Green bundles : a dynamical study

Marie-Claude Arnaud

December 2007

CONTENT

FIRST LECTURE :

Basic results on Hamiltonians, Lagrangians, Lipschitz submanifolds, Hamilton-Jacobi equation and minimization properties.

1 Hamiltonian and Lagrangian formalism

1.1 Symplectic manifold

We assume that M is a d-dimensional manifold endowed with a Riemannian metric and that $\pi : T^*M \to M$ is its cotangent bundle.

We will denote by λ_0 the Liouville 1-form of T^*M ; if $\varphi : U \subset M \to \mathbb{R}^d$ is a chart of M denoted by : $\varphi(q) = (q_1, \dots, q_d)$, the dual chart of T^*M is defined on T^*U by : $\varphi^*(\sum_{i=1}^{d} p_i dq_i) = (q_1, \dots, q_d, p_1, \dots, p_d)$. In such a dual chart (named a *canonical chart*), we have :

$$\lambda_0 = \sum_{i=1}^{d} p_i dq_i.$$

The manifold T^*M is then endowed with the symplectic form $\omega = -d\lambda_0$. In a canonical chart, we have :

$$\omega = \sum_{i=1}^{d} dq_i \wedge dp_i.$$

Remark : (1) If M is an open part of $\mathbb{R}^d$, we may identify T^*M with $M \times \mathbb{R}^d$: for example, for N-body problems, we have global coordinates.

(2) A theoretical result due to A. Weinstein implies that for any compact Lagrangian submanifold[1] $\mathcal{L}$ of a symplectic manifold $\mathcal{S}$, there exists a neighbourhood $\mathcal{N}$ of $\mathcal{L}$ in $\mathcal{S}$ and a neighbourhood $\mathcal{U}$ of the zero section of $T^*\mathcal{L}$ such that $\mathcal{N}$ and $\mathcal{U}$ are diffeomorphic through a symplectic diffeomorphism fixing each point of $\mathcal{L}$. In this sense, T^*M is a model of symplectic manifold.

[1]We will recall later what is a Lagrangian submanifold.

1.2 Hamiltonian equations

A C^2 function $H : T^*M \to \mathbb{R}$ will be called a *Hamiltonian*. The associated *Hamiltonian vector field* X_H is defined by :

$$\forall x \in T^*M, \forall X \in T_x(T^*M), \omega(X_H(x), X) = DH(x).X$$

and the *Hamiltonian flow*, denoted by (φ_t^H) or (φ_t), is the flow of X_H.

Then, we have :

- $\forall t, \varphi_t^* \omega = \omega$, i.e. the Hamiltonian flow is symplectic (i.e. preserves the symplectic form) ;
- $\forall t, H \circ \varphi_t = H$; the flow preserves the level sets of H.

Moreover, in canonical coordinates, we have :

$$\dot{q} = \frac{\partial H}{\partial p}(q, p); \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p).$$

1.3 Introduction to the Lagrangian action

In this section, we don't give any precise proof ; we only try to introduce the Lagrangian action in a natural way.

We consider the action of the Hamiltonian flow on a set Γ of curves drawed on T^*M which are continuous and piecewise C^1 (for example, Γ may be the set of T-periodic loops for a fixed $T > 0$) :

$$\forall \gamma \in \Gamma, \forall \varepsilon \in \mathbb{R}, \Phi_\varepsilon(\gamma)(t) = \varphi_\varepsilon \circ \gamma(t - \varepsilon).$$

We notice that $\gamma \in \Gamma$ is a fixed point of(Φ_ε) iff γ is an orbit of the flow (φ_t).

Let us assume that we have a natural definition of the tangent space to Γ ; then we define on Γ a family (Ω_a^b) of 2-forms by :

$$\forall \gamma \in \Gamma, \forall \delta\gamma, \delta\nu \in T_\gamma\Gamma, \Omega_a^b(\delta\gamma, \delta\nu) = \int_a^b \omega(\gamma(t))(\delta\gamma(t), \delta\nu(t))dt$$

and a family of functionals :

$$\mathcal{H}_a^b(\gamma) = \int_a^b H \circ \gamma - \int_a^b \lambda_0(\gamma)(\dot{\gamma})$$

Then we have :

$$\Omega(\dot{\Phi}(\gamma), \delta\gamma) = d\mathcal{H}_a^b(\gamma)\delta\gamma + [\lambda_0(\gamma)\delta\gamma]_a^b.$$

7

Remark : if Γ is the set of T-periodic loops, then $[\lambda_0(\gamma)\delta\gamma]_0^T = 0$.

Definition : The C^2 Hamiltonian $H \;: T^*M \to \mathbb{R}$ is *monotone* if $\mathcal{L} = H_p \;: T^*M \to TM$ is a C^1-diffeomorphism. Then $\mathcal{L}$ is called the "Legendre map."

