Pencils of skew-symmetric matrices and Jordan-Kronecker invariants

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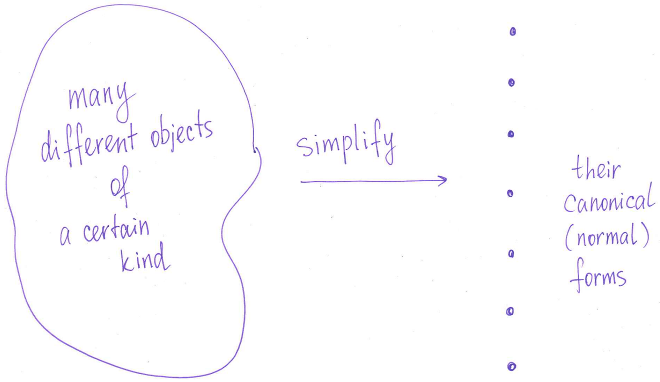
Plan

- What is a normal formal of a “mathematical object”?
- Similarity and Congruence Transformations
- Jordan normal form and simultaneous diagonalisation
- Representation of groups: orbits and invariants

- Pencils of skew-symmetric matrices (forms)
- Jordan-Kronecker decomposition theorem
- Lie algebras and their invariants
- JK invariants of Lie algebras and their applications

Idea of normal (canonical) forms



Jordan normal form

Take a square matrix A and try to simplify (diagonalise?) it!

Theorem

Let A be a $n \times n$ complex matrix. Then there exists an invertible matrix P such that $A' = P^{-1}AP$ takes the following block-diagonal form (called Jordan normal form):

$$A' = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where the blocks J_m are

$$J_m = J(\lambda_m) = \begin{pmatrix} \lambda_m & 1 & & & \\ & \lambda_m & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_m & 1 \\ & & & & \lambda_m \end{pmatrix}$$

Comment: 1×1 blocks $J_m = (\lambda_m)$ are allowed (moreover, they are "typical") and some of λ's may coincide (i.e., $\lambda_i = \lambda_j$ is OK).

Let A and B be two symmetric matrices (symmetric bilinear forms).
Question. To which normal form can one reduce these matrices simultaneously by congruence transformation:

$$A \mapsto PAP^\top, \qquad B \mapsto PBP^\top?$$

Theorem

Assume that A is positive definite, then there exists an invertible matrix P such that

$$A \mapsto P^\top AP = \mathrm{Id}, \quad B \mapsto P^\top BP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Almost equivalent version of this result:

Theorem (Spectral theorem)

Let $\phi : V \to V$ be a symmetric operator on a Euclidean vector space. Then there exists an orthonormal basis that consists of eigenvectors of ϕ.
Equivalently, in matrix form:
Let B be symmetric. Then there exists an orthogonal matrix P such that

$$B \mapsto P^\top BP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Viewpoint of the representation theory

A group G acts on a set S, i.e., each element $g \in G$ is represented as a bijective map $g : S \to S$. (E.g., $S = V$ is a vector space and $g : V \to V$ is an invertible linear operator.)

Definition

The orbit of a point $x \in S$ is

$$\mathcal{O}(x) = \{y = g(x) \in S \mid g \in G\}.$$

S can be presented as the disjoint union of orbits (i.e., can be partitioned into orbits).

Finding "normal form" is just "classification of orbits": for each orbit $\mathcal{O} \subset V$ we wish to find a representative $x \in \mathcal{O}$ (of the simplest possible form).

Definition

A function $f : S \to \mathbb{R}$ (not necessarily $\mathbb{R}$!) is called an invariant of the action if $f(x) = f(g(x))$ for all $x \in S$, $g \in G$. In other words, f is constant on each orbit (but this constant may depend on the orbit).

Similarity transformation $A \mapsto PAP^{-1}$ and congruence transformation $B \mapsto PBP^{\top}$ are two different actions of $GL(n)$ on the space of square matrices.

General problem

Given a (linear) action of G on V, we wish to find the invariants and to describe the orbits.

