Bifurcations, halo orbits and space debris

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Outline

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1. Introduction
2. Around the collinear points
3. Center manifold reduction
4. Lyapunov and Halo orbits
5. Resonant normal form
6. First order estimate
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8. Floquet theory
9. Solar radiation pressure
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1. Introduction

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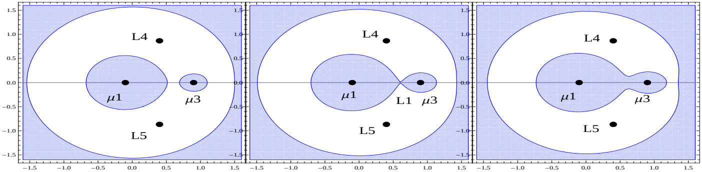
• 1968: C.C. Conley uses the unstable collinear points to construct *low-cost* interplanetary routes. The method relies on Moser’s version of Lyapunov’s theorem and the existence of *transit* orbits between the primaries: increasing slightly the energy, the bottleneck between Hill’s surfaces opens.



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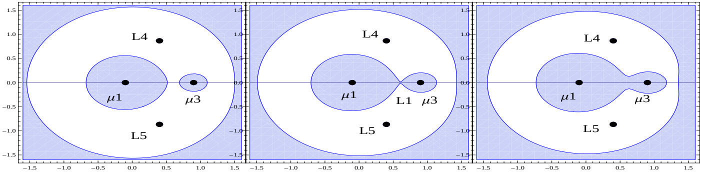

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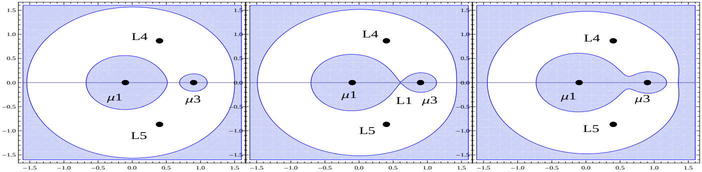

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• Aim of the talk: study of halo orbits arising from bifurcations of Lyapunov orbits. The method makes use of the reduction to the center manifold and a resonant normal form. The technique can also be used to study the generation of equilibria for some lunisolar resonances in space debris dynamics.

Outline

2. Around the collinear points

• SCR3BP: small A influenced by $P(\mu)$ and $S(1-\mu)$. In a synodic reference frame $A = (X, Y, Z) \in \mathbb{R}^3$ defined on the collisionless manifold

$$\mathcal{P}_s \equiv \{(P_X, P_Y, P_Z), (X, Y, Z) \in \mathbb{R}^3 \times \mathbb{R}^3 : \; r_1(X, Y, Z) \neq 0, \; r_2(X, Y, Z) \neq 0\}$$

with

$$r_1 \equiv \sqrt{(X-\mu)^2 + Y^2 + Z^2} \,, \quad r_2 \equiv \sqrt{(X-\mu+1)^2 + Y^2 + Z^2} \,,$$

endowed with the standard symplectic form

$$\omega = dP_X \wedge dX + dP_Y \wedge dY + dP_Z \wedge dZ \,.$$

Hamiltonian in the synodic frame centered at the barycenter of the primaries:

$$H_0(P_X, P_Y, P_Z, X, Y, Z) = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \,.$$

• Translate the origin from the barycenter of the primaries to the collinear point: $(P_X, P_Y, P_Z, X, Y, Z) \to (p_x, p_y, p_z, x, y, z)$. Expand the potential using Legendre polynomials $\mathcal{P}_n$:

$$H_1(p_x, p_y, p_z, x, y, z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + y p_x - x p_y - \sum_{n \geq 2} c_n(\mu) \rho^n \mathcal{P}_n\left(\frac{x}{\rho}\right)$$

for suitable coefficients $c_n(\mu; L_i)$ and $\rho = \sqrt{x^2 + y^2 + z^2}$.

• Linearize H_1 around the equilibrium and, through a symplectic change of variables, reduce the quadratic part to the form

$$H_1^{(qd)}(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z, \tilde{x}, \tilde{y}, \tilde{z}) = \lambda_x \tilde{x} \tilde{p}_x + \frac{\omega_y}{2}(\tilde{y}^2 + \tilde{p}_y^2) + \frac{\omega_z}{2}(\tilde{z}^2 + \tilde{p}_z^2) \qquad (1)$$

with λ_x, ω_y, ω_z real $\Rightarrow$ saddle×center×center.

