right),$$

corresponding to

$$\sqrt{\lambda} = \frac{\omega_1}{\omega_l} \left| \frac{\varpi + 2\pi \frac{m}{s}}{\varpi + 2\pi \frac{r}{s}} \right|.$$

This means that, for all $(\theta, \delta, \beta, \xi) \in \mathbb{R}/s\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \{-1, +1\}$ with

$$\theta - \frac{\beta}{2} - k\eta \frac{\delta}{N} = q \in \mathbb{Z},$$

one has for all $j \in \mathbb{Z}/N\mathbb{Z}$ and for all t, $y_j(t) = e^{2\pi i \alpha} \bar{y}^{\xi}_{\xi(j+\delta)}(\xi(t-\theta))$, where $\alpha = \frac{r}{s}\theta - \frac{\delta}{N} \pmod 1$. As $\xi^2 = 1$, this is equivalent to the following:

$$k_0 + (u_l \zeta^{jl} + v_l \bar{\zeta}^{jl}) e^{i2\pi \frac{m}{s} t} = \left[\bar{k_0}^{\xi} + (\bar{u}_l^{\xi} \zeta^{(j+\delta)l} + \bar{v}_l^{\xi} \bar{\zeta}^{(j+\delta)l}) e^{2\pi i \frac{m}{s}(t-\theta)}\right] e^{2\pi i \alpha}.$$

Let us first take $\xi = 1$ and hence $\bar{u}_l^\xi = u_l$ and $\bar{v}_l^\xi = v_l$. The condition becomes the vanishing of the expression

$$k_0(1 - e^{2\pi i\alpha}) + u_l e^{2\pi i(\frac{jl}{N}+\frac{mt}{s})}\left(1 - e^{2\pi i(\frac{\delta l}{N}+\alpha-\frac{m\theta}{s})}\right)$$
$$+ v_l e^{2\pi i(-\frac{jl}{N}+\frac{mt}{s})}\left(1 - e^{2\pi i(-\frac{\delta l}{N}+\alpha-\frac{m\theta}{s})}\right)$$

for all j and all t. This implies the vanishing of one or the other of the following complex numbers :

$$A = 1 - e^{2\pi i(\frac{\delta}{N}l+\alpha-\frac{m}{s}\theta)}, \quad \text{or} \quad B = 1 - e^{2\pi i(-\frac{\delta}{N}l+\alpha-\frac{m}{s}\theta)}.$$

– First case: $A = 0$, that is $y_j(t) = (k_0 + u_l\zeta^{jl}e^{2\pi i\frac{m}{s}t}, 0)$.

– Second case: $B = 0$, that is $y_j(t) = (k_0 + v_l\bar{\zeta}^{jl}e^{2\pi i\frac{m}{s}t}, 0)$.

In both cases, $k_0 = 0$ unless $\alpha = 0 \mod 2\pi$ but k_0 does not play any role in the determination of λ.

Taking now $\xi = -1$, we see that u_l in the first case, v_l in the second one, must be real. Replacing α by $\frac{r}{s}\theta - \frac{\delta}{N} \mod 1$, we get

$$\frac{\delta}{N}(l-1) + \frac{r-m}{s}\theta \in \mathbb{Z} \quad \text{or} \quad -\frac{\delta}{N}(l+1) + \frac{r-m}{s}\theta \in \mathbb{Z}.$$

Finally, replacing θ by $\frac{\beta}{2} + k\eta\frac{\delta}{N} + q$, we obtain the following conditions:

$$\left[l - 1 + \frac{r-m}{s}k\eta\right]\frac{\delta}{N} + \frac{r-m}{s}(\frac{\beta}{2} + q) \in \mathbb{Z},$$

or

$$\left[-(l+1) + \frac{r-m}{s}k\eta\right]\frac{\delta}{N} + \frac{r-m}{s}(\frac{\beta}{2} + q) \in \mathbb{Z}.$$

Taking $\beta = 0, q = 0, \delta = 1$, this implies that

$l - 1 + \frac{r-m}{s}k\eta = 0 \mod N$ in the first case,

$-(l+1) + \frac{r-m}{s}k\eta = 0 \mod N$ in the second one.

Taking now $\beta = 1$, this implies in both cases that, for any $q \in \mathbb{Z}/s\mathbb{Z}$,

$$\frac{r-m}{s}(\frac{1}{2} + q) = 0 \mod 1, \quad \text{i.e.} \quad \exists p \in \mathbb{Z},\ m = r - 2ps.$$

It follows that $l = 1 - 2pk\eta \mod N$ in the first case, $l = -(1 - 2pk\eta) \mod N$ in the second one. Notice that both cases correspond to the same type of solutions

$$y_j(t) = (k_0 + a_l\zeta^{j(1-2pk\eta)}e^{2\pi i(\frac{r}{s}-2p)t}, 0),$$

(and that, when in addition $1 - 2pk\eta$ is prime to N, these solutions are up to translation and scaling, *a relative equilibrium of the regular N-gon with a different period and a relabelling of the bodies*, that is one of the relative equilibrium solutions found in section 3.5). This shows that the case where $A = B = 0$ will not lead to different solutions and hence need not be studied separately. In both cases, as $\omega_l = \omega_{-l}$,

$$\sqrt{\lambda} = \frac{\omega_1}{\omega_{1-2pk\eta}} \left| \frac{\varpi + 2\pi\frac{r}{s} - 4p\pi}{\varpi + 2\pi\frac{r}{s}} \right| .$$

Notice that the value $p = 0$ corresponds to $\lambda = 1$. Hence, in order to insure that the minimum value of λ is 1, it is enough to insure that λ is always ≥ 1. This amounts to the following conditions, for which *one should remember that the values of* p *must be such that* $1 - 2pk\eta \neq 0 \mod N$.

i) If $p > 0$ and $\varpi + 2\pi\frac{r}{s} \geq 4p\pi$, the condition is

$$(\omega_{1-2pk\eta} - \omega_1)(\varpi + 2\pi\frac{r}{s}) \leq -4p\pi\omega_1,$$

which can never be satisfied because for every l, $\omega_l - \omega_1 \geq 0$; this implies that

$$\varpi + 2\pi\frac{r}{s} < 4\pi;$$

ii) If $p > 0$ and $0 < \varpi + 2\pi\frac{r}{s} < 4p\pi$, the condition is

$$(\omega_1 + \omega_{1-2pk\eta})(\varpi + 2\pi\frac{r}{s}) \leq 4p\pi\omega_1;$$

iii) If $p > 0$ and $\varpi + 2\pi\frac{r}{s} < 0$, the condition is

$$(\omega_1 - \omega_{1-2pk\eta})(\varpi + 2\pi\frac{r}{s}) \leq 4p\pi\omega_1;$$

iv) If $p < 0$ and $\varpi + 2\pi\frac{r}{s} \leq 4p\pi$, the condition is

$$(\omega_1 - \omega_{1-2pk\eta})(\varpi + 2\pi\frac{r}{s}) \leq 4p\pi\omega_1,$$

which can never be satisfied for the same reason as above; and this implies that

$$\varpi + 2\pi\frac{r}{s} > -4\pi;$$

v) If $p < 0$ and $4p\pi \leq \varpi + 2\pi\frac{r}{s} \leq 0$, the condition is

$$-(\omega_1 + \omega_{1-2pk\eta})(\varpi + 2\pi\frac{r}{s}) \leq -4p\pi\omega_1;$$

vi) If $p < 0$ and $\varpi + 2\pi\frac{r}{s} \geq 0$, the condition is

$$(\omega_{1-2pk\eta} - \omega_1)(\varpi + 2\pi\frac{r}{s}) \leq -4p\pi\omega_1.$$

Finally, the horizontal conditions are that $-H- \leq \varpi + 2\pi\frac{r}{s} \leq +H_+$, where

$$H_+ = \inf\left\{4\pi, \inf_{p>0}{}^* \left(\frac{4p\pi\omega_1}{\omega_1 + \omega_{1-2pk\eta}}, \frac{4p\pi\omega_1}{\omega_{1+2pk\eta} - \omega_1}\right)\right\},$$

$$H_- = \inf \left\{ 4\pi, \inf_{p>0}{}^* \left(\frac{4p\pi\omega_1}{\omega_1 + \omega_{1+2pk\eta}}, \frac{4p\pi\omega_1}{\omega_{1-2pk\eta} - \omega_1} \right) \right\},$$

where the $\inf^*$ *means that the index* k *of any* ω_k *involved can never be 0.*

Recalling the vertical condition

$$\left|\varpi + 2\pi\frac{r}{s}\right| \leq V = \inf_{p\geq 0} \frac{\omega_1}{\omega_{(1+2p)k\eta}} |(1+2p)2\pi|,$$

we obtain the

Theorem 23 *The following condition implies that the relative equilibrium solution of the equal mass regular* N*-gon with frequency* $2\pi\frac{r}{s}$ *in a frame rotating with frequency* ϖ *is the sole absolute minimizer of the action among paths which in the rotating frame are* s*-periodic loops with the* $G_{r/s}(N,k,\eta)$*-symmetry of some solution of the vertical variational equation:*

$$-\inf(V, H-) \leq \varpi + 2\pi\frac{r}{s} \leq \inf(V, H_+).$$

When $k = n$, the horizontal contribution disappears; in this case, the corresponding Lyapunov families can be searched for as absolute minimizers of the action with the given symmetry constraints:

Corollary 24 *The following condition implies that the relative equilibrium solution of the equal mass regular* N*-gon with frequency* $2\pi\frac{r}{s}$ *in a frame rotating with frequency* ϖ *is the sole absolute minimizer of the action among paths which in the rotating frame are* s*-periodic loops with the* $G_{r/s}(N,n,\eta)$*-symmetry:*

$$-V \leq \varpi + 2\pi\frac{r}{s} \leq V.$$

Proof. As $\omega_1 \leq \omega_n$, it is enough to prove that

$$\frac{2p}{\omega_{1\pm 2pn} \pm \omega_1} \geq \frac{1}{\omega_n} \quad \text{and} \quad \frac{2p}{\omega_{1\mp 2pn} \pm \omega_1} \geq \frac{1}{\omega_n},$$

in all the cases where no ω_0 is implied. But all these inequalities are implied by $2\omega_n \geq \omega_{1\pm 2pn} + \omega_1$.

