On long-term dynamics of slow-fast systems with passages through resonances

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Based on joint works with A.Artemyev, D. Mourenas, A.Vasiliev

Plan:

I. Passages through resonances, time $1/\varepsilon$.

II. Examples of dynamics on times $\gg 1/\varepsilon$. Kinetic equation.

I. Passages through resonances, time 1/ε.

1. Systems with fast rotating phases

$$\dot{x} = \varepsilon f(x,\varphi,\varepsilon),\ \dot{\varphi} = \Omega(x) + \varepsilon g(x,\varphi,\varepsilon),$$
$$0 < \varepsilon << 1$$

$x = (x_1,\ldots,x_n)$ - slow variables

$\varphi = (\varphi_1,\ldots,\varphi_m)$ - fast angular variables (phases)

Functions f, g are 2π periodic in all phases

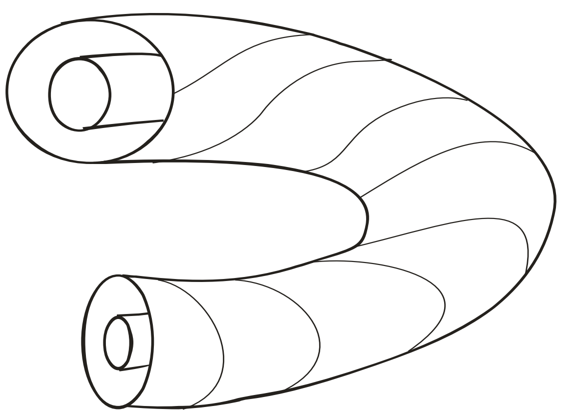
$\Omega = (\Omega_1,\ldots,\Omega_m)$ - unperturbed frequencies

2. Averaging method

Recipe of the averaging method: use solutions of the averaged system for approximate description of slow variables dynamics in the exact system.

$$\dot{\bar{x}} = \varepsilon F(\bar{x}), \quad F(\bar{x}) = \frac{1}{(2\pi)^m} \oint_{\mathbf{T}^m} f(\bar{x}, \varphi, 0) d\varphi$$

$$d\varphi = d\varphi_1 \ldots d\varphi_m$$

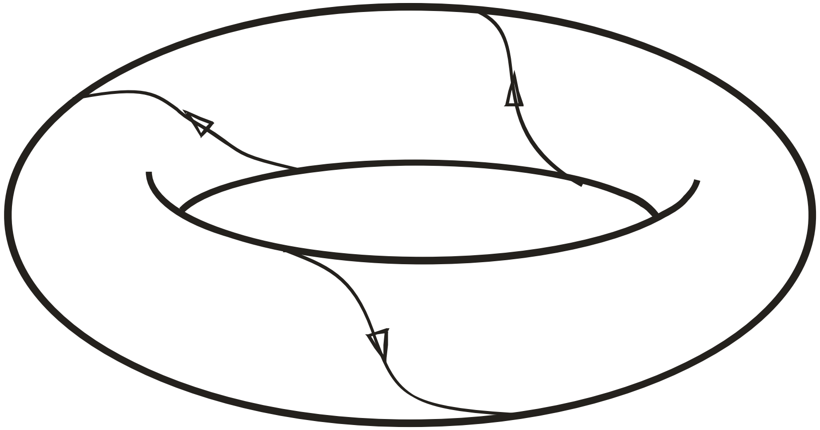


- unperturbed tori

3. Resonances

Resonant relation: $k_1\Omega_1(x) + \ldots + k_m\Omega_m(x) = 0$

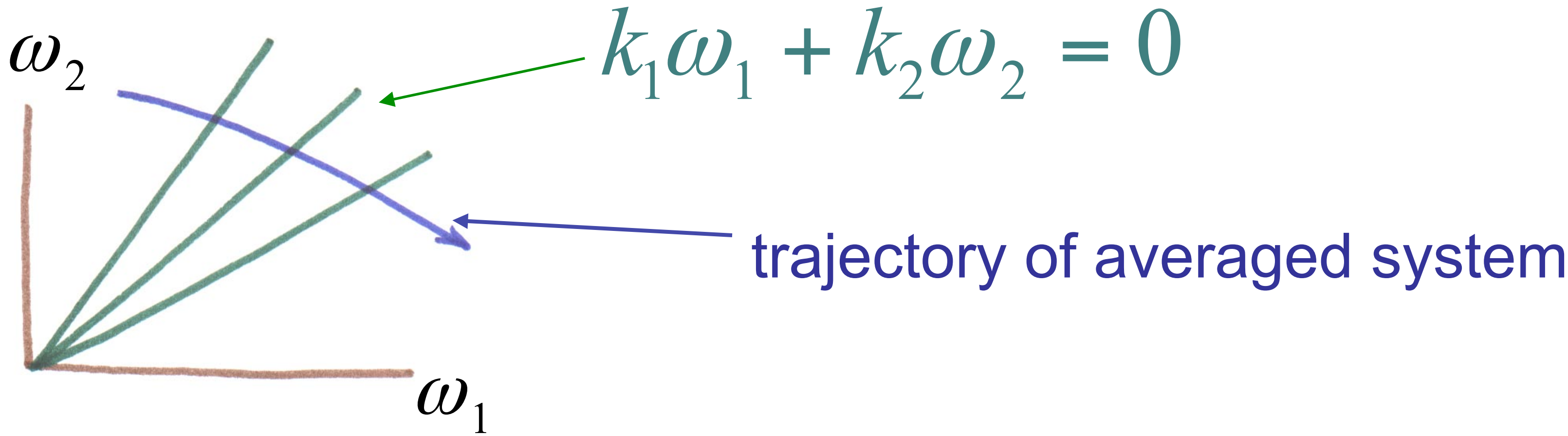
Resonant torus (for *m*=2):



Resonances are obstacles to application of the averaging method.