Outline

3. Center manifold reduction

• Eliminate the hyperbolic component through a canonical transformation using Lie series. From $(\underline{\tilde{p}}, \underline{\tilde{q}})$ to complex coordinates $(\underline{p}, \underline{q})$, (1)+h.o.t, becomes:

$$H_1^{(c)}(\underline{p}, \underline{q}) = \lambda_x q_1 p_1 + i\omega_y q_2 p_2 + i\omega_z q_3 p_3 + \sum_{n \geq 3} H_n(\underline{p}, \underline{q}) \; . \qquad (2)$$

Proposition

There exists a canonical transformation $(\underline{p}, \underline{q}) \to (\underline{P}, \underline{Q})$, *s.t.* (2) *becomes*

$$\begin{aligned} H_{cm}^{(N)}(\underline{P}, \underline{Q}) &= \lambda_x Q_1 P_1 + i\omega_y Q_2 P_2 + i\omega_z Q_3 P_3 \\ &+ \sum_{n=3}^{N} \tilde{H}_n(Q_1 P_1, P_2, P_3, Q_2, Q_3) + R_{N+1}(\underline{P}, \underline{Q}) \; , \end{aligned}$$

where R_{N+1} = *remainder of degree* $N+1$, *polynomials* $\tilde{H}_n$ *depend on* $Q_1\ P_1$.

• Neglecting R_{N+1}, then $I_x \equiv Q_1 P_1 = \text{constant} \Rightarrow H_{cm}^{(N)}$ has 2 d.o.f. and, setting $I_x = 0$, it describes the dynamics in the center manifold up to R_{N+1}.

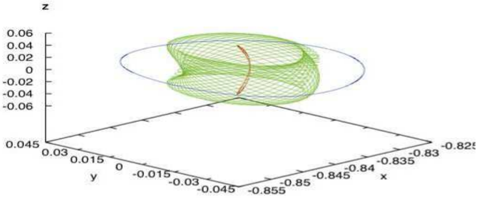
Outline

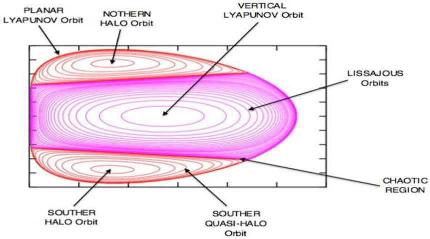
4. Lyapunov and Halo orbits

• **Lyapunov center theorem**. Each equilibrium gives rise to two 1-parameter families of periodic orbits: *planar* and *vertical* **Lyapunov periodic orbits**.

• When the energy increases, the linear stability changes and there appear bifurcating periodic orbits. **Halo orbits:** the family of periodic orbits bifurcating from the family of *planar* Lyapunov orbits (when $\omega_y = \omega_z$).

• Due to the hyperbolic character of the L_i, for energy close to equilibria, the periodic and quasi-periodic orbits in the center manifold inherit hyperbolicity and have stable/unstable manifolds, allowing to construct transfer trajectories and complex interplanetary orbits.





Outline

5. Resonant normal form

- Action-angle coordinates for the quadratic part:

$$\begin{aligned} Q_2 &= -i\sqrt{I_y}e^{i\theta_y} \qquad P_2 = \sqrt{I_y}e^{-i\theta_y} \\ Q_3 &= -i\sqrt{I_z}e^{i\theta_z} \qquad P_3 = \sqrt{I_z}e^{-i\theta_z} \ , \end{aligned}$$

so that the CM-Hamiltonian becomes:

$$H_{cm}^{(N)}(I_y, I_z, \theta_y, \theta_z) = \omega_y I_y + \omega_z I_z + \sum_{n=3}^{N} \tilde{H}_n(I_y, I_z, \theta_y, \theta_z) + R_{N+1} \ ,$$

where $\tilde{H}_n$ is a homogeneous polynomial of degree $n/2$ in the actions.

- Resonant normal form for the synchronous resonance $\omega_y = \omega_z$:

$$\begin{aligned} H_{res}^{(4)}(I_y, I_z, \theta_y, \theta_z) &= \omega_y I_y + \omega_z I_z + \alpha I_y^2 + \beta I_z^2 + \gamma I_y I_z \\ &+ 2 I_y I_z \tilde{\gamma} \cos(2\theta_y - 2\theta_z) + R^{(5)} \ , \end{aligned}$$

where, by construction up to h.o.t., $\dot{I}_y + \dot{I}_z = 0 \Rightarrow \mathcal{E} \equiv I_y + I_z = const.$