In the following sections, where the numerical side plays an important role, we first study the two families associated with the highest frequency ω_n: the Eight families when N is odd and the Hip-Hop families when N is even. The first ones generalize Marchal's P_{12}-family ([CM, Ma2, CFM, CF1, S]) which links the Lagrange equilateral relative equilibrium to the Eight, the second ones generalize the family which links the relative equilibrium of the square to the original Hip-Hop ([CV, TV]). These are precisely the families for which global minimization of the action under the $G_{r/s}(N,k,\eta)$-symmetry constraint may be used for global continuation. But even then, no unicity of minimizers is proved; hence the continuity of the families so obtained is in question, even if global topological continuation results can be used.

We then study the families associated with the lowest frequency ω_1, which lead to Simó's chains with the maximal number of lobes ($N-1$ lobes for N bodies): only numerical results are given here because these families are only local minimizers of the action. This is related to the fact that, starting with $N=4$, there exist isomorphisms between the actions of some of the groups $G_{r/s}(N,k,\eta)$ (see section 3.5). We also give examples of chains with a non maximal number of lobes, where similar phenomena do occur.

Remark It follows from corollary 17 that absolute choreographies are dense in the Lyapunov families as soon as torsion is present (i.e. as soon as ϖ does vary along the family). This results from the fact that a path wich is $G_{r/s}(N,k,\eta)$-symmetric in a frame rotating with frequency ϖ is $G_{r/s+\varpi}(N,k,\eta)$-symmetric in the inertial frame. (In the action of the group, this corresponds to the replacement of α by $\alpha+\varpi\theta$.)

5.2 Families associated with the highest frequency ω_n

They satisfy $V=2\pi\frac{\omega_1}{\omega_n}$ and it follows that the relative equilibrium remains the unique global minimizer as long as

$$-2\pi\frac{\omega_1}{\omega_n}\leq\varpi+2\pi\leq 2\pi\frac{\omega_1}{\omega_n},$$

i.e. until one reaches the vertical bifurcation of the corresponding family.

5.2.1 The Eight families: $G=G_2(2n+1,n,-1)$

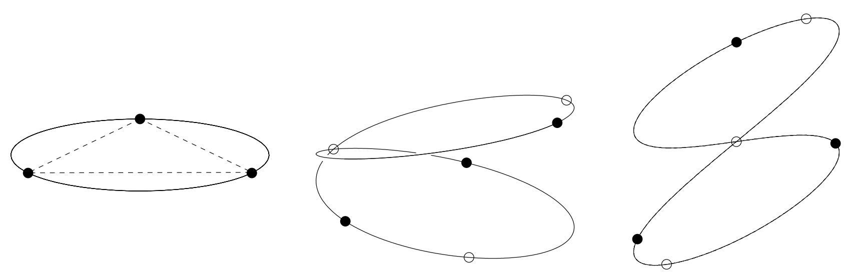


Figure 8: The P_{12} family. Filled and hollow circles represent bodies at times $t=0$ and $t=T/12$

When $n=1$ one gets the original P_{12}-family of Christian Marchal (see figure 8). In this case, the minimization property of the relative equilibrium family was already proved in [BT] under the weaker assumption of the choreography symmetry. Note that for planar solutions, 2π is replaced by 4π. We leave to the reader the pleasure to check that this was a priori obvious.

When $n\geq 2$, a proof of the existence of a $G_2(2n+1,n,-1)=D_{2n}$-symmetric Eight is still missing. The existence of a D_n-symmetric Eight is proved in [FT] but, while highly probable, the fact that it is automatically D_{2n}-symmetric is not proved. Figures 12 and 18 depict the case $n = 2$.

5.2.2 The Hip-Hop families: $G = G_1(2n, n, \pm 1)$

The original Hip-Hop (figures 9 and 16) corresponds to the case $n = 2$. For all values of n, the existence of the Hip-Hop family is proved in [TV], with the usual proviso that no uniqueness, hence no continuity, is proved. It seems very likely that the natural end of the family is a pair of simultaneous n-tuple collisions for $\varpi = \omega_2$, but this not proved either. The heuristic explanation is that the phenomenon is the same as for a fixed center pb with angle $\alpha = 2\pi$: in the inertial frame, each body must turn by exactly 2π during time $T/2$). Moreover, all along the family, the minimizing solutions should possess the brake symmetry.

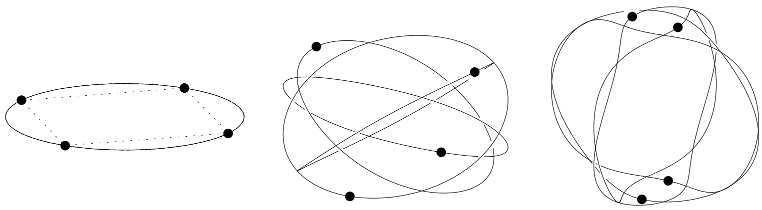


Figure 9: The original Hip-Hop family (Hip-Hop in the middle)

Remark Illustrating the remark at the end of 5.1, figure 16 shows the two simplest spatial choreographies (in the inertial frame) in the Hip-Hop family for $n = 2$. First described by S. Terracini and A. Venturelli, they correspond to the symmetry groups $G_{\frac{3}{2}}(4, 2, \pm 1)$ and $G_{\frac{5}{6}}(4, 2, \pm 1)$. On the other hand, figure 10 shows the original Hip-Hop in a frame rotating with its frequency ω_2.

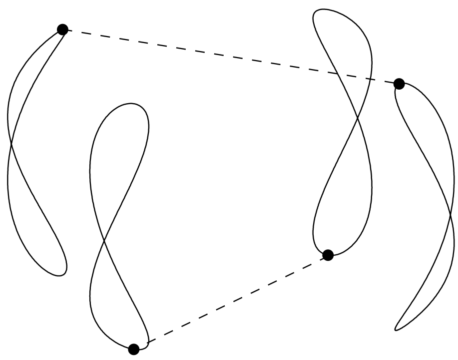


Figure 10: The Hip-Hop in a frame rotating with its very frequency

5.3 Chains families: the role of angular momentum

The chains are planar choreographies of the equal-mass N-body problem, similar to the Eight but with a number $\ell \geq 2$ of lobes. The first example with $\ell > 2$ is Gerver's chain with 3 lobes and 4 bodies [CGMS]. Because of their symmetries, chains with an even number ℓ of lobes have a vanishing angular momentum while this is not the case when ℓ is odd. This fundamental difference accounts for a totally different behaviour of the corresponding Lyapunov families. We have studied numerically the cases of 4 and 5 bodies and have observed:

– when ℓ is even, a complete unfolding of the corresponding unchained polygon into a vertical chain;

– when ℓ is odd, a complete unfolding into a horizontal chain (in a frame which is still rotating), followed by a plane family which continues to a horizontal chain.

Due to the existence of isomorphisms of the actions of different symmetry groups $G_{r/s}(n, k, \eta)$, global minimization of the action may succeed only for the Eights ($\ell = 2$).

5.3.1 Maximal chain families: $G = G_{N-1}(N, 1, -1)$, $N = 2n + 1$

When observed in a frame which rotates $N - 1$ times per period in the negative direction, the non-trivial solutions of the (VVE) corresponding to the frequency ω_1 give rise, to an infinitesimal choreography which unfolds the $N - 1$ circles. When $N = 2n + 1$ is odd, the full family hopefully continues up to the fixed frame into a vertical zero angular momentum planar chain with $N - 1$ lobes. We recall that it was proved in section 3.5 that the action of $G_{N-1}(N, 1, -1)$ is, for $N = 2n+1$, isomorphic to the one of the Eight with N bodies $G_2(N, n, -1)$.

We study now the simplest case where the maximal chain family and the Eight family differ.

5.3.2 The 4-lobe chain and the Eight for 5 bodies: two isomorphic actions of the symmetry groups

Four-lobe chain: $G = G_4(5, 1, -1)$ When $N = 5$, we have checked numerically that, indeed, the $G_4(5, 1-1)$ family continues up to a vertical planar chain with four lobes. During the unfolding, the two central lobes become smaller and flatten more rapidly while the the two exterior lobes remain during a long time almost vertical before flattening down as the Big Ears of Big Brother.

We have

$$V = 2\pi, \quad H_+ = 4\pi\frac{\omega_1}{\omega_1 + \omega_2},\ H_- = \inf\left(2\pi, 4\pi\frac{\omega_1}{\omega_2 - \omega_1}\right) = 2\pi.$$

Hence, on the left hand side ($\varpi + 8\pi < 0$), the estimate goes all the way till the bifurcation of the chain family but on the right hand side, it stops before: $-2\pi \leq \varpi + 8\pi \leq 4\pi\frac{\omega_1}{\omega_1+\omega_2}$. This is due to the existence of the Eight family (see figure 12) which, as we have recalled, has the same symmetries up to reordering (see figure 18).

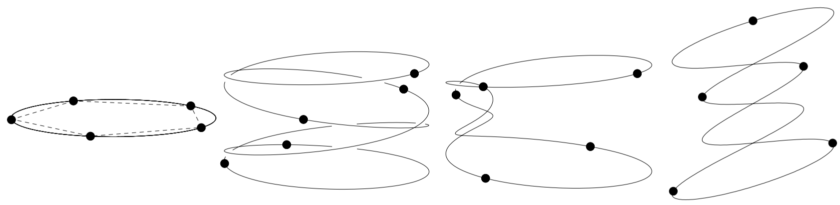


Figure 11: The family of the 5-body chain with 4 loops

Five-body Eight: $G = G_2(5, 2, -1)$ The solutions of the (VVE) corresponding to the 2 families with frequency ω_2 (recall that they differ only by the replacement of ζ^k by $\bar{\zeta}^k$) give rise to chains with respectively 3 lobes and 2 lobes, the order of the bodies in the choreography being respectively the one of the retrograde and the direct stellated pentagon. We are interested now in the second one.