The fixed points of the flow (Φ_ε), i.e. the orbits of (φ_t) satisfy :

$$0 = \frac{\partial q_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} = -\dot{q} + H_p(q,p); \quad 0 = \frac{\partial p_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} = -\dot{p} - H_q(q,p).$$

If H is monotone, the radially transformed subset of Γ is :

$$\mathcal{R} = \{\gamma = (q,p); \dot{q} = H_p(q,p)\} = \{(q,p); p = \mathcal{L}^{-1}(q,\dot{q})\}.$$

If $(q,p) \in \mathcal{R}$, we have : $\mathcal{H}_a^b(q,p) = -\int_a^b L(q,\dot{q})$ where L is the *Lagrangian* associated to H, defined on TM by : $L(q,v) = \mathcal{L}^{-1}(q,v).v - H(\mathcal{L}^{-1}(q,v))$.

The quantity $A_L(q) = \int_a^b L(q,\dot{q})$ is called the *Lagrangian action* of q (restricted to $[a,b]$). We have seen that : $A_L(q) = -\mathcal{H}_a^b(\mathcal{L}^{-1}(q,\dot{q}))$, hence the Lagrangian action represents the restriction of the "Hamiltonian" $\mathcal{H}_a^b$ to the radially transformed set.

Let us recall some classical properties of the Lagrangian function and action :

Property *if H is monotone and $(q,v) = \mathcal{L}(q,p)$, then :*

$$v = H_p(q,p); \quad H_q(q,p) = -L_q(q,v); \quad p = L_v(q,v).$$

We deduce immediately from these properties that : L is C^k iff H is C^k.

The Euler-Lagrange flow (f_t^L) is defined on TM by : $f_t^L = \mathcal{L} \circ \varphi_t^H \circ \mathcal{L}^{-1}$. An orbit of this flow is $(q,\dot{q})$ where $q \;: I \to M$ satisfies the so-called Euler-Lagrange equations :

$$\frac{d}{dt}\left(L_v(q,\dot{q})\right) = L_q(q,\dot{q}).$$

Then $(q,\dot{q})$ is an orbit of the Euler-Lagrange flow iff γ is a critical point of the Lagrangian action among the C^1 arcs which have the same ends. Such a solutions is always C^2.

1.4 More on the monotony property

We only state :

Proposition : *(A. Fathi) Let $H : T^*M \to \mathbb{R}$ be a monotone Hamiltonian and L be the corresponding Lagrangian. Let $K \subset M$ be a compact subset of M and $C > 0$ be a constant.*
Then there exists $\varepsilon > 0$ such that, for every $q \in K$ and every $t \in]-\varepsilon, \varepsilon[\backslash\{0\}$: $\pi \circ f_t^L(B_{T_qM}(0, 2C)) \supset B(q, C|t|)$ and $\pi \circ f^L_{t|B_{T_qM}(0,2C)}$ is a C^1 diffeomorphism.

2 Lipschitz Lagrangian submanifolds, images of such manifolds and minimization properties

2.1 Lagrangian submanifolds

Lagrangian submanifold : a C^1 submanifold N of T^*M is a Lagrangian submanifold of T^*M if $\dim N = \dim M = d$ and $\omega_{|TN} = 0$; it is equivalent to : for every C^1 loop γ drawed on N and homotopic in N to a point, $\int_\gamma \lambda_0 = 0$; this last equality being also true for Lipschitz arcs, we define :

Lipschitz Lagrangian submanifold : it is a d-dimensional Lipschitz submanifold N such that, for every Lipschitz loop drawed on N and homotopic in N to a point : $\int_\gamma \lambda_0 = 0$. Such a manifold is *exact Lagrangian* if the same equality is true for every Lipschitz loop (not necessarily homotopic to a point) drawed on N. We recall :

Lipschitz submanifold : N is a d-dimensional Lipschitz submanifold of T^*M if for every $x \in M$, there exists a neighbourhood U of x in T^*M, a neighbourhood V of 0 in $\mathbb{R}^d$, a Lipschitz map $\psi : V \to V$ and a C^∞ diffeomorphism $F : U \to V \times V$ such that $F(N \cap U) = \text{graph}(\psi)$.

Lipschitz graph : it is a Lipschitz section of $\pi : T^*M \to M$. Let us notice that this notion is stronger than "Lipschitz submanifold which is a section."

Definition : Let $\eta : M \rightarrow T^*M$ a continuous 1-form. This 1-form is *closed* (in the distribution sense) if for every Lipschitz loop drawed on M which is homotopic to a point, we have : $\int_\gamma \eta = 0$. In this definition, we may replace "Lipschitz loop" by "C^1-loop".
The Lipschitz 1-form η is *exact* if for every Lipschitz loop drawed on M, we have : $\int_\gamma \eta = 0$.

Proposition : *Let $\eta : M \rightarrow \mathbb{R}$ be a Lipschitz 1-form. We denote by $\mathcal{G}_\eta$ the graph of η. Then, $\mathcal{G}_\eta$ is Lipschitz Lagrangian iff η is closed. Moreover, $\mathcal{G}_\eta$ is exact Lagrangian iff η is exact.*

Remark : in a similar way, we may define the "C^0 Lagrangian graphs", initially introduced by M. Herman.

Definition : let $\mathcal{G}_\eta$ be a Lipschitz Lagrangian graph. The *cohomological class* of $\mathcal{G}_\eta$ is the cohomological class of the 1-form η.

