Reduction and relative equilibria for the 2-body problem in spaces of constant curvature

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Introduction

The study of the dynamics of material points and rigid bodies in spaces of constant curvature has been a popular subject of research in the past two decades. For recent advances in this area, see the review [6, 4] and the book by Diacu [11].

In this paper we focus on the 2-body problem on a (complete and simply connected) two dimensional space of constant non-zero curvature. Contrary to the situation in flat space, the system is not equivalent to the corresponding generalisation of the Kepler problem: it is nonintegrable and exhibits chaotic behaviour. Recent papers like [6, 20, 7, 2, 14, 21, 13] have considered the reduction by symmetries and some qualitative aspects of the problem. Libration points and choreographies are treated in [14, 16, 7] and the restricted two-body problem is considered in [5, 10].

Our main contributions consist of using the explicit form of the reduced system to classify all relative equilibria (RE) of the problem and to study their stability. We describe this in more detail below.

Reduction

As mentioned above, the reduction of the problem has been considered before. We have nevertheless included a self-contained presentation of the reduction for completeness.

For both the positive and negative curvature cases, the unreduced system is a four degree of freedom symplectic Hamiltonian system, the symmetry group is three dimensional and acts freely and properly. The reduced system is a five dimensional Poisson Hamiltonian system, whose generic symplectic leaves are the four dimensional level sets of a Casimir function.

We first deal with the case of positive curvature and consider the general N-body problem on the 2-dimensional sphere S^2. We perform the reduction of the problem by the action of $SO(3)$ that simultaneously rotates all of the masses. We proceed by introducing a moving coordinate frame whose axes are aligned according to the configuration of the first two bodies in a convenient way. The Hamiltonian of the system may then be written in terms of generalised coordinates for the positions of the masses with respect to the moving frame, their corresponding generalised momenta, and the vector of angular momentum $\boldsymbol{m}$ written in the moving frame. These quantities do not depend on the orientation of the fixed frame and may therefore be used as coordinates on the reduced space.

This approach is inspired by the reduction of the free rigid body problem and, just as it happens for that problem, the Euclidean squared norm of $\boldsymbol{m}$ passes down to the quotient space as a Casimir function whose level sets are the symplectic leaves of the reduced space.

We apply an analogous reduction scheme in the case of negative curvature by considering the action of $SO(2,1)$ on the pseudo-sphere L^2. This time, the Casimir function C on the reduced space is the squared norm of the momentum vector $\boldsymbol{m}$ with respect to the Minkowski metric.

Classification and stability of relative equilibria

We classify the RE of the problem by finding all of the extrema of the reduced Hamiltonian restricted to the the level sets of the corresponding Casimir function. In this way we recover the results of [8, 12, 13] in a systematic and elementary fashion. Moreover, in this manner, we arrive at a convenient position to analyse their stability.

The case of negative curvature

As it is known [12, 13], there are two families of RE, ***hyperbolic*** and ***elliptic***. The former are unbounded solutions that do not have an analog in euclidean space, while the latter are periodic solutions that generalise the RE in euclidean space. This classification becomes very transparent in our treatment: hyperbolic RE correspond to extrema of the reduced Hamiltonian restricted to negative values of the Casimir function C, whereas elliptic RE correspond to positive ones. The (nonlinear) stability properties of these RE was first established in [13] by working on a symplectic slice for the unreduced system since the reduced equations were not known. With the reduced system at hand, we are able to recover these results in elementary terms by directly analysing the signature of the Hessian of the reduced Hamiltonian at the RE. Hyperbolic RE are always unstable, whereas elliptic RE are stable if the masses are sufficiently close. However, as the distance between them grows, the family undergoes a saddle-node bifurcation and the elliptic RE become unstable.

All of the information of the RE of the system is conveniently illustrated in the Energy-Momentum bifurcation diagram 4, that is presented here for the first time. This kind of diagram, as well as the underlying topological considerations of the analysis, goes back to Smale and has been been developed in detail for integrable systems by Bolsinov, Borisov and Mamaev [8]. The application of this kind of analysis to nonintegrable systems, as the one considered in this paper, is not very common[1].

[1] We only mention the paper [9] on the Conley index (where new isosceles vortex configurations were found and their stability was established using topological methods).

The case of positive curvature

In this case all RE are periodic solutions in which the masses rotate about an axis that passes through the shortest geodesic that joins them. As indicated first in [7], the classification of RE in this case is more intricate than for negative curvature since it depends on how the masses of the bodies compare to each other:

1. If the masses are different, there are two disjoint families of RE that we term ***acute*** and ***obtuse***, according to the (constant) value of the angle between the masses along the motion. For acute RE, it is the heavier mass that is closer to the axis of rotation and hence these are a natural generalisation of the RE of the problem in euclidean space. On the other hand, for obtuse RE it is the lighter mass which is closer to the axis of rotation, and these RE to not have an analog in euclidean (or hyperbolic) space.
2. If the masses are equal, there are two families of RE. We call ***Isosceles*** RE those for which the axis of rotation bisects the arc that joins the two masses, and ***right angle*** RE those for which the angle between the masses is $\pi/2$ and the axis of rotation is located anywhere between them. These two families intersect when the axis of rotation of right angle RE bisects the arc between the masses, and a bifurcation of the RE of the system takes place.

As for the case of negative curvature, we compute the signature of the Hessian of the reduced Hamiltonian at the RE in an attempt to establish stability results. Via this analysis it is possible to conclude the instability of certain RE of the system. However, contrary to the case of negative curvature, this is insufficient to prove any kind of nonlinear stability results of the RE since the Hessian matrix is not definite and hence the reduced Hamiltonian may not be used as a Lyapunov function of the system. This surprising feature of the problem was also found in [18] by working on a symplectic slice of the unreduced system.

In view of the above considerations, we took an analytical approach to the study of the nonlinear stability of certain RE of the problem. By using Birkhoff normal forms and applying KAM theory, we are able to show that, in the case of different masses, the generic acute RE of the problem are stable.

Outline of the paper

The reduction of the problem on S^2 and L^2 is respectively presented in Sections 1 and 2. The case of the positive curvature is presented first since it is more natural. We then proceed to classify the RE of the problem and study their stability. We first deal with the case of negative curvature in section 3 and then with the case of positive curvature in section 4. We have chosen to present first the negative curvature results since, as was discussed above, the analysis is more straightforward. Finally, some related open problems are described at the end.

1. REDUCTION IN THE CASE S^2

1.1 Group parameterization of configurations

Consider the N-body problem on a two-dimensional sphere $S^2 \subset \mathbb{R}^3$. Let $OXYZ$ be a fixed coordinate system and let $\boldsymbol{R}_\alpha = (X_\alpha, Y_\alpha, Z_\alpha)$ be the Cartesian coordinates of the point mass μ_α, $\alpha = 1, \ldots, N$.

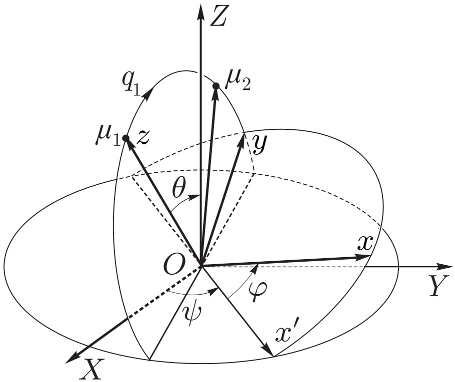


Fig. 1. Euler angles for the 2-body configuration on S^2.

Let us choose a pair of particles μ_1, μ_2 and suppose that a moving orthogonal coordinate system $Oxyz$ is attached to them in such a way that the axis Oz passes through the point μ_1 and the plane Oyz contains both masses μ_1, μ_2 (see Fig. 1). In this case, the radius vectors of the point masses $\boldsymbol{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ in the moving axes $Oxyz$ characterize the *relative* position of the particles (i.e., the configuration of the bodies irrelative to its position on the sphere S^2). Let the corresponding generalized (local) coordinates, which completely parameterize the relative position (configuration) of the particles, be denoted by $\boldsymbol{q} = (q_1, \dots, q_n)$ in the case of N material points $n = 2N - 3$. We will describe the orientation of the moving axes relative to the fixed axes by the Euler angles θ, φ, ψ so that the position of the particles in the fixed axes is described by

$$\boldsymbol{R}_\alpha(\theta, \varphi, \psi, \boldsymbol{q}) = \mathbf{Q}(\theta, \varphi, \psi) \cdot \boldsymbol{r}_\alpha(\boldsymbol{q}),$$

$$\mathbf{Q} = \mathbf{Q}_\psi \mathbf{Q}_\theta \mathbf{Q}_\varphi = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & \sin\theta\sin\psi \\ \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & -\sin\theta\cos\psi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{pmatrix},$$

$$\mathbf{Q}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \ \mathbf{Q}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{Q}_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\mathbf{Q}$ is the matrix of the direction cosines.

Assuming that the forces of interaction are potential, we construct the Lagrangian of the system

$$L = T - U,$$

where T and U are the kinetic and potential energy, respectively. We use a relation (well-known in rigid body dynamics) for the angular velocity matrix:

$$\hat{\boldsymbol{\omega}} = \mathbf{Q}^{-1}\dot{\mathbf{Q}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -\dot{\psi}\cos\theta - \dot{\varphi} & \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \dot{\psi}\cos\theta + \dot{\varphi} & 0 & -\dot{\psi}\sin\theta\sin\varphi - \dot{\theta}\cos\varphi \\ -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi & \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi & 0 \end{pmatrix}, \tag{2}$$

where $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ are the projections of the angular velocity of the moving frame onto the moving axes $Oxyz$, and we represent the kinetic energy of the system as

$$T = \sum_\alpha \mu_\alpha \big(\dot{\boldsymbol{R}}_\alpha, \dot{\boldsymbol{R}}_\alpha\big) = \frac{1}{2}\big(\omega, \mathbf{I}(\boldsymbol{q})\omega\big) + \big(\omega, \xi(\boldsymbol{q}, \dot{\boldsymbol{q}})\big) +$$

$$+\frac{1}{2}\sum_{i,j} G_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j,$$

$$\mathbf{I}(\boldsymbol{q}) = \sum_\alpha \mu_\alpha \left(\boldsymbol{r}_\alpha^2 \mathbf{E} - \boldsymbol{r}_\alpha \otimes \boldsymbol{r}_\alpha\right), \quad \boldsymbol{\xi} = \sum_\alpha \mu_\alpha \boldsymbol{r}_\alpha \times \dot{\boldsymbol{r}}_\alpha = \sum_{\alpha,i} \mu_\alpha \boldsymbol{r}_\alpha \times \frac{\partial \boldsymbol{r}_\alpha}{\partial \boldsymbol{q}_i}\,\dot{q}_i,$$

$$G_{ij} = \sum_\alpha \mu_\alpha \left(\frac{\partial \boldsymbol{r}_\alpha}{\partial \boldsymbol{q}_i}, \frac{\partial \boldsymbol{r}_\alpha}{\partial \boldsymbol{q}_j}\right),$$

where $\boldsymbol{q} = (q_1, \dots, q_n)$, the notation $(\cdot,\cdot)$, $\times$ and $\otimes$ corresponds to the scalar, vector and tensor product in $\mathbb{R}^3$ and $\mathbf{E}$ is the identity matrix. Since the particles interact only with each other, the potential energy of the system does not depend on the Euler angles:

$$U = U(\boldsymbol{q}).$$

1.2 Integrals of motion and reduction

As is well known, due to the invariance of the Lagrangian under rotations (i.e., under the change of fixed axes) the projections of the angular momentum of the system onto the fixed axes are preserved:

$$\boldsymbol{M} = \sum_\alpha \mu_\alpha \boldsymbol{R}_\alpha \times \dot{\boldsymbol{R}}_\alpha = \mathbf{Q}\Big(\mathbf{I}(\boldsymbol{q})\boldsymbol{\omega} + \boldsymbol{\xi}\Big) = \text{const.}$$