Outline

6. First order estimate

- Change of coordinates:

$$\mathcal{E} = I_y + I_z \qquad \mathcal{R} = I_y \qquad \nu = \theta_z \qquad \psi = \theta_y - \theta_z$$

and let $\delta = (\omega_y - \omega_z)/\omega_z$ be the *detuning*. Then, up to h.o.t.:

$$H_{new}(\mathcal{E}, \mathcal{R}, \nu, \psi) = \mathcal{E} + \delta\mathcal{R} + a\mathcal{R}^2 + b\mathcal{E}^2 + c\mathcal{E}\mathcal{R} + d(\mathcal{R}^2 - \mathcal{E}\mathcal{R})\cos(2\psi) \ .$$

Proposition (C-Pucacco-Stella, JNS 2015)

To first order in δ, the energy level at which a bifurcation to halo orbits occurs is given by

$$E = \frac{\omega_z^2 \delta}{\gamma - 2(\alpha + \tilde{\gamma})} \ .$$

Proof. Given that $\dot{\mathcal{E}} = 0$, we have a 1 d.o.f. system in $(\mathcal{R}, \psi)$, whose fixed points give periodic orbits in the original system:

$$\dot{\mathcal{R}} = 2s\mathcal{R}(\mathcal{R} - \mathcal{E})\sin(2\psi) \qquad \dot{\psi} = \delta + 2a\mathcal{R} + c\mathcal{E} + d(2\mathcal{R} - \mathcal{E})\cos(2\psi) \ .$$

For $\mathcal{R} = \mathcal{E}$ we have a normal mode along the y-axis, i.e. a *planar* Lyapunov orbit; for $\mathcal{R} = 0$ we have a normal mode along the z-axis, i.e. a *vertical* Lyapunov orbit.

• Halo orbits arise from bifurcations of the normal modes when entering the synchronous resonance. We have $\dot{\mathcal{R}} = 0$ for $\psi = 0, \pi, \pm\frac{\pi}{2}$, while $\dot{\psi} = 0$ for

$$\mathcal{R}|_{(0,\pi)} = -\frac{\delta + (c-d)\mathcal{E}}{2(a+d)}\,, \qquad \mathcal{R}|_{\pm\frac{\pi}{2}} = -\frac{\delta + (c+d)\mathcal{E}}{2(a-d)}\,.$$

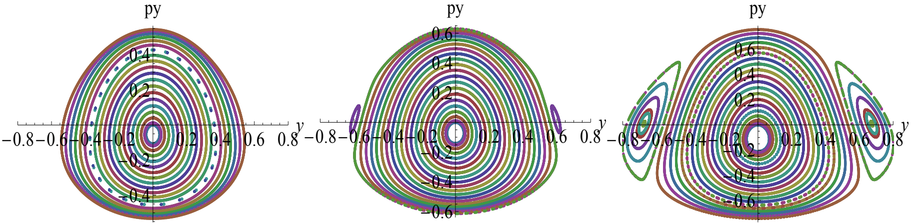
From $I_y, I_z \geq 0$ and $\mathcal{E} = I_y + I_z$ we have $0 \leq I_y, I_z \leq \mathcal{E}$, namely $0 \leq \mathcal{R} \leq \mathcal{E}$, which gives the constraints

$$\mathcal{E} \geq \mathcal{E}_{iy} \equiv -\frac{\delta}{2a+c+d} \qquad \text{or} \qquad \mathcal{E} \geq \mathcal{E}_{iz} \equiv \frac{\delta}{-c+d}$$

$$\mathcal{E} \geq \mathcal{E}_{\ell y} \equiv -\frac{\delta}{2a+c-d} \qquad \text{or} \qquad \mathcal{E} \geq \mathcal{E}_{\ell z} \equiv \frac{\delta}{-c-d}\,,$$

where $\mathcal{E}_{\ell y}$ is the threshold for the bifurcation of the halo family from the planar Lyapunov orbit. From $E_1 = \omega_z \mathcal{E}_1$ one obtains the proof.

• Earth–Moon Poincaré maps for $E = 0.3, 0.4, 0.6$.



	numerical	I order	II order
L_1	0.3069	0.3069	0.3069
L_2	0.3557	0.3636	0.3555
L_3	0.2985	0.3354	0.2965

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7. Second order estimate

- Central manifold & normal form up to 6^{th} order:

$$\begin{aligned} H^{(6)}(I_y, I_z, \theta_y, \theta_z) &= H_{res}^{(4)} + \alpha_{3300} I_y^3 + \alpha_{0033} I_z^3 + \alpha_{1122} I_y I_z^2 + \alpha_{2211} I_y^2 I_z \\ &+ 2\alpha_{2013} I_y I_z^2 \cos(2(\theta_y - \theta_z)) + 2\alpha_{3102} I_y^2 I_z \cos(2(\theta_y - \theta_z)) \end{aligned}$$