Example

$SL(2,\mathbb{R})$ acts on $sl(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$ by conjugation $A \mapsto CAC^{-1}$.

$f(A) = \det A = -a^2 - bc$ is an invariant of this action.

The orbits are "connected components" of f-levels: $-a^2 - bc = \text{const}$.

Example

$GL(2,\mathbb{R})$ acts on the space of symmetric matrices $V = \left\{ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\}$ by:

$A \mapsto CAC^\top$.

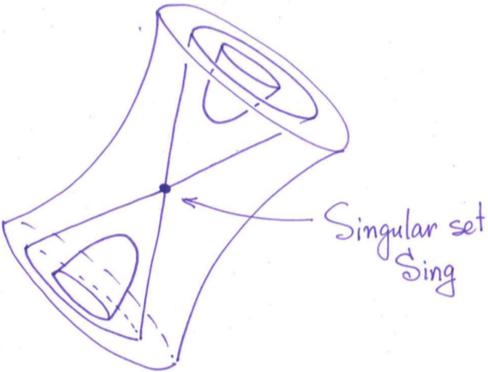
There are no smooth invariants.

There are 6 distinct orbits represented by the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Orbits of $SL(2,\mathbb{R})$ acting on $sl(2,\mathbb{R})$

$X \longmapsto PXP^{-1}$



Orbit space

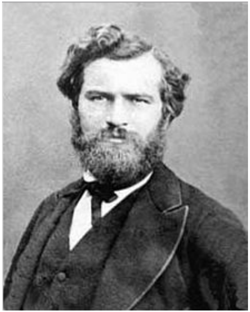
$\approx \mathbb{R}$

Skew-symmetric pencils

Now consider two skew-symmetric matrices A and B ($A^\top = -A$ and $B^\top = -B$) and the pencil $\mathcal{P} = \{A + \lambda B\}$ generated by them. Can we say anything similar about simultaneous canonical form in this case?

Karl Weierstrass (1815 -1897), Leopold Kronecker (1823-1891), Camille Jordan (1838-1922)





Felix Gantmacher (1908 - 1964) and his famous book *Theory of Matrices (1953)*.

Jordan-Kronecker decomposition theorem

Theorem

Let A and B be skew-symmetric $n \times n$ matrices. Then there exists an invertible P such that the transformation $A \mapsto PAP^\top$, $B \mapsto PBP^\top$ gives the following normal form

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \qquad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where pairs of the corresponding blocks A_i and B_i can be of the following types (see next slide)

Blocks in the skew-symmetric case

	A	B
Jordan type ($\lambda_m \in \mathbb{C}$)	$\begin{pmatrix} & J(\lambda_m) \\ -J^\top(\lambda_m) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$
	here $J(\lambda_m)$ is standard Jordan λ-block	

Kronecker type

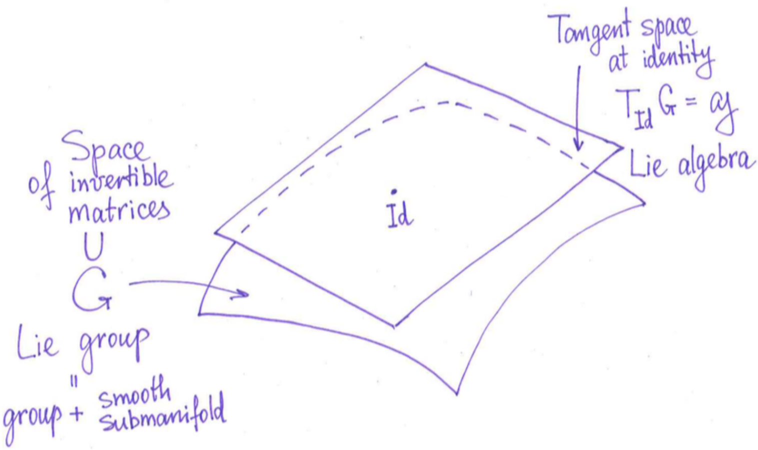
$$A = \begin{pmatrix} & \begin{array}{|cccc|} \hline 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ \hline \end{array} \\ \begin{array}{|ccc|} \hline -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \\ \hline \end{array} & \end{pmatrix}$$

$$B = \begin{pmatrix} & \begin{array}{|cccc|} \hline 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \hline \end{array} \\ \begin{array}{|ccc|} \hline 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \\ \hline \end{array} & \end{pmatrix}$$

