Questions of autonomy

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Everything will be in the book

Generating maps, invariant manifolds and related topics

Beginning of the story (1993), suggested by Albert Fathi

Sternberg's theorem for contractions. If two invertible germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $\rho\big(Df(0)\big) < 1$ are formally conjugate to a high enough order $k > 0$, i.e., g and $\psi \circ f \circ \psi^{-1}$ have k^{th} order contact at 0 for some diffeomorphism germ $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, then there is a unique conjugacy of f to g having k^{th} order contact with ψ at 0

(Alexander Shoshitaishvili) A map $h : X \to Y$ semi-conjugates $f : X \to X$ to $g : Y \to Y$ ($i.e.$, $g \circ h = h \circ f$) iff $\mathrm{graph}(h)$ is invariant (stable) under $f \times g : (x, y) \mapsto \big(f(x), g(y)\big)$.

Sternberg revisited: If the image under $f \times g$ of the submanifold germ $V := \mathrm{graph}\,\psi$ has k^{th} order contact with V at 0, then there exists a unique truly $f \times g$–invariant submanifold germ W having k^{th} order contact with V at 0.

The spectra of the automorphisms of T_0V and $\mathbb{R}^{2n}/T_0V$ induced by $D(f \times g)(0)$ both equal the spectrum of $Df(0)$.

The virtue of boredom (1993)

Board of the French Mathematical Society

Let M be C^r manifold, $h : (M, a) \to (M, a)$ a C^r map germ, $r \geq 1$, and $E = T_aM$. Assume that $L = Dh(a) : E \to E$ preserves a vector subspace S, that the induced endomorphism A of S satisfies $\rho(A) < 1$, that the induced endomorphism B of E/S is invertible and $r > \ell := \dfrac{\ln \rho(B^{-1})}{-\ln \rho(A)}$. Set

$$k := \max\{1, [\ell]\} \text{ and } m := \begin{cases} 1 & \text{for } \ell < 1 \\ \min\{k+1, r\} & \text{otherwise} \end{cases}$$

(hence $\ell < m \leq r$).

Then, for each C^r submanifold germ V of M at a such that $T_aV = S$, formally h-invariant to order k, there is a unique h-invariant C^m submanifold germ W at a having k^{th} order contact with V at a such that $T_aW = S$, and W is C^r.

References, first examples and sketch of the proof

Variétés stables et formes normales, *C. R. Acad. Sc. Paris* **317**, *Série 1* (1993), 87–92
(Invariant manifolds revisited, *Proceedings of the Steklov Institute* **236** (2002), 415–433)

Sternberg: up to linear conjugacy, $A = B = Df(0)$, hence $\ell = \frac{\ln \rho(A^{-1})}{-\ln \rho(A)} \geq k = [\ell] \geq 1$. For $r > 1$, the C^r contraction germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is C^r-conjugate to its k^{th} order Taylor polynomial $j_0^k f$ via a unique φ such that $j_0^k \varphi = j_0^k \mathrm{Id}$. Hence, for $r > 1$, the C^r contraction germ $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is C^r-linearisable. False if $r = 1$ (Sternberg).

The proof. Local chart $(M, a) \to (S \times U, 0)$ in which $V = S \times \{0\}$. Implicit equation $z_{n+1} = h(z_n)$ at 0 in a space of sequences $(z_n) = (x_n, y_n)$ such that x_n and y_n tend to 0 as the components of an orbit in W should. The graph of the y_0 component of the implicit function germ $(x_{n+1}, y_n)_{n \geq 0} = \Phi(x_0)$ thus obtained is W.

How I failed to escape poverty (1993) but survived (1995)

Oberwolfach, Summer '93
Jean-Pierre Eckmann

Geneva colloquium, Fall '93
Vaughan Jones

(with F. Coudray): Invariant manifolds, conjugacies and blow-up. *Ergodic Theory and Dynamical Systems* **17** (1997), 783–791.

"Preparation Lemma": put a diffeomorphism or vector field germ into normal form all along the stable and/or unstable subspaces of its linear part.