2.2 Images of Lipschitz Lagrangian submanifolds

Proposition : *(1) If N is a Lipschitz Lagrangian submanifold of T^*M, then for every t, $\varphi_t(N)$ is a Lipschitz Lagrangian submanifold of T^*M ;*
(2) if N is the Lipschitz Lagrangian graph of η and if for a $t \in \mathbb{R}$ $N_t = \varphi_t(N)$ is the Lipschitz Lagrangian graph of η_t, then the cohomological classes are equal : $[\eta] = [\eta_t]$.

Proposition : *Let $\mathcal{G}$ be a Lipschitz graph (of η) above a compact part K of M. Then there exists $\varepsilon > 0$ such that for every $t \in]-\varepsilon, \varepsilon[$, the set $\mathcal{G}_t = \varphi_t(\mathcal{G})$ is a Lipschitz graph above a compact part of M.*

2.3 Hamilton-Jacobi equation and some properties of minimization

Situation : we assume that $U \subset M$ is open and $u : U \to \mathbb{R}$ is a $C^{1,1}$ function ; then the graph $\mathcal{G}$ of du is an exact Lipschitz Lagrangian graph. We assume that J is an interval of $\mathbb{R}$ containing 0 and that, for any $t \in J$, $\mathcal{G}_t = \varphi_t(\mathcal{G})$ is a Lipschitz (and then exact Lagrangian) graph.
We denote by U_t the set $\pi(\mathcal{G}_t)$ and we write : $q_t = \pi \circ \varphi_t(du(q))$.

Hamilton-Jacobi equation : if $u_t : U_t \to \mathbb{R}$ is defined by :

$$u_t(q_t) = u(q) + \int_0^t L(q_s, \dot{q}_s)ds,$$

then :

- $\mathcal{G}_t$ is the graph of du_t ;
- (H-J) : $\dot{u}_t + H(., du_t(.)) = 0$.

Hence the Hamilton-Jacobi equation is a partial differential equation (PDE) describing the evolution of an exact Lagrangian graph under the Hamiltonian flow.

Remarks :

1. We may see in the proof that if $\mathcal{U} = \bigcup_{t \in J} \{t\} \times U_t$ and $\tilde{u}_t : \mathcal{U} \to \mathbb{R}$ is defined by $\tilde{u}(t, x) = u_t(x)$, then $\mathcal{U}$ is an open subset of $J \times M$ and $\tilde{u}$ is $C^{1,1}$.
2. If we are interested in Lipschitz Lagrangian graphs which are not exact, for example in those which are in the cohomological class of $\nu : M \to T^*M$, we obtain that $\mathcal{G}_t$ is the graph of $\nu + du_t$ where the definition of u_t and the Hamilton-Jacobi equation are valuable for the modified Hamiltonian : $H_\nu(q, p) = H(q, p + \nu(q))$ (in this case the modified Lagrangian is : $L_\nu(q, v) = L(q, v) - \nu(q)(v)$).
3. A. Fathi proved that every C^1-solution of the H.-J. equation is in fact $C^{1,1}$.

We will know assume some additional hypotheses :

Definition : Let $H : T^*M \to \mathbb{R}$ be C^2. We say that H is a Tonelli Hamiltonian if :

1. H is superlinear in the fiber :
$$\forall q \in M, \lim_{\|p\| \to \infty} \frac{H(q, p)}{\|p\|} = +\infty;$$
2. H is strictly convex in the fiber : $\forall (q, p) \in T^*M, H_{p,p}(q, p)$ is positive definite.

Proposition : *If H is a Tonelli Hamiltonian, then it is monotone.*

Proposition : *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let $\mathcal{G}$ be a Lipschitz exact Lagrangian graph above an open subset U of M such that, for every $t \in [0, \Delta]$, the set $\mathcal{G}_t = \varphi_t(\mathcal{G})$ is a Lipschitz graph.*
then, for every $q_0 \in U$, if $\gamma_0(s) = \pi \circ \varphi_s(q)$, for every continuous $\gamma : [0, \Delta] \to M$ which is piecewise C^1 and such that :

- *$\gamma(0) = \gamma_0(0)$ and $\gamma(\Delta) = \gamma_0(\Delta)$;*
- *$\forall t \in [0, \Delta], \gamma(t) \in U_t = \pi(\mathcal{G}_t)$;*

we have : $A_L(\gamma_0) \leq A_L(\gamma)$, with equality iff $\gamma = \gamma_0$.

Corollary 1 : *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian , let* $\mathcal{G}$ *be an exact Lagrangian Lipschitz graph above* M *and let* I *be an interval of* $\mathbb{R}$ *containing* 0 *such that for every* $t \in J$*,* $\varphi_t(\mathcal{G})$ *is a Lipschitz graph. Then every arc of orbit* $(\varphi_t(x_0))_{t\in I}$ *with initial contition* x_0 *in* $\mathcal{G}$ *is strictly minimizing with fixed ends.*

Corollary 2 : *(Weierstrass) Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian, let* $K \subset M$ *be a compact subset of* M *and let* $C > 0$ *be a constant.*
There exists $\varepsilon > 0$ *such that, for every* $q_0 \in K$*, every* $t \in]0, \varepsilon[$*, every* $q \in M$ *such that* $d(q_0, q) \leq Ct$*, there exists a strict minimizer of the Lagrangian action joining* $(0, q_0)$ *to* (t, q)*. Moreover, such a* γ *is a solution of the Euler-Lagrange equations.*

SECOND LECTURE :

Lagrangian subbundles.

