Inclined co-orbitals in the three-body problem

From Marchal's family to P_{12}

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LTE Observatoire de Paris | PSL

Introduction

The three-body problem: generalities

- Problem definition: 3 masses and gravity

$$\mathcal{H}(\boldsymbol{u}, \tilde{\boldsymbol{u}}) = \underbrace{\frac{\tilde{\boldsymbol{u}}_{\boldsymbol{0}}^{\ 2}}{2M} + \sum_{i=1}^{2} \left(\frac{m_0 + m_i}{2 m_0 m_i} \tilde{\boldsymbol{u}}_{\boldsymbol{i}}^{\ 2} - \mathcal{G} \frac{m_0 m_i}{|\boldsymbol{u}_{\boldsymbol{i}} - \boldsymbol{u}_{\boldsymbol{0}}|} \right)}_{\mathcal{H}_0} + \underbrace{\underbrace{\frac{\tilde{\boldsymbol{u}}_{\boldsymbol{1}} . \tilde{\boldsymbol{u}}_{\boldsymbol{2}}}{m_0}}_{T_1} \underbrace{- \mathcal{G} \frac{m_1 m_2}{|\boldsymbol{u}_{\boldsymbol{2}} - \boldsymbol{u}_{\boldsymbol{1}}|}}_{U_1}}_{\mathcal{H}_1}$$

The three-body problem: generalities

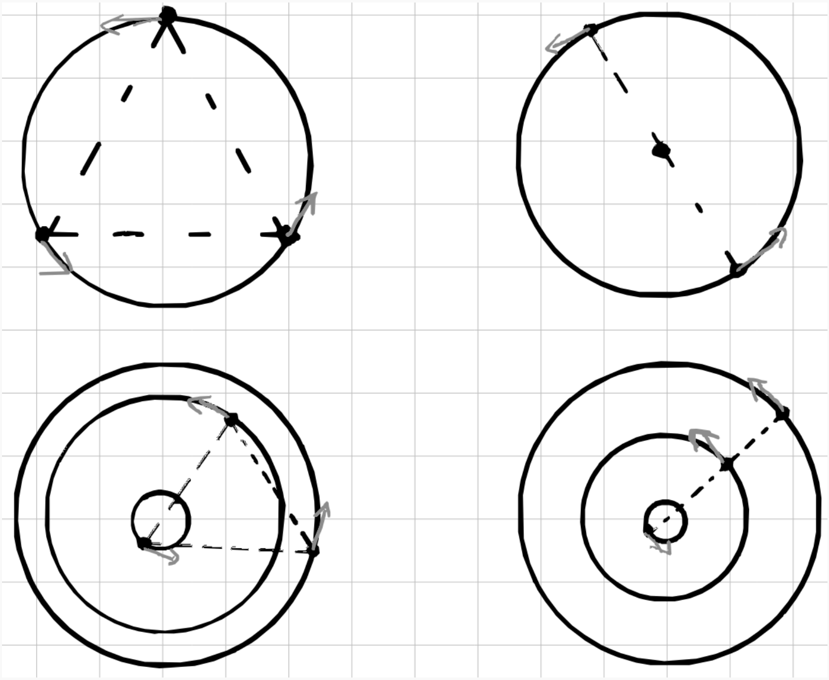


Figure 1: Lagrange (left) and Euler (right) periodic orbits.

The three-body problem: generalities

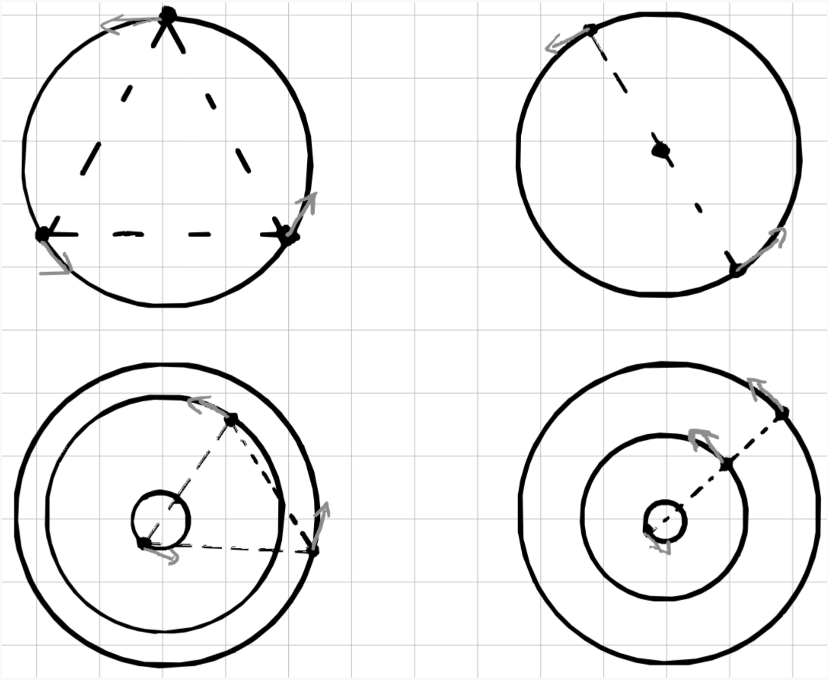
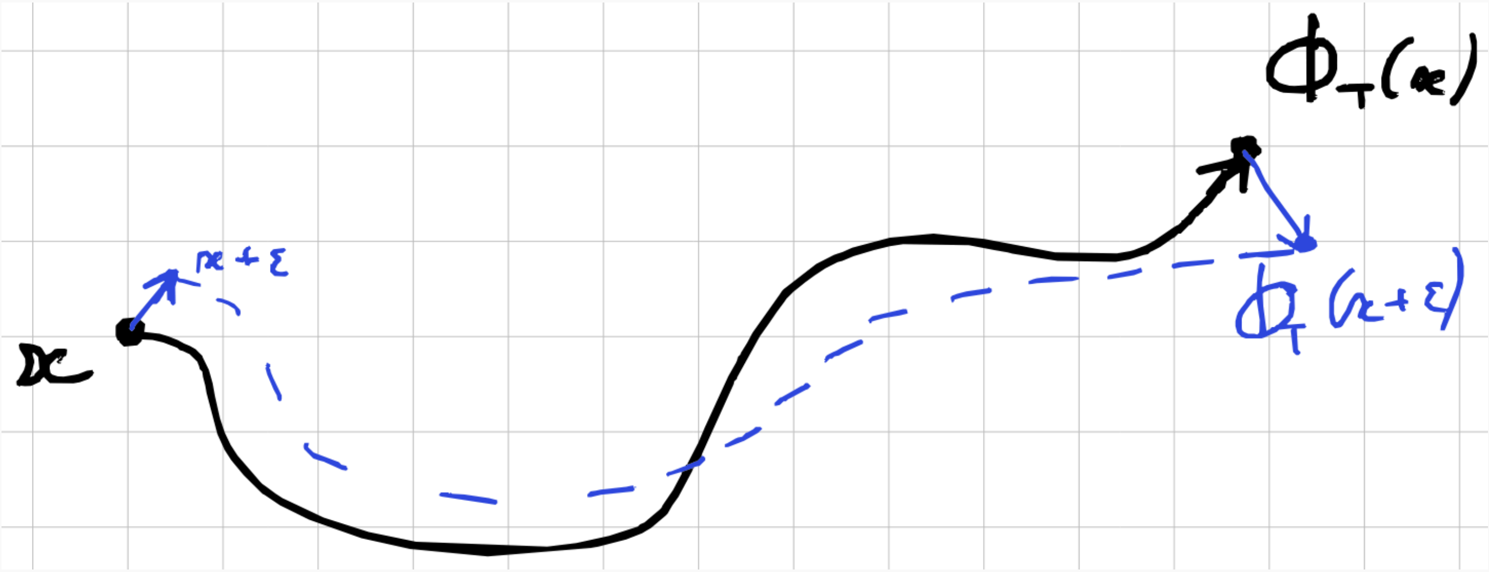


Figure 2: Lagrange (left) and Euler (right) periodic orbits.