We have

$$V = 2\pi\frac{\omega_1}{\omega_2}, \quad H_+ = \inf(4\pi, 8\pi\frac{\omega_1}{\omega_2 - \omega_1}, 12\pi\frac{\omega_1}{\omega_1 + \omega_2}) = 4\pi, \ H_- = 8\pi\frac{\omega_1}{\omega_1 + \omega_2}\cdot$$

It follows that $\lambda^G_\varpi = 1$ as long as $-2\pi\frac{\omega_1}{\omega_2} \leq \varpi + 4\pi \leq 2\pi\frac{\omega_1}{\omega_2}$, i.e. until we reach the vertical bifurcation of the Eight family (see figure 18).

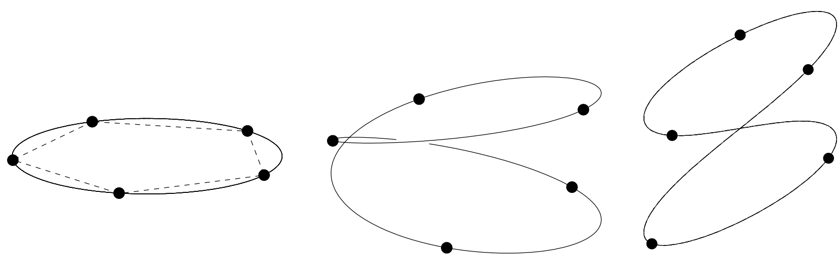


Figure 12: The family of the 5-body Eight.

Remarks. 1) The case of $G = G_4(5, 1, -1)$ is the first one where $H_+ < V$. This is because the second frequency ω_2, while ruled out from the G-invariant vertical solutions, is present into the G-invariant horizontal solutions.

2) The estimate H_+ yields a lower bound of the action of the member of the Lyapunov family which bifurcates at $\varpi = (-2\omega_2 - \omega_1)\frac{\omega_1}{\omega_2}$. In particular, at this point, the action is higher than it is at the bifurcation point, because, $\frac{\omega_1}{\omega_2} < \frac{2\omega_1}{\omega_1 + \omega_2}$.

5.3.3 The 3-lobe chains for 4 or 5 bodies

Four bodies When $N = 4$, the non-trivial solutions of the (VVE) corresponding to the frequency ω_1, more precisely to the group $G_3(4, 1, \pm 1)$, give rise to 3 lobes chains when one starts with a frame which rotates two full turns in the negative direction. When the rotation decreases, the chain starts opening but the central lobe decreases and when it reaches approximately -0.9 turns by period, an almost collision occurs. After that, the solution flattens to the horizontal plane (with ϖ not yet vanishing) where a second bifurcation leads to the Gerver solution (figures 13 and 17).

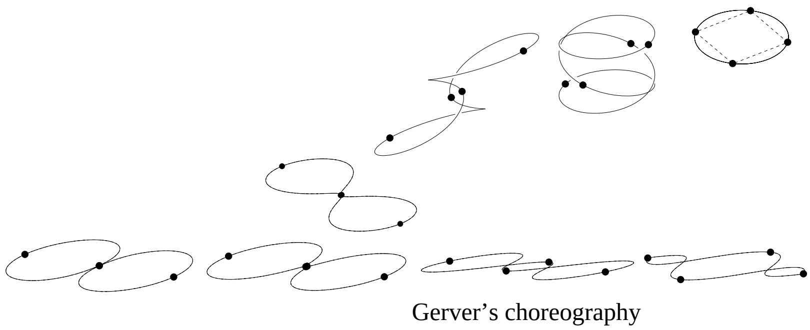


Figure 13: The planar Gerver family as a secondary bifurcation

Five bodies An analogous scenario is observed for $N = 5$ for the frequency ω_2, more precisely with the group $G_3(5, 2, 1)$. but in this case, the family never gets close to collision (figures 14 and 19) and the central lobe is bigger than the two extreme ones.

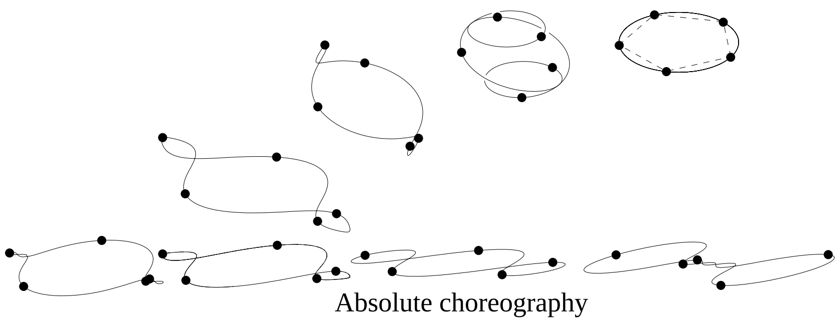


Figure 14: The 5-body 3-lobe chain family and its planar secondary bifurcation

5.4 Action diagrams

Figures 15–19 display the action of families of solutions as a function of the frequency ϖ of the rotation of the frame. Each figure corresponds to a fixed symmetry in the rotating frame (recall that the isomorphisms between group actions are associated with relabelling of the bodies as described in theorem 23):

Figure 15	$G_2(3,1,-1)$	$\equiv$	$G_{-2}(3,1,1)$
Figure 16	$G_1(4,2,\pm 1)$		
Figure 17	$G_3(4,1,-1)$	$\equiv$	$G_1(4,1,1)$
Figure 18	$G_4(5,1,-1)$	$\equiv$	$G_2(5,2,-1)$
Figure 19	$G_3(5,2,1)$	$\equiv$	$G_1(5,1,1)$

The solutions are represented in perspective in $\mathbb{R}^3$. This holds in particular for solutions lying in the horizontal plane (relative equilibria and horizontal secondary families in figures 17 and 19) or in a vertical plane (Eights or chains in figures 15 and 18).

Display of the implications of theorem 23 Fat segments on the ϖ-axis and the corresponding fat part on the graph of the action indicate intervals on which the relative equilibrium of the regular N-gon is the unique absolute minimizer for the given symmetry in the rotating frame. It appears that the theorem detects global phenomena i.e., the presence of different branches of Lyapunov families.

Symmetry with respect to $\varpi = 0$ of figure 15 This symmetry corresponds to the isomorphism $G_2(3,1,-1) \equiv G_{-2}(3,1,1)$. Analogous symmetries could have been shown on the other figures.

On the $\varpi < 0$-half (resp. $\varpi > 0$-half) of the figure, the configuration of the relative equilibrium is a positively (resp. negatively) oriented equilateral triangle. In the rotating frame, it makes two turns in the positive (resp. negative) direction. Notice that in the inertial frame the four possible combinations of orientation of the configuration and direction of motion occur at the four bifurcation points $-3\omega_1$, $-\omega_1$, ω_1 and $3\omega_1$.

Also, in figure 15, the continuity of the Lyapunov family implies a change of sign of the x-coordinate of the body 0 when ϖ goes through 0. This is consistent with the $G_r(N,k,\eta)$-symmetry: if a loop is $G_r(N,k,\eta)$-symmetric, the same is true for its image under the rotation of angle π around the vertical axis.

Cases without torsion The $G_{N-1}(N,1,-1)$-Lyapunov family starting at $\varpi = -N\omega_1$ (resp. $+N\omega_1$) from the relative equilibrium which completes minus (resp. plus) one turn per period in the inertial frame has no torsion: indeed, it is merely the family obtained by rotating the horizontal relative equilibrium around the y-axis. As this family has constant action, this fact accounts for the end of the action diagrams in figures 15, 17 and 18.

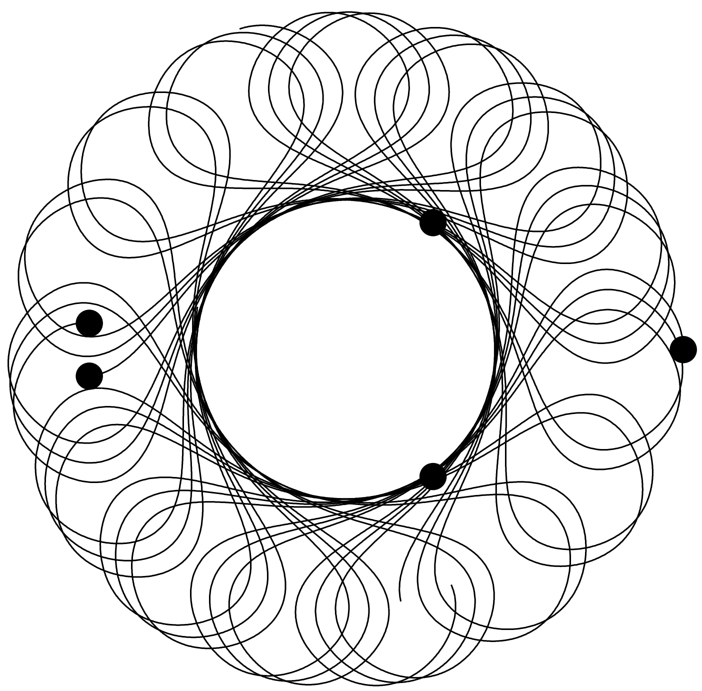
About figure 18 The family of solutions which in figure 18 bifurcates at $\varpi = (-2\omega_2 - \omega_1)\frac{\omega_1}{\omega_2}$ is, up to scaling, symmetric of that which, in figure 19, bifurcates at $\varpi = \omega_1 - 3\omega_1$. More precisely, the latter family transforms into the former one by:

1) reflection about $\varpi = 0$, which corresponds to the isomorphism $G_3(5, 2, 1) \equiv G_{-3}(5, 2, -1)$;

2) translation of $-5\omega_2$ along the ϖ-axis, which transforms the $G_{-3}(5, 2, -1)$-symmetry into the symmetry group $G_2(5, 2, -1)$ of figure 18 (see the remark at the end of 5.1);

3) scaling by ω_1/ω_2.

In the rotating frame, the family starts as the P_{12}-family but, at some point, two more loops develop on the supporting curve which then flattens to a horizontal planar (relative) choreography with two loops, described twice per period. The figure below shows this (quasiperiodic) solution in the inertial frame.