3 Links with the Lagrangian bundles

We fix a Lagrangian subbundle of $T(T^*M)$, called the *vertical bundle* : $\forall x \in T^*M, V(x) = \ker D\pi^*(x)$.

3.1 On the Lagrangian bundles which are transverse to the vertical

Notation : $\mathcal{L}_V$ is the bundle of the Lagrangian linear spaces wich are transverse to the vertical.
Introduction of an order relation : if $x \in T^*M$ and $L_1, L_2 \in \mathcal{L}_V(x)$, the *height* of L_2 above L_1 (relatively to $V(x)$) is the quadratic form $Q(L_1, L_2)$ defined on $E_x = T_x(T^*M)/V(x)$ (which is isomorphic to $T_{\pi(x)}M$) by :

$$\forall X \in E_x, Q(L_1, L_2)(X) = \omega((p_{|L_1})^{-1}(X), (p_{|L_2})^{-1}(X))$$

where $p\ : T(T^*M) \to E$ is the projection.
Then :

- we say : L_2 is *above* L_1 and write $L_1 \leq L_2$ when $Q(L_1, L_2)$ is positive ;
- we say : L_2 is *strictly above* L_1 and write $L_1 < L_2$ when $Q(L_1, L_2)$ is positive definite.

Proposition : *We have :*

- $\forall L_1, L_2 \in \mathcal{L}_V(x), Q(L_1, L_2) = -Q(L_2, L_1)$;
- $\forall L_1, L_2, L_3 \in \mathcal{L}_V(x), Q(L_1, L_2) + Q(L_2, L_3) = Q(L_1, L_3)$.

We deduce frome this result that $\leq$ is a preorder relation on $\mathcal{L}_V(x)$ (i.e. reflexive and transitive).

Let us denote the set of quadratic forms of E by $\mathcal{Q}(E)$.

Proposition : *Let us fix $L \in \mathcal{L}_V(x)$. Then the map $Q(L, .)\ : \mathcal{L}_V(x) \to \mathcal{Q}(E_x)$ is a homeomorphism. Moreover :*

$$\forall L' \in \mathcal{L}_V(x), L \cap L' = (p_{|L})^{-1}(\ker Q(L, L')) = (p_{|L'})^{-1}(\ker Q(L, L')).$$

Hence $\leq$ is an order relation (antisymmetric).

We assume that K is a subset of T^*M and that $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}_V(K)$ are two Lagrangian subbundles of $T(T^*M)$ above K, we write $\mathcal{L}_1 < \mathcal{L}_2$ (resp. $\mathcal{L}_1 \leq \mathcal{L}_2$) if for every $x \in K$, we have : $\mathcal{L}_1(x) < \mathcal{L}_2(x)$ (resp. $\mathcal{L}_1(x) \leq \mathcal{L}_2(x)$).

First examples of Lagrangian bundles : If K is a C^1 Lagrangian submanifold of T^*M, its tangent bundle is a Lagrangian subbundle of $T(T^*M)$.
If K is the graph of a Lipschitz closed 1-form η, the set D of differentiability points of η is a dense subset of M ; let us define $K_D = \eta(D)$. Then at every point of K_D there exists a tangent space to K_D, which belongs to $\mathcal{L}_V(K_D)$.

Definition Let $\mathcal{L} \in \mathcal{L}_V(K)$. We say thet $\mathcal{L}$ is *upper semi-continuous* (u.s.c.) (resp. *lower semi-continuous* (l.s.c.)) if for every $\mathcal{L}_1 \in \mathcal{L}_V(K)$ which is continuous, then $\{x \in K; \mathcal{L}(x) < \mathcal{L}_1(x)\}$ (resp. $\{x \in K; \mathcal{L}_1(x) < \mathcal{L}(x)\}$) is open in K.

Proposition *Let $\mathcal{L} \in \mathcal{L}_V(K)$. Then :*

- *the bundle $\mathcal{L}$ is continuous iff it is u.s.c. and l.s.c ;*
- *if $\mathcal{L}$ is the (simple) limit of an increasing sequence of l.s.c. bundles of $\mathcal{L}_V(K)$, then $\mathcal{L}$ is l.s.c.*

Proposition *Let $\mathcal{L}_-, \mathcal{L}_+ \in \mathcal{L}_V(K)$ such that :*

- *the bundle $\mathcal{L}_+$ is u.s.c. ;*
- *the bundle $\mathcal{L}_-$ is l.s.c ;*
- $\mathcal{L}_- \leq \mathcal{L}_+$.