Using the Noether theorem, we obtain the corresponding expressions for the components of this vector in the Euler angles:

$$\begin{gathered} M_X = \cos\psi \frac{\partial L}{\partial \dot\theta} + \frac{\sin\psi}{\sin\theta}\left(\frac{\partial L}{\partial \dot\varphi} - \cos\theta \frac{\partial L}{\partial \dot\psi}\right), \\ M_Y = \sin\psi \frac{\partial L}{\partial \dot\theta} - \frac{\cos\psi}{\sin\theta}\left(\frac{\partial L}{\partial \dot\varphi} - \cos\theta \frac{\partial L}{\partial \dot\psi}\right), \\ M_Z = \frac{\partial L}{\partial \dot\psi}. \end{gathered} \tag{3}$$

We define the generalized momenta of the system in a standard way:

$$
\begin{gathered}
P_\theta = \frac{\partial L}{\partial \dot\theta}, \quad P_\varphi = \frac{\partial L}{\partial \dot\varphi}, \quad P_\psi = \frac{\partial L}{\partial \dot\psi}, \\
p_i = \frac{\partial L}{\partial \dot q_i}, \quad i = 1, \ldots, n.
\end{gathered} \tag{4}
$$

Then the equations of motion can be represented in the canonical Hamiltonian form:

$$
\begin{gathered}
\dot\theta = \frac{\partial H}{\partial P_\theta}, \quad \dot P_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot\varphi = \frac{\partial H}{\partial P_\varphi}, \quad \dot P_\varphi = -\frac{\partial H}{\partial \varphi}, \quad \dot\psi = \frac{\partial H}{\partial P_\psi}, \quad \dot P_\psi = -\frac{\partial H}{\partial \psi}, \\
\dot q_i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q_i}.
\end{gathered} \tag{5}
$$

The Hamiltonian function is expressed in a natural way in terms of the projections of the angular momentum vector onto the moving axes $\boldsymbol{m} = (m_x, m_y, m_z)$:

$$H = \frac{1}{2}\Big(\boldsymbol{m}, \mathbf{A}(\boldsymbol{q})\boldsymbol{m}\Big) + \Big(\boldsymbol{m}, \boldsymbol{k}(\boldsymbol{q}, \boldsymbol{p})\Big) + \frac{1}{2}\sum_{i,j} C_{ij}(\boldsymbol{q}) p_i p_j + U(\boldsymbol{q}),$$

$$m_x = \frac{\sin\varphi}{\sin\theta}(P_\psi - P_\varphi \cos\theta) + P_\theta \cos\varphi, \quad m_y = \frac{\cos\varphi}{\sin\theta}(P_\psi - P_\varphi \cos\theta) - P_\theta \sin\varphi,$$

$$m_z = P_\varphi,$$

$$\boldsymbol{k} = \mathbf{B}(\boldsymbol{q})\boldsymbol{p}, \tag{6}$$

where $\mathbf{A}(\boldsymbol{q})$, $\mathbf{B}(\boldsymbol{q})$, $\mathbf{C}(\boldsymbol{q})$ are 3×3, $3 \times n$ and $n \times n$ matrices which are the blocks of the $(3+n) \times (3+n)$ matrix arising when the quadratic form corresponding to the kinetic energy is inverted:

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{C} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I} & \left\| \dfrac{\partial \xi_i}{\partial \dot{q}_j} \right\| \\ \hline \left\| \dfrac{\partial \xi_i}{\partial \dot{q}_j} \right\|^T & \mathbf{G} \end{array}\right)^{-1}.$$

In order to obtain a reduced system, we pass from the canonical momenta P_θ, P_φ, P_ψ to the variables m_x, m_y, m_z. It turns out that the set of variables $\boldsymbol{m}$, $\boldsymbol{q}$, $\boldsymbol{p}$ is closed relative to the Poisson bracket:

$$\{m_i, m_j\} = -\varepsilon_{ijk} m_k, \quad \{q_i, p_j\} = \delta_{ij}.$$

This Poisson bracket is degenerate and possesses the Casimir function

$$C_0 = m_x^2 + m_y^2 + m_z^2. \tag{7}$$

Since the Hamiltonian (6) is expressed only in terms of these variables, we obtain the closed system of equations

$$\dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{8}$$

which defines the reduced system for this problem (since the variables $\boldsymbol{m}$, $\boldsymbol{q}$, $\boldsymbol{p}$ are invariant under the left action of the group $SO(3)$, i.e., under the change of the fixed axes).

As is well known [1], in order to obtain a reduced system in canonical variables, it is necessary to define on the level set of the integral (7)

$$C_0 = M_0^2$$

the cylindrical coordinates (Andoyer variables)

$$m_x = p_0, \quad m_y = \sqrt{M_0^2 - p_0^2}\sin q_0, \quad m_z = \sqrt{M_0^2 - p_0^2}\cos q_0,$$

$$q_0 \in [0, 2\pi), \quad p_0 \in [-M_0, M_0],$$

which commute canonically:

$$\{q_0, p_0\} = 1.$$

1.3 Reconstruction

Assume that we are given a solution to the system (8)

$$\boldsymbol{m}(t), \quad \boldsymbol{q}(t), \quad \boldsymbol{p}(t).$$

We need to determine the time dependence for the Euler angles.

1. As a first step, we choose the fixed axes $OXYZ$ in such a way that $\boldsymbol{M} || OZ$. Hence, the following relations hold:

$$M_X = 0, \quad M_Y = 0, \quad M_Z = P_\psi = M_0.$$

Using them, we find from (3) that

$$P_\theta = 0, \quad P_\varphi - P_\psi \cos\theta = m_z(t) - M_0 \cos\theta = 0.$$

Finally, taking into account (6), we find

$$P_\theta = 0, \quad P_\varphi = m_z(t), \quad P_\psi = M_0 = \text{const},$$

which leads to the relations

$$m_x = M_0 \sin\theta \sin\varphi, \quad m_y = M_0 \sin\theta \cos\varphi, \quad m_z = M_0 \cos\theta. \tag{9}$$

So finally we obtain

$$\cos\theta = \frac{m_z(t)}{M_0}, \quad \tan\varphi = \frac{m_x(t)}{m_y(t)}. \tag{10}$$

2. Using these relations, we obtain the quadrature for the angle ψ:

$$\begin{gathered}\dot{\psi} = \frac{\sin\varphi\omega_x + \cos\varphi\omega_y}{\sin\theta} = \frac{M_0\left(m_x(t)\omega_x(t) + m_y(t)\omega_y(t)\right)}{M_0^2 - m_z^2(t)},\\ \omega_x = \frac{\partial H}{\partial m_x}, \quad \omega_y = \frac{\partial H}{\partial m_y}.\end{gathered} \tag{11}$$

1.4 Example — the 2-body problem on S^2

In this case, the system has only one mutual variable, which is the angle between the radius vectors of the points (see Fig. 1)

$$q_1 \in (0, \pi).$$

If we denote the radius of the sphere by a, the radius vectors of the particles in the moving coordinate system $Oxyz$ are

$$\boldsymbol{r}_1 = (0, 0, a), \quad \boldsymbol{r}_2 = (0, a\sin q_1, a\cos q_1). \tag{12}$$

Performing the above operations, we obtain the Hamiltonian of the system (6) in the form

$$H = \frac{1}{2a^2\mu_1}\left((\boldsymbol{m}, \mathbf{A}(q_1)\boldsymbol{m}) + 2m_x p_1 + \left(1 + \frac{\mu_1}{\mu_2}\right)p_1^2\right) + U(q_1), \tag{13}$$

$$\mathbf{A}(q_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathcal{B} \\ 0 & \mathcal{B} & \mathcal{C} \end{pmatrix}, \quad \mathcal{B} = \frac{\cos q_1}{\sin q_1}, \quad \mathcal{C} = \frac{\mu_1 + \mu_2\cos^2 q_1}{\mu_2 \sin^2 q_1}.$$

Using (11) we obtain the quadrature for the angle of precession

$$\mu_1 a^2 \dot{\psi} = M_0\left(1 + \frac{m_x p_1 + \mathcal{B} m_y m_z}{M_0^2 - m_z^2}\right). \tag{14}$$

2. REDUCTION IN THE CASE L^2

2.1 Group parameterization of configurations

Let us consider a three-dimensional Minkowski space and attach the fixed coordinate system $OXYZ$ to it. The scalar product is given by

$$\langle \boldsymbol{R}, \boldsymbol{R} \rangle_{\mathrm{g}} = (\boldsymbol{R}, \mathrm{g}\boldsymbol{R}), \quad \mathrm{g} = \mathrm{diag}(1, 1, -k^2). \tag{15}$$

REMARK 1. If we formally substitute $k = i$, then we obtain a standard Euclidean scalar product and all subsequent formulas will turn into the formulas of the previous section.