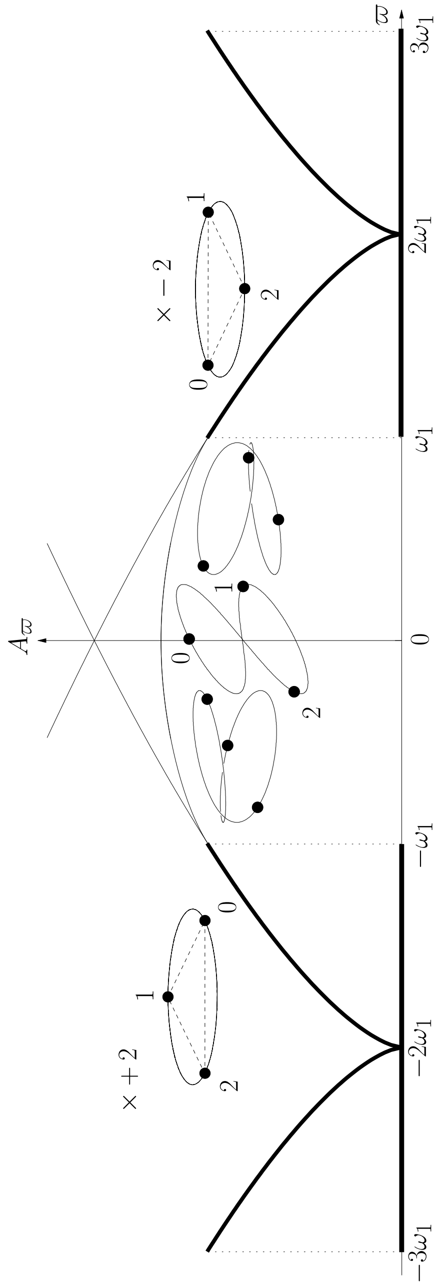


Figure 15: The action of the P_{12} family and of two times Lagrange solution in the rotating frame

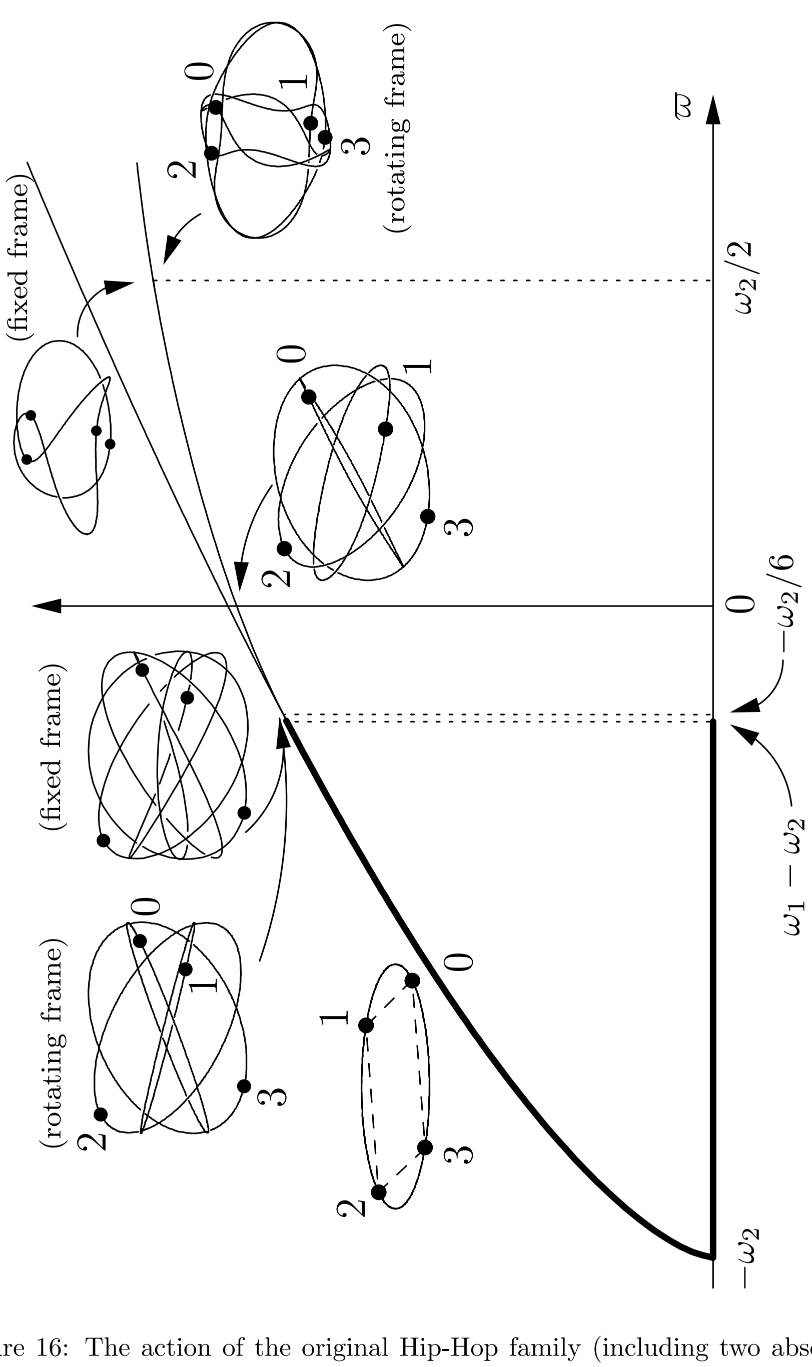


Figure 16: The action of the original Hip-Hop family (including two absolute choreographies discovered in [TV])

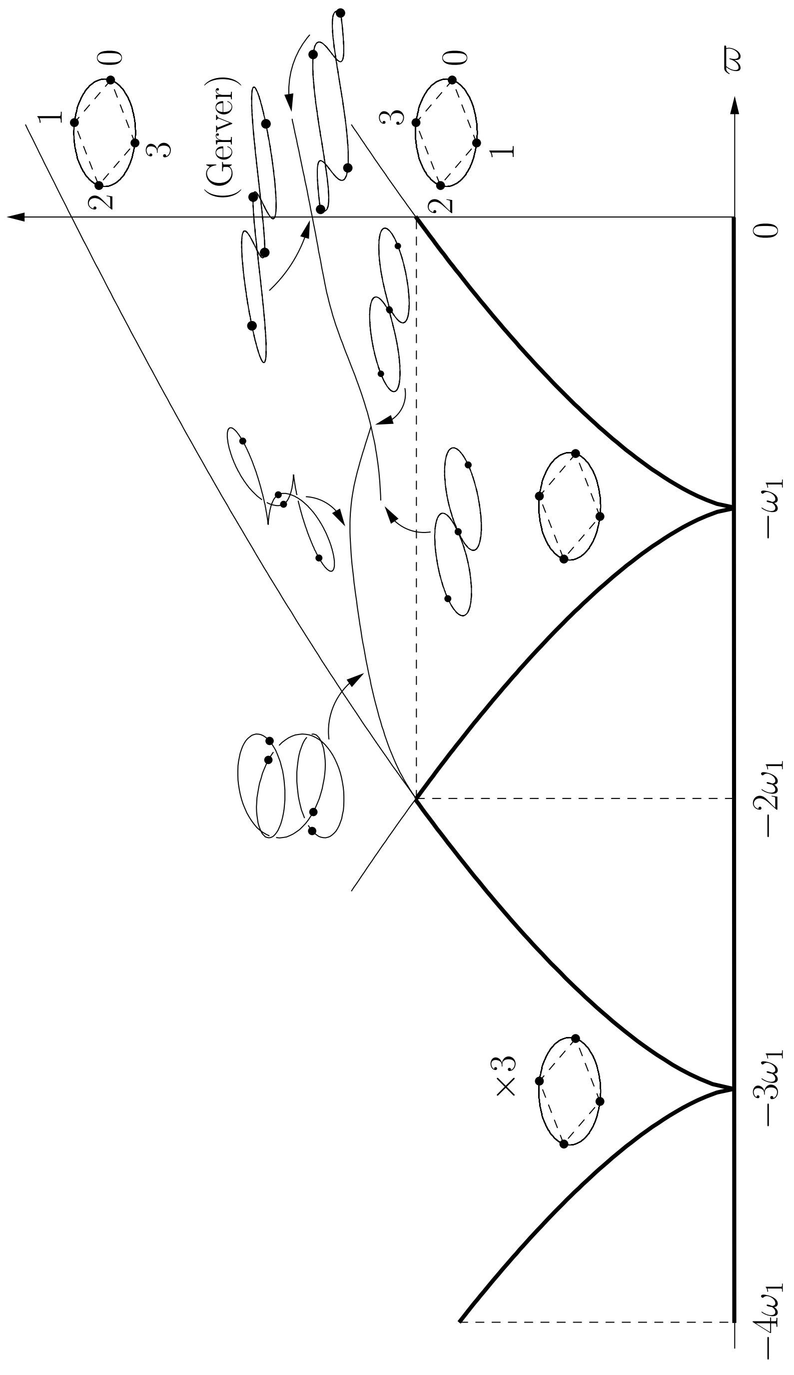


Figure 17: Action of the 4-body, 3-loop chain family, and of the planar Gerver family