Then $G = \{x \in K; \mathcal{L}_-(x) = \mathcal{L}_+(x)\}$ is a G_δ subset of K. Moreover, if $\mathcal{L} \in \mathcal{L}_V(K)$ is such that $\mathcal{L}_- \leq \mathcal{L} \leq \mathcal{L}_+$, then $\mathcal{L}$ is continuous at every point of G.

3.2 Images of the vertical in the convex case

Proposition *Let $K \subset T^*M$ be compact and $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Then there exists $\varepsilon > 0$ such that, for every $x_0 \in K$, for every $t \in]-\varepsilon, \varepsilon[\backslash\{0\}$, the Lagrangian subspace $G_t(x_0) = D\varphi_t.V(\varphi_{-t}x_0)$ is transverse to the vertical $V(x_0)$ and such that :*

$$\forall -\varepsilon \leq s' < s < 0 < t < t' \leq \varepsilon, \quad G_s(x_0) < G_{s'}(x_0) < G_{t'}(x_0) < G_t(x_0).$$

The proof use the following result :
Lemma *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let $x_0 \in T^*M$ and $I = [-T, T']$ be such that :*

$$\forall t, s \in I, t \neq s \quad \text{and} \quad t.s \geq 0 \Rightarrow G_t(x_0) \cap G_s(x_0) = \{0\}.$$

Then :

$$\forall -T \leq s' < s < 0 < t < t' \leq T', G_s(x_0) < G_{s'}(x_0) < G_{t'}(x_0) < G_t(x_0).$$

3.3 The notion of conjugate points

Definition Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. The points $(x_1; t_1)$, $(x_2; t_2) \in T^*M \times \mathbb{R}$ are *conjugate* if :

$$D\varphi_{t_2-t_1}V(x_1) \cap V(x_2) \neq \{0\}.$$

In other words, $(x_1; t_1)$ and $(x_2; t_2)$ are conjugate if $\varphi_{t_2-t_1}(x_1) = x_2$ and there exists an infinitesimal orbit $\delta x = (\delta q, \delta p)$ ("infinitesimal" means for the linearized flow) along $(\varphi_t(x_1))_{t\in[0,t_2-t_1]}$ which is not the zero infinitesimal solution and such that $\delta q(0) = \delta q(t_2 - t_1) = 0$.

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. If $(x_0; t_0) \in T^*M \times \mathbb{R}$ and if I is an interval containing t_0, the two following sentences are equivalent :*

1. *there is no pair of conjugate points on $(\varphi_{t-t_0}(x_0), t)_{t\in I}$;*
2. *forall $t \neq s$ in $\mathbb{R}^*$ such that $t_0 - t, t_0 - s \in I$, then $G_t(x_0) \cap G_s(x_0) = \{0\}$.*

Theorem *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian. We consider* $x_0 \in T^*M$ *and an interval* $I \subset \mathbb{R}$*. The three first following propositions are equivalent; if* I *is open, the fourth proposition is equivalent to the others :*

1. *for every* $[a, b] \subset I$*, there exists a Lagrangian bundle along* $(\varphi_t(x_0), t)_{t\in[a,b]}$ *which is invariant under* $(D\varphi_t)$ *and transverse to the vertical;*
2. *for every* $[a, b] \subset I$*, there exists a family of exact Lagrangian* C^1 *graphs* $(\mathcal{G}_t)_{t\in[a,b]}$ *above some open subsets* U_t *of* M *such that :* $\forall t \in [a, b], \varphi_t(x_0) \in \mathcal{G}_t$ *and* $\forall s, t, \varphi_{t-s}(\mathcal{G}_s) = \mathcal{G}_t$ *;*
3. *there is no conjugate point along* $(\varphi_t(x_0); t)_{t\in[a,b]}$ *;*
4. *for every* $[a, b] \subset I$*, the orbit* $(\varphi_t(x_0))_{t\in[a,b]}$ *is locally minimizing, i.e. if* $\gamma_0(t) = \pi \circ \varphi_t(x_0)$*, there exists a neighbourhood* U_0 *of* γ_0 *in* C^0 *topology such that, for every* $\gamma : [a, b] \to M$ *in* U_0 *which is continuous and piecewise* C^1 *and has the same ends as* γ_0 *:*

$$\int_a^b L(\gamma_0(t), \dot\gamma_0(t))dt \le \int_a^b L(\gamma(t), \dot\gamma(t))dt$$

with equality if and only if $\gamma_0 = \gamma$*.*

Proposition *We assume that* H *satisfies the four equivalent assumptions of the theorem. We assume that* $[a, b] \subset I$ *and that* $(L^t)_{t\in[a,b]}$ *is an invariant Lagrangian bundle along* $(\varphi_t(x_0))_{t\in[a,b]})$ *which is transverse to the vertical. Then :*

$$\forall a \le t_1 < t < t_2 \le b, G_{t-t_2}(\varphi_t(x_0)) < L^t < G_{t-t_1}(\varphi_t(x_0)).$$

Hence in this case the images of the vertical allows us to bound L^t from above and below.

THIRD LECTURE :

Green bundles.