The pseudo-sphere L^2 (the Lobachevsky plane) in this case is represented as one of the connectedness components of the two-sheet hyperboloid

$$\langle \boldsymbol{R}, \boldsymbol{R} \rangle_{\mathrm{g}} = X^2 + Y^2 - k^2 Z^2 = -a^2 k^2, \tag{16}$$

where a is some real-valued parameter. For definiteness we set $X_3 > 0$. Choosing the local coordinates on (16) in the form

$$X = ka \sinh\theta \cos\varphi, \quad Y = ka \sinh\theta \sin\varphi, \quad Z = a \cosh\theta,$$

we obtain a standard expression for the metric (obtained under the restriction (15)):

$$ds^2 = a^2 k^2 (d\theta^2 + \sinh^2\theta d\varphi^2).$$

Its Gaussian curvature is constant and equal to

$$K = -\frac{1}{a^2 k^2}.$$

We denote the coordinates of the particles in the fixed axes again as $R_\alpha = (X_\alpha, Y_\alpha, Z_\alpha)$ and their masses as μ_α, $\alpha = 1 \dots N$.

As in the previous case, we choose a pair of particles μ_1, μ_2 and attach to them a moving coordinate system $Oxyz$ which is orthogonal in the metric (15) and for which both these points lie in the plane Oyz and the axis Oz passes through the point μ_1. In these moving axes the radius vectors of the point masses $\boldsymbol{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ define the configuration of the bodies; they depend only on the internal generalized coordinates $\boldsymbol{q} = (q_1, \dots, q_n)$.

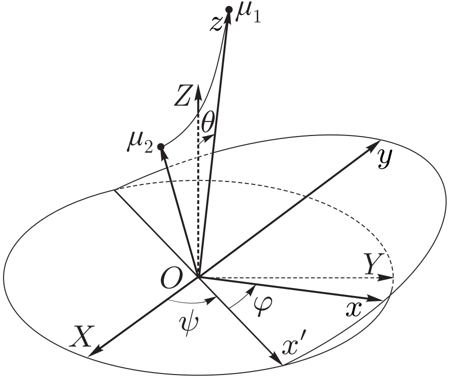


Fig. 2. Generalized Euler angles for the 2-body configuration on L^2.

As is well known, the transformation from the moving axes Oxy to the fixed axes $OXYZ$ is given by the matrix $\mathbf{Q} \in SO(2,1)$:

$$\boldsymbol{R}_\alpha = \mathbf{Q}\boldsymbol{r}_\alpha, \quad \alpha = 1 \dots N,$$

which satisfies the equation

$$\mathbf{Q}^T\mathbf{g}\mathbf{Q} = \mathbf{g}.$$

To parameterize the group $SO(2,1)$, we choose $\mathbf{Q}$ as a sequence of 3 rotations, one of which is hyperbolic:

$$\mathbf{Q}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & k\sinh\theta \\ 0 & \frac{1}{k}\sinh\theta & \cosh\theta \end{pmatrix}, \quad \mathbf{Q}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{Q}_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{Q} = \mathbf{Q}_\psi\mathbf{Q}_\theta\mathbf{Q}_\varphi =$$

$$= \begin{pmatrix} \cos\varphi\cos\psi - \cosh\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi - \cosh\theta\cos\varphi\sin\psi & -k\sinh\theta\sin\psi \\ \cos\varphi\sin\psi + \cosh\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \cosh\theta\cos\varphi\cos\psi & k\sinh\theta\cos\psi \\ \frac{1}{k}\sinh\theta\sin\varphi & \frac{1}{k}\sinh\theta\cos\varphi & \cosh\theta \end{pmatrix}$$

(to pass to $SO(3)$, we have to set $k = i$, $\theta \to i\theta$).

The matrix of the left-invariant angular velocities is given by the following relation, analogous to (2):

$$\widehat{\boldsymbol{\omega}} = \mathbf{Q}^{-1}\dot{\mathbf{Q}} = \begin{pmatrix} 0 & -\omega_z & -k^2\omega_y \\ \omega_z & 0 & k^2\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -\dot{\varphi}-\dot{\psi}\cosh\theta & k(\dot{\theta}\sin\varphi-\dot{\psi}\sinh\theta\cos\varphi) \\ \dot{\varphi}+\dot{\psi}\cosh\theta & 0 & k(\dot{\theta}\cos\varphi+\dot{\psi}\sinh\theta\sin\varphi) \\ \frac{1}{k}(\dot{\theta}\sin\varphi-\dot{\psi}\sinh\theta\cos\varphi) & \frac{1}{k}(\dot{\theta}\cos\varphi+\dot{\psi}\sinh\theta\sin\varphi) & 0 \end{pmatrix}.$$

2.2 The 2-body problem

Here we shall not develop a general construction for an arbitrary number of particles as we did in the previous section, but consider in more detail the 2-body problem on L^2. In the chosen moving coordinate system $Oxyz$ the position of the bodies is given by the radius vectors

$$r_1 = (0, 0, a), \quad r_2 = (0, ak\sinh q_1, a\cosh q_1),$$

where $q_1 \in \mathbb{R}^+ = (0, \infty)$ because in the case where $a = k = 1$ it corresponds to the distance between the particles in L^2.

In this case, the Lagrangian function is

$$L = T - U(q_1),$$

where $U(q_1)$ is the potential energy of interaction and the kinetic energy T can be represented as

$$T = \frac{1}{2}\sum_{\alpha} \mu_\alpha \langle \dot{\boldsymbol{R}}_\alpha, \dot{\boldsymbol{R}}_\alpha \rangle_{\mathrm{g}} = \frac{1}{2}\sum_{\alpha} \mu_\alpha \langle \hat{\boldsymbol{\omega}} \boldsymbol{r}_\alpha + \dot{\boldsymbol{r}}_\alpha, \hat{\boldsymbol{\omega}} \boldsymbol{r}_\alpha + \dot{\boldsymbol{r}}_\alpha \rangle_{\mathrm{g}} =$$

$$= \frac{a^2k^4}{2}\Big((\mu_1 + \mu_2)\omega_x^2 + (\mu_1 + \mu_2\cosh^2 q_1)\omega_y^2 + \frac{\mu_2 \sinh^2 q_1}{k^2}\omega_z^2 +$$

$$2\frac{\mu_2 \sinh q_1 \cosh q_1}{k}\omega_y\omega_z + \frac{\mu_2}{k}\omega_x\dot{q}_1 + \frac{\mu_2}{k^2}\dot{q}_1^2\Big).$$

The projections of the angular momentum vector to the axes attached to the body and the momentum corresponding to the coordinate q_1 are given by

$$m_x = \frac{\partial L}{\partial \omega_x} = k\Big(P_\theta \cos\varphi - P_\varphi \frac{\cosh\theta\cos\varphi}{\sinh\theta} + P_\psi \frac{\sin\varphi}{\sinh\theta}\Big),$$

$$m_y = \frac{\partial L}{\partial \omega_y} = k\Big(-P_\theta \sin\varphi - P_\varphi \frac{\cosh\theta\cos\varphi}{\sinh\theta} + P_\psi \frac{\cos\varphi}{\sinh\theta}\Big),$$

$$m_z = \frac{\partial L}{\partial \omega_z} = P_\varphi, \quad p_1 = \frac{\partial L}{\partial \dot{q}_1}.$$

The Poisson bracket of these functions has the form

$$\{m_x, m_y\} = k^2 m_z, \quad \{m_y, m_z\} = -m_x, \quad \{m_z, m_x\} = -m_y, \qquad (17)$$
$$\{q_1, p_1\} = 1.$$

The Hamiltonian of the system is defined by

$$H = \Big(\sum_i m_i\omega_i + p_1\dot{q}_1 - L\Big)\Big|_{\boldsymbol{\omega}\to\boldsymbol{m},\dot{q}_1\to p_1} =$$

$$= \frac{1}{2a^2k^4\mu_1}\left(m_x^2 + m_y^2 - 2\mathcal{B}m_ym_z + \mathcal{C}m_z^2 - 2km_xp_1 + k^2\Big(1+\frac{\mu_1}{\mu_2}\Big)p_1^2\right) + U(q_1)$$

$$\mathcal{B} = k\frac{\cosh q_1}{\sinh q_1}, \quad \mathcal{C} = k^2\frac{\mu_1 + \mu_2\cosh^2 q_1}{\mu_2\sinh^2 q_1}. \qquad (18)$$

The equations of motion are written in vector form as follows:

$$\dot{\boldsymbol{m}} = (\mathrm{g}\boldsymbol{m}) \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}. \qquad (19)$$

3. RELATIVE EQUILIBRIA FOR THE 2-BODY PROBLEM ON THE LOBACHEVSKY PLANE

3.1 Existence and classification of relative equilibria

We denote $p_1 = p$, $q_1 = q \in \mathbb{R}^+ = (0, +\infty)$. Let $\mu = \dfrac{\mu_1}{\mu_2}$ be the quotient between the masses. The reduced Hamiltonian (18) can be written as

$$H(\boldsymbol{m}, q, p) = \frac{1}{2\mu_1 a^2}\Big((\boldsymbol{m}, \mathbf{A}(q)\boldsymbol{m}) - 2km_x p + k^2(1+\mu)p^2\Big) + U(q), \qquad (20)$$

where

$$\mathbf{A}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -k\dfrac{\cosh q}{\sinh q} \\ 0 & -k\dfrac{\cosh q}{\sinh q} & \dfrac{k^2(\mu + \cosh^2 q)}{\sinh^2 q} \end{pmatrix}.$$

In our analysis we shall assume that the potential $U(q)$ is purely attractive, meaning that $U'(q) > 0$. This situation is encountered in the gravitational case where

$$U(q) = -\frac{G\mu_1\mu_2}{\tanh q}, \qquad (21)$$

where $G > 0$ is proportional to the gravitational constant.

The Poisson bracket (17) has generic rank 4 (its rank drops to 2 when $\boldsymbol{m} = 0$). Its 4-dimensional symplectic leaves are the regular level sets of the Casimir function

$$C(\boldsymbol{m}) = -\langle \boldsymbol{m}, \boldsymbol{m} \rangle_{\mathrm{g}} = -m_x^2 - m_y^2 + k^2 m_z^2,$$

that is a first integral of the reduced equations of motion (19).

The relative equilibria of the problem correspond to equilibrium points on the reduced space. These are the local extrema of H on the level sets of C.

Critical points of H on the level sets of C are characterised as the solutions to the following set of equations:

$$\frac{\partial H}{\partial p} = 0, \tag{22a}$$

$$\frac{\partial H}{\partial \boldsymbol{m}} \times (\mathrm{g}\boldsymbol{m}) = \boldsymbol{0}, \tag{22b}$$

$$\frac{\partial H}{\partial q} = 0, \tag{22c}$$

where the matrix g is defined in (15) and $\times$ is the vector product in $\mathbb{R}^3$. The condition (22a) yields

$$p = \frac{m_x}{k(1+\mu)}. \tag{23}$$

Substituting this expression into (22b) yields two possibilities:

1) $m_y = m_z = 0$. In this case (22c) implies $U'(q) = 0$, and there is no solution for q by our assumption that U is strictly attractive.
2) $m_x = 0$. We analyse this case in what follows assuming that m_y and m_z do not vanish simultaneously, since otherwise we are back in case (i).

