1. **LEMMA** BOUNDED STABLE PART

$\{\xi^\alpha\}_{\alpha\in a}$ Bdd stably hyperbolic

$\Rightarrow \exists \epsilon>0,\ \exists K>0$ s.t.

$\{\eta^\alpha\}_{\alpha\in a}$ periodically equiv. family

$d(\eta,\xi)<\epsilon \Rightarrow \forall \alpha\in a \quad \forall i\in\mathbb{Z}$

$$\Big\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\big|_{E^s(\eta^\alpha)} \Big\| < K, \quad m = Per(\eta^\alpha).$$

2 **LEMMA** CONTRACTING STABLE PART AT PERIOD

$\{\xi^\alpha\}_{\alpha\in a}$ Bdd stably hyperbolic

$\exists \epsilon>0 \quad K>0 \quad 0<\lambda<1$ s.t.

$\{\eta^\alpha\}_{\alpha\in a}$ Per. equiv fam. $d(\eta,\xi)\le\epsilon$

$\Rightarrow$

$$\forall \alpha\in a \quad \forall i\in\mathbb{Z} \quad \Big\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\big|_{E^s_i(\eta^\alpha)} \Big\| < K\lambda^m$$

$m = Per(\eta^\alpha)$.

3. LEMMA (BOUNDED ANGLE)

$\{\xi^\alpha\}_{\alpha\in a}$ Bdd stably hyperbolic

$\exists\, \epsilon > 0 \quad \exists\, \gamma > 0 \quad \exists\, N_0 \in \mathbb{Z}^+$

$\{\eta^\alpha\}_{\alpha\in a}$ per. equiv. fam. $d(\eta,\xi)\le\epsilon$

$$\Longrightarrow \quad \measuredangle\left(E_i^s(\eta^\alpha), E_i^u(\eta^\alpha)\right) > \gamma$$

$\forall i\in\mathbb{Z} \quad \forall \alpha\in a$ with minimal period $> N_0$.

SKETCH OF PROOF OF THM

$$\boxed{\left.\begin{array}{l}\text{STABLE HYPERBOLIC}\\ \text{FAMILY}\\ +\\ \text{SYMPLECTIC.}\end{array}\right\} \Rightarrow \text{HYPERBOLICITY.}}$$

① ONCE WE KNOW THE ANGLES ARE UNIFORMLY BOUNDED BELOW FOR ANY PERTURBATION, WE CAN ASSUME THAT E^s AND E^u ARE ORTHOGONAL.

i.e. the metric adapted to splitting $E^s\oplus E^u$:

$$[\![\textstyle\sum x_i e_i + y_i f_i]\!] = \sum |x_i|^2 + \sum |y_i|^2$$

$\{e_i\}$ basis E^s, $\{f_i\}$ basis E^u

Norms of matrices in $[\![\ ]\!]$ are comparable to the norms on usual metric.

② If a seq. does not uniformly contract E^s

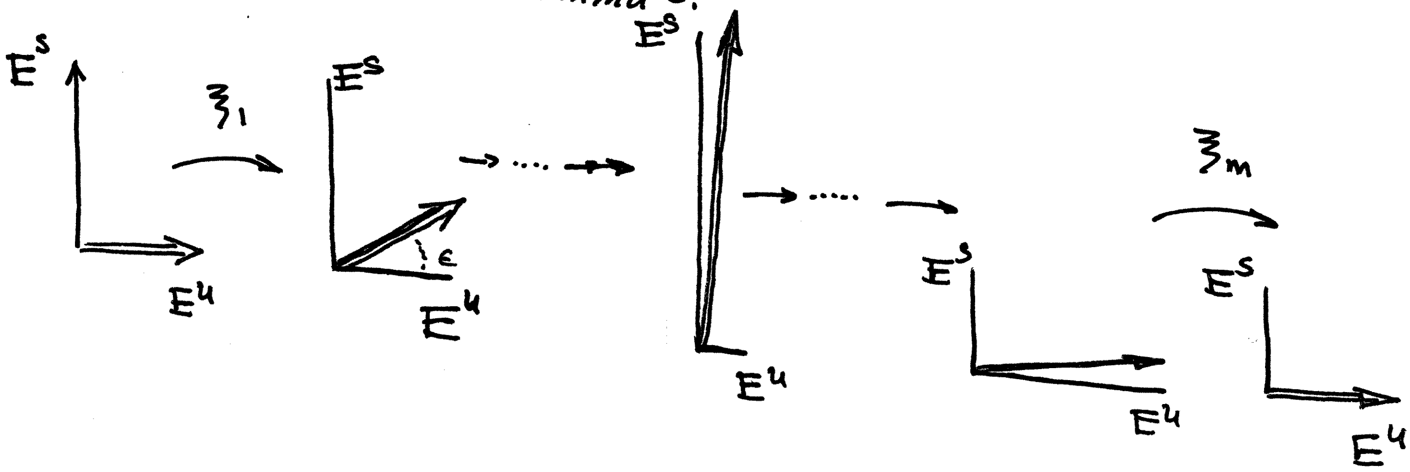
$$\left[\text{say } \left\| \prod_{i=1}^{k} \xi_i \Big|_{E^s} \right\| \geq \frac{1}{2} \right]$$