Gascheau's criterium:

$$\frac{m_0 m_1 + m_0 m_2 + m_1 m_2}{(m_0 + m_1 + m_2)^2} > \frac{1}{27}$$

Some concepts



- $\boldsymbol{x}$ = (positions, momenta)
- $\Phi_T : \boldsymbol{x} \mapsto$ evolution after a time T
- Eigenvalues of $D\Phi_T$: describe how $\Phi_T(\boldsymbol{x} + \boldsymbol{\varepsilon})$ will behave

The eigenvalues of Lagrange for the planetary problem

Family	Ending
Homographics	Triple collisions

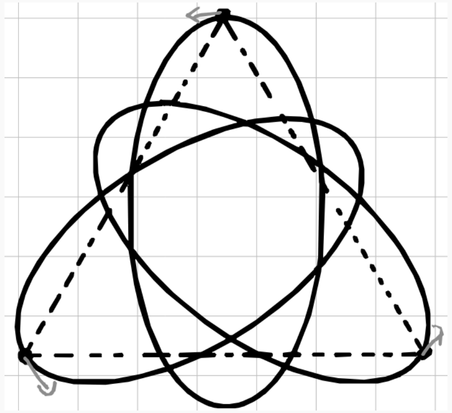


Figure 4: Homographic, anti-Lagrange, and tadpole/horseshoe orbits. Cf. (Robutel and Pousse, 2013).

The eigenvalues of Lagrange for the planetary problem

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Anti-Lagrange	Triple collisions (unproven)

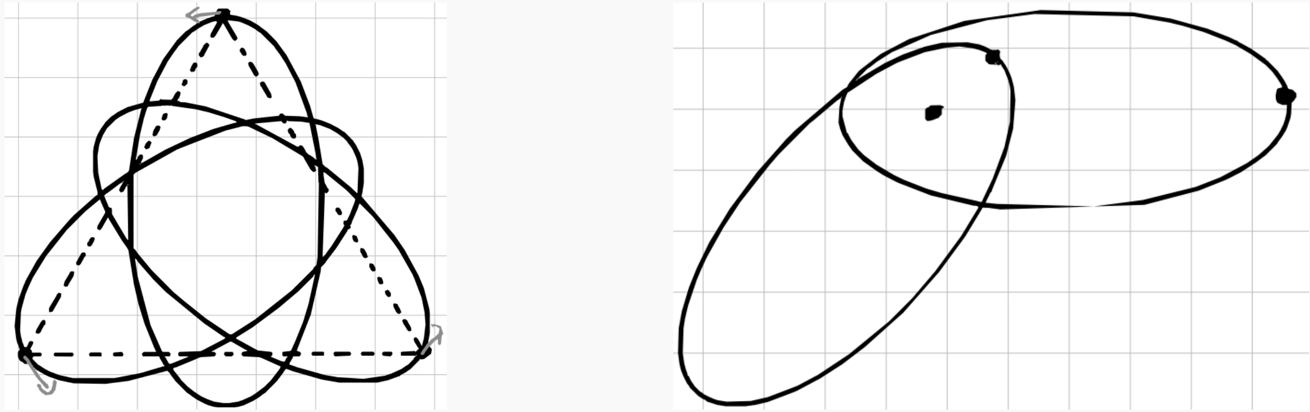

Figure 5: Homographic, anti-Lagrange, and tadpole/horseshoe orbits. Cf. (Robutel and Pousse, 2013).

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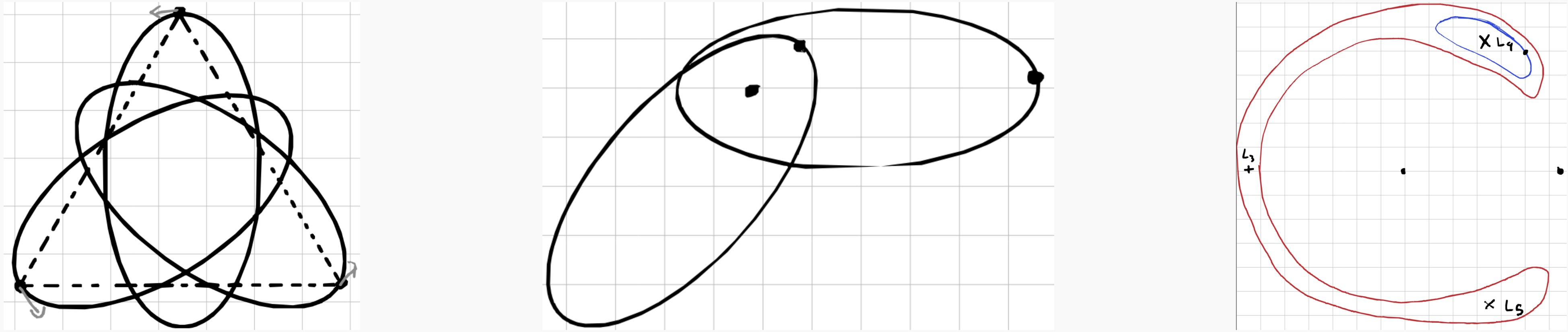


Figure 6: Homographic, anti-Lagrange, and tadpole/horseshoe orbits. Cf. (Robutel and Pousse, 2013).

The eigenvalues of Lagrange for the planetary problem

Family	Ending
Homographics	Triple collisions
Anti-Lagrange	Triple collisions (unproven)
Tadpole	Horseshoe orbits
Vertical	Unknown (depending on the masses)

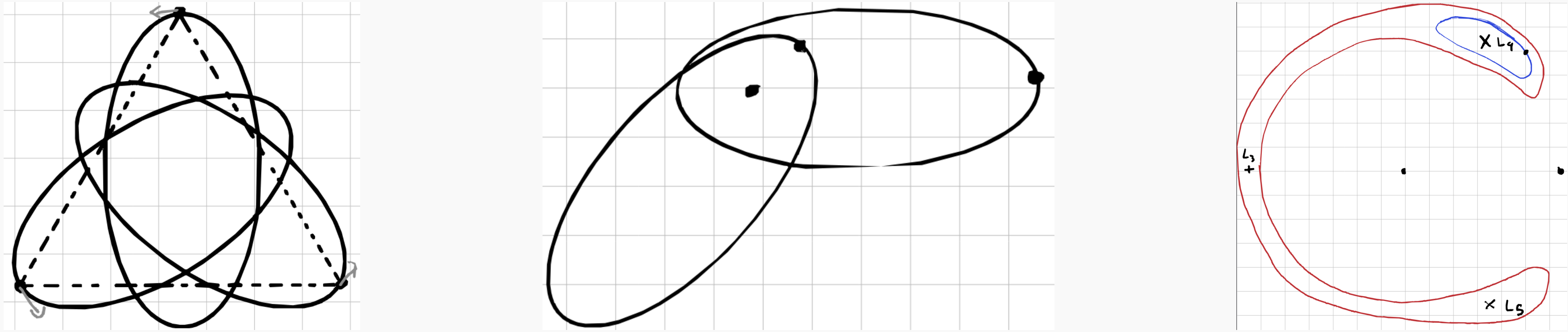


Figure 7: Homographic, anti-Lagrange, and tadpole/horseshoe orbits. Cf. (Robutel and Pousse, 2013).