On the unreduced space, the qualitative properties of these solutions depend on the sign that C takes along them:

1. Relative equilibria having $C > 0$ are periodic solutions and are termed ***elliptic relative equilibria***. Their existence is due to a balance between centrifugal and gravitational forces. These type of solutions also exist in the positive and zero curvature cases.
2. Relative equilibria having $C < 0$ are unbounded solutions and are termed ***hyperbolic relative equilibria***. This type of relative equilibrium does not exist in the positive or zero curvature case. As explained in [13] their existence is due to the property of the Lovachevsky space that makes parallel geodesics "separate". This separating effect is balanced by the gravitational attraction in a very delicate manner. As we shall see, these relative equilibria are all unstable.
3. Relative equilibria having $C = 0$ do not exist as we shall see below. If they did exist, they would be called ***parabolic relative equilibria*** in a natural analogy with the terminology for quadrics on the plane.

Existence and classification of elliptic relative equilibria. Recall that we have assumed that $m_x = 0$. We parametrise the open region of the reduced phase space having $C(\boldsymbol{m}) > 0$ with the parameters $M \neq 0$ and $\alpha \in \mathbb{R}$, by putting

$$m_y = M \sinh \alpha, \qquad m_z = \frac{M}{k} \cosh \alpha. \tag{24}$$

Then $C(\boldsymbol{m}) = M^2$ and (22b) is satisfied provided that

$$\sinh 2(q - \alpha) = \mu \sinh 2\alpha. \tag{25}$$

This necessary condition is equivalent to the one given in [13] by introducing the concept of the hyperbolic centre of mass. Equation (25) admits the unique solution for α

$$\alpha = \frac{q}{2} + \frac{1}{4} \ln \left(\frac{\mu + e^{2q}}{1 + \mu e^{2q}} \right), \tag{26}$$

With the above value of α, equation (22c) is satisfied provided that M is such that

$$M^2 = \frac{\mu_1 a^2 \sinh^3 q U'(q)}{\mu \cosh^2 \alpha \cosh q + \cosh \alpha \cosh(q - \alpha)}. \tag{27}$$

For an attractive potential, like the gravitational (20), the right hand side of this expression is positive.

The choice of sign for M corresponds to two solutions related by reversing the direction of time. We shall not distinguish these two. Therefore, we have shown the following.

Proposition 1. *There exists a unique family of elliptic relative equilibria that is parametrised by q. This family has $m_x = p = 0$, and m_y and m_z defined by (24), where α and $|M|$ are determined by (26) and (27).*

This family of relative equilibria corresponds to the bifurcation curve with a cusp on the semi-plane $C > 0$ on the energy – momentum diagram (see Fig. 4).

Existence and classification of hyperbolic relative equilibria. The analysis is analogous to the above. This time we put

$$m_y = M\cosh\alpha, \qquad m_z = \frac{M}{k}\sinh\alpha, \tag{28}$$

so $C(\boldsymbol{m}) = -M^2 < 0$. Taking into account that $m_x = 0$, then (22b) is satisfied provided that (25), and hence also (26), hold. The condition for M in this case is

$$M^2 = \frac{\mu_1 a^2 \sinh^3 q U'(q)}{\mu \sinh^2\alpha \cosh q - \sinh\alpha \sinh(q-\alpha)}. \tag{29}$$

It can be shown, using (26), that the right hand side of this expression is positive. Therefore, upon the same considerations as above when counting the number of relative equilibria, we conclude

Proposition 2. *There exists a unique family of hyperbolic relative equilibria that is parametrised by q. This family has $m_x = p = 0$, and m_y and m_z defined by* (24) *where α and $|M|$ are determined by* (26) *and* (29).

This family of relative equilibria corresponds to the smooth bifurcation curve on the semi-plane $C < 0$ on the energy – momentum diagram (see Fig. 4).

Non-existence of parabolic relative equilibria. Finally we show that there are no solutions of (22b) having $C(\boldsymbol{m}) = 0$. Substituting $m_x = 0$ and $m_y = \pm k m_z$ into the first component of (22b) yields, after a simple calculation,

$$\cosh 2q \pm \sinh 2q + \mu = 0,$$

which clearly has no solutions for q since $\mu > 0$.

This means that the bifurcation curves corresponding to elliptic and hyperbolic relative equilibria meet each other at the punctured point when $C = 0$, see Fig. 4.

3.2 Stability analysis of the relative equilibria

Denote the level surface of Casimir function by

$$\mathcal{M}_{M_0} = \{(\boldsymbol{m}, q, p) | \, C(\boldsymbol{m}) = M_0^2\}$$

and let the restriction of the Hamiltonian (20) be $\tilde{H} = H|_{\mathcal{M}_{M_0}}$. The relative equilibria found above (Propositions 1, 2) are critical points of $\tilde{H}$.

According to Lyapunov's Theorem, if a relative equilibrium is a local maxima or minimum of $\tilde{H}$, it is nonlinearly stable. This happens in particular if the Hessian matrix of $\tilde{H}$ at the equilibrium is positive or negative definite. On the other hand, if the Hessian matrix of $\tilde{H}$ at an equilibrium has an odd number of negative eigenvalues, then the linearised system has at least one eigenvalue with positive real part and the equilibrium is unstable.

We now compute the signature of the Hessian matrix of $\tilde{H}$ at the relative equilibria found in the previous section. This will give definite information on the nonlinear stability of these solutions. The results in this section were obtained previously in [13] by working on a symplectic slice in the unreduced system, since the reduced equations of motion were not known at that time. The two approaches are in fact equivalent.

Analysis for the elliptic relative equilibria. Fix $q_0 \in \mathbb{R}^+$. According to Proposition 1, there is a unique elliptic relative equilibrium $(\boldsymbol{m}, q, p) = (\boldsymbol{m}_0, q_0, 0)$ associated to q_0, with the vector $\boldsymbol{m}_0$ given by

$$\boldsymbol{m}_0 = \left(0, M_0 \sinh \alpha_0, \frac{M_0}{k} \cosh \alpha_0\right).$$

Here α_0 and $M_0 \neq 0$ are the values taken by α and M in (26) and (27) when one puts $q = q_0$.[2] Moreover, we have $C(\boldsymbol{m}_0) = M_0^2$. The relative equilibrium $(\boldsymbol{m}_0, q_0, 0)$ is a critical point of

$$H_{\lambda_0}(\boldsymbol{m}, q, p) = H(\boldsymbol{m}, q, p) - \frac{\lambda_0}{2} C(\boldsymbol{m}),$$

where the Lagrange multiplier is given by

$$\lambda_0 = \frac{\cosh(q_0 - \alpha_0)}{\mu_1 a^2 \sinh \alpha_0 \sinh q_0}.$$

[2]there is an unessential choice in the sign of M_0 that we continue to ignore.

It can be shown that after an appropriate choice of coordinates the Hessian matrix $\mathbf{N}$ of $\tilde{H}$ coincides with the restriction of the quadratic form defined by the Hessian matrix $D^2 H_{\lambda_0}(\boldsymbol{m}_0, q_0, 0)$ to the tangent space $T\mathcal{M}_{M_0}$. If we choose the following basis vectors for $T\mathcal{M}_{M_0}$:

$$\boldsymbol{e}_1 = (0,0,0,0,1), \qquad \boldsymbol{e}_2 = (1,0,0,0,0),$$
$$\boldsymbol{e}_3 = (0, k\cosh\alpha_0, \sinh\alpha_0, 0, 0), \qquad \boldsymbol{e}_4 = (0,0,0,1,0),$$

then the resulting matrix $\mathbf{N}$ has the 2×2 block form

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}^{(1)} & 0 \\ 0 & \mathbf{N}^{(2)} \end{pmatrix}$$

where

$$\mathbf{N}^{(1)} = \frac{1}{\mu_1 a^2} \begin{pmatrix} k^2(\mu+1) & -k \\ -k & \dfrac{\cosh\alpha_0 \cosh q_0}{\sinh\alpha_0 \sinh q_0} \end{pmatrix}.$$

It is straightforward to check that $\mathbf{N}^{(1)}$ is positive definite (recall that $q_0, \alpha_0 > 0$).

Using (25) the entries of the matrix $\mathbf{N}^{(2)}$ are written as

$$N_{11}^{(2)} = \frac{k^2 \cosh q_0}{\mu_1 a^2 \sinh \alpha_0 \sinh q_0 \cosh \alpha_0},$$

$$N_{12}^{(2)} = N_{21}^{(2)} = \frac{kM_0 \big(\sinh(q_0 - 2\alpha_0) - \sinh(3q_0 - 2\alpha_0) \big)}{2\mu_1 a^2 \sinh^3 q_0},$$

$$N_{22}^{(2)} = U''(q_0) + \frac{M_0^2 \cosh \alpha_0 \left(\mu \cosh \alpha_0 (2\cosh^2 q_0 + 1) + \cosh(\alpha_0 - 2q_0) + 2\cosh \alpha_0 \right)}{\mu_1 a^2 \sinh^4 q_0}.$$

A numerical study shows that for the gravitational potential (20), the trace of the the matrix $\mathbf{N}^{(2)}$ is positive for any value of q_0. On the other hand, its determinant is positive for for small q_0 and negative for large q_0 for any value of μ. We conclude[3] that the signature of the Hessian at elliptic relative equilibria is $(+,+,+,+)$ for small q_0 and $(+,+,+,-)$ for larger values of q_0. The former equilibria are Lyapunov stable while the latter are unstable. The system undergoes a saddle-node bifurcation. This corresponds to the cusp in Fig. 4.

As shown in Fig. 4, for a given μ the change of stability of the elliptic relative equilibria occurs at a maximum value of M^2 given by (29). An implicit plot of $\dfrac{dM^2}{dq}(q, \mu) = 0$ leads to the diagram 3 for the stability of elliptic relative equilibria.

[3] For an analytical proof see [13].