multiply its stable component by $(1+\epsilon)^m$
unstable " " $(1+\epsilon)^{-m}$

so that at some iterate, say k, it expands E^s & contracts E^u.

Then the following perturbation of only the maps ξ_1 and ξ_m, $m = \mathrm{Per}(\xi^\alpha)$, obtains a small angle $\measuredangle(E^s, E^u)$ at the k-th iterate.

This contradicts Lemma 3.



ARC SPACES

X = algebraic variety on $\mathbb{R}^N$ (= zeroes of polynomial eqs).

Path space on X

$$\mathcal{C}(X) := \left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{R}^N \;\middle|\; \begin{array}{l} \exists \gamma \in C^\infty(\mathbb{R},\mathbb{R}^N) \\ \gamma(\mathbb{R}) \subset X \\ \frac{1}{n!}\gamma^{(n)}(0) = a_n \ \forall n \in \mathbb{N} \end{array} \right\}$$

$F = (f_1, \dots, f_q)$ generators of the ideal $I(X) = \{ f \in \mathbb{R}[x_1,\dots,x_N] \mid f|_X \equiv 0 \}$

Arc space $\mathcal{L}(X)$:

$$\mathcal{L}_\bullet(X) := \{ (a_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}}^{\bullet} \mathbb{R}^N \mid F(\sum_{k=0}^{\infty} a_k t^k) \equiv 0 \}$$

where $\equiv$ is equality as formal power series.

$\mathcal{L}_n(X)$

$\dim_{\mathbb{R}} (X \cap \mathbb{R}) \cancel{\leq}$

$\mathcal{L}_n(X)$ is an algebraic variety

$\pi : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$ the projection $(a_k)_{k\in\mathbb{N}} \longrightarrow (a_k)_{k=0}^{n}$

$\pi_n(\mathcal{L}(X))$ is a constructible set in $\mathcal{L}_n(X)$

(finite union of algebraic sets)

$\overline{\pi_n(\mathcal{C}(X))}$:= Zariski closure of $\pi_n(\mathcal{C}(X))$ = minimal alg var. containing

PROPOSITION

(a) $\dim \overline{\pi_n(\mathcal{C}(X))} \leq (n+1)\dim X$

(b) The fibers of $\pi_{n+1}(\mathcal{C}(X)) \longrightarrow \pi_n(\mathcal{C}(X))$ have dimension $\leq \dim X$.

PROOF

(1) Enough to prove for an algebraic variety X in $\mathbb{C}^N$
[Because $\dim_{\mathbb{R}}(X \cap \mathbb{R}) \leq \dim_{\mathbb{C}} X$]

(2) Obs: (b) $\Rightarrow$ (a)

(3) Fix $\bar{a} = (a_0, \dots, a_n) \in \pi_n(\mathcal{C}(X))$

$$Z_{n+1} := \{(t,x) \in \mathbb{C} \times \mathbb{C}^N \mid F(a_0 + \dots + a_n t^n + t^{n+1} x) = 0\}$$

For $t \in \mathbb{C}$ let

$$Z_{n+1}(t) := \{x \in \mathbb{C}^N \mid (t,x) \in Z_{n+1}\}$$

The limit W_{n+1} at $t=0$ of the 1-parameter family of varieties $Z_{n+1}(t)$ exists. ["1-dim. families are flat" alg. closed field.]

