# Lagrangian submanifolds in elliptic and log symplectic manifolds

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Geometry and Dynamics in Interaction

15.01.18

# Generalized complex geometry

$M^{2n}$ smooth manifold
$H \in \Omega^3_{cl}(M)$

**Generalized Geometry** (Hitchin, Gualtieri, 2000's):
$TM \rightsquigarrow \mathbb{T}M := TM \oplus T^*M$

## Generalized complex structures

A *GC structure* on $M$ is $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$ s.t.

- $\mathcal{J}^2 = -\mathbb{1}$.
- $\mathcal{J}$ orthogonal w.r.t $\langle X + \xi, Y + \eta \rangle = \eta(X) + \xi(Y)$.
- $+i$-eigenbundle $L \subset \mathbb{T}_{\mathbb{C}}M$ is *integrable* w.r.t.

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + L_X\eta - i_Y \, \mathrm{d}\xi + i_Y i_X H$$

**Examples**

- $I$ complex structure
  $$\Rightarrow \mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix} : \begin{pmatrix} TM \\ T^*M \end{pmatrix} \to \begin{pmatrix} TM \\ T^*M \end{pmatrix} \text{ is GC.}$$
- $\omega$ symplectic form $\Rightarrow \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is GC.

**In general:**

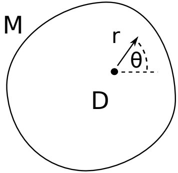
$$\mathcal{J} = \begin{pmatrix} * & Q \\ * & * \end{pmatrix}, Q \text{ Poisson.}$$

# Outline

# Stable GC structures

- Cavalcanti, Gualtieri 2006/2015
- Examples: GC, neither complex nor symplectic
- (Up to gauge equivalence) determined by Poisson structure $Q$.

$Q$ non-denerate except on $D =$ codim-$2$ submanifold. (On $D$: rank $\dim M - 4$)
Write: $\omega = Q^{-1}$



# Logarithmic and elliptic symplectic geometry

| **Logarithmic** | **Elliptic** |
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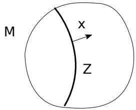
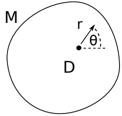
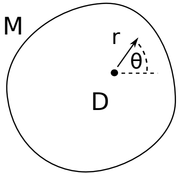
|   |   |
| $Z =$ codim. 1<br>$(x, y_2, \dots, y_{2n}), Z = \{x = 0\}$<br>$TM(-\log Z)$<br>$= \left\langle x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{2n}} \right\rangle$<br>$T^*M(\log Z)$<br>$= \left\langle \frac{dx}{x}, dy_2, \dots, dy_{2n} \right\rangle$ | $D =$ codim. 2<br>$(r, \theta, y_3, \dots), D = \{r = 0\}$<br>$TM(-\log\lvert D\rvert)$<br>$= \left\langle r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y_3}, \dots, \frac{\partial}{\partial y_{2n}} \right\rangle$<br>$T^*M(\log\lvert D\rvert)$<br>$= \left\langle \frac{dr}{r}, d\theta, dy_3, \dots, dy_{2n} \right\rangle$ |
| $\omega \in \Gamma(\wedge^2 T^*M(\log Z))$<br>non-degenerate and $d\omega = 0$ | $\omega \in \Gamma(\wedge^2 T^*M(\log\lvert D\rvert))$<br>non-degenerate and $d\omega = 0$ |

# Stable GC structures

- Cavalcanti, Gualtieri 2006/2015
- Examples: GC, neither complex nor symplectic
- (Up to gauge equivalence) determined by Poisson structure $Q$.

Write: $\omega = Q^{-1}$

$D$ co-oriented, $\omega$ **elliptic symplectic** with $\mathrm{res}_{ell}\,\omega = 0$.



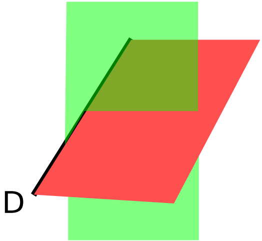

$$\omega = \frac{\mathrm{d}r}{r} \wedge \Omega_I + \mathrm{d}\theta \wedge \Omega_R + \sigma, \;\; \Omega_I, \Omega_R, \sigma \in \Omega^\bullet(D)$$

$$\xcancel{\frac{\mathrm{d}r}{r} \wedge \mathrm{d}\theta}$$

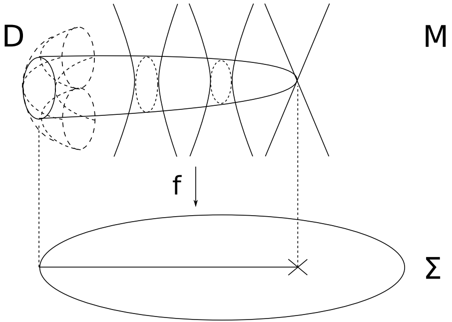
$L^n \subset M^{2n}$ **Lagrangian** $\Leftrightarrow L \cap D$ smooth,
$T(L \cap D) = TL \cap TD|_{L\cap D}$ and

$$\iota_L^* \omega = 0.$$

- **Case 1:** $L \pitchfork D, \dim(L \cap D) = n - 2.$
  elliptic structure,
  "Generalized complex brane"
- **Case 2:**
  $L \cap D = \partial L, \dim(L \cap D) = n - 1.$
  logarithmic structure,
  "Lagrangian brane with boundary"