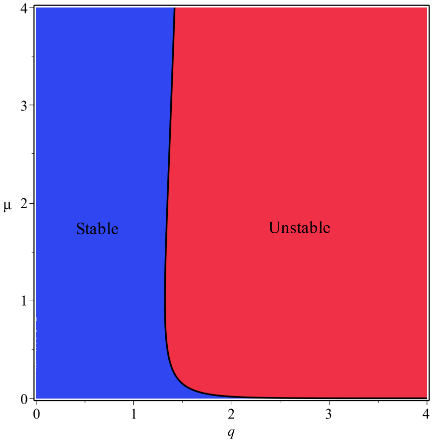


Fig. 3. Stability region for elliptic relative equilibria as functions of q and the mass ratio $\mu = \mu_1/\mu_2$. For a given mass ratio μ, the equilibria are stable for small q and unstable for large q.

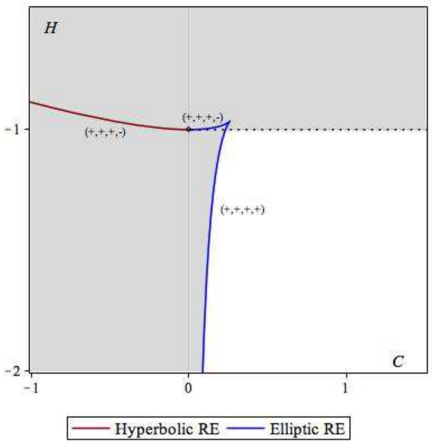


Fig. 4. Energy-Momentum bifurcation diagram of relative equilibria for the gravitational potential with $G = 2$, $a = 1$, $k = 1$, $\mu_1 = 0.5$, $\mu_2 = 1$. The shaded area on the C-H plane shows all possible values of (C, H). We also indicate the signature of the Hessian matrix of the Hamiltonian along each branch of relative equilibria. Notice the change in signature at the cusp of the elliptic relative equilibria where a saddle-node bifurcation takes place.

Analysis of the hyperbolic relative equilibria. Proceeding in a completely analogous way as for the elliptic relative equilibria, one obtains the corresponding matrix $\mathbf{N}$ that is block diagonal with the blocks $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$. This time $\mathbf{N}^{(1)}$ is given by

$$\mathbf{N}^{(1)} = \frac{1}{\mu_1 a^2} \begin{pmatrix} k^2(\mu+1) & -k \\ -k & \dfrac{\cosh q_0 \sinh \alpha_0}{\sinh q_0 \cosh \alpha_0} \end{pmatrix},$$

that may be shown to be positive definite using (26).

On the other hand, using (26), the entries of $\mathbf{N}^{(2)}$ can be written as

$$N_{11}^{(2)} = \frac{k^2 \cosh q_0}{\mu_1 a^2 \sinh q_0 \sinh \alpha_0 \cosh \alpha_0},$$

$$N_{12}^{(2)} = N_{21}^{(2)} = \frac{k M_0 \big(\cosh q_0 - \cosh(q_0 - 4\alpha_0) - \cosh 3q_0 + \cosh(3q_0 - 4\alpha_0)\big)}{4\mu_1 a^2 \sinh^3 q_0 \sinh 2\alpha_0},$$

$$N_{22}^{(2)} = U''(q_0) + \frac{M_0^2 \sinh \alpha_0 \left(\mu \sinh \alpha_0 (2\cosh^2 q_0 + 1) + \sinh \alpha_0 - 2\cosh q_0 \sinh(q_0 - \alpha_0)\right)}{\mu_1 a^2 \sinh^4 q_0}$$

A numerical study[4] shows that for the gravitational potential the determinant of $\mathbf{N}^{(2)}$ is negative for any value of q_0. Therefore, its eigenvalues have opposite sign. In conclusion, the signature of the Hessian matrix along the branch of hyperbolic relative equilibria is $(+,+,+,-)$ and hence, they are all unstable.

[4] An analytic proof is given in [13].

Energy – momentum diagram. In the case of the gravitational potential (21), our analysis is summarised in the bifurcation diagram given in Fig. 4. The shaded region corresponds to the image of the energy momentum mapping (C, H) from the phase space of the system into $\mathbb{R}^2$. The elliptic relative equilibria are indicated in blue. Those having signature $(+,+,+,+)$ are stable and correspond to small values of q (the particles are close). Eventually, when the distance between the particles is sufficiently large, the momentum C of the elliptic relative equilibria achieves a maximum, and there is a change in the stability. This corresponds to the cusp in the figure. The elliptic RE with signature $(+,+,+,-)$ correspond to large values of q and are all unstable. They approach the axis $C=0$ as $q\to\infty$. The hyperbolic RE are illustrated in red. They have signature $(+,+,+,-)$ and are unstable. They also approach the axis $C=0$ as $q\to\infty$. Notice that the families seem to meet when $C=0$ but that there is no relative equilibrium at this point, since, as we have shown, there do not exist parabolic relative equilibria.

The value of H where the two families seem to meet, and that creates the boundary of the shaded area for large $C>0$ is equal to the limit of the potential U as $q\to\infty$.

3.3 Topology of the Energy-Momentum level surfaces

Denote by $\mathcal{Z}$ the reduced phase space. $\mathcal{Z} \cong \mathbb{R}^4 \times \mathbb{R}^+$ with global coordinates $\boldsymbol{m}, p, q$. The map $(C, H) : \mathcal{Z} \to \mathbb{R}^2$ is not proper. It has the following properties:

Proposition 3. *Let $\zeta_n = (\boldsymbol{m}_n, p_n, q_n)$ be a sequence on $\mathcal{Z}$ and suppose that $(C, H)(\zeta_n)$ converges to $(c_0, h_0) \in \mathbb{R}^2$. Then*

1. *If $q_n \to \infty$ then $h_0 \geqslant \lim_{q\to\infty} U(q)$.*
2. *If $q_n \to 0$ then $c_0 \leqslant 0$.*
3. *If $\varepsilon \leqslant q_n \leqslant \frac{1}{\varepsilon}$ for some $\varepsilon > 0$, then $\boldsymbol{m}_n$ and p_n are bounded.*

Proof.

1. Since the kinetic energy is positive, we have $H(\zeta_n) \geqslant U(q_n)$, and the result follows by letting $n \to \infty$.
2. Suppose $-m_x^2 - m_y^2 + k^2 m_z^2 = c_0 > 0$. Then, for n large enough, $k^2 (m_z)_n^2 \geqslant \dfrac{c_0}{2}$ and we have
$$H(\zeta_n) \geqslant \frac{c_0 \mu}{4\mu_1 a^2 \sinh^2 q_n} - G\mu_1\mu_2 \coth q_n$$
that grows without bound as $q_n \to 0$. This contradicts our hypothesis that $H(\zeta_n)$ converges to h_0.
3. For $\varepsilon \leqslant q \leqslant \frac{1}{\varepsilon}$ it is possible to bound H from below by a constant, positive definite quadratic form on $\boldsymbol{m}_n$ and p_n with constant coefficients, plus a constant value (the minimum of U for $\varepsilon \leqslant q \leqslant \frac{1}{\varepsilon}$). Hence, the only way in which H can remain bounded if q is bounded is if $\boldsymbol{m}_n$ and p_n are also bounded.

∎

One can show that the fibre over the point (c_0, h_0) is compact only for $c_0 > 0$ and $h_0 < \lim_{q\to\infty} U(q)$. In this case it topologically consists of the disjoint union of two 3-spheres. We can in fact describe the topology of the fibre for all other values (c_0, h_0) having $c_0 > 0$. For the small region inside the cusp it is these two 3-spheres with two unbounded 3-dimensional balls, and for (c_0, h_0) the upper region of the C-H plane it is just the two unbounded 3-dimensional balls.

We were unable to determine the topology of the unbounded fibres having $c_0 < 0$. The transition from $c_0 > 0$ to $c_0 < 0$ is surprisingly complicated.

4. RELATIVE EQUILIBRIA FOR THE 2-BODY PROBLEM ON THE SPHERE

Denoting $\mu = \dfrac{\mu_1}{\mu_2}$, $q_1 = q$ and $p_1 = p$, the reduced Hamiltonian (13) takes the form

$$H = \frac{1}{2a^2\mu_1}\left((\boldsymbol{m}, \mathbf{A}(q)\boldsymbol{m}) + 2m_x p + (1+\mu)p^2\right) + U(q), \tag{30}$$

where $q \in (0, \pi)$ is the angle between the radius vector of the two particles and

$$\mathbf{A}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{\cos q}{\sin q} \\ 0 & \dfrac{\cos q}{\sin q} & \dfrac{\mu + \cos^2 q}{\sin^2 q} \end{pmatrix}.$$

Since the symmetry group $SO(3)$ is compact and has rank 1, all of the relative equilibria of the problem for $(\boldsymbol{m}, \boldsymbol{m}) \neq 0$ are periodic solutions. The 2 masses simultaneously rotate about a fixed axis of rotation at a steady angular speed ω.

Relative equilibria are equilibrium points of the reduced system and are characterised as critical points of H on a level set $(\boldsymbol{m}, \boldsymbol{m}) = M_0^2$. Therefore, at such solution the following equations must be satisfied:

$$\frac{\partial H}{\partial p} = 0, \tag{31a}$$

$$\frac{\partial H}{\partial \boldsymbol{m}} \times \boldsymbol{m} = \boldsymbol{0}, \tag{31b}$$

$$\frac{\partial H}{\partial q} = 0. \tag{31c}$$

The condition (31a) yields

$$p = -\frac{m_x}{1+\mu}. \tag{32}$$

Substituting (32) into (31b) one obtains two possibilities:

1. $m_y = m_z = 0$. In this case (31c) implies $\dfrac{dU}{dq} = 0$. If the potential is attractive there is no solution.
2. $m_x = 0$. We focus on this case in what follows.

Introduce the angle θ by setting

$$m_z = M_0 \cos\theta, \qquad m_y = M_0 \sin\theta. \tag{33}$$

In order to interpret θ, consider the reconstructed motion along the relative equilibrium. As in subsection 17, assume that the fixed axes $OXYZ$ are chosen in such a way that $\boldsymbol{M}||OZ$. Then, in view of (9) and given that $m_x = 0$, the angle θ coincides with the Euler angle θ of the matrix $\mathbf{Q}$ if we take the Euler angle $\varphi = 0$[5]. As it follows from (10), both of these angles remain constant along the relative equilibrium. On the other hand we have $\dot{\psi} = \omega$ where the constant angular velocity ω is computed from (14) to be

$$\omega = \frac{M_0 \cos(q-\theta)}{a^2 \mu_1 \sin q \sin\theta}. \tag{34}$$

Now recall from (12) that the positions of the masses on the moving frame are $\boldsymbol{r}_1 = (0, 0, a)$ and $\boldsymbol{r}_2 = (0, a\sin q, a\cos q)$. Assuming $\psi(0) = 0$, the positions $\boldsymbol{R}_\alpha = \mathbf{Q}\boldsymbol{r}_\alpha$ of the particles on the fixed axes are

$$\boldsymbol{R}_1(t) = a \begin{pmatrix} \sin\omega t \sin\theta \\ -\cos\omega t \sin\theta \\ \cos\theta \end{pmatrix}, \qquad \boldsymbol{R}_2(t) = a \begin{pmatrix} -\sin\omega t \sin(q-\theta) \\ \cos\omega t \sin(q-\theta) \\ \cos(q-\theta) \end{pmatrix}.$$

Therefore, the two particles rotate about the OZ axis with angular velocity ω as illustrated in Fig. 5 below, and θ is the angle between the particle μ_1 and the fixed axis of rotation. Hence, without loss of generality we may assume that $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

[5] The other possibility, namely that $\varphi = \pi$, simply changes the sign of θ.