The vertical family – equal masses: P_{12}

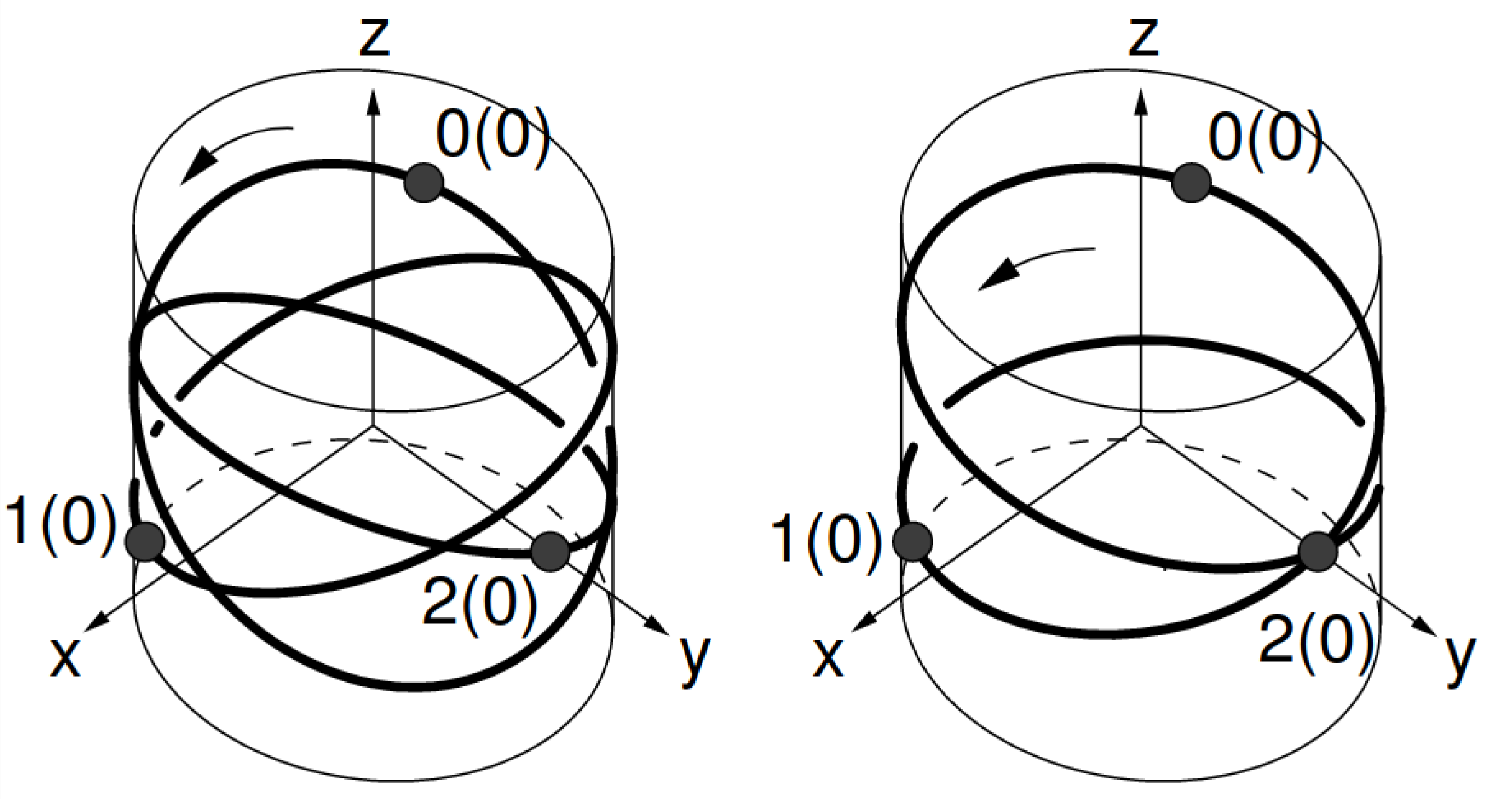


Figure 8: Figure 1 of (Chenciner and Fejoz, 2008). Also studied in (Chenciner, Féjoz and Montgomery, 2005; Chenciner and Fejoz, 2008; Calleja *et al.*, 2024)...

The vertical family – restricted problem: Marchal 2009

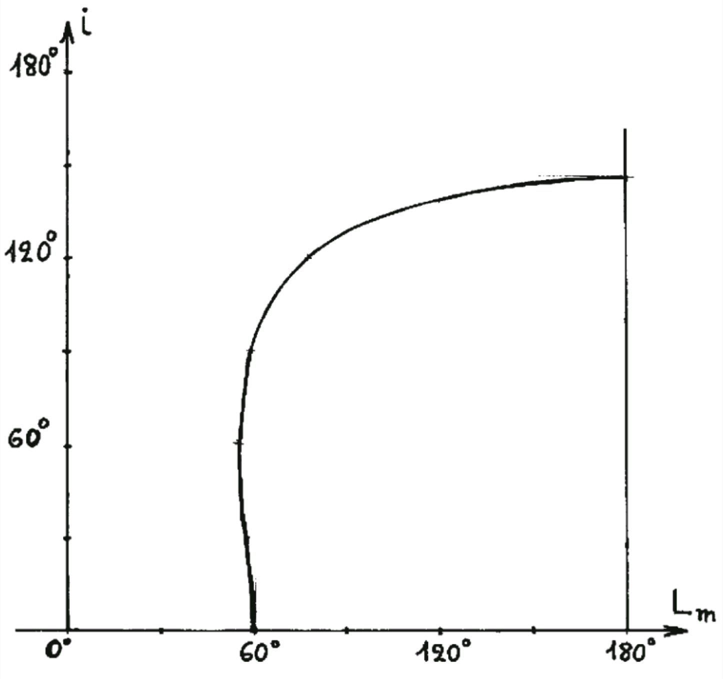


Figure 9: Fig. 3 of (Marchal, 2009) showing the vertical family in the restricted problem, with masses $(1 - \varepsilon, \varepsilon, 0)$.

The vertical family – restricted problem: Marchal 2009

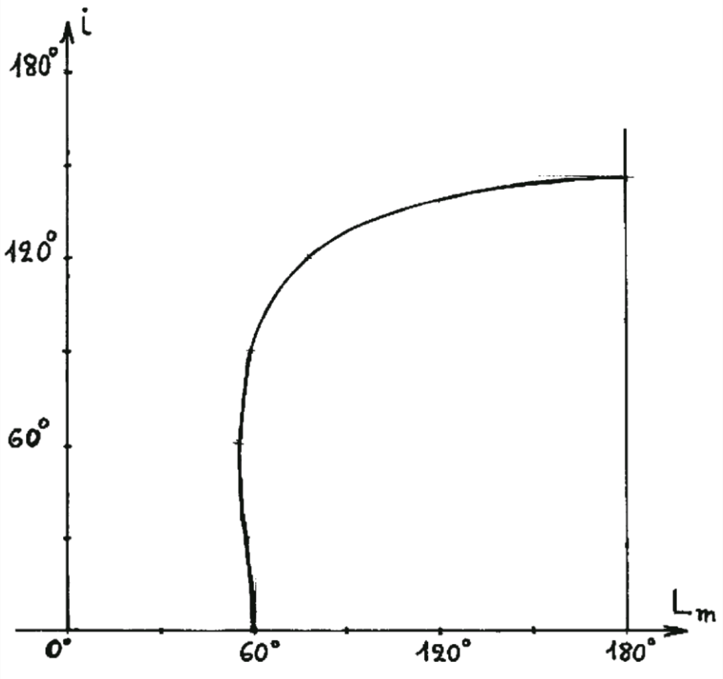


Figure 10: Fig. 3 of (Marchal, 2009) showing the vertical family in the restricted problem, with masses $(1 - \varepsilon, \varepsilon, 0)$.

"These periodic orbits belong to a three parameter family of similar periodic orbits of the general three body problem, two parameters being the mass ratios of the three bodies and the third a generalization of the inclination i. [...] The one parameter family P_{12} of Marchal (2000) is another limit case of this three parameter family: the case of three equal masses."

Agenda

1. Introduction
2. How to chase a quasi-periodic solution
3. Marchal's family for small masses
4. From Marchal to P_{12}: varying the masses
5. Further work