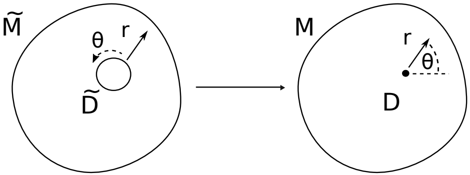


# Examples: Lefschetz thimbles in stable GC Lefschetz fibrations



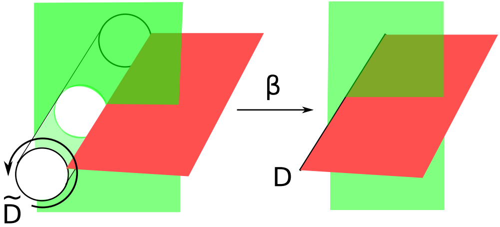

Boundary Lefschetz fibration (Cavalcanti, Klaasse '17):
$D$ fibres over boundary $\partial\Sigma$.
Well-defined notion for *stable GC Lefschetz fibration*:
$\omega$ compatible with $f$.
Can extend Lefschetz thimbles into $D$.

# Real oriented blow-up




| | |
|---|---|
| $\beta\|_{\tilde{D}} : \tilde{D} \to D$ is $U(1)$-principal bundle | $ND$ oriented $\Leftrightarrow$ $ND$ complex. |
| {log vector fields & forms} $\leftrightarrow$ | {elliptic vector fields & forms} |
| $\tilde{\omega} = \beta^*(\omega)$ log symplectic with: <br>• $i_{\frac{\partial}{\partial\theta}} \operatorname{res} \tilde{\omega} = 0$ <br>• $\mathrm{d}i_{\frac{\partial}{\partial\theta}} \tilde{\omega}\|_{\tilde{D}} = 0$ | $\omega$ stable GC |

# Branes under real oriented blow-up



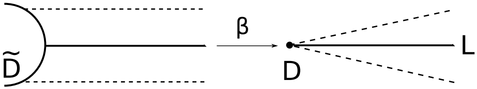

$\tilde{L} \subset \tilde{M}$ Lagrangian, $\tilde{L} \pitchfork \tilde{D}, \tilde{L} \cap \tilde{D} = \partial \tilde{L}$.

- **Case 1:** $\frac{\partial}{\partial \theta} \in T(\tilde{L} \cap \tilde{D}) \Rightarrow L := \beta(\tilde{L}) \pitchfork D$, smooth brane without boundary.
- **Case 2:** $\frac{\partial}{\partial \theta}$ *not* tangent to $\tilde{L} \cap \tilde{D}$, $\beta|_{\tilde{L}}$ injective.
  $\Rightarrow L := \beta(\tilde{L})$ *brane with boundary*.

# Local neighbourhoods and small deformations

**Lagrangian neighbourhood theorems:**

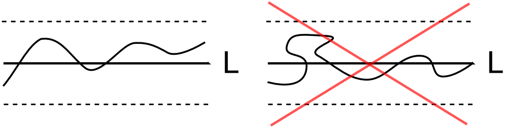
- *log symplectic geometry:* $(\tilde{L}, \partial\tilde{L}) \subset (\tilde{M}, \tilde{D}, \tilde{\omega})$ compact, $\tilde{L} \pitchfork \tilde{D}$. $\Rightarrow (\tilde{U}, \tilde{U} \cap \tilde{D}) \cong (T^*\tilde{L}(\log \partial\tilde{L}), \tilde{\omega}_0)$.
- *elliptic symplectic geometry:* $L \subset M$ compact, $L \pitchfork D$. $\Rightarrow (U, U \cap D) \cong (T^*L(\log|L \cap D|), \omega_0)$.
- *wedge neighbourhood for Lag. branes with boundary:* $(L, \partial L) \subset (M, D, \omega)$ compact. $\Rightarrow \beta(\tilde{U}), (\tilde{U}, \tilde{U} \cap \tilde{D}) \cong T^*L(\log \partial L)$ "wedge neighbourhood".



# Small deformations of Lagrangians

"Small" = Inside Lagrangian neighbourhood, graph of one-form

Small deformations (up to Ham. isotopy) $\cong$ First cohomology



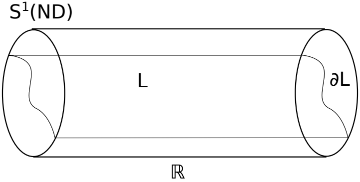

- log Lagrangians: $H^1(\Omega^\bullet(L, \log \partial L))$
- elliptic Lagrangian, $L \pitchfork D$ : $H^1(\Omega^\bullet(L, \log|L \cap D|))$
- Lagrangian brane with boundary: $H^1(\Omega^\bullet(L, \log \partial L))$

Natural differential complex associated to brane.

# Fukaya category for stable GC manifolds?

**Stable Hamiltonian system in neighbourhood of $D$**
Neighbourhood of puncture in $M \setminus D$:

$$\omega = \frac{\mathrm{d}r}{r} \wedge \alpha + \beta, \ \alpha, \beta \in \Omega^\bullet(\tilde{D}), \alpha \wedge \beta^{n-1} \neq 0$$




$\Rightarrow$Well-defined wrapped Fukaya category?
How to take $D$ and full GC structure into account?

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