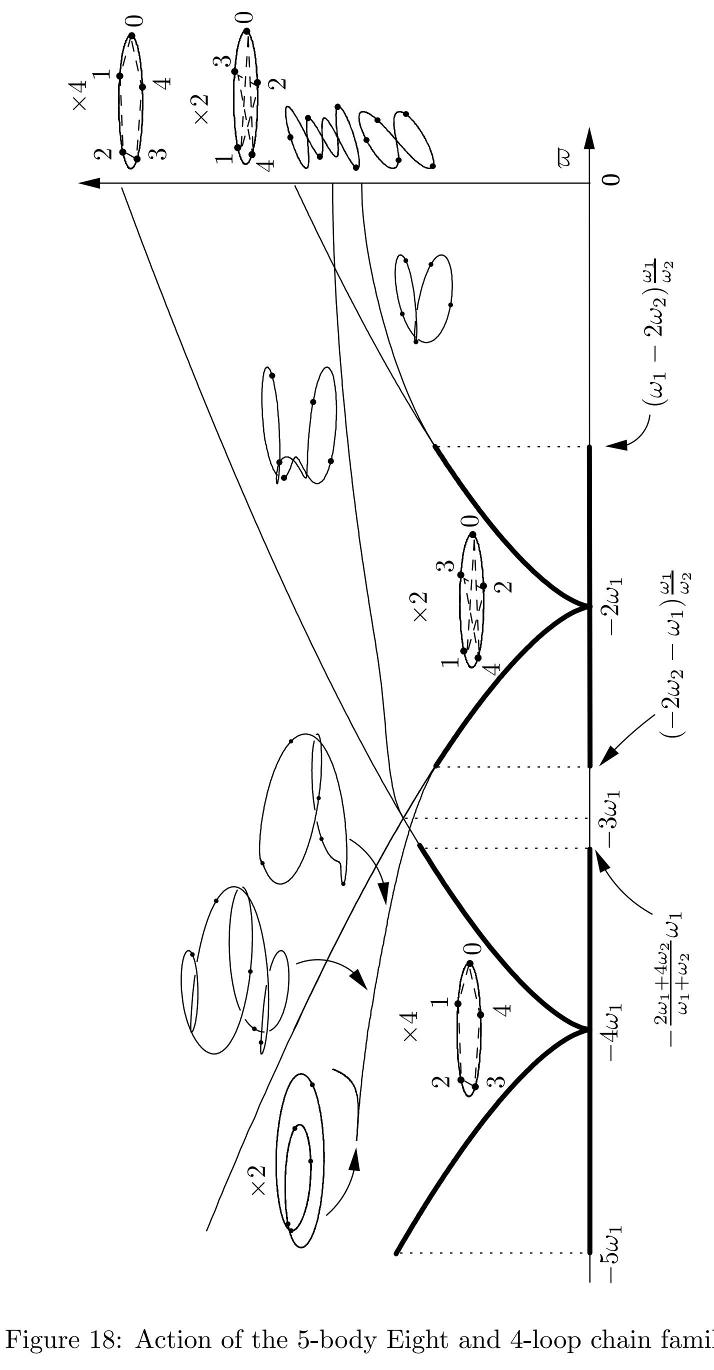


Figure 18: Action of the 5-body Eight and 4-loop chain families

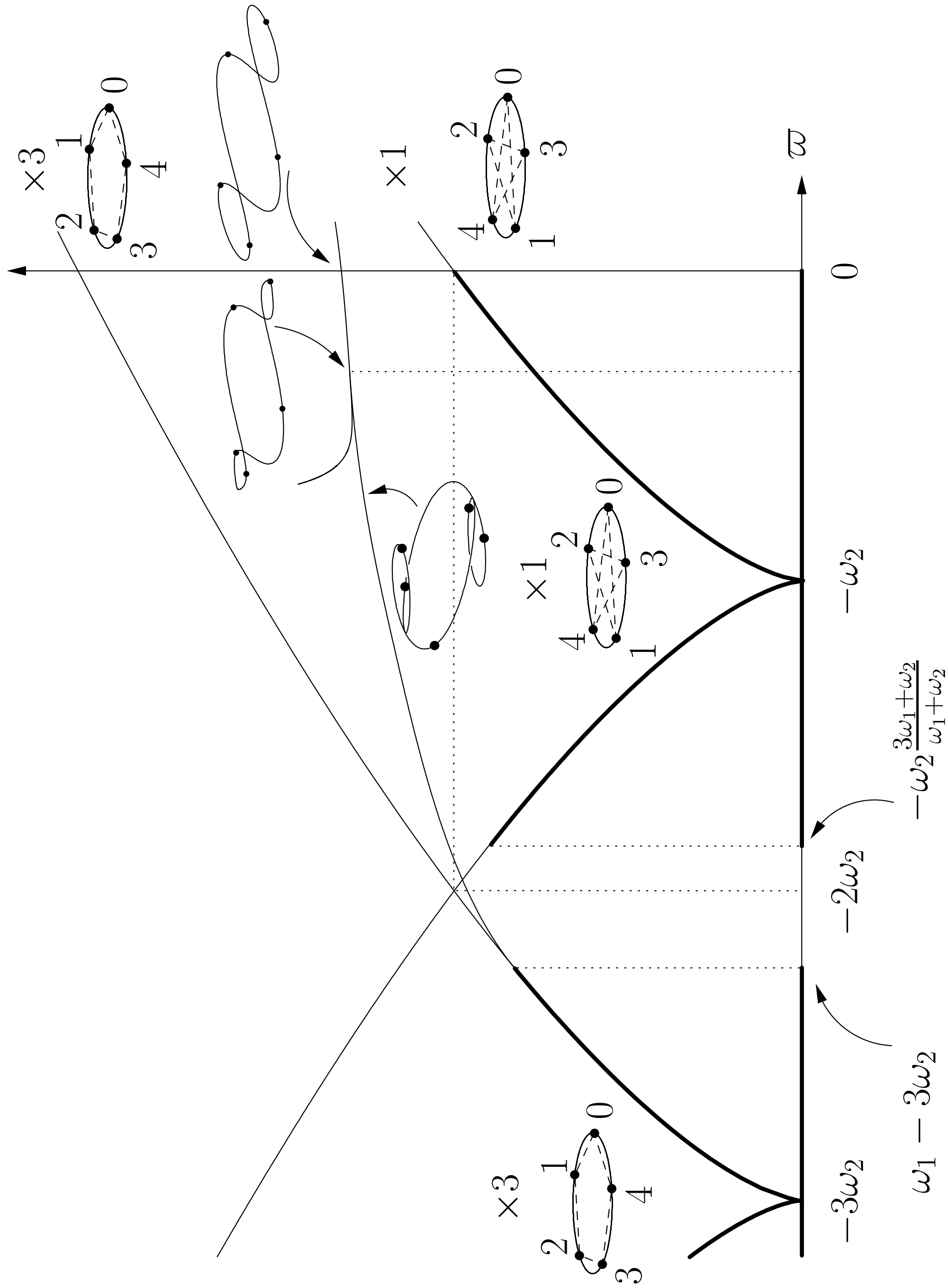


Figure 19: Action of the 5-body, 3-loop chain family, and of the corresponding planar family (cf. [S])

6 Appendix: Fourier expansions and the torsion

Approximate Fourier expansions of $G_{r/s}(N,k,\eta)$-symmetric Lyapunov families, in the same spirit as Marchal's computations in [Ma1, Ma2], allow to evaluate the torsion under a regularity hypothesis.

6.1 The symmetry ansatz

We are looking for local one-parameter families of solutions of the N-body problem which, in a family of frames rotating with frequency

$$\varpi = \omega_1 - \frac{r}{s}\omega_k + \tilde{\varpi},$$

are periodic of period $T = s\frac{2\pi}{\omega_k}$. *We will suppose that $\omega_k = 2\pi$, and hence $T = s$.*

Such solutions are of the form

$$x_j(t) = (h_j(t), z_j(t)),\ h_j(t) = e^{i(\omega_1 - \frac{r}{s}\omega_k + \tilde{\varpi})t}\tilde{h}_j(t), \quad j = 0, \cdots, N-1,$$

with

$$\tilde{h}_j(t) = \sum_{l=-\infty}^{+\infty} a_l^j e^{i2\pi\frac{l}{s}t},\ z_j(t) = \mathrm{Re}\big(\sum_{l=-\infty}^{+\infty} b_l^j e^{i2\pi\frac{l}{s}t}\big).$$

Moreover, we ask the solutions in the rotating frame $\tilde{x}_j(t) = (\tilde{h}_j(t), z_j(t))$ to be symmetric under the action of $G_{r/s}(N,k,\eta)$ described in section 3. Recall that an s-periodic loop of configurations $x(t) = \big(x_1(t), \cdots, x_N(t)\big)$ is invariant under the action of $G_{r/s}(N,k,\eta)$ if and only if, for every $(\theta,\delta,\beta,\xi) \in G_2$ representing an element of $G_{r/s}(N,k,\eta)$, i.e. such that $\theta - \frac{\beta}{2} - k\eta\frac{\delta}{N} = m \in \mathbb{Z}$, one has

$$\forall j \in \mathbb{Z}/N\mathbb{Z},\ x_j(t) = \rho x_{\xi(j+\delta)}\big(\xi(t-\theta)\big),$$

where the action of ρ on $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R}$ is defined by

$$\rho(h,z) = (e^{i2\pi\alpha}\bar{h}^\xi, e^{i\pi\beta}z) \quad \text{with} \quad \alpha = \frac{r}{s}\theta - \frac{\delta}{N} \pmod 1,$$

where $\bar{h}^\xi = h$ if $\xi = +1$ and $\bar{h}^\xi = \bar{h}$ if $\xi = -1$.

Translated in terms of Fourier coefficients, this reads:

$$a_l^j = e^{i2\pi(\alpha - \frac{l}{s}\theta)}\overline{a_l^{\xi(j+\delta)}}^{\xi},$$

for the horizontal components and

$$\mathrm{Re}\,(b_0^j - e^{i\pi\beta\xi}\overline{b_0^{\xi(j+\delta)}}^{\xi}) = 0 \quad \text{and} \quad b_l^j = e^{i(\pi\beta\xi - 2\pi\frac{l}{s}\theta)}\overline{b_l^{\xi(j+\delta)}}^{\xi} \quad \text{if} \quad l \neq 0$$

for the vertical components, implying restrictions on the coefficients a_l^j and b_l^j:

Horizontal coefficients We have

$$\alpha - \frac{l}{s}\theta = \frac{r-l}{s}\theta - \frac{\delta}{N} = \frac{r-l}{s}(\frac{\beta}{2}+m) + \left(\frac{r-l}{s}k\eta - 1\right)\frac{\delta}{N}.$$

Fixing ξ and δ and changing β, the angle $e^{i2\pi(\alpha-\frac{l}{s}\theta)}$ takes at least two different values as soon as $\frac{r-l}{s}$ is not even. Hence

$$a_l^j = 0 \quad \text{if} \quad \frac{r-l}{s} \quad \text{is not an even integer.}$$

If $\frac{r-l}{s} = 2p$ is even, the symmetry conditions become

$$a_l^j = e^{i2\pi(2pk\eta-1)\frac{\delta}{N}}\overline{a_l^{\xi(j+\delta)}}^{\xi}.$$

Applying these identities with both values $\xi = 1$ and $\xi = -1$, we get by difference

$$\forall p, j, a_{r-2ps}^j = \overline{a_{r-2ps}^{-j}}, \quad \text{hence} \quad \forall p,\ a_{r-2ps}^0 = \overline{a_{r-2ps}^0}.$$