Proposition (C-Pucacco-Stella, JNS 2015)

To second order in δ, the energy level at which a bifurcation to halo orbits occurs is given by

$$E = \frac{\omega_z^2 \delta}{\gamma - 2(\alpha + \tilde{\gamma})} + \left[\frac{\gamma - \alpha - 2\tilde{\gamma}}{(\gamma - 2(\alpha + \tilde{\gamma}))^2} - \omega_z \frac{\alpha_{2211} - 3\alpha_{3300} - 2\alpha_{3102}}{(\gamma - 2(\alpha + \tilde{\gamma}))^3} \right] \delta^2 .$$

Proof. Let $\mathcal{R}_s$ be s.t. $\frac{\partial H^{(6)}(\mathcal{R}_s, \psi = \pi/2)}{\partial \mathcal{R}} = 0$. Expand $\mathcal{R} = A_0 + A_1 \delta + A_2 \delta^2$ for some real A_i; from $\mathcal{R} \leq \mathcal{E}$ we get

$$\mathcal{E} > \mathcal{E}_2 = \mathcal{E}_1 - \frac{\alpha_{2211} - 3\alpha_{3300} - 2\alpha_{3102}}{(\gamma - 2(\alpha + \tilde{\gamma}))^3} \delta^2 ,$$

which expressed in terms of the energy E of the normal mode gives the proof.

Outline

8. Floquet theory

• Small variations around the planar Lyapunov orbit $I_z = 0$ (or $\mathcal{R} = \mathcal{E}$).
• Introduce complex coordinates (v, w) in place of (I_z, θ_z):

$$v = \sqrt{2I_z}\, i\, e^{-i\theta_z} \,, \qquad w = -\sqrt{2I_z}\, i\, e^{i\theta_z} \,.$$

• Compute the transformed Hamiltonian and write the variational dynamics on the energy shell $\mathcal{E} = I_y + I_z = I_y + \frac{vw}{2}$.
• Up to 2^{nd} order in v, w, one gets the Hamiltonian parametrized by $\mathcal{E}$:

$$K^{(2)}(v, w, \theta_y) = (\omega_z+\delta)\mathcal{E}+\alpha\mathcal{E}^2-(\delta + (2\alpha - \gamma)\mathcal{E})\,\frac{vw}{2}-\frac{\tilde{\gamma}}{2}\mathcal{E}\left(v^2 e^{2i\theta_y} + w^2 e^{-2i\theta_y}\right)$$

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- Compute the evolution through the Floquet matrix:

$$\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = M(t) \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} e^{-i\theta_y} & 0 \\ 0 & e^{i\theta_y} \end{pmatrix} e^{tF} \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} \text{ where}$$

$$F = -i \begin{pmatrix} \delta + (2\alpha - \sigma)\mathcal{E} & 2\tau\mathcal{E} \\ -2\tau\mathcal{E} & -\delta - (2\alpha - \sigma)\mathcal{E} \end{pmatrix}.$$

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• Denoting by T_y the period of the planar Lyapunov orbit, transition stability/instability occurs when $\mathrm{Trace}(M(T_y)) = 2 \Rightarrow \mathcal{E}_{\ell y}$.

Outline

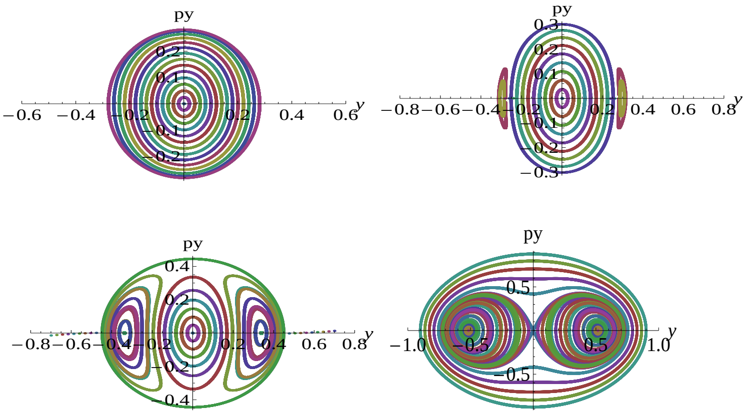
9. Solar radiation pressure

$$H_0(P_X, P_Y, P_Z, X, Y, Z) = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{(1-\beta)(1-\mu)}{r_1} - \frac{\mu}{r_2} \,,$$

$\beta = (L_S Q / (4\pi\, c\, \mathcal{G} M_S))\, A/m$,
L_S =Sun luminosity, $Q = 1$+reflectivity, c =speed of light, M_S =Sun mass,
A/m =area-to-mass ratio.