$W_{n+1} := \lim_{t\to 0} Z_{n+1}(t)$

i.e. if $Z^*_{n+1} := \{(t,x) \in Z_{n+1} \mid t \neq 0\}$

then $Z^*_{n+1} \cup W_{n+1}$ is the Zariski closure of Z^*_{n+1}.

$\mathbb{F}_{\bar{a}} := \Theta_n^{-1}(a)$ fiber of $\Theta_n : \pi_{n+1}(\mathcal{G}(X)) \longrightarrow \pi_n(\mathcal{G}(X))$ over $\bar{a}$

Claim 1: $\mathbb{F}_{\bar{a}} \subset W_{n+1}$

$\vdash$ Let $a_{n+1} \in \mathbb{F}_{\bar{a}}$

Since $(a_0, \dots, a_n, a_{n+1}) \in \pi_{n+1}(\mathcal{G}(X))$

$\exists \gamma \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ s.t. $F \circ \gamma \equiv 0$

$\gamma(t) = a_0 + \dots + a_n t^n + a_{n+1} t^{n+1} + \theta(t^{n+2})$, $t \in \mathbb{R}$

Let $x_t := \frac{1}{t^n}\left[\gamma(t) - \sum_{k=0}^{n} a_k t^k\right] = a_{n+1} + \theta(t) \in Z_{n+1}(t) \subset \mathbb{C}^N$

This implies $a_{n+1} \in W_{n+1}$ //

Claim 2: $\dim W_{n+1} \leq \dim X$ $\quad \boxed{\text{claim 2} \Rightarrow \text{PROPO}}$

$\vdash$

① For $t \neq 0$, variety $Z_{n+1}(t) \overset{\text{iso}}{\approx} X$ by invertible change of variables

$Z_{n+1}(t) \ni z \longleftrightarrow x \in X$

$x = a_0 + a_1 t + \dots + a_n t^n + t^{n+1} z$

$\therefore \dim Z_{n+1}(t) = \dim X$ when $t \neq 0$.

② Need projective varieties

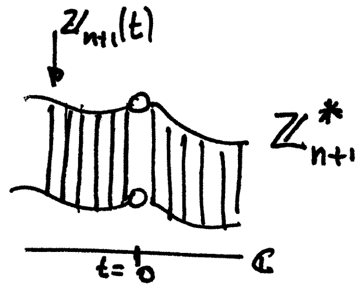
consider $\mathbb{C}^N = \mathbb{C}^N \times \{1\} \subset \mathbb{CP}^N \subset \mathbb{C}^N \cup \mathbb{CP}^{N-1}$

and corresp. varieties:

$$\mathbb{Z}_{n+1}(t), \quad \mathbb{Z}^*_{n+1} = \bigcup_{t \neq 0} \mathbb{Z}_{n+1}(t)$$

$$\mathbb{W} = \lim_{t \to 0} \mathbb{Z}_{n+1}(t)$$

Then $\quad W_{n+1} = \mathbb{W}_{n+1} \cap \mathbb{C}^N.$



- Since for generic fiber $t \neq 0$ $\quad \dim \mathbb{Z}_{n+1}(t) = \dim X$

  we have $\quad \dim \mathbb{Z}^*_{n+1} = \dim X + 1$

If $\quad \dim W_{n+1} > \dim X$

$\Rightarrow \quad \dim \mathbb{W}_{n+1} \geq \dim W_{n+1} \geq \dim X + 1$

$\Rightarrow$ $\mathbb{W}_{n+1}$ contains an irreducible component of $\overline{\mathbb{Z}^*_{n+1}}$

this is incompatible with $\mathbb{W}_{n+1} = \lim_{t \to 0} \mathbb{Z}_{n+1}(t)$

$\boxtimes$