So, the coefficients a_{r-2ps}^0 are real and

$$\forall p, j,\ a_{r-2ps}^j = e^{-i2\pi(2pk\eta-1)\frac{j}{N}}a_{r-2ps}^0.$$

Finally, as the center of mass is at the origin, we have that for all l, $\sum_{j=0}^{N-1} a_l^j = 0$, hence

$$\left(\sum_{j=0}^{N-1} e^{-i2\pi(2pk\eta-1)\frac{j}{N}}\right) a_{r-2ps}^0 = 0,$$

which implies

$$a_{r-2ps}^j = 0 \quad \text{for all} \quad p \quad \text{such that} \quad 2pk\eta - 1 = 0 \pmod N.$$

To summarize,

$$a_l^0 = 0 \quad \text{unless possibly for} \quad l = r - 2ps \quad \text{with} \quad 2pk\eta - 1 \neq 0 \pmod N$$

$$\forall p, j,\ a_{r-2ps}^j = e^{-i2\pi(2pk\eta-1)\frac{j}{N}}a_{r-2ps}^0.$$

Vertical coefficients Giving successively its two possible values 0 and 1 to β, we get that

$$\forall j,\ \operatorname{Re} b_0^j = 0.$$

Moreover,

$$\frac{\beta}{2}\xi - \frac{l}{s}\theta = \frac{\beta}{2}(\xi - \frac{l}{s}) - m\frac{l}{s} - k\eta\frac{l}{s}\frac{\delta}{N}$$

takes two different values for $\beta = 0$ and $\beta = 1$ as long as $\xi - \frac{l}{s}$ is not an even integer, which implies that, for $l \neq 0$, $b_l^j = 0$ can be different from 0 only if l is of the form $l = (2p+1)s$, with p an integer.

Now, if $l \neq 0$, choosing $\delta = -j$ and giving its two possible values ± 1 to ξ and noticing that $\beta\xi = \beta \pmod 2$, we get

$$b_l^j = e^{i2\pi(\frac{\beta}{2}(1-\frac{l}{s})-m\frac{l}{s}+k\eta\frac{l}{s}\frac{j}{N})}b_l^0 = e^{i2\pi(\frac{\beta}{2}(1-\frac{l}{s})-m\frac{l}{s}+k\eta\frac{l}{s}\frac{j}{N})}\overline{b_l^0},$$

hence $b_l^0 \in \mathbb{R}$. Finally, as the center of mass is at the origin, $\forall l,\ \sum_{j=0}^{N-1} b_l^j = 0$, that is

$$e^{i2\pi(\frac{\beta}{2}(1-\frac{l}{s})-m\frac{l}{s})}\left(\sum_{j=0}^{N-1} e^{i2\pi k\eta\frac{l}{s}\frac{j}{N}}\right) b_l^0 = 0.$$

Finally, b_l^0 (and hence all b_l^j) must vanish unless $k\eta l \neq 0 \pmod{sN}$.

To summarize,

$$b_l^0 = 0 \quad \text{unless possibly for} \quad l = (2q+1)s \quad \text{with} \quad (2q+1)k\eta \neq 0 \pmod N,$$

$$\forall q, j,\ \ b^j_{(2q+1)s} = e^{i2\pi k\eta(2q+1)\frac{j}{N}}b^0_{(2q+1)s}.$$

In particular,

$$z_j(t) = \sum_{q\geq 0} c_{2q+1}\cos 2\pi(2q+1)(t + k\eta\frac{j}{N}),$$

where we have used the notation

$$c_{2q+1} = b^0_{(2q+1)s} + b^0_{-(2q+1)s}, \quad q = 0, 1, \cdots$$

For example, these conditions for the P_{12} family (that is for the group $G_2(3,1,-1)$) become

$$a_l^0 \in \mathbb{R}, \quad a_l^j = 0 \quad \text{if either} \quad l = 1 \pmod 2 \quad \text{or} \quad l = 0 \pmod 3,$$

$$b_l^0 \in \mathbb{R}, \quad b_l^j = 0 \quad \text{if either} \quad l = 0 \pmod 2 \quad \text{or} \quad l = 0 \pmod 3,$$

which coincide with the conditions found by Marchal in [Ma1].

6.2 The regularity ansatz

In the rotating frame, the solutions we are interested in are the ones tangent to the cylinder of solutions of (VVE)

$$(A_0\zeta^j e^{i2\pi\frac{r}{s}t}, \epsilon\mathrm{Re}\,\zeta^{k\eta j}e^{i2\pi t}),\ j = 0, \cdots, N-1,$$

where as usual $\zeta = e^{i\frac{2\pi}{N}}$.

In the inertial frame, these solutions are of the form

$$x_j(t) = (h_j(t), z_j(t)) \qquad j = 0, \cdots, N-1,$$

$$h_j(t) = e^{i(\omega_1+\tilde{\varpi})t} \sum_p a^0_{r-2ps} e^{i2\pi[-2pt-(2pk\eta-1)\frac{j}{N}]},$$

$$z_j(t) = \sum_{q\geq 0} c_{2q+1} \cos 2\pi(2q+1)(t + k\eta \frac{j}{N}),$$

where the coefficients $a^0_l(\epsilon)$ and $c_l(\epsilon)$ are real and the summations are respectively over all integers $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$ such that

$$2pk\eta - 1 \neq 0 \pmod N \quad \text{and} \quad (2q+1)k\eta \neq 0 \pmod N.$$

We will assume that, in the reduced phase space where the relative equilibrium becomes an equilibrium, the one-parameter family of s-periodic solutions we are looking at generates an analytic foliation of an analytic 2-dimensional surface, more precisely, that it is the image under an analytic embedding of the trivial planar family $u(t) = \epsilon \cos 2\pi t/s$, $v(t) = \epsilon \sin 2\pi t/s$. As $(\cos^k 2\pi t/s)(\sin^l 2\pi t/s)$ has no harmonics of order larger than $k+l$, this implies in particular that the Fourier expansion of the coefficient of ϵ^n in $x(t)$ does not contain harmonics of order larger than n. This justifies taking

$$\epsilon = c_1$$

as our main local parameter and making the following ansatz:

$$a^0_{r-2ps} = A_{2p}\epsilon^{|2p|} + O(\epsilon^{|2p|+2}), \quad c_{2q+1} = C_{2q+1}\epsilon^{2q+1} + O(\epsilon^{2q+3}),$$

which in the case of the P_{12} family coincides with the one made in [Ma1].

Using both ansatz, an identification in the equations of motion allows us to determine the leading coefficients a^0_l and c_l, as well as the dominant term in $\tilde{\varpi}$, in terms of ϵ. The dominant terms in the components of $z_j(t)$ are

$$z_j(t) = \epsilon \cos 2\pi(t + k\eta \frac{j}{N}) + C_3\epsilon^3 \cos 6\pi(t + k\eta \frac{j}{N}) + O(\epsilon^5),$$

while those of $h_j(t)$ are

$$h_j = e^{i(\omega_1+\tilde{\varpi})t} \sum_{p=0,\pm 1} a^0_{r-2ps} e^{i2\pi(-2pt+\frac{j[p]}{N})} + O(\epsilon^4),$$

that is

$$h_j = e^{i(\omega_1+\gamma\epsilon^2)t} \begin{bmatrix} (A_0 + \alpha\epsilon^2)e^{i2\pi\frac{j}{N}} + \\ \left(A_2 e^{i2\pi(-2t+\frac{j[1]}{N})} + A_{-2}e^{i2\pi(2t+\frac{j[-1]}{N})}\right)\epsilon^2 \end{bmatrix} + O(\epsilon^4),$$

where we have used the following notations (we have directly set to zero the coefficient of ϵ in a^0_r and $\tilde{\varpi}$ because this is an immediate consequence of the equations in the next section):

$$a^0_r = A_0 + \alpha\epsilon^2 + O(\epsilon^4), \quad \tilde{\varpi} = \gamma\epsilon^2 + O(\epsilon^4).$$

The Eight families $G_2(N=2n+1,n,-1,)$

$$h_j(t) = e^{i(\omega_1+\tilde{\varpi})t}\left(a_2^0 e^{i2\pi\frac{j}{N}} + a_4^0 e^{i2\pi(2t+(-2n+1)\frac{j}{N})} + \cdots\right),$$

The Hip-Hop families $G_1(N=2n,n,\pm1)$

$$h_j(t) = e^{i(\omega_1+\tilde{\varpi})t}\left(a_1^0 e^{i2\pi\frac{j}{N}} + a_3^0 e^{i2\pi(2t+(2n+1)\frac{j}{N})} + a_{-1}^0 e^{i2\pi(-2t-(2n-1)\frac{j}{N})} + \cdots\right),$$

The maximal chain families $G_{2n}(2n+1,1,-1)$

$$h_j(t) = e^{i(\omega_1+\tilde{\varpi})t}\left(a_{2n}^0 e^{i2\pi\frac{j}{N}} + a_{2n+2}^0 e^{i2\pi(2t-\frac{j}{N})} + a_{2n-2}^0 e^{i2\pi(-2t+3\frac{j}{N})} + \cdots\right),$$

except if $N=3$ where the last term is absent.

6.3 Identification of dominant coefficients

It is remarkable how well the two ansätze above make it possible to identify Fourier expansions of the Lyapunov families: the regularity ansatz makes the system of equations block-triangular, while the symmetry ansatz makes the solution unique for a given choice of group.

Notations

$$e^{i2\pi\frac{j}{N}} - e^{i2\pi\frac{l}{N}} = \vec{r}_{jl} = u_{jl} + iv_{jl} = \rho_{jl}e^{i4\pi\theta_{jl}}, \quad -(2pk\eta-1)j = j[p].$$

$$A_{jl} = A_0\rho_{jl}, \quad B_{jl} = \frac{1}{A_{jl}}\sin^2 2\pi(k\eta\frac{j-l}{2N}), \quad \Theta_{jl;p} = \theta_{jl} - \theta_{j[p]l[p]}.$$

Finally, we recall the restrictions on the values of p and q which come into the sums; in particular, $\sum_{p=\pm1}$ will mean the sum restricted to those values of $p=\pm1$ such that $2pk\eta-1\neq 0 \pmod N$.