4 Green bundles

4.1 Construction of the Green bundles

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let $x_0 \in T^*M$. Then :*

- *if $(\varphi_t(x_0))_{t>0}$ has no conjugate point, then for every $s > 0$, $(G_{-t}(\varphi_s(x_0)))_{t>0}$ is a strictly increasing family of Lagrangian subspaces which are transverse to the vertical; moreover, we can define $G_-(\varphi_s(x_0)) = \lim_{t\to+\infty} G_{-t}(\varphi_s(x_0))$; then G_- is a Lagrangian subbundle which is transverse to the vertical; on its set of definition, G_- is l.s.c;*
- *if $(\varphi_t(x_0))_{t<0}$ has no conjugate point, then for every $s < 0$, $(G_t(\varphi_s(x_0)))_{t>0}$ is a strictly decreasing family of Lagrangian subspaces which are transverse to the vertical; moreover, we can define $G_+(\varphi_s(x_0)) = \lim_{t\to+\infty} G_t(\varphi_s(x_0))$; then G_+ is a Lagrangian subbundle which is transverse to the vertical; on its set of definition, G_- is u.s.c;*
- *if $\mathcal{C} = \{x; (\varphi_t(x))_{t\in\mathbb{R}}$ has no conjugate point$\}$, G_- and G_+ are defined on $\mathcal{C}$, $G_- \leq G_+$ and G_- and G_+ are invariant by $(D\varphi_t)$.*

Definition The bundles G_- and G_+ are the *Green bundles*.

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Then :*

1. *if there exists a Lagrangian subbundle $\mathcal{L}$ of $T(T^*M)$ along $(\varphi_t(x_0); t)_{t>0}$ which is transverse to the vertical and invariant under $(D\varphi_t)$, then :*

$$\forall t > 0, G_-(\varphi_t(x_0)) \leq \mathcal{L}(\varphi_t(x_0); t);$$

2. *if there exists a Lagrangian subbundle $\mathcal{L}$ of $T(T^*M)$ along $(\varphi_t(x_0); t)_{t<0}$ which is transverse to the vertical and invariant under $(D\varphi_t)$, then :*

$$\forall t < 0, \mathcal{L}(\varphi_t(x_0); t) \leq G_+(\varphi_t(x_0)).$$

4.2 A dynamical criterion

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. We consider $x \in T^*M$ whose orbit is relatively compact, $v \in T_x(T^*M)$ and $\varepsilon > 0$. Then :*

- *if $(\varphi_t(x))_{t>-\varepsilon}$ has no conjugate point, if $v \notin G_-(x)$, then :*

$$\lim_{t\to+\infty} \|D(\pi \circ \varphi_t)(x)v\| = +\infty;$$

- *if $(\varphi_t(x))_{t<\varepsilon}$ has no conjugate point, if $v \notin G_+(x)$, then :*

$$\lim_{t\to+\infty} \|D(\pi \circ \varphi_{-t})(x)v\| = +\infty.$$

Consequences : 1) Let us assume that $K \subset T^*M$ is invariant under (φ_t), has no conjugate point and is such that $(D\varphi_{t|T_K(T^*M)})$ is partially hyperbolic with a decomposition :

$$T_x(T^*M) = E^s(x) \oplus E^c(x) \oplus E^u(x).$$

Then $E^s(x) \subset G_-(x)$ and $E^u(x) \subset G_+(x)$.

2) If the orbit of x is relatively compact, with no conjugate point and non critical, then :

$$\mathbb{R}X_H(x) \subset G_-(x) \cap G_+(x).$$

4.3 The reduced Green bundles

Let us introduce some notations/assumptions :

• we consider a level set $\mathcal{E} = H^{-1}(c)$ and a subset $\mathcal{F} \subset \mathcal{E}$ which is invariant and such that : $\forall x \in \mathcal{F}, X_H(x) \notin V(x)$.
We define a bundle F above $\mathcal{F}$ whose fiber is $F(x) = T_x\mathcal{E}/\mathbb{R}X_H(x)$. The corresponding projection is denoted by $p\ : F \to \mathcal{F}$.

• The symplectic product Ω is defined on F by :

$$\forall u, v \in T_x\mathcal{E}, \Omega(p(u), p(v)) = \omega(u, v).$$

The vertical is $v(x) = p(V(x) \cap T_x\mathcal{E})$ and is Lagrangian (because X_H is not vertical). Then we will be interested in the heights relatively to $v(x)$.

• As we have : $D\varphi_t \mathbb{R}X_H(x) = \mathbb{R}X_H(\varphi_t x)$, we may define the reduced cocycle M_t on $\mathcal{F}$.
We assume that $x \in \mathcal{F}$; we have : $\forall t, X_H(x) \notin G_t(x)$ (because X_H is not vertical) ; hence, $g_t(x) = p(G_t(x) \cap T_x\mathcal{E})$ is a Lagrangian subspace of $F(x)$. Moreover, we have : $g_t(x) = M(\varphi_{-t}x)v(\varphi_{-t}x)$.

Lemma *Let $x \in \mathcal{F}$ be such that its orbit has no conjugate point ; then, for every $t \neq 0$, $g_t(x)$ is transverse to $v(x)$.*

A direct consequence is that, with the same assumptions as in the lemma, for every $t \neq t'$, the two spaces $g_t(x)$ and $g_{t'}(x)$ are transverse.

