# Necessary and sufficient conditions for minimum-time affine control problems in space mechanics

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We adress the issue of $\pi$-singularities, a phenomenon occuring when one is dealing with minimum time control of a mechanical system. More detailes can be found in [?, ?].

Minimum time control of planar two and restricted three body problems comes down to tackling the following affine controlled dynamics:

$$\begin{cases} \dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x), x \in M, \|u\| \leq 1, \\ x(0) = x_0, \\ x(t_f) = x_f, \\ t_f \to \min . \end{cases} \tag{1}$$

where $M$ is the 4-dimensional phase space, $u = (u_1, u_2)$ is the control, $F_0$ is the drift coming from the gravitational potential, and $F_1$, $F_2$ are the two orthogonal vector fields supporting the control. This implies the following convenient property on the vector fields and their Lie brackets:

$$(A_1) \qquad \forall x \in M, \quad \mathrm{rank}(F_1(x), F_2(x), [F_0, F_1](x), [F_0, F_2](x)) = 4.$$

This dynamics defines a non Hamiltonian system, but necessary conditions coming from the Pontryagin Maximum Principle lead to study the flow of a singular Hamiltonian problem given by

$$H(x,p) = H_0(x,p) + \sqrt{H_1^2(x,p) + H_2^2(x,p)},$$

with $H_i(x,p) = \langle p, F_i(x) \rangle, i = 0,1,2$, being the Hamiltonian lift of the vector fields. It also provides the control feedback $u = \frac{(H_1,H_2)}{\|(H_1,H_2)\|}$ outside of the singular locus $\Sigma = \{H_1 = H_2 = 0\}$. The local flow is regular on a stratification, where existence and uniqueness hold.

The controlled Kepler and restricted three body problems have a simpler structure and we have the additional hypothesis :

$$(A_2) \qquad [F_1, F_2] \equiv 0.$$

From the feedback given above and the involution condition $(A_2)$, it appears that when $\Sigma$ is crossed out, we get an instant rotation of angle $\pi$ on the control (the so-called $\pi$-singularities).

One can also investigate integrability properties of such systems. It turns out that, as opposed to other similar problems, the extremal flow of the minimum time Kepler problem is non integrable in the class of meromorphic functions.

The last part of the talk will concern on going work about sufficient conditions for local optimality of minimum time extremals, via symplectic geometry technics. The difficulty is once again the lack of regularity, but some progress can be made due to the fact that the extremal flow is a symplectic homeomorphism.

# References


[1] J.-B. Caillau, M. Orieux On the extremal flow of the minimum time Kepler problem (In preparation)

[2] J.-B. Caillau, B. Daoud. Minimum time control of the restricted three-body problem *SIAM J. Control Optim*, 50:3178-3202, 2012.