Mutual distances With the notations above,

$$h_j - h_l = e^{i(\omega_1+\tilde{\varpi})t}\sum_p a^0_{r-2ps}e^{-i4\pi pt}\vec{r}_{j[p]l[p]}\ ;$$

$$z_j - z_l = \sum_{q\geq 0} c_{2q+1}\left[\cos 2\pi(2q+1)(t+k\eta\frac{j}{N}) - \cos 2\pi(2q+1)(t+k\eta\frac{l}{N})\right].$$

Mod $O(\epsilon^4)$ we obtain

$$||h_j - h_l||^2 = (A_0\rho_{jl})^2 + 2A_0\left[\rho_{jl}\alpha + \sum_{p=\pm1} A_{2p} < \vec{r}_{jl}, e^{-i4\pi pt}\vec{r}_{j[p]l[p]} >\right]\epsilon^2;$$

$$|z_j - z_l|^2 = \left[\cos 2\pi(t + k\eta\frac{j}{N}) - \cos 2\pi(t + k\eta\frac{l}{N})\right]^2 \epsilon^2,$$

that is

$$||h_j - h_l||^2 = (A_0\rho_{jl})^2 + 2A_0\rho_{jl}\left[\alpha + \sum_{p=\pm1} A_{2p}\rho_{j[p]l[p]}\cos 4\pi(pt + \theta_{jl} - \theta_{j[p]l[p]})\right]\epsilon^2 \;;$$

$$|z_j - z_l|^2 = 4\sin^2 2\pi(t + k\eta\frac{j+l}{2N})\sin^2 2\pi(k\eta\frac{j-l}{2N})\epsilon^2,$$

which, using the notations above, becomes

$$||h_j - h_l||^2 = A_{jl}^2\left[1 + \frac{2}{A_{jl}}\left(\alpha + \sum_{p=\pm1} A_{2p}\rho_{j[p]l[p]}\cos 4\pi(pt + \Theta_{jl;p})\right)\epsilon^2\right];$$

$$|z_j - z_l|^2 = 2A_{jl}B_{jl}\left(1 - \cos 4\pi(t + k\eta\frac{j+l}{2N})\right)\epsilon^2.$$

Finally, $\|x_j - x_l\|^{-3}$ equals

$$A_{jl}^{-3}\left[1 - \frac{3}{A_{jl}}\left(\begin{array}{l} \alpha + \sum_{p=\pm1} A_{2p}\rho_{j[p]l[p]}\cos 4\pi(pt + \Theta_{jl;p}) + \\ B_{jl}\left(1 - \cos 4\pi(t + k\eta\frac{j+l}{2N})\right) \end{array}\right)\epsilon^2\right] + O(\epsilon^4).$$

We will now plug in the obtained expressions into the equations of motion

$$\ddot{h}_j = \sum_{l\neq j}\frac{h_l - h_j}{||x_l - x_j||^3}, \quad \ddot{z}_j = \sum_{l\neq j}\frac{z_l - z_j}{||x_l - x_j||^3}.$$

Horizontal equations Recall that

$$h_j = e^{i(\omega_1 + \tilde{\varpi})t}\sum_{p=0,\pm1} a^0_{r-2ps} e^{i2\pi(-2pt + \frac{j[p]}{N})} + O(\epsilon^4),$$

hence

$$\ddot{h}_j = e^{i(\omega_1 + \tilde{\varpi})t}\sum_{p=0,\pm1}\left(\begin{array}{c} a^0_{r-2ps} e^{i2\pi(-2pt + \frac{j[p]}{N})}\times \\ \left[-(\omega_1 + \tilde{\varpi})^2 + 8\pi p(\omega_1 + \tilde{\varpi}) - 16\pi^2 p^2\right] \end{array}\right) + O(\epsilon^4),$$

that is

$$e^{-i(\omega_1 + \tilde{\varpi})t}\ddot{h}_j = -(A_0 + \alpha\epsilon^2)(\omega_1 + \gamma\epsilon^2)^2 e^{i2\pi\frac{j}{N}}$$

$$+ \sum_{p=\pm1} A_{2p}\left[-(\omega_1 + \gamma\epsilon^2)^2 + 8\pi p(\omega_1 + \gamma\epsilon^2) - 16\pi^2 p^2\right] e^{i2\pi(-2pt + \frac{j[p]}{N})}\epsilon^2 + O(\epsilon^4),$$

or

$$\begin{array}{rcl} e^{-i(\omega_1 + \tilde{\varpi})t}\ddot{h}_j & = & -\omega_1^2 A_0 e^{i2\pi\frac{j}{N}} + \\ & & \left[\begin{array}{l} -\omega_1(2A_0\gamma + \omega_1\alpha)e^{i2\pi\frac{j}{N}} + \\ \sum_{p=\pm1} A_{2p}\left[-\omega_1^2 + 8\pi p\omega_1 - 16\pi^2 p^2\right] e^{i2\pi(-2pt + \frac{j[p]}{N})} \end{array}\right]\epsilon^2 + \\ & & O(\epsilon^4), \end{array}$$

Finally, as

$$e^{-i(\omega_1+\tilde{\varpi})t}(h_l-h_j)=(A_0+\alpha\epsilon^2)\vec{r}_{lj}+\sum_{p=\pm 1}A_{2p}\epsilon^2 e^{-i4\pi pt}\vec{r}_{l[p]j[p]}+O(\epsilon^4),$$

the horizontal part of the equations splits into the 0-th order equation, which is nothing but the equation satisfied by the relative equilibrium:

$$-\omega_1^2 A_0 e^{i2\pi\frac{j}{N}}=\sum_{l\neq j}\frac{A_0}{(A_0\rho_{jl})^3}\vec{r}_{lj}\ ,$$

and the second order equation

$$-\omega_1(2A_0\gamma+\omega_1\alpha)e^{i2\pi\frac{j}{N}}+\sum_{p=\pm 1}A_{2p}\left[-\omega_1^2+8\pi p\omega_1-16\pi^2p^2\right]e^{i2\pi(-2pt+\frac{j[p]}{N})}=$$

$$\sum_{l\neq j}A_{lj}^{-3}\left(\alpha\vec{r}_{lj}+\sum_{p=\pm 1}A_{2p}e^{-i4\pi pt}\vec{r}_{l[p]j[p]}\right)+$$
$$-3A_0\sum_{l\neq j}A_{jl}^{-4}\begin{pmatrix}\alpha+\sum_{p=\pm 1}A_{2p}\rho_{j[p]l[p]}\cos 4\pi(pt+\Theta_{jl;p})+\\ B_{jl}\left(1-\cos 4\pi(t+k\eta\frac{j+l}{2N})\right)\end{pmatrix}\vec{r}_{lj}.$$

We will call (H_j) this equation.

Now, because of the symmetry ansatz, the N equations (H_j) are equivalent to one of them, for example (H_0), which is of the following form

$$U+Ve^{i4\pi t}+We^{-i4\pi t}=0,$$

where the complex numbers U,V,W are affine functions of the 4 real unknowns

$$\alpha,\ A_2,\ A_{-2},\ \gamma.$$

Moreover, it turns out that, because of the invariance of the expressions $A_{0l},B_{0l},\cdots$ under the change of l into $-l$, the coefficients of U,V,W are indeed real. As (H_0) has to be satisfied for all values of t, it is equivalent to the three real affine equations

$$U=V=W=0.$$

Vertical equations At the order of approximation $O(\epsilon^4)$, we have

$$z_j(t)=\epsilon(\cos 2\pi(t+k\eta\frac{j}{N})+C_3\epsilon^3\cos 6\pi(t+k\eta\frac{j}{N})+O(\epsilon^5),$$

hence

$$z_l-z_j=\epsilon\left[u_{jl}\sin 2\pi(t+k\eta\frac{j+l}{2N})+C_3v_{jl}\epsilon^2\sin 6\pi(t+k\eta\frac{j+l}{2N})+O(\epsilon^4)\right],$$

where

$$u_{jl} = -2\sin 2\pi(k\eta\frac{j-l}{2N}), \quad v_{jl} = -2\sin 6\pi(k\eta\frac{j-l}{2N}).$$

The vertical equation (V_j) is obtained by identifying the ϵ^2-terms in the identity (mod $O(\epsilon^4)$)

$$-4\pi^2\cos 2\pi(t+k\eta\frac{j}{N}) - 36\pi^2 C_3\epsilon^2\cos 6\pi(t+k\eta\frac{j}{N}) =$$

$$\sum_{l\neq j}||x_l-x_j||^{-3}\left(u_{jl}\sin 2\pi(t+k\eta\frac{j+l}{2N}) + C_3 v_{jl}\epsilon^2\sin 6\pi(t+k\eta\frac{j+l}{2N}) + O(\epsilon^4)\right)$$

The identification of the terms of order 0 give the vertical variational equation (VVE) which is already satisfied by $z_j(t) = \cos 2\pi(t+k\eta\frac{j}{N})$.
The identification of the terms of order 2 gives the two remaining relations beween the five unknowns $\alpha, A_2, A_{-2}, C_3, \gamma$. They are of the form

$$\mathrm{Re}\,(X_j e^{i2\pi t}) = 0, \quad \mathrm{Re}\,(Y_j e^{i6\pi t}) = 0,$$

where X_j is an affine function of α, A_2, A_{-2} and Y_j is an affine function of A_2, A_{-2}, C_3. As above, the symmetry ansatz implies that the N equations (V_j) are equivalent to one of them, for example (V_0) and the invariance of the expressions $A_{0l}, B_{0l}, \cdots$ under the change of l into $-l$ implies that X_0 and Y_0 are real, hence that (V_0) reduces to exactly two real equations.

Identification of coefficients Finally, we have 5 real equations which are affine in the 5 unknowns

$$\alpha, A_2, A_{-2}, C_3, \gamma.$$

They can be solved in the following order (assuming non degeneracies which will prove true in the cases investigated below):

– In (H_0) the term in ϵ^0 depends only on A_0, which it determines.

– In (H_0) the term U in ϵ^2 constant with respect to time depends only on α and γ and can be used to eliminate α.