Proposition *We assume that $x \in \mathcal{F}$ has no conjugate point ; Then :*

1. $\forall t \in \mathbb{R}^*, g_t(x)$ *is transverse to* $v(x)$ *;*
2. $\forall s' < s < 0 < t < t', g_s(x) < g_{s'}(x) < g_{t'}(x) < g_t(x)$.

As in the non reduced case, we deduce that if $x \in \mathcal{F}$ has no conjugate point :

• then $g_-(x) = \lim\limits_{t \to -\infty} g_t(x)$ and $g_+(x) = \lim\limits_{t \to +\infty} g_t(x)$ are two Lagrangian subspaces of $F(x)$ such that $g_- \leq g_+$ and : $\forall t, M_t(g_\pm(x)) = g_\pm(\varphi_t x)$.

• if $K \subset \mathcal{F}$ is invariant under (φ_t) and has no conjugate point, then g_+ is u.s.c, g_- is l.s.c. and $\mathcal{G} = \{x \in K; g_-(x) = g_+(x)\}$ is a G_δ subset of K such that g_- and g_+ are continuous at every point of $\mathcal{G}$. Moreover, if g is any Lagrangian subbundle of F above K such that $g_- \leq g \leq g_+$, then g is continuous at every point of $\mathcal{G}$.

• Let g be a Lagrangian subbundle of F above $\{\varphi_t x; t \in \mathbb{R}\}$ such that :

1. $\forall t \in \mathbb{R}, g(\varphi_t x)$ is transverse to $v(\varphi_t x)$;
2. $\forall t \in \mathbb{R}, g(\varphi_t x) = M_t(g(x))$.

Then : $\forall t \in \mathbb{R}, g_-(\varphi_t x) \leq g(\varphi_t x) \leq g_+(\varphi_t x)$.

We have a dynamical criterion too :

Proposition *We assume that $\mathcal{F} \subset \mathcal{E}$ is invariant by the Hamiltonian flow, has no conjugate point, is compact and such that the angle between the Hamiltonian vectorfield X_H and the vertical is uniformly bounded from below when it is defined (i.e. when $X_H \neq 0$). Then, for every $x \in \mathcal{F}$ and $v \in T_x\mathcal{E}$:*

- *if $v \notin G_-(x)$,then : $\lim\limits_{t\to+\infty} \|p(D\varphi_t(x)v)\| = +\infty$;*
- *if $v \notin G_+(x)$,then : $\lim\limits_{t\to+\infty} \|p(D\varphi_{-t}(x)v)\| = +\infty$.*

4.4 A characterization of hyperbolicity

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and K be a compact subset invariant by (φ_t) which is contained in an energy level $\mathcal{E}$, with no conjugate point and such that : $\forall x \in K, X_H(x) \notin V(x)$. Then the two following properties are equivalent :*

- *$(D\varphi_t)$ restricted to $T\mathcal{E}_{|K}$ is hyperbolic ;*
- *On K, g_- and g_+ are transverse.*

FOURTH LECTURE :

Regularity of Lipschitz Lagrangian invariant graphs.

5 Regularity of the Lipschitz Lagrangian invariant graphs

5.1 Generalized tangent vectors and cones

Definition Let $U \subset \mathbb{R}^d$ be open and $h : U \to \mathbb{R}^n$ be a topological embedding. If $x \in U$, $w \in \mathbb{R}^n$ is a generalized tangent vector to h at x if there exists some sequences $(x_k) \in U$, $(t_k) \in \mathbb{R}_+^*$ such that $\lim\limits_{k\to\infty} x_k = x$, $\lim\limits_{k\to\infty} t_k = 0$ and :

$$w = \lim_{k\to\infty} \frac{1}{t_k}(h(x_k) - h(x)).$$

The set of those vectors is denoted by $T_x^G h$, it is a cone named *tangent cone* at x.

Proposition *If* $\psi : (\mathbb{R}^n, h(x)) \to (\mathbb{R}^n, \psi(h(x)))$ *is a diffeomorphism and* $\varphi : (\mathbb{R}^d, \varphi^{-1}(x)) \to (\mathbb{R}^d, x)$ *is a homeomorphism, then :*

$$T^G_{\varphi^{-1}(x)}(\psi \circ h \circ \varphi) = D\psi(h(x)) T_x^G h.$$

Definition Let us assume that N is the graph of a Lipschitz 1-form λ of M. We define : $T_x^G N = T_{\pi(x)}^G \lambda$ is the *generalized tangent space to* N at x. An element of $T_x^G N$ is called a "generalized tangent vector" to N at x.
Let us notice that in general, $T_x^G N$ is a cone, not a linear space.

