# Small divisors for interval exchange maps

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The simplest setting for this kind of results is the linearization problem for circle diffeomorphisms.

# Linearization of circle diffeomorphisms

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$DC(\gamma, \tau) = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall q \in \mathbb{Z}_{>0},\ \forall p \in \mathbb{Z}, |q\alpha - p| \geqslant \gamma q^{-1-\tau}\}.$

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In other terms, in the group of $C^\infty$ orientation-preserving diffeomorphisms of the circle, the conjugacy class of a diophantine rotation is a codimension-one submanifold.

# The linear equation

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$$RT := \bigcap_{\tau>0} \left(\cup_{\gamma>0} DC(\gamma, \tau)\right).$$

This is a full measure set which contain any irrational algebraic number (Roth 1955).

# Roth type rotations

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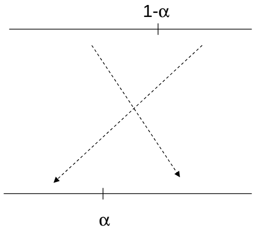
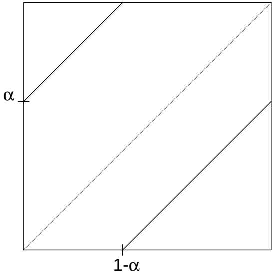
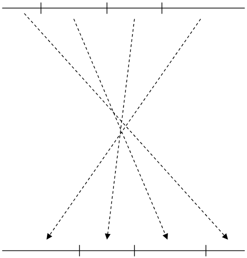
- Let $r > 2$. For any $f \in \mathrm{Diff}^r_+(\mathbb{T})$, $(\star)$ has a unique solution $(t, h)$ with $h(0) = 0$ and $h \in \mathrm{Diff}^s_+(\mathbb{T})$ for all $s < r - 1$.

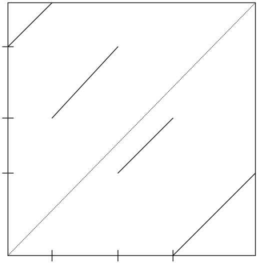
# Circle rotations



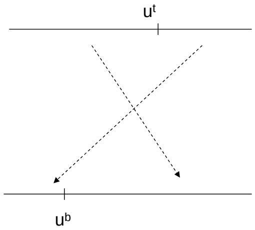



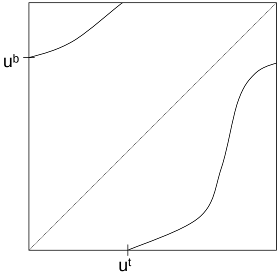
# Standard interval exchange maps



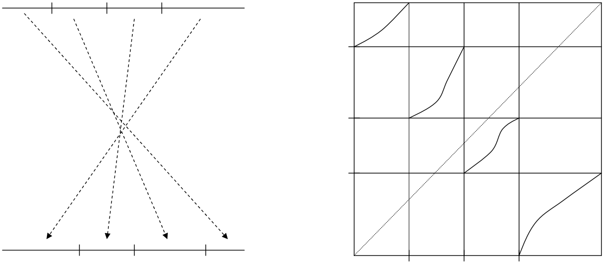


# Circle homeomorphisms and diffeomorphisms



# Generalized interval exchange maps



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Let $T$ be a standard i.e.m. on $d$ intervals. Let $u_1 < \ldots < u_{d-1}$ be the singularities of $T$ and $v_1 < \ldots < v_{d-1}$ be the singularities of $T^{-1}$.

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**Theorem** (Keane) A standard i.e.m with no connection is minimal: every orbit is dense.

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The set of i.e.m with given combinatorics is parametrized by the lengths $(u_i - u_{i-1})_{1 \leqslant i \leqslant d}$, i.e a $(d-1)$-dimensional simplex.

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The restricted Roth type condition is defined from the Rauzy-Veech algorithm, a generalization for i.e.m.'s with no connection of the continued fraction algorithm for irrational numbers.

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# Linearization of generalized i.e.m

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Thanks for your attention