How to chase a quasi-periodic solution

Rephrasing the problem

18 dims

Start: 18 dimensional-problem, 3 positions+velocities for 3 bodies

Rephrasing the problem

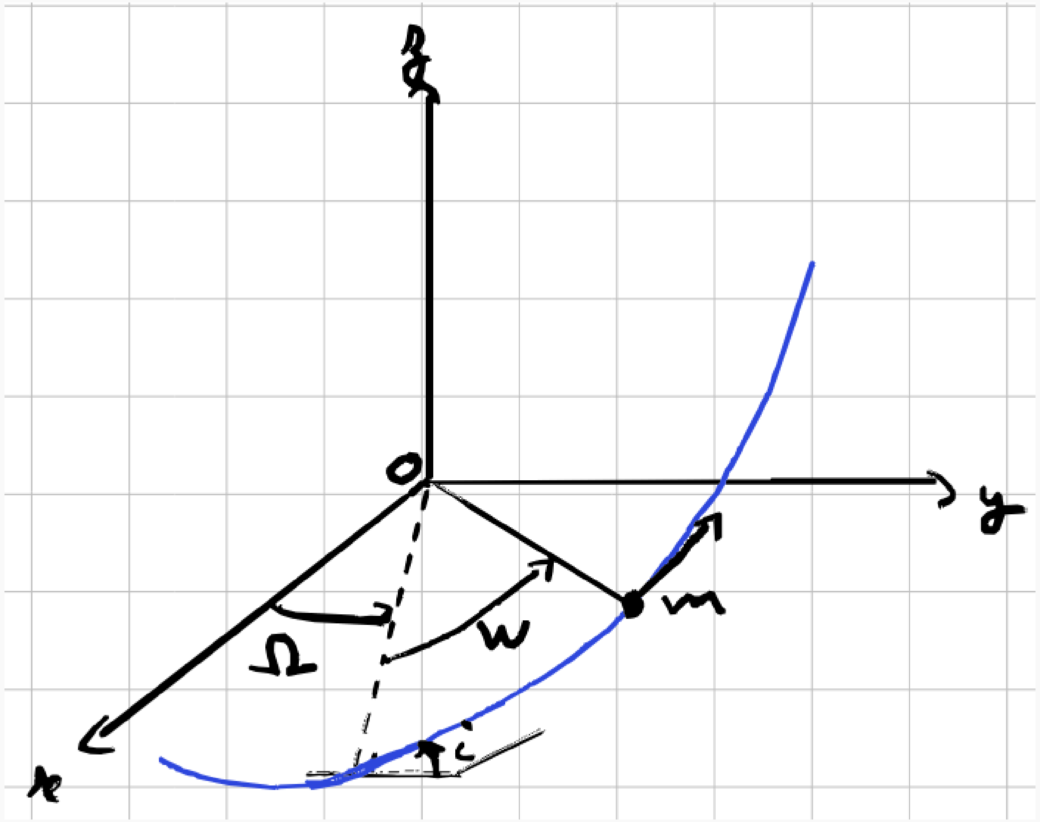
12 dims

Start: 18 dimensional-problem, 3 positions+velocities for 3 bodies

Fix one body at the origin: “lose one body” → 12 dimensions.

- Canonical heliocentric coordinates
  - Heliocentric positions, barycentric momenta

Rephrasing the problem



12 dims

Hill coordinates (Laskar, 2017)

- r_i radii, $\tilde{r}_i$ radial velocities
- w_i true anomalies, G_i angular momenta
- Ω_i node longitudes, $G_i \cos(i_i)$ vertical angular momenta
- C total angular momentum

Reduction of the nodes

$$\mathscr{H}\Big(\Omega_1 - \Omega_2, \cancel{\Omega_1 + \Omega_2}, \underbrace{i_1 + i_2}_{J}, \cancel{i_1 - i_2} \dots\Big)$$

12 dims

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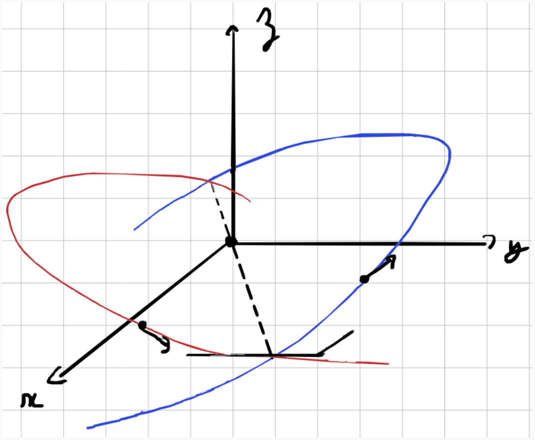
$$\mathscr{H}\left(\Omega_1 - \Omega_2, \cancel{\Omega_1 + \Omega_2}, \underbrace{i_1 + i_2}_{J}, \cancel{i_1 - i_2}\dots\right)$$

10 dims



Reduction of the nodes

9 dims

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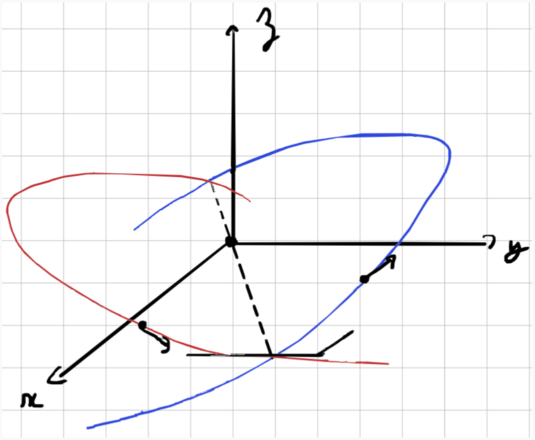

But $\Omega_1 - \Omega_2 = \pi$

We can gain 3 dimensions ! (Robutel, 1995; Mastroianni and Efthymiopoulos, 2019)

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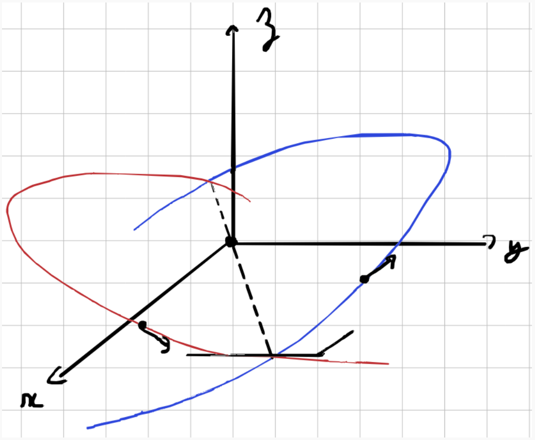

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Draconitic frame

- 9 dimensions : $(r_i, w_i, \tilde{r}_i, G_i)_{i=1,2}, C$

The numerical method

The problem :

- Fixed : masses, period (space-time invariance)
- 9-dimensional search space : 8 coordinates and C
- We are looking for periodic orbits... in a rotating frame

$$\Phi_T(\boldsymbol{x}) - \boldsymbol{x} = \boldsymbol{0} \rightarrow \text{Newton-Raphson algorithm}$$

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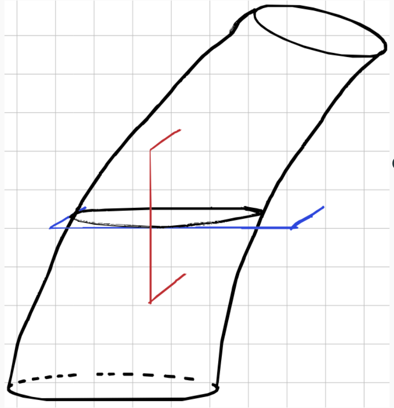

- Lack of local unicity: add sections
  - One for the family, one for the orbit

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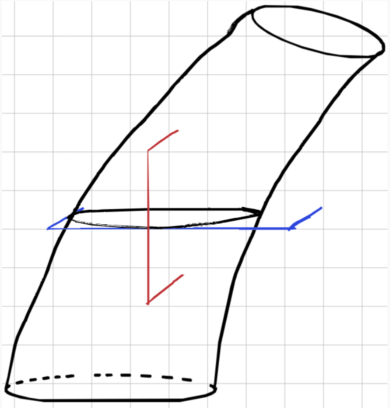
$$\begin{pmatrix} \Phi_T(\boldsymbol{x}) - \boldsymbol{x} \\ \sigma(\boldsymbol{x}) \end{pmatrix} = \mathbf{0} \rightarrow \text{Newton-Raphson algorithm}$$



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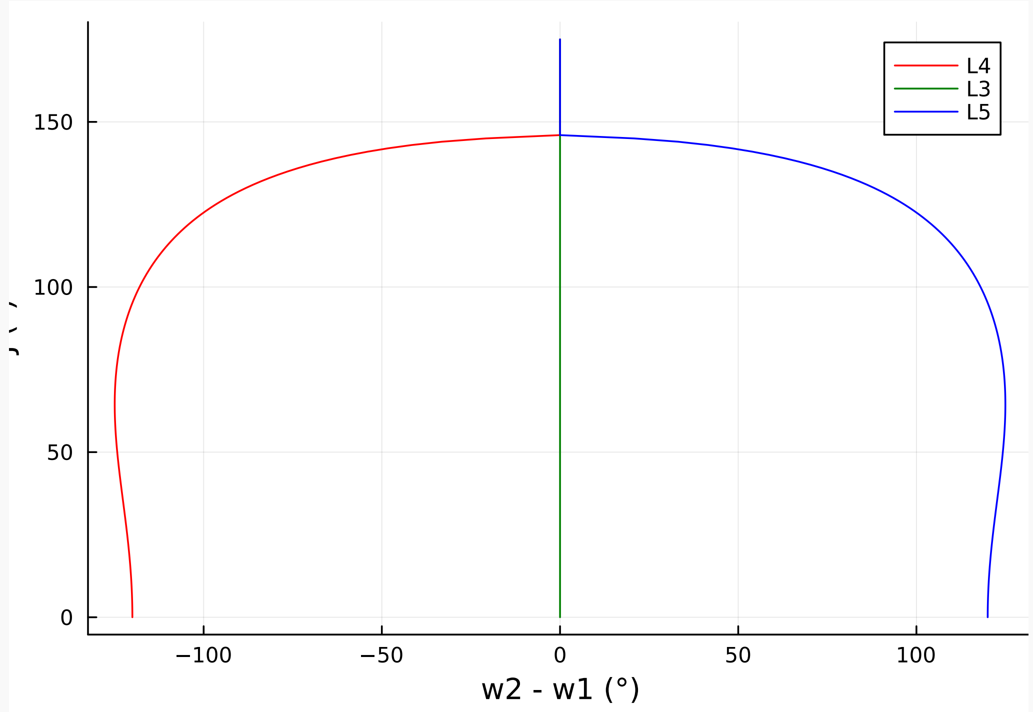
  - One for the family, one for the orbit