Corollary *If λ is a Lipschitz 1-form of M, if ϕ is a diffeomorphism of T^*M such that $\phi(N) = N$, then :*

$$\forall q \in M, T^G_{\phi(\lambda(q))} N = D\phi(\lambda(q)) T^G_{\lambda(q)} N.$$

Proposition *Let N be the Lipschitz graph of $\lambda : M \to T^*M$. Then :*

1. *$\forall x \in N, D\pi(T_x^G N) = T_{\pi(x)} M$;*
2. *if $q \in M$ is such that $T^G_{\lambda(q)} N$ is contained in a d-plane P, then λ is differentiable at q and $T_{\lambda(q)} N = P$.*

5.2 Generalized derivative and generalized tangent planes

Definition Let M, N be two Riemannian manifolds and $\lambda : M \to N$ be a Lipschitz map. Then the set D of points of M where λ has a derivative is a dense subset of M. If $x \in M$, the *generalized derivative* of λ at x is the convex hull of the limits of the sequences $(D\lambda(x_k))$ where $x_k \in D$ and $\lim\limits_{k\to\infty} x_k = x$. This set is denoted by $D^G\lambda(x)$.

Then $D^G\lambda(x)$ is compact, convex and non empty.

Proposition *Let $\lambda : M \to N$ be bi-Lipschitz and let $q \in M$. Then :*

$$T_q^G \lambda \subset \{Lv; v \in T_q M, L \in D^G\lambda(q)\} = \bigcup_{L \in D^G\lambda(q)} L(T_q M).$$

Corollary *Let $\lambda : M \to T^*M$ be a Lipschitz 1 form. The two following assertions are equivalent :*

(i) $D^G\lambda(q)$ has only one element;

(ii) λ is differentiable at q and q is a point of continuity of λ.

Definition we assume that $\mathcal{G} \subset T^*M$ is the graph of $\lambda : M \to T^*M$ which is Lipschitz. A *generalized tangent plane* to $\mathcal{G}$ at $\lambda(q)$ is $\mathrm{Im}(L) = L(T_qM)$ where $L \in D^G\lambda(q)$.

Proposition *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let $\lambda : M \to T^*M$ be a Lipschitz closed 1-form of M whose graph is invariant by the Hamiltonian flow (φ_t) of H. Then :*

$$\forall q \in M, \forall t \in \mathbb{R}_+^*, \forall L \in D^G\lambda(x), G_{-t}(\lambda(q)) < L(T_qM) < G_t(\lambda(q)).$$

5.3 Regularity of the invariant graphs for $d = 2$

Definition if $X_H(x) = 0$, the singularity x is non degenerate if $DX_H(x)$ has no double eigenvalue.

Theorem *Let M be a compact surface and $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Let $\mathcal{G}$ be the Lipschitz Lagrangian graph of λ which is invariant by the Hamiltonian flow of H.*
We assume that every singularity of H belonging to $\mathcal{G}$ is non degenerate.
Then , there is a dense G_δ subset D of M which has full Lebesgue measure and is such that at every point of D, λ is C^1.

Remarks : we proved a similar result for the symplectic twists of $\mathbb{T} \times \mathbb{R}$: every invariant C^0 graph is C^1 on a set with full measure.

5.4 More results on the regularity

Theorem *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{G}$ be a Lipschitz Lagrangian graph invariant by the Hamiltonian flow (φ_t) of H. We assume that there exists $(t_k)_{k\in\mathbb{Z}}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} t_{-n} = -\infty$ and $(\varphi_{t_k|\mathcal{G}})_{k\in\mathbb{Z}}$ is equi-Lipschitz. Then $\mathcal{G}$ is C^1.*

Corollary *Let $H : T^*\mathbb{T}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{G}$ be a Lipschitz Lagrangian graph invariant by the Hamiltonian flow (φ_t) of H. We assume that $\varphi_{1|\mathcal{G}}$ is bi-Lipschitz conjugate to a rotation of $\mathbb{T}^d$. then $\mathcal{G}$ is C^1.*

Definition a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ is C^0-*integrable* if there exists a partition $\mathcal{P}$ of T^*M in C^0-Lagrangian graphs invariant by the Hamiltonian flow such that the map $\mathcal{P} \to H^1(M)$ is surjective.

Theorem *Let $H : T^*M \to \mathbb{R}$ be a C^0-integrable Tonelli Hamiltonian and let $\Lambda_1 \subset \Lambda_1(M)$ be such that $\{\mathcal{G}_\lambda; \lambda \in \Lambda_1\}$ is a partition of T^*M in C^0 Lagrangian invariant graphs. Then there exists a dense G_δ subset $G(H)$ of Λ_1 such that every $\lambda \in G(H)$ is C^1.*

Bibliography

M. .C. Arnaud preprints
http ://www.univ-avignon.fr/fr/recherche/annuaire-chercheurs/membrestruc/personnel/arnaud-marie-claude.html

A. Chenciner, *Calculus of variations in the convex case : an introduction to Fathi's weak KAM theory and Mather's theory of minimal invariant measures* 5 conférences à l'Universitat Politcnica de Catalunya, Barcelone, 26 juin – 2 juillet 2004
http ://www.imcce.fr/fr/presentation/equipes/ASD/person/chenciner/chenciner.html

Gonzalo Contreras & Renato Iturriaga, *Convex Hamiltonians without conjugate points* Ergodic Theory Dynam. Systems **19** , no. 4, 901–952 (1999).

A. Fathi, *Weak KAM theorems in Lagrangian dynamics*, livre en préparation.