[Bucciarelli-Ceccaroni-C-Pucacco, Ann. Mat. Pura Applicata 2015]

- Location of the collinear points + CM reduction + 1^{st} order estimate.
- Poincaré maps + frequency analysis + FLI.
- Sun-Vesta: $\mu = 1.3574\, 10^{-10}$, $\beta = 10^{-2}$.
- A different sequence of bifurcations, due to the complicated interplay between the small mass of Vesta and SRP.



- $E = 0.05$: first bifurcation ($\mathcal{E}_{\ell y}$) to Halo orbits
- $E = 0.1$: second bifurcation ($\mathcal{E}_{iy}$) of anti-halo orbits; the planar Lyapunov orbit (outermost curve) regains stability
- $E = 0.4$: third bifurcation ($\mathcal{E}_{iz}$) with instability of the vertical Lyapunov orbit.
- Analytical vs. numerical results: agreement to $2^{nd} - 3^{rd}$ digit.

Outline

10. Space debris lunisolar resonances

- Secular resonances induced by Sun and Moon on space debris.
- Hamiltonian $H = H_{Kep} + R_{geo} + R_{Moon} + R_{Sun}$, where H_{Kep} is the Keplerian part, R_{geo} is the J_2-contribution of the geopotential, R_{Moon}, R_{Sun} are the modified Lane/Giacaglia/Hughes, averaged over the mean anomalies of debris, Moon, Sun.
- Secular resonance induced by the body b:

$$k_1\dot{\omega} + k_2\dot{\Omega} + k_3\dot{M}_b + k_4\dot{\omega}_b + k_5\dot{\Omega}_b = 0 \ .$$

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  - $\dot{\Omega} = 0$ at polar orbits;
  - $\alpha\dot{\omega} + \beta\dot{\Omega} = 0$ for some $\alpha, \beta \in \mathbb{Z}$, precisely
    - $\dot{\omega} + \dot{\Omega} = 0$ at $i = 46.4^o$ or $i = 106.9^o$;
    - $-\dot{\omega} + \dot{\Omega} = 0$ at $i = 73.2^o$ or $i = 133.6^o$;
    - $-2\dot{\omega} + \dot{\Omega} = 0$ at $i = 69.0^o$ or $i = 123.9^o$;
    - $2\dot{\omega} + \dot{\Omega} = 0$ at $i = 56.1^o$ or $i = 111.0^o$.

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• SAMPLE CASE: $-\dot{\omega} + \dot{\Omega} = 0$.

• Resonant variables: $\psi = -\omega + \Omega$, $\mathcal{R} = -G$, $\nu = \Omega$, $\mathcal{E} = H + G$ with ν fast variable $\Rightarrow$ average over $\nu \Rightarrow$ 1 d.o.f. $(\mathcal{R}, \psi)$ with $\mathcal{E}$ constant.

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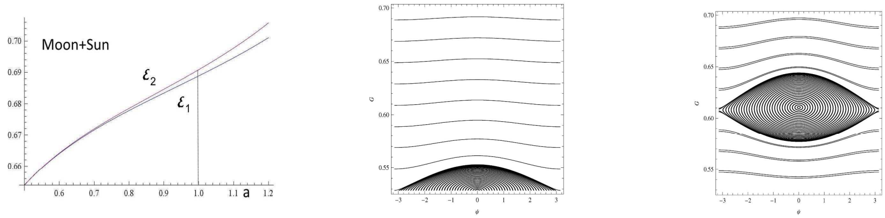
► Generation of 1 or 2 equilibria are given by the bifurcation curves $\mathcal{E}_1$, $\mathcal{E}_2$ for Moon+Sun case, $a = 1$ ([C-Gales-Pucacco, Preprint 2015]).



• **References:**

► C. A., Pucacco G., Stella D., "Lissajous and Halo orbits in the restricted three-body problem", J. Nonlinear Science, vol. 25 (2015)

► Bucciarelli S., Ceccaroni M., C. A., Pucacco G., "Qualitative and analytical results of the bifurcation thresholds to halo orbits", Annali di Matematica Pura e Applicata (2015)

► Ceccaroni M., C. A., Pucacco G., "Halo orbits around the collinear points of the restricted three-body problem", Preprint 2015

► C. A., Gales C., Pucacco G., "Bifurcation of lunisolar secular resonances for space debris orbits", Preprint 2015