– In (H_0) the coefficients V and W of $\epsilon^2 e^{\pm i4\pi t}$ allow to eliminate $A_{\pm 2}$. (For the Eight families, it can help to remember or it can be checked, that $a_0^0 = A_2\epsilon^2 + ... = 0$.)

– In (V_0), the coefficient of $\epsilon^2 e^{i6\pi t}$ (after simplification by ϵ) allows to eliminate C_3. (In the case of the P_{12} family, $C_3 = 0$.)

– In (V_0), the coefficient of $\epsilon^2 e^{i2\pi t}$ eventually allows to compute γ.

Remark The fact that, in the rotating frame, the horizontal period is half the vertical one follows from the invariance of the problem under the symmetry with respect to the horizontal plane (materialized in the action of β). This fact accounts for the generic aspect displayed by the solutions in a frame which accompanies the rotation of the regular N-gon, as exemplified on figure 10. Notice that the same type of behaviour, with the same explanation, is observed in the restricted problem for the vertical solutions originating from the Lagrange points (see [Za]).

6.4 Torsion of the first cases

Recall that we are computing the torsion of the Lyapunov families which bifurcate at $\varpi = -\frac{r}{s}\omega_k + \omega_1$. Since $\omega_1 > 0$, γ is necessarily ≥ 0.

Three bodies

The family of rotated Lagrange solutions $(S_2(3,1,-1))$

$$\gamma = 0 \quad \text{as expected.}$$

The P_{12}-family $(S_2(3,1,-1))$ $\quad \gamma = \frac{12}{19}\left(6\pi^7\right)^{1/3} \simeq 16.589$

This value is consistent with Marchal's computation [Ma1]. Indeed, writing with primes quantities introduced in his book, he finds that the frequency shift of the P_{12}-family is

$$\frac{\alpha'}{2\pi} + h.o.t. = \frac{3}{19}c_1'^2 + h.o.t.,$$

to be compared with our $\gamma\epsilon^2$. But $\omega_1' = 1$ while $\omega_1 = 2\pi$. Hence

$$2\pi\frac{3}{19}c_1'^2 = \gamma\epsilon^2$$

(that Marchal's masses are not the same as ours does not come into play in this equality between frequencies). Besides, his c_1' is the amplitude of oscillations of $z_1 - z_2$ at the first order and the edges of the equilateral triangle he considers have length 1. On the other hand, our ϵ is the amplitude of oscillations of z_1 and we consider a triangle whose edges have length $a_0\sqrt{3}$, with $a_0^{-3} = 4\sqrt{3}\pi^2$. Hence

$$\left(c_1'/\tfrac{1}{\sqrt{3}}\right) : \tfrac{1}{\sqrt{3}} = \epsilon : a_0,$$

whence the above expression of γ.

Four bodies

The 4-body, 3-lobe chain family $(S_2(4,1,-1))$

$$\gamma = \frac{48\,\pi^{7/3}}{41 \times 2^{1/3}7^{2/3}}\left(9\sqrt{2} - 11\right)\left(2\sqrt{2}+1\right)^{2/3} > 9$$

The 4-body Hip-Hop family $(S_1(4,2,\pm 1))$

$$\gamma = \frac{4692\,\pi^{7/3}}{2446096 \times 2^{1/4}\sqrt{2\sqrt{2}+1}}\left(440\sqrt{2}+981\right) > 19$$

Five bodies

The 5-body, 4-lobe chain family ($S_4(5, 1, -1)$)

$$\gamma = \frac{6\,2^{\frac{1}{3}}\,\left(17995 - 7823\,\sqrt{5}\right)\,\pi^{\frac{7}{3}}}{5981 \times 5^{\frac{1}{3}}} > 5.$$

The 5-body Eight family ($S_2(5, 2, -1)$)

$$\gamma = \frac{\pi^{\frac{7}{3}}(1+\sqrt{5})^{1/6}(1+3\sqrt{5})^{1/6}}{2^{11} \times 5^{1/3} \times 11^{55/6}\,761\,809} \times$$

$$\left(\begin{array}{l} \left(230356305630646272\,\sqrt{5} + 954953942246092800\right)\sqrt{\sqrt{5}-1}\,\sqrt{3\,\sqrt{5}-1} \\ +10427244637440119808\,\sqrt{5} - 35321080493656320000 \end{array} \right) > 19.$$

The 5-body, 3-lobe chain family ($S_3(5, 2, 1)$)

$$\begin{aligned} \gamma \quad = \quad & \frac{\pi^{\frac{7}{3}}\left(\sqrt{5}+1\right)^{\frac{1}{6}}\left(3\,\sqrt{5}-1\right)^{\frac{1}{6}}}{69129994912 \times 5^{1/3} \times 11^{11/6}} \times \\ & \left(\begin{array}{l} \left(522541527\,\sqrt{5} - 177004875\right) - \\ \left(11543868\,\sqrt{5} + 47855700\right)\sqrt{\sqrt{5}-1}\,\sqrt{3\,\sqrt{5}-1} \end{array} \right) > 12 \end{aligned}$$

Six bodies

The 6-body, 5-loop chain ($S_5(6, 1, -1)$)

$$\gamma = \frac{48\,\pi^{7/3}}{781199 \times 6962^{1/3}}(32634\sqrt{3} - 39889)(15 - 4\sqrt{3})^{2/3} > 3$$

A 6-body Hip-Hop ($S_1(6, 2, -1)$) $\quad \gamma > 14$ (long expression)

Acknowledgments. The authors warmly thank Carles Simó for his invaluable advice on numerical computations, and Richard Montgomery for his comments and questions. Thanks also to Michel Hénon who made us aware of his researches [H] on families of spatial periodic solutions of the restricted circular three-body problem which bifurcate from planar solutions. In the same way as his families were continued to the full problem, it would be natural to continue ours by varying the masses.

References

[AC] A. Albouy and A. Chenciner, Le problème des n corps et les distances mutuelles, *Inventiones Mathematicæ* (1998) **131** 151–184

[AY] J. C. Alexander and J. Yorke, Global bifurcations of periodic orbits, *Amer. J. Math.* **100:2** (1978), 263–292

[BT] V. Barutello and S. Terraccini, Action minimizing orbits in the n-body problem with simple choreography constraint, preprint (2007)

[BCPS] E. Barrabés, J. M. Cors, C. Pinyol y J. Soler , Hip-Hop solutions of the 2N-body problem, *Celestial Mech. Dynam. Astronom.* (2006) **95** 55–66

[C1] A. Chenciner, Simple non-planar periodic solutions of the n-body problem *NDDS Conference, Kyoto* (August 2002)

[C2] A. Chenciner, Symmetric relative equilibria as absolute minimizers (variations on a theorem of V. Barutello and S. Terracini), *manuscript* (December 2007)

[CF1] A. Chenciner and J. Féjoz, L'équation aux variations verticales d'un équilibre relatif comme source de nouvelles solutions périodiques du problème des N corps, *C. R. Math. Acad. Sci. Paris* **340:8** (2005) 593–598.

[CF2] A. Chenciner and J. Féjoz, The flow of the equal-mass spatial 3-body problem in the neighborhood of the equilateral relative equilibrium, *Discrete and Continuous Dynamical Systems, Series B*, Special Issue dedicated to Carles Simó on the occasion of his 60th anniversary 10:2-3 (2008) 421–438

[CFM] A. Chenciner, J. Féjoz and R. Montgomery, Rotating Eights I: the three Γ_i families, *Nonlinearity* **18** (2005) 1407–1424

[CGMS] A. Chenciner, J. Gerver, R. Montgomery and C. Simó, Simple choreographic motions of N bodies: a preliminary study, *Geometry, mechanics, and dynamics*, 287–308, Springer, New York, 2002

[CM] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* **152:3** (2000) 881–901

[CV] A. Chenciner & A. Venturelli, Minima de l'intégrale d'action du Problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites "hip-hop", *Celestial Mech. Dynam. Astronom.* (2000) **77** 139–152

[CMP] S. N. Chow and J. Mallet-Paret, The Fuller index and global Hopf bifurcation, *J. Differential Equations* **29:1** (1978), 66–85.

[FT] D. Ferrario and S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invent. Math.* **155:2** (2004), 305–362

[GSL] M. Galassi et al, GNU Scientific Library Reference Manual (2nd Ed.), `http://www.gnu.org/software/gsl/`

[HNW] E. Hairer, S. P. Nørsett & G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer, 1993

[H] M. Hénon, Vertical stability of periodic orbits in the restricted problem. 1 and 2, *Astron. Astrophys.* **28** (1973) 415–426 and **30** (1974) 317–321

[Ma1] C. Marchal, *The 3-body problem*, Elsevier, 1990, paragraph 10.8.2. Russian translation, Institut Kompiuternikh issliedovannii, Moskva, Ijevsk (2004)

[Ma2] Marchal C., The family P_{12} of the three-body problem. The simplest family of periodic orbits with twelve symmetries per period, *Celestial Mech. Dynam. Astronom.* **78** (2000) 279–298

[Mo] R. Moeckel, Linear stability analysis of some symmetrical classes of relative equilibria, *Hamiltonian dynamical systems (Cincinnati, OH, 1992)*, 291–317, IMA Vol. Math. Appl., **63**, Springer, New York, 1995

[MS] K. Meyer & D. Schmidt, Libration of central configurations and braided Saturn rings, *Celestial Mech. and Dynamical Astronomy* **55** (1993) 289–303

[Mos] J. Moser, Periodic orbits near an equilibrium and a theorem by Alan Weinstein, *Comm. Pure Appl. Math.* **29:6** (1976) 724–747

[PW] L. M. Perko & E. L. Walter, Regular polygon solutions of the n-body problem, *Proc. AMS* 94:2 (1985) 301–309

[S] C. Simó, *New families of Solutions in N-Body Problems*, Progr. Math. **201** (2001) 101–115

[TV] S. Terracini and A. Venturelli, Symmetric trajectories for the $2N$-body problem with equal masses, *Arch. Ration. Mech. Anal.* **184:3** (2007) 465–493

[Za] C.G. Zagouras, Three-dimensional periodic orbits about the triangular equilibrium points of the restricted problem of three bodies, *Celestial Mech. Dynam. Astronom.* **37** (1985) 27-46