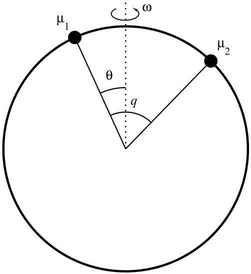


Fig. 5. $0 \leqslant \theta \leqslant \pi/2$ is the angle that the first particle makes with the axis of rotation.

Substitution of (33) into (31b) yields the necessary condition

$$\mu \sin(2\theta) - \sin(2(q - \theta)) = 0, \tag{35}$$

that is equivalent to the "law of lever" requirement given in [7]. We will analyse separately the case of equal and different masses.

4.1 The case of different masses Assume that the masses are different and, without loss of generality, that $\mu_1 < \mu_2$ (i.e. $0 < \mu < 1$). With our assumption that $0 \leqslant \theta \leqslant \frac{\pi}{2}$ equation (35) implicitly defines θ as a smooth function of q on the open intervals $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$. Figure 6 shows a graph of $\theta = \theta(q)$.

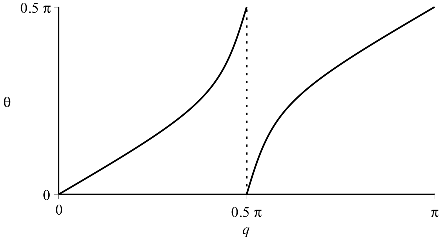


Fig. 6. The angle θ as a function of q for $\mu = 0.7$.

The interpretation is the following: suppose that we are given a relative configuration specified by an angle $q \in (0, \pi)$, $q \neq \frac{\pi}{2}$ and we ask ourselves if there is a relative equilibrium at this relative configuration. Below we show that such relative equilibrium does exist, it is unique (up to the sense of rotation), and that equation (35) specifies the unique angle $\theta \in (0, \frac{\pi}{2})$ between the particle μ_1 and the corresponding axis of rotation. The relative configurations having $q = \frac{\pi}{2}$ require a more careful analysis. As we shall see, for the gravitational potential, they do not admit relative equilibria.

Substitution of (33) and (32) into (31c) leads to

$$M_0^2 = \frac{\mu_1 \mu_2 a^2 \sin^3 q \, U'(q)}{\cos\theta(\mu_1 \cos q \cos\theta + \mu_2 \cos(q - \theta))}. \tag{36}$$

For an attractive potential we have $U'(q) > 0$ so, for any $q \in (0, \pi)$, $q \neq \frac{\pi}{2}$, equation (36) determines a unique value of $|M_0|$ that specifies the speed of rotation ω by (34) (in all expressions $\theta = \theta(q)$ as explained above).

Once again, the choice of sign for M_0 corresponds to two solutions related by reversing the direction of time. We shall not distinguish these two. Therefore, we have shown the following.

Proposition 4. *Suppose that $\mu_1 \neq \mu_2$. For any relative configuration $q \in (0, \pi)$, $q \neq \frac{\pi}{2}$, there exists a unique relative equilibrium; it has axis of rotation specified by the unique angle $\theta \in (0, \pi/2)$ satisfying* (35)*, and a unique speed of rotation ω specified by* (34) *where M_0^2 is given by* (36)*. Along these solutions $m_x = 0$ and $p = 0$.*

Relative equilibria with relative configuration $q \in (0, \pi/2)$ will be called ***acute relative equilibria***. For these we may write $q = \theta + \frac{1}{2}\arcsin(\mu(\sin 2\theta))$ and the inequality $\theta > \frac{q}{2}$ holds. Therefore, they are characterised by the condition that the heavier mass is closer to the axis of rotation. They correspond to the smooth bifurcation curve in the Energy-Momentum diagram 7(a).

Relative equilibria having relative configuration with $q \in (\pi/2, \pi)$ will be called ***obtuse relative equilibria***. For these we have $q = \theta + \frac{\pi}{2} - \frac{1}{2}\arcsin(\mu(\sin 2\theta))$ and they satisfy $\theta < \frac{q}{2}$. For these, it is the the lighter mass that is closer to the axis of rotation. These relative equilibria correspond to the bifurcation curve with the cusp in the Energy-Momentum diagram 7(a).

For the gravitational potential $U(q) = -G\mu_1\mu_2\cot(q)$, for a constant $G > 0$, and equation (36) becomes

$$M_0^2 = \frac{G\mu_1^2\mu_2^2 a^2 \sin q}{\cos\theta(\mu_1\cos\theta\cos q + \mu_2\cos(\theta - q))}. \tag{37}$$

In view of (35) the angle θ approaches $\pi/2$ as $q \to \pi/2$ from the left and it approaches 0 as $q \to \pi/2$ from the right. In both cases, according to (37), $M_0^2 \to \infty$. Therefore, in the gravitational case there do not exist relative equilibria having $q = \frac{\pi}{2}$. It follows that for the gravitational potential, acute and obtuse relative equilibria are two separate branches. Acute relative equilibria resemble their euclidean counterpart, whereas obtuse relative equilibria do not exist in the zero curvature case.

4.2 The case of equal masses

If $\mu_1 = \mu_2$ then $\mu = 1$ and equation (35) is only satisfied in the following two different situations that lead to two families of relative equilibiria.

1. The first family is labeled[6] by $\theta \in (0, \pi/2)$ and has $q = \pi/2$. We will refer to the members of these family as ***right-angle relative equilibria***.
2. For the second family $q = 2\theta$. These will be called ***isosceles relative equilibria***. Depending on the value of $q \in (0, \pi)$, we speak of acute, obtuse or right-angle isosceles relative equilibria.

For both families, the corresponding value of M_0^2 is determined by substituting the values of q and θ in (37). Note that both families intersect at the right angle-isosceles relative equilibrium as illustrated in the Energy-Momentum diagram in Fig. 7(b). These considerations, together with the analysis of the previous section allow us to conclude the following.

[6] It could be labeled by $\theta \in (0, \pi/4]$ by introducing a $\mathbb{Z}_2$ particle-relabeling symmetry.

Proposition 5. *Suppose that* $\mu_1 = \mu_2$.

1. *Consider the relative configuration* $q = \frac{\pi}{2}$. *Given any* $\theta \in (0, \pi/2)$ *there exists a unique relative equilibrium in which the first mass makes an angle* θ *with the axis of rotation.*
2. *For any relative configuration* $q \in (0, \pi)$, $q \neq \frac{\pi}{2}$, *there exists a unique relative equilibrium. The axis of rotation in such relative equilibrium makes an angle* $\theta = q/2$ *with any of the particles.*

In both cases, the unique speed of rotation ω *is specified by* (34) *where* M_0^2 *is given by* (36). *Along both of these families* $m_x = 0$ *and* $p = 0$.

As usual, in the statement of the proposition, we do not distinguish between relative equilibria having opposite directions of rotation.

4.3 The signature of the Hessian of the reduced Hamiltonian along the relative equilibria

Let

$$H_\lambda(\boldsymbol{m}, q, p) = H(\boldsymbol{m}, q, p) - \frac{\lambda}{2}(\boldsymbol{m}, \boldsymbol{m}).$$

A relative equilibrium $(\boldsymbol{m}_0, q_0, 0) = (0, M_0 \sin\theta_0, M_0 \cos\theta_0, q_0, 0)$ found in the previous section is a critical point of H_λ if the Lagrange multiplier

$$\lambda = \lambda_0 = \frac{\cos(q_0 - \theta_0)}{\mu_1 a^2 \sin\theta_0 \sin q_0}.$$

To compute the Hessian matrix of the restriction of H to the level set $(\boldsymbol{m}, \boldsymbol{m}) = M_0^2$ at the relative equilibrium, we proceed as in section 33. Namely, we consider the restriction of the quadratic form defined by the Hessian matrix $D^2 H_{\lambda_0}(\boldsymbol{m}_0, q_0, 0)$ to the tangent space of the surface $(\boldsymbol{m}, \boldsymbol{m}) = M_0^2$ at the critical point. In terms of the basis

$$\boldsymbol{e}_1 = (0,0,0,1,0), \qquad \boldsymbol{e}_2 = (1,0,0,0,0),$$
$$\boldsymbol{e}_3 = (0, -\cos\theta_0, \sin\theta_0, 0, 0), \qquad \boldsymbol{e}_4 = (0,0,0,0,1),$$

the resulting matrix $\mathbf{N}$ has the 2×2 block form

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}^{(1)} & 0 \\ 0 & \mathbf{N}^{(2)} \end{pmatrix}$$

where

$$\mathbf{N}^{(1)} = \frac{1}{\mu_1 a^2} \begin{pmatrix} (\mu + 1) & 1 \\ 1 & -\dfrac{\cos\theta_0 \cos q_0}{\sin\theta_0 \sin q_0} \end{pmatrix}.$$

Assume for the moment that the masses are different. It is straightforward to check that $\det(\mathbf{N^{(1)}}) < 0$ for acute relative equilibria since for these $0 < \theta_0, q_0 < \frac{\pi}{2}$. The same inequality holds for obtuse relative equilibria as can be shown using (35) and the estimate $q - \frac{\pi}{2} < \theta < \frac{q}{2}$ that follows from (35) for $q \in (\pi/2, \pi)$, and $\theta \in (0, \frac{\pi}{2})$. Therefore, the signature of $\mathbf{N^{(1)}}$ is $(+,-)$ both for the obtuse and the acute relative equilibria.