Marchal’s family for small masses

General properties

- Two small masses (10^{-3}) and one big mass ($1 - 2 * 10^{-3}$)



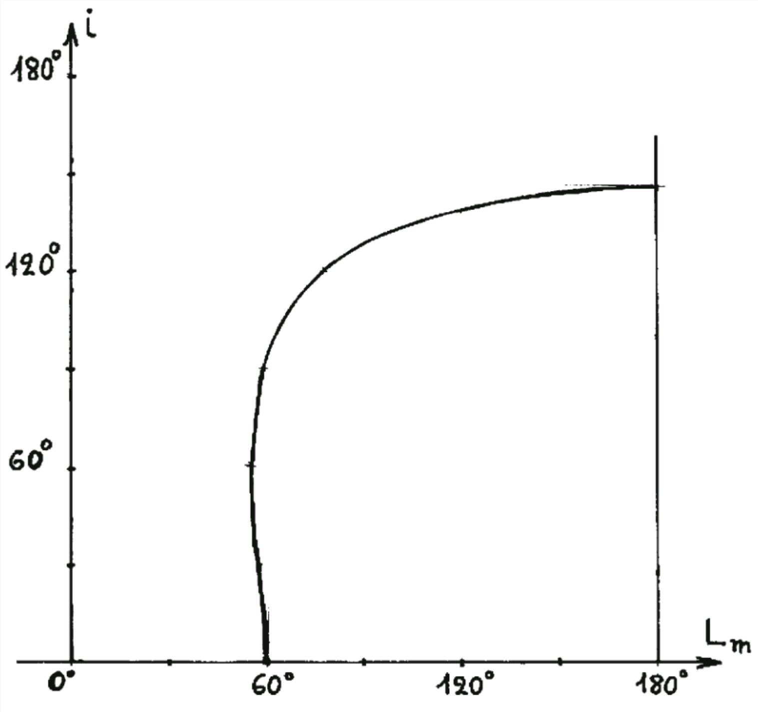


Figure 19: Lagrange and Euler families

Stability

- Linearly stable until ~ 60°

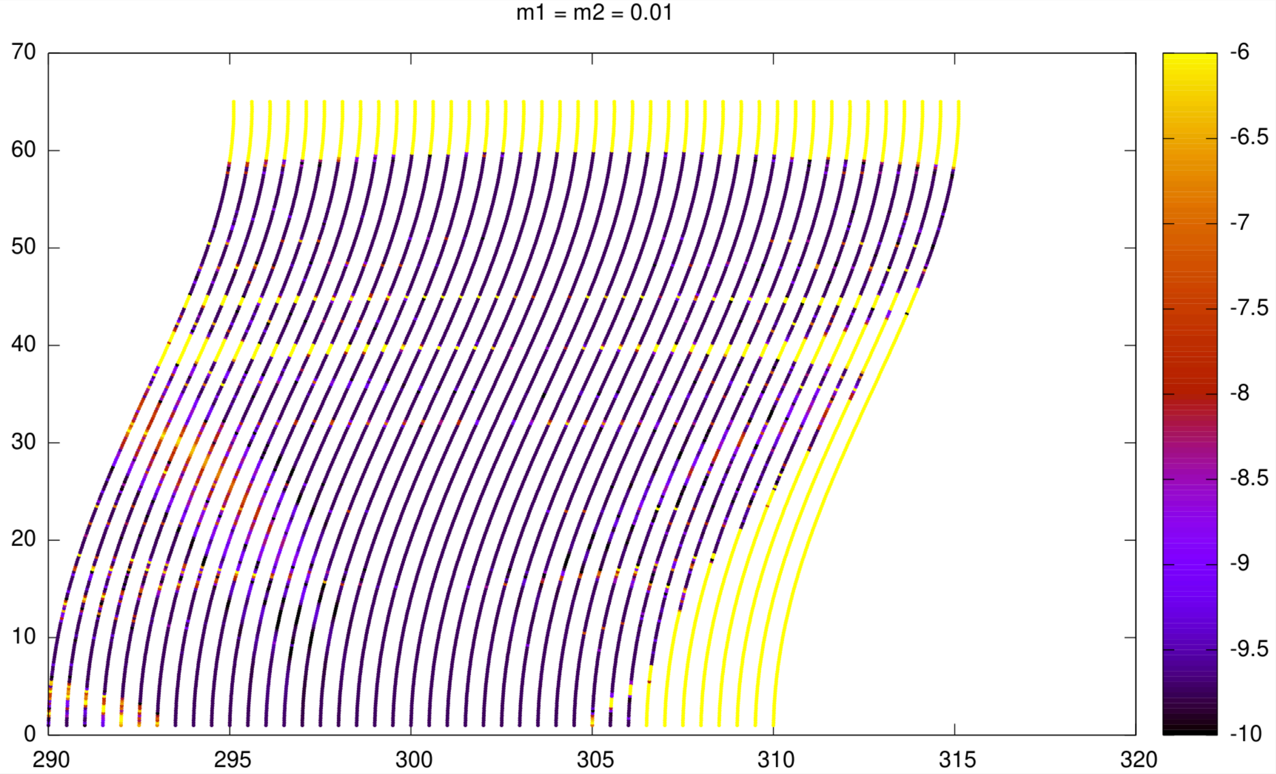


Figure 20: Full stability around the limit for $\varepsilon_m = 0.01$.

Symetries

$$\Phi_{\frac{T}{2}}(\boldsymbol{x}) = -\boldsymbol{x}$$

- Italian symetry
  - Cut integration time in half
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$$R_y(\pi)\boldsymbol{x} = \sigma_{1\leftrightarrow 2}(\boldsymbol{x})$$

- Second symetry
  - Fixes position on the orbit
  - Adds three more sections.

From Marchal to P_{12}: varying the masses

Stability: over Gascheau's limit

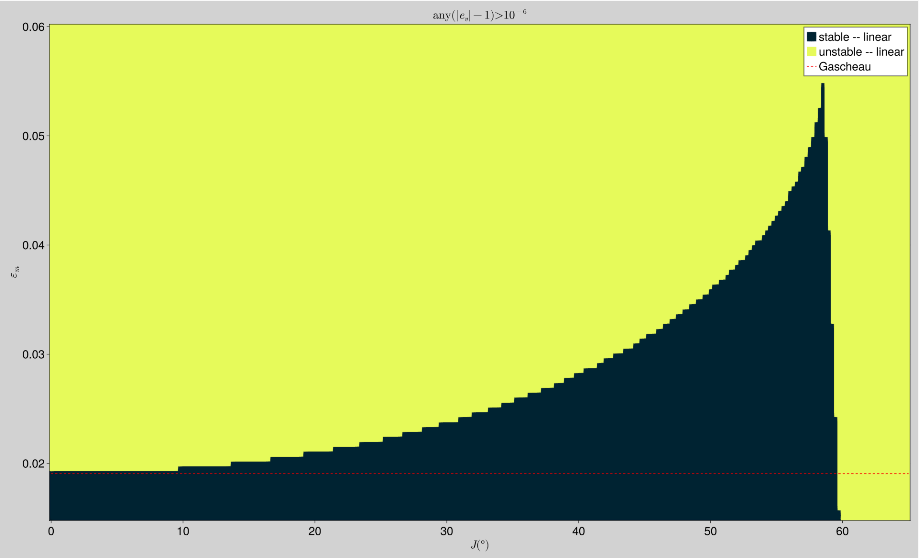


Figure 21: Linear and full stability along the family for various masses.