"Blow-up" $y = zx^k$: losing uniqueness and requiring too much differentiability, my '93 theorem follows from the stable manifold theorem in (x, z)-space

and the C^m Sternberg-Chen theorem, $1 \leq m < \infty$, from the pseudo-stable manifold theorem.

Sternberg-Chen. Let $h_0 : (M_0, a_0) \to (M_0, a_0)$ be a *hyperbolic* smooth diffeomorphism germ. Then, for each integer $m \geq 1$, there is an integer $m_1 \geq m$ with the following property: whenever h_0 and a smooth $h_1 : (M_1, a_1) \to (M_1, a_1)$ are formally conjugate to order m_1, they are C^m conjugate.

One can assume (Preparation Lemma) that h_0 and h_1 have ℓ^{th} order contact along the unstable manifold W^u of h_0: the germ $V := \Delta M_0$ is invariant to order ℓ under $h := h_0 \times h_1$ along $W_0 := \Delta W^u$.

Let the C^∞ germ $h : (M, a) \to (M, a)$ preserve a C^∞ submanifold germ W_0; assume $L := Dh(a)$ preserves a vector subspace S of $E := T_a M$ containing $S_0 := T_a W_0$ and induces endomorphisms A, A_1, B of S, $S_1 := S/S_0$, E/S with $\rho(A_1) < 1$, $\rho(A) > 1$ and B invertible.
Then, for each integer $k \geq 1$, there is an integer $k_1 \geq k$ with the following property: for every C^∞ submanifold germ V with $T_p V = S$, formally h–invariant along W_0 to order k_1, there exists an h–invariant C^k submanifold germ W having k^{th} order contact with V along W_0.

Extension du domaine de la lutte (2000-2002)

Proof. In a chart $(M, V, W_0, a) = (S_0 \times S_1 \times E/S, S_0 \times S_1, S_0, 0)$, blow-up $(\theta, x, z) \mapsto (\theta, x, zx^{k_1})$.

True: For each C^∞ submanifold germ V with $T_pV = S$, formally h–invariant along W_0, there is an h–invariant C^∞ submanifold germ W having infinite contact with V along W_0 (hence the C^∞ Sternberg-Chen theorem).

But non-uniqueness, so one cannot just let $k \to \infty$.

Invariant manifolds revisited, *Proceedings of the Steklov Institute* **236** (2002), 415–433 (second half)

Stable manifolds and the Perron-Irwin method. *Ergodic Theory and Dynamical Systems* **24** (2004), 1359–1394

Reprinted in A. Fathi, J.-C.Yoccoz (ed): *Dynamical Systems: Michael Herman Memorial Volume*, Cambridge University Press, 2006.

Invert the last component with respect to the last variable

Given $h = (f, g)$, find (in a suitable space) the sequences $(z_n) = (x_n, y_n)$ such that $z_{n+1} = h(z_n)$ (orbits of h) as solutions of $x_{n+1} = f(z_n)$ and $y_n = G(x_n, y_{n+1})$.

México (UNAM) colloquium, September '05

Write the previous equations under the more symmetric form $x_{n+1} = F(x_n, y_{n+1}) := f\big(x_n, G(x_n, y_{n+1})\big)$, $y_n = G(x_n, y_{n+1})$.

I call (F, G) a generating map of h.

The Lipschitzian core of some invariant manifold theorems. *Ergodic Theory and Dynamical Systems* **28** (2008), 1419–1441.

R. McGehee, E. A. Sander: A new proof of the stable manifold theorem, *Z. Angew. Math. Phys.* **47**, no. 4 (1996), 497–513.

Generating maps

$E = E_s \times E_u$ (Banach spaces), $H = (f, g) : (E, 0) \to (E, 0)$ local C^r map, $r \geq 1$, $DH(0) = A_s \times A_u$, A_u invertible, $|A_s| < 1 < |A_u^{-1}|^{-1}$.
$(x, y) \mapsto \big(x, g(x, y)\big)$ local diffeo, inverse $(x, y') \mapsto \big(x, G(x, y')\big)$.
If $F(x, y') := f\big(x, G(x, y')\big)$ then, near 0,
$\operatorname{graph} H = \{(x, y, x', y') : x' = F(x, y'), y = G(x, y')\}$.