The entries of the matrix $\mathbf{N^{(2)}}$ are

$$N_{11}^{(2)} = \frac{-2\cos q_0}{\mu_2 a^2 \sin(2(q_0-\theta))\sin q},$$

$$N_{12}^{(2)} = N_{21}^{(2)} = \frac{-M_0\left(\mu \sin 2\theta_0 \cos q_0 + \sin(2\theta_0 - q_0)\right)}{\mu_1 a^2 \sin^3 q_0},$$

$$N_{22}^{(2)} = U''(q_0) + \frac{M_0^2 \cos\theta_0 \left((\mu+1)\cos\theta_0(2\cos^2 q_0 + 1) + \sin\theta \sin 2q_0\right)}{\mu_1 a^2 \sin^4 q_0}.$$

where we have used (35) to simplify the expression of $N_{11}^{(2)}$.

A numerical study of the signature of $\mathbf{N^{(2)}}$ for the gravitational potential is easily performed by writing q_0 in terms of θ_0 with the corresponding expressions for acute or obtuse relative equilibria given above. Such study shows that the determinant of $\mathbf{N^{(2)}}$ is always negative for acute relative equilibria. On the other hand, for obtuse relative equilibria, the determinant of $\mathbf{N^{(2)}}$ passes from being negative for small θ to being positive for larger values of θ. The trace of $\mathbf{N^{(2)}}$ in this case is positive, so we conclude that the signature of $\mathbf{N^{(2)}}$ for obtuse RE changes from $(+,-)$ to $(+,+)$ as θ increases.

Summarising, our numerical studies show that the signature of the restricted Hessian for all acute relative equilibria is $(+,+,-,-)$. For the obtuse relative equilibria, given that θ is an increasing function of q, we conclude the signature is $(+,+,-,-)$ for $\frac{\pi}{2} < q < q_*$ and $(+,+,+,-)$ for $q_* < q < \pi$ for a certain obtuse critical angle q_* that only depends on the ratio between the masses.[7] The change of signature corresponds to the cusp in the bifurcation curve of the Energy-Momentum diagram 7.

The analysis is of the signature of the matrices $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ simplifies significantly if the masses are equal. For the gravitational potential, we have for the right-angle relative equilibria

$$\det(\mathbf{N}^{(1)}) = \frac{-1}{a^4\mu_1^2} < 0, \qquad \det(\mathbf{N}^{(2)}) = \frac{-2G\mu_1\cos^2 2\theta}{a^2\sin 2\theta}.$$

Note that $\det(\mathbf{N}^{(2)}) < 0$ for all $0 < \theta < \pi/2$ except for $\theta = \frac{\pi}{4}$ where it vanishes. On the other hand, for the isosceles relative equilibria one obtains

$$\det(\mathbf{N}^{(1)}) = \frac{-\cot^2(q/2)}{a^4\mu_1^2} < 0,$$

$$\det(\mathbf{N}^{(2)}) = \frac{-G\mu_1(3-2\cos^2(q/2))(2\cos^2(q/2)-1)}{16a^2\cos^5(q/2)\sin^5(q/2)},$$

$$\mathrm{trace}(\mathbf{N}^{(2)}) = \frac{G\mu_1^3a^2 - 4\sin(q/2)\cos^2(q/2)(2\cos^2(q/2)-1)}{8a^2\mu_1\cos^5(q/2)\sin^3(q/2)}.$$

It is clear that $\det(\mathbf{N}^{(2)})$ is negative for $q \in (0,\pi/2)$ and positive for $q \in (\pi/2,\pi)$. For the latter values of q, we have $\mathrm{trace}(\mathbf{N}^{(2)}) > 0$.

[7] An analytic proof of of this was recently obtained in [18].

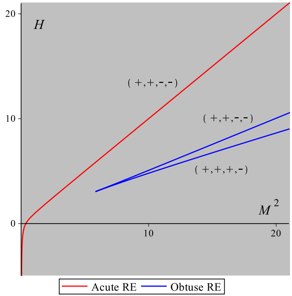
Therefore, the signature of the restricted Hessian for acute isosceles relative equilibria is $(+,+,-,-)$, while obtuse ones have $(+,+,+,-)$. The change occurs at the right-angle isosceles relative equilibrium where the family of isosceles relative equilibria intersects the family of right-angle relative equilibria, see Fig. 7(b). Away from this intersection point, the signature of right-angle relative equilibria is $(+,+,-,-)$.

Stability. All relative equilibria of the problem for which the signature of the restricted Hessian is $(+,+,+,-)$ are unstable. The question remains to determine the stability of those having signature $(+,+,-,-)$. All of these are elliptic equilibria (see [18] and section 58 ahead) so they are candidates to be stable solutions.

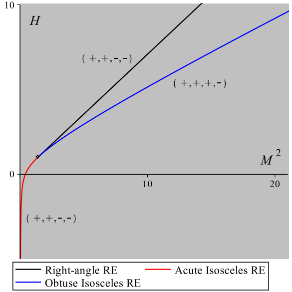
In section 58 below we will perform an analysis of stability for acute RE that involves Birkhoff normal forms and the use of KAM techniques. Our treatment suggests that generic acute RE are nonlinearly stable.

Energy-Momentum Diagram. Below we reproduce the energy-momentum diagram of the system given in [7] for the gravitational potential. We complement the diagram by illustrating the image of the energy-momentum map (M^2, H) and the signature of the restriction of the Hessian to the constant momentum surfaces along each family of relative equilibria. In view of the discussion above, such diagram must be done separately for the cases of equal and different masses.

REMARK 2. Considering that the gravitational potential U satisfies $\lim_{q\to 0} U(q) = -\infty$, it is easy to construct a sequence q_n, p_n with $q_n \to 0$, and $p_n \to \infty$, such that H evaluated at $(\boldsymbol{m}, q, p) = (0, M_0, 0, q_n, p_n)$ converges to any arbitrary value $h_0 \in \mathbb{R}$. This shows that the energy-momentum map (M^2, H) is not proper and that all of its fibres are non-compact.



(a) The case of different masses, $\mu_1 = 0.5$, $\mu_2 = 1$. Notice the change in signature at the cusp of the obtuse relative equilibria



(b) The case of equal masses, $\mu_1 = \mu_2 = 1/\sqrt{2}$. There is a change of signature passing from acute to obtuse isosceles RE when the two families of RE intersect.

Fig. 7. Energy-Momentum bifurcation diagram of relative equilibria for the gravitational potential with $G = 2$, $a = 1$, $\mu_1 = 0.5$, $\mu_2 = 1$. The shaded area on the M^2-H plane show all possible values of (M^2, H).

4.4 Stability of acute relative equilibria in the case of different masses

The analysis presented in the previous section shows that one cannot prove stability of RE for the 2-body problem on S^2 under the gravitational potential relying on purely topological methods. However, the restriction of the reduced equations of motion to the surface of constant momentum $(\boldsymbol{m}, \boldsymbol{m}) = M^2 > 0$, that we denote by $\mathcal{M}_{M^2}$, is a two-degree of freedom Hamiltonian system. Even if topological methods are not applicable, sufficient conditions for nonlinear stability of equilibria for these kind of systems may be obtained by computing Birkhoff normal forms and applying KAM techniques as we now recall.

Consider a RE of the problem that projects to an (isolated) equilibrium point on $\mathcal{M}_{M^2}$. Our investigation of its stability will proceed by checking that the following three conditions are satisfied:

1°. It is an *elliptic* equilibrium point of the restriction of the reduced system to $\mathcal{M}_{M^2}$. Namely, the eigenvalues of the linearized system are purely imaginary

$$\lambda_1 = i\Omega_1, \quad \lambda_2 = -i\Omega_1, \quad \lambda_3 = i\Omega_2, \quad \lambda_4 = -i\Omega_2, \quad 0 < \Omega_1 < \Omega_2.$$

It is well known that this is a necessary necessary condition for stability (in the absence of zero eigenvalues). Next we will check that

2°. there are no second or third-order resonances:

$$\Omega_2 \neq 2\Omega_1, \quad \Omega_2 \neq 3\Omega_1.$$

Under this condition one may put the Hamiltonian (restricted to $\mathcal{M}_{M^2}$) in Birkhoff normal form

$$H = \frac{1}{2}\sum_{j=1}^{2} \alpha_j I_j + \frac{1}{4}\sum_{j,k=1}^{2} \beta_{jk} I_j I_k + O_5, \quad I_j = x_j^2 + y_j^2, \quad |\alpha_i| = \Omega_i. \qquad (38)$$

Here, x_j and y_j are suitable canonical coordinates on a neighbourhood of the equilibrium on $\mathcal{M}_{M^2}$ (i.e., $\{x_j, y_k\} = \delta_{jk}$) with the equilibrium located at $x_j = y_j = 0$, β_{jk} are constants, and O_5 denotes a power series containing terms of order no less that 5 in x_j, y_j.

If conditions 1° and 2° are satisfied, a sufficient condition for nonlinear stability (under perturbations within $\mathcal{M}_{M^2}$) may be given in terms of the nonlinear terms in (38). Specifically, one requires that

3°. the *Arnold determinant* is different from zero

$$D := \det\begin{pmatrix} \beta_{11} & \beta_{12} & \alpha_1 \\ \beta_{12} & \beta_{22} & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{pmatrix} = 2\beta_{12}\alpha_1\alpha_2 - \beta_{11}\alpha_2^2 - \beta_{22}\alpha_1^2 \neq 0.$$

This nonlinear condition allows one to apply the KAM theorem in such a way that the invariant tori act as boundaries for the flow on each constant energy surface, leading to Lyapunov stability of the equilibrium (see e.g. [17], §35 in [19], or Section 13 in [15] for proofs and details).

REMARK 3. If the Arnold determinant $D = 0$, one may still obtain sufficient conditions for stability by considering higher order terms in the normal form expansion (38) (see e.g. [15]). On the other hand, the presence of second or third-order resonances may lead to instability. We shall not consider any of these possibilities here.

REMARK 4. We emphasise that the above analysis ensures nonlinear stability of RE only with respect to perturbations on the initial conditions that lie on the momentum surface $\mathcal{M}_{M^2}$.

Below we analyse these conditions for the acute RE presented in Section 47. We give numerical evidence that suggests that they are generically nonlinearly stable.

In our analysis we assume that the constants $\varkappa_1 := G\mu_1\mu_2$ and $\varkappa_2 := a^2\mu_1$ equal one. In this way, the gravitational potential $U(q) = -\cot(q)$, and the Hamiltonian (30) depends on the parameters of the problem only through the mass ratio μ that we will continue to assume to be $0 < \mu < 1$.[8] Therefore, in view of Proposition 4, the stability of the RE of the problem depends on two essential parameters. An internal parameter that labels the RE of the problem, and the external parameter μ.

[8] The assumption that $\varkappa_1$ and $\varkappa_2$ equal 1 is done without loss of generality since one may eliminate these quantities from the equations of motion (8) by rescaling time $t \to \frac{\varkappa_2 t}{\sqrt{\varkappa_1}}$, and the momenta $p \to \sqrt{\varkappa_1}p,\ \boldsymbol{m} \to \sqrt{\varkappa_1}\boldsymbol{m}$.