Stability: over Gascheau's limit

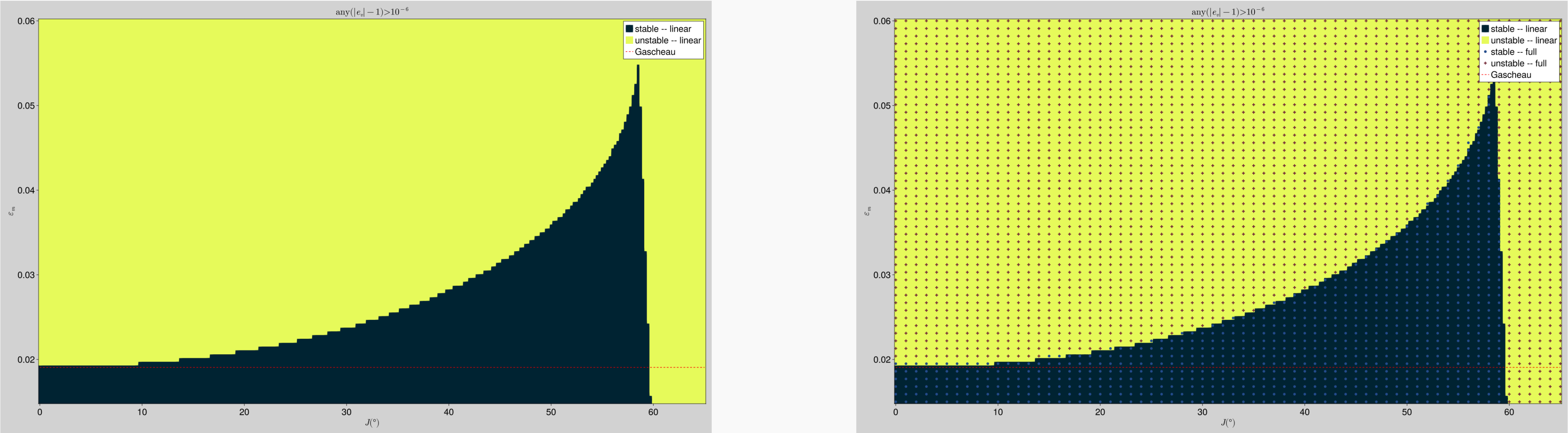


Figure 22: Linear and full stability along the family for various masses.

Stability: over Gascheau's limit

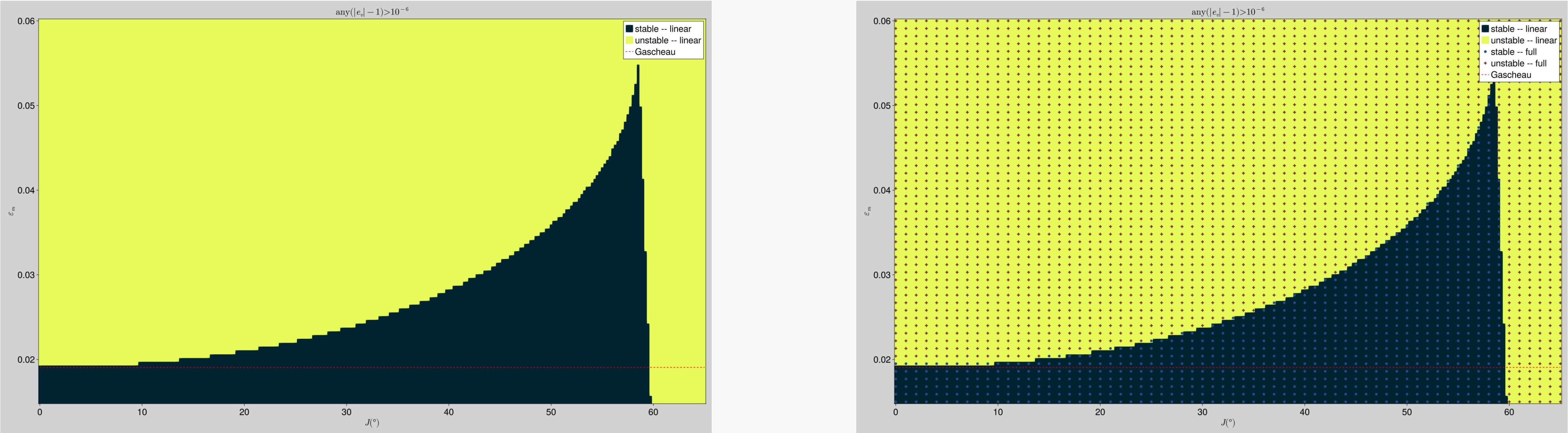


Figure 23: Linear and full stability along the family for various masses.

Similar results obtained for elliptic Lagrange (Roberts, 2002)

Approaching the equal masses

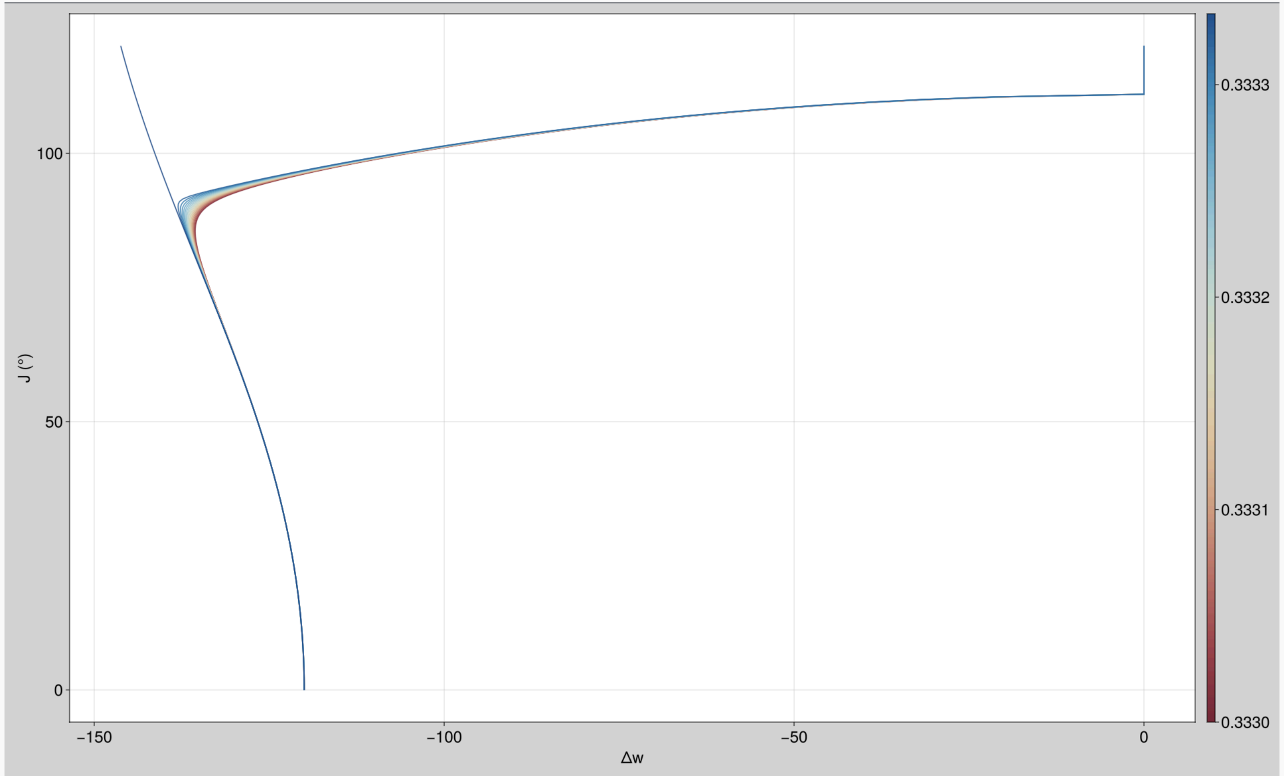


Figure 24: Start of Marchal family for $\varepsilon_m \in \left[0.3330, \frac{1}{3}\right]$ in the $(\Delta w, J)$ plane.