Some comments

- The characteristic number λ_m plays the same role as an "eigenvalue" in the case of linear operators. More precisely, λ_m are those numbers for which the rank of $A_\lambda = A + \lambda B$ for $\lambda = \lambda_m$ is not maximal.
- If $\mu \neq \lambda_m$, then $A_\mu = A + \mu B$ is called regular (in the pencil $\mathcal{P} = \{A_\lambda\}$).
- Characteristic numbers are the roots of the characteristic polynomial $\mathrm{f}_{\mathcal{P}}(\lambda) = \gcd\{\mathrm{Pf}_{i_1 \dots i_{2k}}(A + \lambda B)\}$, $2k = \operatorname{rank} P$.
- The sizes of Kronecker blocks are odd $2k_i - 1$, the sizes of Jordan blocks are even $2j_m$. The numbers k_i and j_m are called Kronecker and Jordan indices of the pencil.
- If a pencil $P = \{A + \lambda B\}$ is pure Kronecker (i.e. no Jordan blocks), then all matrices in the pencil are of the same rank.
- The number of Kronecker blocks equals the corank of the pencil $\mathcal{P}$, i.e., $n - \operatorname{rank} \mathcal{P}$.
- The number of all characteristic numbers (with multiplicities) equals the degree of the characteristic polynomial $\mathrm{f}_{\mathcal{P}}(\lambda)$

Lie groups and Lie algebras



$GL(n) = \{A \text{ invertible}\}$ and $gl(n) = \{\text{all matrices}\}$,
$SL(n) = \{A, \det A = 0\}$ and $sl(n) = \{X, \operatorname{tr} X = 0\}$,
$SO(n) = \{A, A^\top = A^{-1}\}$ and $so(n) = \{X, X^\top = -X\}$,

Definition

A Lie algebra $\mathfrak{g}$ is a vector space endowed with a bilinear operation (Lie bracket) $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

- $[\xi, \eta] = -[\eta, \xi]$ (skew symmetry),
- $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$ (Jacobi identity).

In the case of matrix Lie algebras: $[X, Y] = XY - YX$.

Each Lie algebra $\mathfrak{g}$ can be defined by means of its structure constants. Take a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$. Then

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k, \qquad (c_{ij}^k \text{ are structure constants of } \mathfrak{g})$$

They depend on the choice of a basis. Invariants of $\mathfrak{g}$ are “functions” of c_{ij}^k which do not depend on this choice.

First we simplify (c_{ij}^k) by transferring it to a pair of skew symmetric matrices:

$$\mathcal{A}_x = \left(\sum_k c_{ij}^k x_k\right) \quad \text{and} \quad \mathcal{A}_a = \left(\sum_k c_{ij}^k a_k\right)$$

for some $x = (x_1, \dots, x_n) \in \mathfrak{g}^*$ and $a = (a_1, \dots, a_n) \in \mathfrak{g}^*$. Now $\mathcal{A}_x$ and $\mathcal{A}_a$ are two skew symmetric forms defined on $\mathfrak{g}$:

$$\mathcal{A}_x(\xi, \eta) = \langle x, [\xi, \eta]\rangle \quad \text{and} \quad \mathcal{A}_a(\xi, \eta) = \langle a, [\xi, \eta]\rangle.$$

Consider the pencil $\mathcal{A}_{x+\lambda a}$ generated by these two forms and its algebraic type defined by the Jordan-Kronecker decomposition: Kronecker and Jordan indices, multiplicities of characteristic numbers and so on... everything except specific values of characteristic numbers. This algebraic type, of course, depends on x and a, but...

