Real Geometric Quantization

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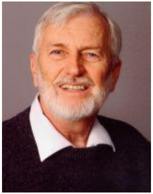
Contents

Bertram Kostant.

Nick Woodhouse.



Jędrzej Śniatycki.

Eva Miranda.

Mark Hamilton.

Romero Solha.

Fran Presas.

Setup

Classical mechanics

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Not unique: Planck's constant $\hbar$ parameter.

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(M, ω)

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    2. $\mathcal{P}^J$ then $H^0(M; \mathcal{J}) \equiv \{\text{Holomorphic sections}\}$.
- H Hamiltonian, $s \in \mathcal{H}$.

$$\hat{H}s = -i\hbar \nabla_{X_H} s + Hs.$$

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Theorem (Ṡniatycki)

$(f_1,\ldots,f_n) : (M^{2n},\omega) \to B^n \subset \mathbb{R}^n$ integrable system, then $L^2H^0(M;\mathcal{J}) = \mathcal{Q}(M)$.

Computation Kit

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$$\cdots \to H^j(M; \mathcal{J}) \to H^j(M_1; \mathcal{J}) \oplus H^j(M_2; \mathcal{J}) \to H^j(M_1 \cap M_2; \mathcal{J}) \to H^{j+1}(M; \mathcal{J}) \to \cdots$$

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5. Künneth (Miranda, P):

$$H^j(M_1 \times M_2; \mathcal{J}_1 \boxtimes \mathcal{J}_2) \cong \bigoplus_{j=p+q} H^p(M_1;\mathcal{J}_1) \oplus H^q(M_2;\mathcal{J}_2)$$

whenever M_1 admits a good cover, the geometric quantization associated to $(M_2, \mathcal{J}_2)$ has finite dimension and M_2 is a submanifold of a compact manifold.

Regular integrable system

$I_j = (-\varepsilon, \varepsilon), j = 1, 2.$

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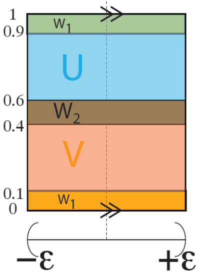
- Consider $I_1 \times \mathbb{S}^1_2 = U \cup V = (I_1 \times (0.4, 1.1)) \cup (I_1 \times (-0.1, 0.6))$.



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Thus

$$H^1(I_1 \times \mathbb{S}^1_2) = \begin{cases} 0 & \text{if non BS,} \\ \mathbb{C} & \text{if there is one BS.} \end{cases}$$

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Computation 4:

$$\mathcal{Q}(M^{2n}_{Tor,Reg}; \mathcal{P}(Torus)) = \bigoplus_{j=1}^{n} H^j(M; \mathcal{J}) = \mathbb{C}^b, \;\; b = \#\text{BS}.$$

Semitoric systems and almost toric manifolds

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Geometric quantization of these models gives us all the possible combinations.

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Lego land provides any answer.

$K3$ surface

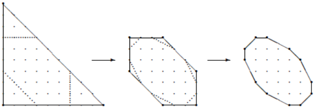

Figure: Delzant polytopes of $\mathbb{C}P^2$, $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$.

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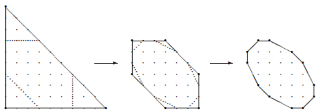

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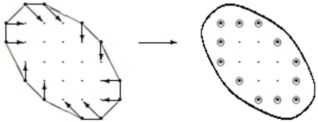

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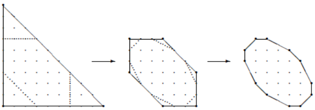

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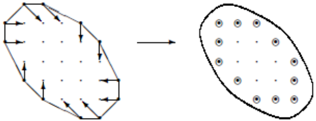

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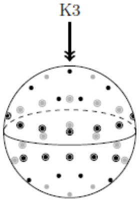
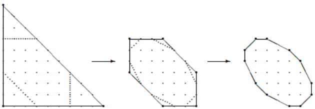


Figure: $K3$ surface as a singular fiber bundle over the sphere.

$K3$ surface



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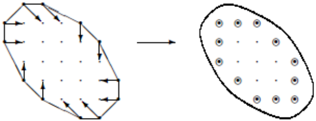

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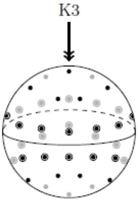

Figure: $K3$ surface as a singular fiber bundle over the sphere.

Thus, we obtain a $K3$ with up to 24 BS focus-focus fibers.

$$\mathcal{Q}(K3) \cong \mathbb{C}^{14} \oplus \bigoplus_{j=1}^{24} C^{\infty}(\mathbb{R};\mathbb{C}).$$

Other examples

1. Spin-spin system.

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2. Spherical pendulum and the spin-oscillator system.

Thanks

Thanks for listening!

Special thanks to Eduardo Fernández for helping with these slides