Approaching the equal masses

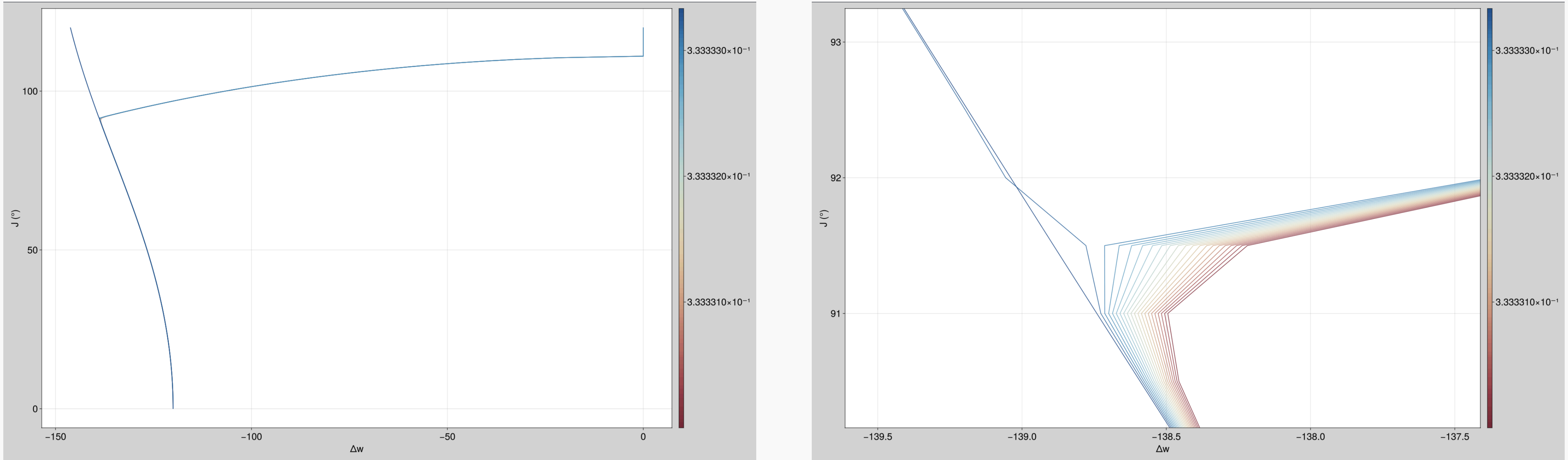


Figure 25: Zooming on the bifurcation for $\varepsilon_m \in \left[0.333330; \frac{1}{3}\right]$.

P_{12}: the bifurcations

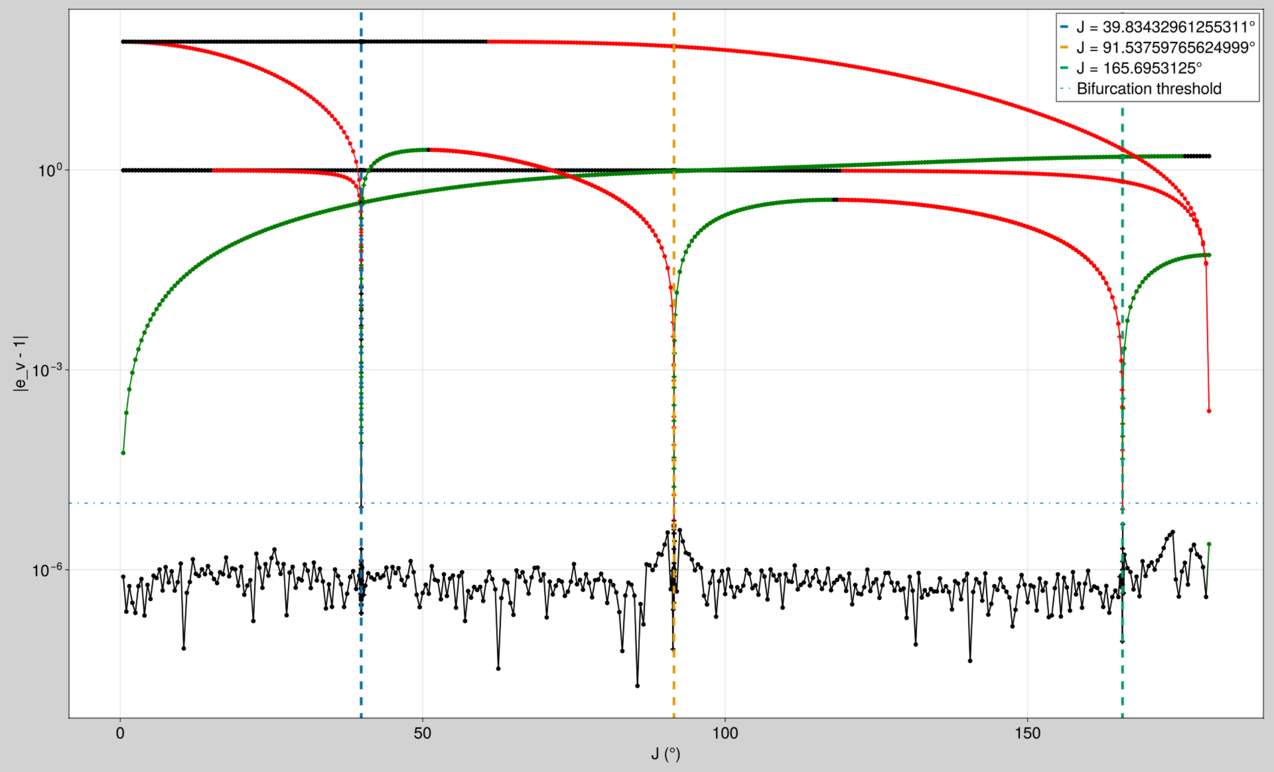


Figure 26: Distance to 1 of monodromy matrix eigenvalues along P_{12}.

Marchal and P_{12}

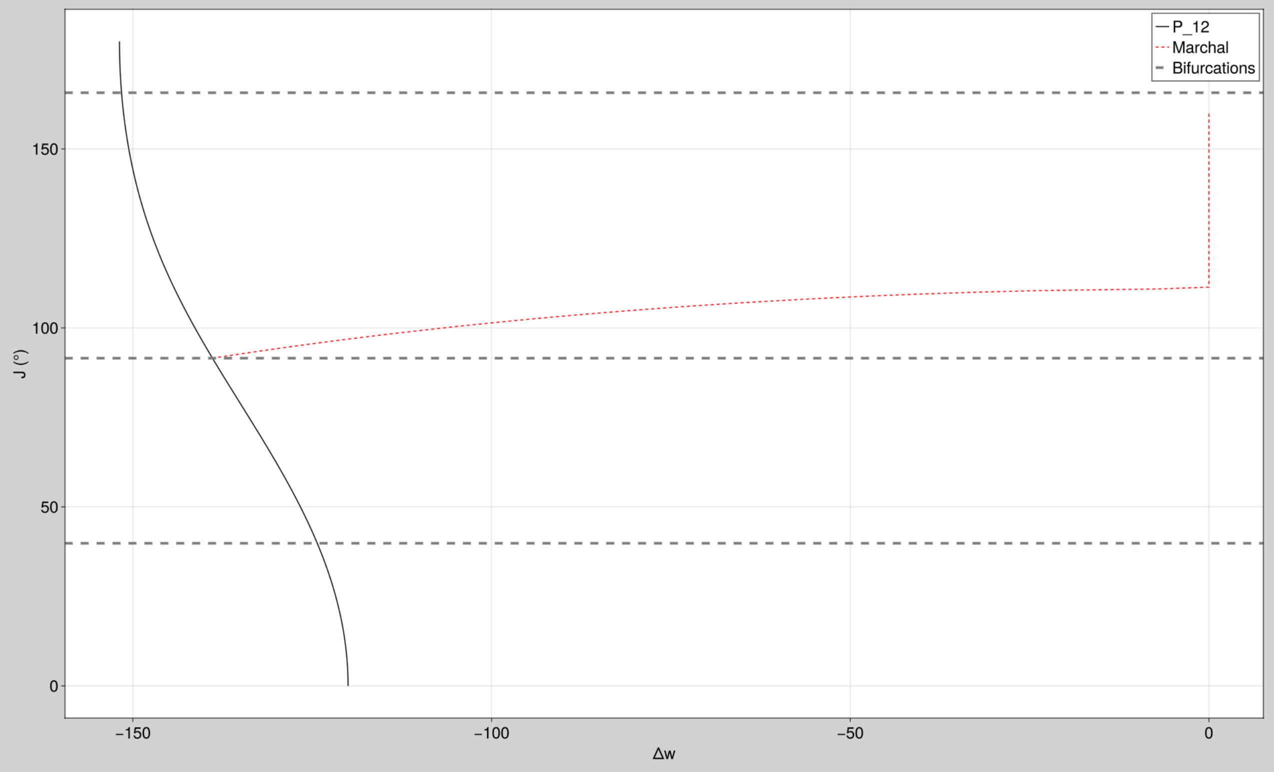


Figure 27: Marchal and P_{12} for equal masses.