Shut yourself in a small box Z: $\rho > 0$ small,
$Z = X \times Y = \big\{(x, y) \in E_s \times E_u : \max\{|x|, |y|\} \leq \rho\big\}$.
$\operatorname{Lip} F|_Z (\simeq |A_s|) \leq \lambda < 1$, $\operatorname{Lip} G|_Z (\simeq |A_u^{-1}|) < \mu < 1$ for small enough ρ, hence $(F, G)(Z) \subset Z$.

$(F, G) : Z \to Z$ generating map of $H|_{Z \cap h^{-1}(Z)}$ viewed as the correspondence h of Z into itself (map $Z \to \mathcal{P}(Z)$) given by $h(z) = \{H(z)\}$ if $H(z) \in Z$, $h(z) = \emptyset$ otherwise, i.e.:

$$\begin{aligned} \operatorname{graph} h &:= \bigcup_{z \in Z} \{z\} \times h(z) \\ &= \{(x, y, x', y') \in Z^2 : x' = F(x, y'), y = G(x, y')\}. \end{aligned}$$

Generating maps

For all $\ell \in \mathbb{N}$, the ℓ^{th} iterate h^ℓ of h has a C^r generating map $(F_\ell, G_\ell) : Z \to Z$.

For all sequences (y_ℓ) in Y, the sequence of maps $x \mapsto G_\ell(x, y_\ell)$ converges to the same C^r function $\varphi : X \to Y$.

Its graph is the local stable manifold of $H|_{Z \cap H^{-1}(Z)}$ at 0.

Geometrically, the projection of the graph of h^ℓ onto the first factor of Z^2 (i.e., the set $\operatorname{dom} h^\ell$ of those $z \in Z$ such that $H^j(z)$ remains in Z for $0 \leq j \leq \ell$) becomes thinner and thinner when ℓ increases, and tends to the stable manifold.

No functional analysis involved.

ICMAT Madrid, November 2013

Generating maps, invariant manifolds, conjugacy. *Journal of Geometry and Physics* **87** (2015), 76–85.

My '93 theorem follows from a non-autonomous version of the stable manifold theorem (yielding Pesin's, etc.).

One invariant manifold theorem to rule them all

(Lipschitzian part) I intersection of $\mathbb{Z}$ with an interval, unbounded from above. For all $i \in I$, $Z_i = X_i \times Y_i$ product of two complete metric spaces, h_{i+1} correspondence of Z_i into Z_{i+1} admitting a generating map (F_{i+1}, G_i).

Assume there are positive numbers λ, μ such that, for all $i \in I$, $\mathrm{Lip}\, F_{i+1} \leq \lambda$, $\mathrm{Lip}\, G_i \leq \mu$, $\lambda\mu < 1$ and $\lim_{\ell \to \infty} \mu^\ell \,\mathrm{diam}\, Y_\ell = 0$.

Then, for each $j \in I$, the set W_j^s of those $z \in Z_j$ such that there exists an orbit $(z_i)_{i \geq j}$ of $(h_{i+1})_{i \geq j}$ with $z_j = z$ is the graph of a function $\varphi_j : X_j \to Y_j$ with $\mathrm{Lip}\, \varphi_j \leq \mu$.

One has $h_{j+1}^{-1}(W_{j+1}^s) = W_j^s$ and the correspondences $W_j^s \ni z \mapsto h_{j+1}(z) \cap W_{j+1}^s$ are *maps* $h_{j+1}^s : \big(x, \varphi_j(x)\big) \mapsto \big(f_{j+1}(x), \varphi_{j+1} \circ f_{j+1}(x)\big)$ with $\mathrm{Lip}\, f_{j+1} \leq \lambda$.

Every orbit $(z_i)_{i \leq j}$ of $(h_i)_{i \leq j}$ satisfies $d(z_i, W_i^s) \leq \mu^{j-i} \,\mathrm{diam}\, Y_j$.

Vaughan Jones was not so wrong after all.