Linear analysis

Introduce cylindrical coordinates (z,θ) on $\mathcal{M}_{M^2}$

$$m_x = z, \quad m_y = \sqrt{M^2 - z^2}\sin\theta, \quad m_z = \sqrt{M^2 - z^2}\cos\theta.$$

Then (q,p,θ,z) are Darboux coordinates and the restriction of the reduced equations of motion to $\mathcal{M}_{M^2}$ takes the canonical form

$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q}, \quad \dot\theta = \frac{\partial H}{\partial z}, \quad \dot z = -\frac{\partial H}{\partial \theta},$$

where, in view of (30),

$$H = \frac{1}{2}\Bigg[\Big((1+\mu)p^2 + 2pz - \frac{\cos 2q + \cos 2(q-\theta) + \mu(1+\cos 2\theta)}{2\sin^2 q} z^2\Big) - \frac{1}{\sin^2 q}\Big(\sin 2q - \frac{M^2}{2}\big(1 + \cos 2(q-\theta) + \mu(1+\cos 2\theta)\big)\Big)\Bigg].$$

Of course, in the above equations, M^2 should be treated as a constant.

The RE of the problem are the equilibria of the above equations. As it was shown in Section 47, these occur at points where $p=0$, $z=0$, θ and q are related by (35). In addition, in view of (37), q and θ should be such that

$$M^2 = \frac{\sin q}{\cos\theta\big(\cos(q-\theta) + \mu\cos\theta\cos q\big)}. \tag{39}$$

Consider the RE described above, and a small deviation from it within $\mathcal{M}_{M^2}$ given by the vector $\mathbf{\Delta} = (\Delta z, \Delta p, \Delta\theta, \Delta q)$. The linearized system has the form

$$\dot{\mathbf{\Delta}} = \mathbf{A}\mathbf{\Delta}, \quad \mathbf{A} = \begin{pmatrix} 0 & \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} & 0 \end{pmatrix},$$

where $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are symmetric 2×2 matrices. The entries of $\mathbf{A}^{(1)}$ may be written as

$$\begin{aligned} \mathbf{A}^{(1)}_{11} &= \frac{M^2(\cos(2(q-\theta)) + \mu\cos 2\theta)}{\sin^2 q}, \qquad \mathbf{A}^{(1)}_{22} = -\frac{M^2(1+\mu)\cos^2\theta}{\sin^4 q}, \\ \mathbf{A}^{(1)}_{12} &= \mathbf{A}^{(1)}_{21} = \frac{M^2(\sin q\cos 2\theta - (1+\mu)\cos q\sin 2\theta)}{\sin^3 q}, \end{aligned} \tag{40}$$

where we have used (39) to simplify $\mathbf{A}^{(1)}_{22}$. On the other hand

$$\mathbf{A}^{(2)} = \begin{pmatrix} -\dfrac{\cos\theta\big(\cos\theta\cos 2q + \sin\theta\sin 2q + \mu\cos\theta\big)}{\sin^2 q} & 1 \\ 1 & 1+\mu \end{pmatrix}. \tag{41}$$

The eigenvalues of of the matrix $\mathbf{A}$ are the roots of its characteristic polynomial, which due to the Hamiltonian nature of the problem turns out to be biquadratic:

$$P(\lambda) = \lambda^4 + a\lambda^2 + b.$$

Proposition 6. *All acute RE are elliptic.*

Proof.

The ellipticity condition 1° may be written in terms of the coefficients a and b of the characteristic polynomial P as

$$a > 0, \quad b > 0, \quad R_1 := \frac{1}{4}a^2 - b > 0. \tag{42}$$

Using the expressions given above for the entries of **A** and (35) and (39) we can simplify

$$a = \frac{M^2(1+\cos(2(q-\theta)))}{\sin^2 q \sin^2 \theta}, \qquad b = \frac{a^2}{4}\big(1 - 4\sin^2\theta\sin^2(q-\theta)\big), \tag{43}$$
$$R_1 = a^2 \sin^2\theta \sin^2(q-\theta).$$

Recall from section 47 that acute RE correspond to the solutions of (35) that satisfy

$$q = \theta + \frac{1}{2}\arcsin(\mu(\sin 2\theta)), \tag{44}$$

with $\theta \in (0, \pi/2)$. Using this, it is immediate to see that the inequality $a > 0$, and hence also $R_1 > 0$, hold. On the other hand, the inequality $b > 0$ in (42) is equivalent to

$$f := 1 - 4\sin^2\theta \sin^2(q-\theta) > 0. \tag{45}$$

To show that this inequality holds, we write, using (44),

$$f = 1 - 2\sin^2\theta\Big(1 - \sqrt{1 - \mu^2 \sin^2 2\theta}\Big).$$

Since $\mu^2 \sin^2 2\theta < \sin^2 2\theta$ for all $\mu \in (0, 1)$, we find that

$$f > 1 - 2\sin^2\theta(1 - |\cos 2\theta|) = \begin{cases} 1 - 4\sin^4\theta & 0 < \theta \leqslant \pi/4, \\ \cos^2 2\theta & \pi/4 \leqslant \theta < \pi/2. \end{cases}$$

The above function is everywhere greater than zero except at the point $\theta = \pi/4$, but, as can be verified,

$$f\Big|_{\theta=\pi/4} = \sqrt{1 - \mu^2} > 0.$$

■

REMARK 5. An analogous analysis may be performed to show that the obtuse RE with signature $(+,+,-,-)$ are elliptic. For these one has

$$q = \theta + \frac{\pi}{2} - \frac{1}{2}\arcsin(\mu(\sin 2\theta)), \tag{46}$$

and, using (43), one may easily verify that $a > 0$ (and hence also $R_1 > 0$) for all $\theta \in (0, \pi/2)$, $\mu \in (0,1)$. On the other hand, the corresponding expression for f is $f = 1 - 2\sin^2\theta\left(1 + \sqrt{1-\mu^2\sin^2 2\theta}\right)$. This is a strictly decreasing function of θ on the interval $(0, \pi/2)$, that is positive for $0 < \theta < \theta_*$ and negative for $\theta_* < \theta < \pi/2$. Here θ_* is the unique root of the equation

$$\cos 2\theta = 2\sin^2\theta\sqrt{1-\mu^2\sin^2 2\theta}$$

on the interval $(0, \pi/2)$. This value of θ_* corresponds to the cusp in the bifurcation diagram Fig. 7(a), where the signature of the obtuse RE changes.

Figure 8 below illustrates the plane a-b of coefficients of the characteristic polynomial of $\mathbf{A}$. The curves σ_1 and σ_2 respectively correspond to the values of (a, b) attained at the acute and obtuse RE of the problem for the fixed value of $\mu = 0.95$. These RE are conveniently parametrised by $\theta \in (0, \pi/2)$ by respectively using (44) and (46). The figure also illustrates the parabolae corresponding to the zero loci of R_1, and of the second and third order resonance polynomials R_2 and R_3 introduced below.

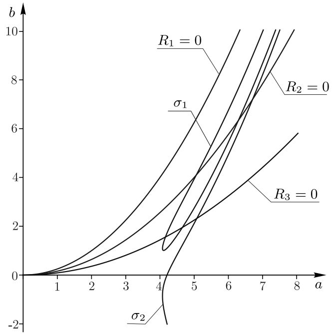


Fig. 8. Curves σ_1, σ_2, corresponding to acute and obtuse RE (respectively given by (44) and (46)) on the coefficient plane (a, b) for $\mu = 0.95$.

Resonances

Condition 2° that requires that there are no second or third-order resonances is written in terms of the coefficients a and b of the characteristic polynomial of $\mathbf{A}$ as

$$R_2 := \frac{4}{25}a^2 - b \neq 0, \quad R_3 := \frac{9}{100}a^2 - b \neq 0. \tag{47}$$

An analytic investigation of these conditions involves very heavy calculations so we present numerical results. We restrict our attention to the acute RE that may be parametrised by θ using (44), and we present our stability results in terms of the parameters $(\mu, \theta) \in (0, 1) \times (0, \pi/2)$.

One can express R_2 and R_3 as functions of (μ, θ) by using (43), (39) and the relation (44) for acute RE. The zero loci of R_2 and R_3 on the θ-μ-plane are the two curves illustrated in Fig. 9.

Analysis of the Arnold determinant

As for the resonance condition, we only present numerical results for our investigation of condition 3° for the acute RE. By using (39) and (44), we express $D = D(\mu, \theta)$.

The zero locus of D on the plane θ-μ consists of the two curves illustrated in Fig. 9.

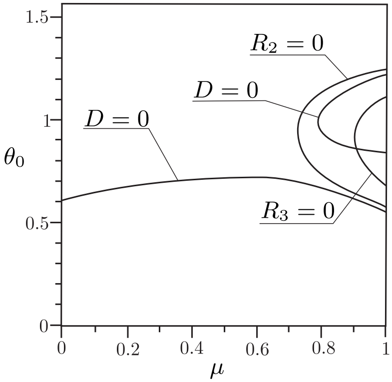


Fig. 9. Curves on the plane μ-θ_0 plane corresponding to RE with second and third order resonances (respectively $R_2 = 0$ and $R_3 = 0$) and where the Arnold determinant vanishes ($D = 0$).

Nonlinear stability of acute RE

According to the discussion above, the acute RE that correspond to parameter values (μ, θ) lying outside of the curves in Fig. 9 are nonlinearly stable. Therefore, *we have provided numerical evidence to show that generic*[9] *acute RE are stable in the sense of Lyapunov*.

The stability analysis for the RE corresponding to the exceptional parameter values (μ, θ) that lead to resonances $R_2 = 0$ and $R_3 = 0$, or the vanishing of the Arnold determinant $D = 0$, has to be done separately.

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[9] for an open dense set of parameter values.

Open problems To conclude, we point out a number of open problems concerning the two-body problems in spaces of constant curvature:

- In the case of different masses in S^2, investigate the nonlinear stability of acute RE for which there are resonances ($R_2 = 0$, $R_3 = 0$) and/or the Arnold determinant vanishes ($D = 0$).
- In the case of different masses in S^2, investigate the nonlinear stability of obtuse RE for which the signature of the reduced Hamiltonian is $(+, +, -, -)$.
- In the case of equal masses in S^2, investigate the nonlinear stability of acute-isosceles RE and right-angle RE.
- Classify and investigate the stability of all RE for the spatial two-body problem on S^3 and L^3.

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