P_{12}: the bifurcations

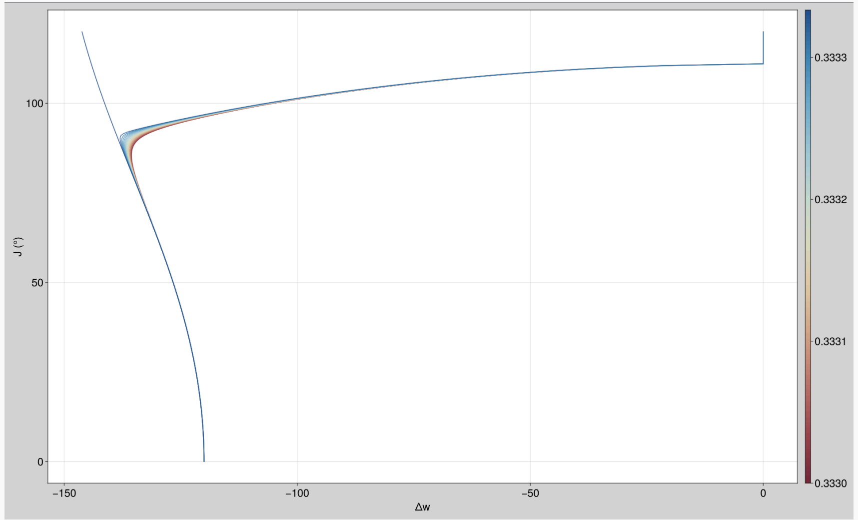
Figure 28: The first bifurcation was followed by (Calleja *et al.*, 2021) to a rotating Eight. The second is the separation with Marchal's family. The third one appears to be new.

Further work

Digging into the bifurcations

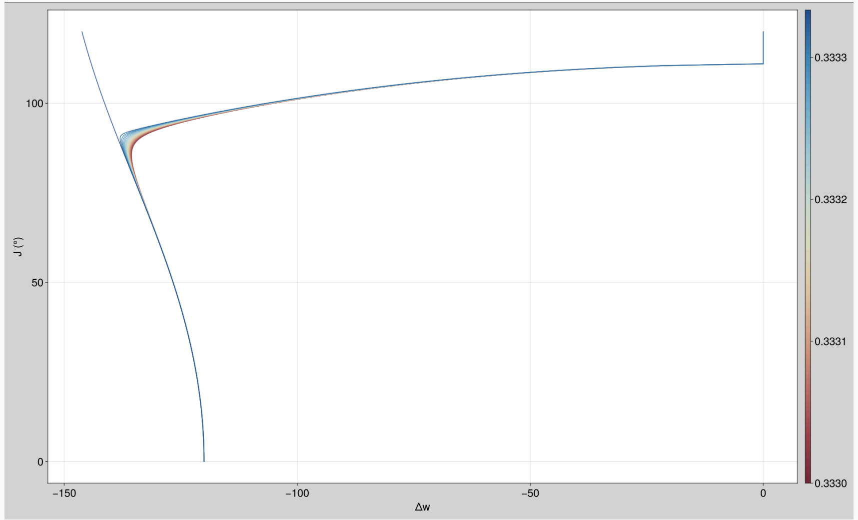
- Understand which bifurcations are crossings with families
- Which of these preserve symetries

The P_{12} branch for unequal masses



Does it exist for unequal masses? Where does it go?

The P_{12} branch for unequal masses



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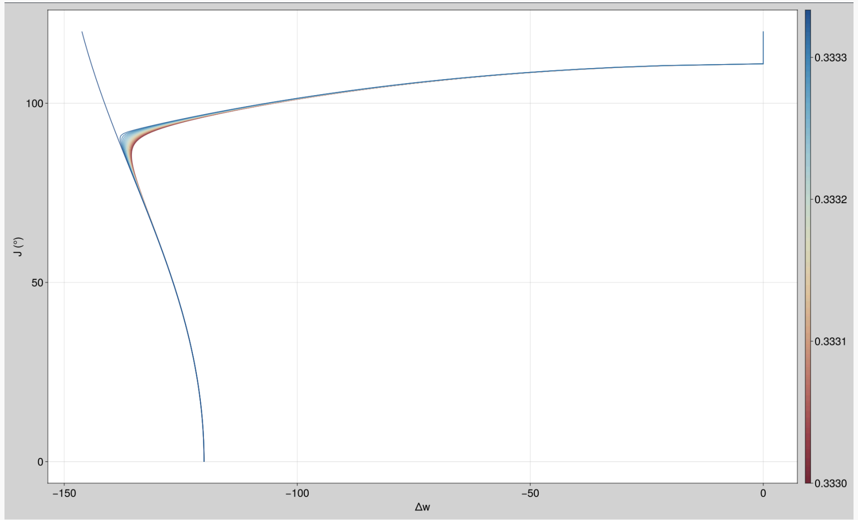
(Doedel *et al.*, 2003) continues the Eight for masses $(1-2m, m, m), m > \frac{1}{3}$

- Wild conjecture: could the P_{12} branch end at the Doedel family?

The P_{12} branch for unequal masses



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What happens to the other bifurcations at unequal masses?

The Euler family at equal masses

- Currently failing (probably for numerical reasons, not mathematical ones)

Conclusion

We...

Confirmed part of Marchal's conjecture by linking Marchal's family in the restricted case to the P_{12} family in the equal masses case

- ... Up to a point: bifurcation

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Confirmed part of Marchal's conjecture by linking Marchal's family in the restricted case to the P_{12} family in the equal masses case

- ... Up to a point: bifurcation

Developed numerical tools to find periodic orbits

- Not necessarily in the three-body problem!

Bibliography

Calleja, R. *et al.* (2024) “From the Lagrange Triangle to the Figure Eight Choreography: Proof of Marchal's Conjecture.” arXiv. Available at: https://doi.org/10.48550/arXiv.2406.17564.

Calleja, R. *et al.* (2021) “From the Lagrange Polygon to the Figure Eight I: Numerical Evidence Extending a Conjecture of Marchal,” *Celestial Mechanics and Dynamical Astronomy*, 133(3). Available at: https://doi.org/10.1007/s10569-021-10009-9.

Chenciner, A. and Fejoz, J. (2008) “The Flow of the Equal-Mass Spatial 3-Body Problem in the Neighborhood of the Equilateral Relative Equilibrium,” *Discrete and Continuous Dynamical Systems - B*, 10(2 & 3), pp. 421–438. Available at: https://doi.org/10.3934/dcdsb.2008.10.421.

Chenciner, A., Féjoz, J. and Montgomery, R. (2005) “Rotating Eights: I. The Three \Gammai Families”, *Nonlinearity*, 18(3), p. 1407. Available at: https://doi.org/10.1088/0951-7715/18/3/024.

Doedel, E.J. *et al.* (2003) “Computation of Periodic Solutions of Conservative Systems with Application to the 3-Body Problem,” *International Journal of Bifurcation and Chaos*, 13(6), pp. 1353–1381. Available at: https://doi.org/10.1142/S0218127403007291.

Laskar, J. (2017) “Andoyer Construction for Hill and Delaunay Variables,” *Celestial Mechanics and Dynamical Astronomy*, 128(4), pp. 475–482. Available at: https://doi.org/10.1007/s10569-017-9761-0.

Marchal, C. (2009) “Long Term Evolution of Quasi-Circular Trojan Orbits,” *Celestial Mechanics and Dynamical Astronomy*, 104(1), pp. 53–67. Available at: https://doi.org/10.1007/s10569-009-9195-4.

Mastroianni, R. and Efthymiopoulos, C. (2019) “Secular Dynamics in Extrasolar Systems with Two Planets in Mutually Inclined Orbits,” *Proceedings of the International Astronomical Union*, 15(S364), pp. 191–196. Available at: https://doi.org/10.1017/S1743921321001368.

Roberts, G.E. (2002) “Linear Stability of the Elliptic Lagrangian Triangle Solutions in the Three-Body Problem,” *Journal of Differential Equations*, 182(1), pp. 191–218. Available at: https://doi.org/10.1006/jdeq.2001.4089.

Robutel, P. (1995) “Stability of the Planetary Three-Body Problem,” *Celestial Mechanics and Dynamical Astronomy*, 62(3), pp. 219–261. Available at: https://doi.org/10.1007/BF00692089.

Robutel, P. and Pousse, A. (2013) “On the Co-Orbital Motion of Two Planets in Quasi-Circular Orbits,” *Celestial Mechanics and Dynamical Astronomy*, 117(1), pp. 17–40. Available at: https://doi.org/10.1007/s10569-013-9487-6.