For almost all x and a (in other words, for a generic pair (x, a)) the algebraic type of the pencil $\mathcal{A}_{x+\lambda a}$ will be one and the same.

Definition

The algebraic type (in the sense of Jordan-Kronecker canonical form) of the pencil $\mathcal{A}_{x+\lambda a}$ for a generic pair $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$, is called the Jordan-Kronecker invariant of $\mathfrak{g}$.

The Kronecker and Jordan indices of a generic pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ are said to be the Kronecker and Jordan indices of $\mathfrak{g}$.

More examples, properties and applications can be found in

Bolsinov, A. V., Zhang P., *Jordan–Kronecker invariants of finite-dimensional Lie algebras*, Transformation Groups 21 (2016) 1, 51–86.

Kronecker indices and Ad*-invaraint polynomials

Let $f : \mathfrak{g}^* \to \mathbb{C}$ be a polynomial on $\mathfrak{g}^*$. Using the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$, we may think of f as a formal polynomial in $e_1, \dots, e_n$ (basis of $\mathfrak{g}$). For instance, $f = e_1^2 + 3e_2e_3$. The corresponding Lie group G acts on $\mathfrak{g}$ in the adjoint way:

$$\xi \mapsto \mathrm{Ad}_g\, \xi, \quad \xi \in \mathfrak{g},\ g \in G.$$

The polynomial f is called Ad*-invariant if it does not change under the natural transformation induced by the adjoint action, i.e.,

$$f(e_1, \dots, e_n) = f(\mathrm{Ad}_g\, e_1, \dots, \mathrm{Ad}_g\, e_n), \quad \text{for all } g \in G.$$

All together Ad*-invariants form an algebra $\mathrm{I}(\mathfrak{g})$ called the algebra of Ad*-invariants.

How complicated is this algebra $\mathrm{I}(\mathfrak{g})$?

Theorem

Let $f_1(x), f_2(x), \dots, f_s(x) \in P(\mathfrak{g})$ be algebraically independent polynomial Ad*-invariants of $\mathfrak{g}$, $s = \mathrm{ind}\, \mathfrak{g}$, and $m_1 \le m_2 \le \cdots \le m_s$ be their degrees, $m_i = \deg f_i$. Then

$$m_i \ge k_i,$$

where $k_1 \le k_2 \le \cdots \le k_s$ are Kronecker indices of the Lie algebra $\mathfrak{g}$.

In the semisimple case (but not only!): $m_i = k_i$.

Let $f_1, \ldots, f_s$, $s = \operatorname{ind}\mathfrak{g}$, be algebraically independent Ad^*-invariant polynomials. Then

$$\sum_{i=1}^{s} \deg f_i \geq \sum_{i=1}^{s} k_i = \frac{1}{2}(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g}) - \deg \mathrm{f}_{\mathfrak{g}}.$$

For many classes of Lie algebras, this estimate becomes an equality (known as a *sum rule*).

Theorem

Let $k_1 \leq \cdots \leq k_s$ be the Kronecker indices of $\mathfrak{g}$ and $f_1, \ldots, f_s \in \mathrm{I}(\mathfrak{g})$ be algebraically independent Ad^*-invariant polynomials with $\deg f_1 \leq \deg f_2 \leq \cdots \leq \deg f_s$, $s = \operatorname{ind}\mathfrak{g}$. Assume that $\mathfrak{g}$ is unimodular and $\mathrm{f}_{\mathfrak{g}} \in \mathrm{I}(\mathfrak{g})$. Then the following conditions are equivalent:

1. $k_i = \deg f_i$, $i = 1. \ldots, s$;
2. $\sum_{i=1}^{s} \deg f_i = \frac{1}{2}(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g}) - \deg \mathrm{f}_{\mathfrak{g}}$;
3. $\mathrm{I}(\mathfrak{g})$ *is polynomial on* $f_1, \ldots, f_s$.

THANK YOU FOR YOUR ATTENTION