4. Two-frequency systems:

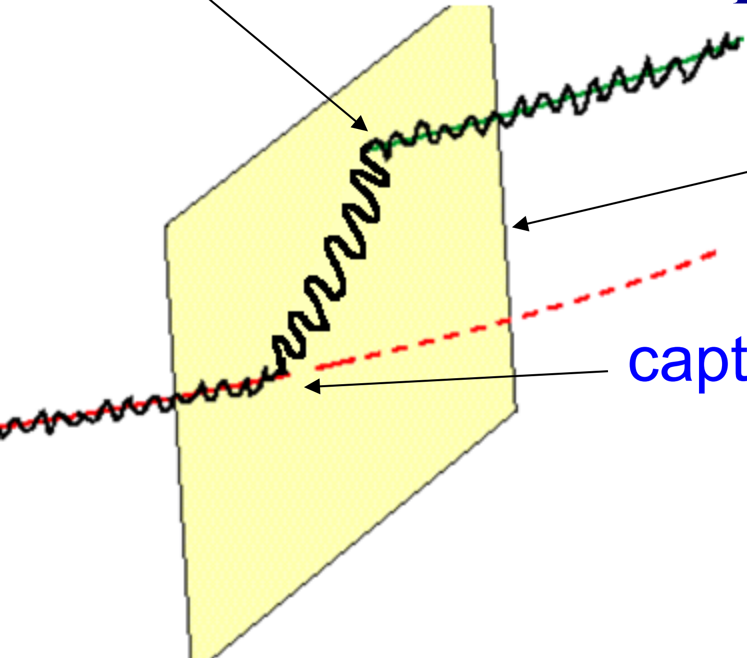
$$\varphi = (\varphi_1, \varphi_2), \quad \omega = (\omega_1, \omega_2)$$



Effect of each resonance can be studied separately.

Principal phenomena associated with effect of a single resonance are capture into resonance and scattering on resonance.

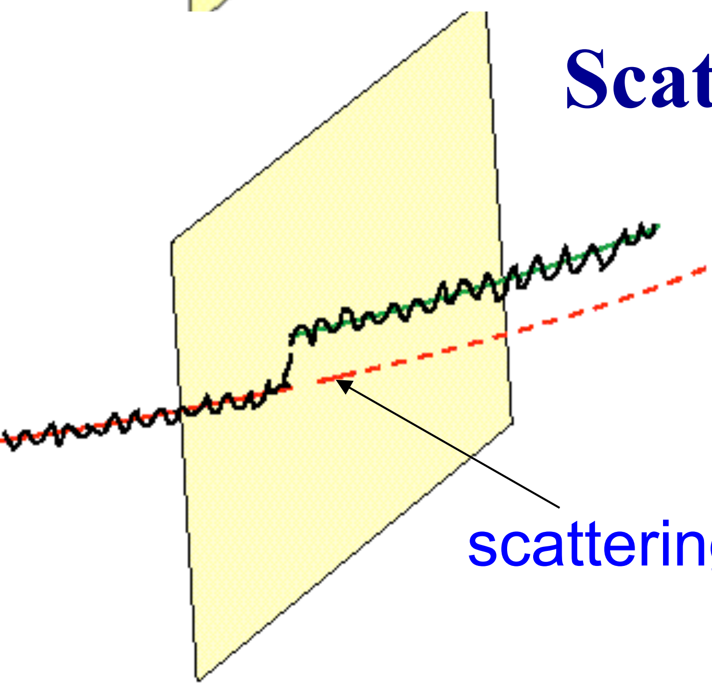
Capture into resonance



mes(capture) $\cong \sqrt{\varepsilon}$

(P.Goldreich, S.Peale, 1966)

Scattering on resonance



Amplitude of scattering $\cong \sqrt{\varepsilon}$

(B.V.Chirikov, 1959)

5. Slow–fast Hamiltonian system:

$E = E(p,q,y,x)$ - Hamiltonian

$0 < \varepsilon << 1$

$$\dot{p} = -\frac{\partial E}{\partial q}, \quad \dot{q} = \frac{\partial E}{\partial p}, \quad (p,q) \in \mathbf{R}^4$$ (fast variables)

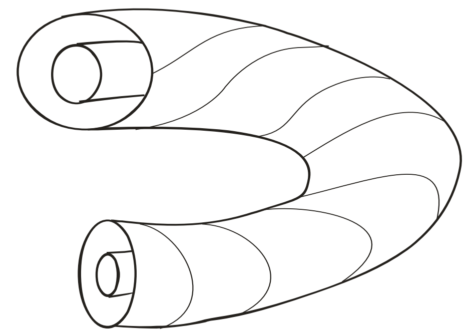
$$\dot{y} = -\varepsilon \frac{\partial E}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial E}{\partial y}, \quad (y,x) \in \mathbf{R}^{2m}$$ (slow variables)

Symplectic structure $dp \wedge dq + \frac{1}{\varepsilon} dy \wedge dx$

For frozen slow variables:

$$\dot{p} = -\frac{\partial E}{\partial q}, \quad \dot{q} = \frac{\partial E}{\partial p}, \quad y, x = \text{const}$$

Let this system be integrable



I, φ - action-angle variables

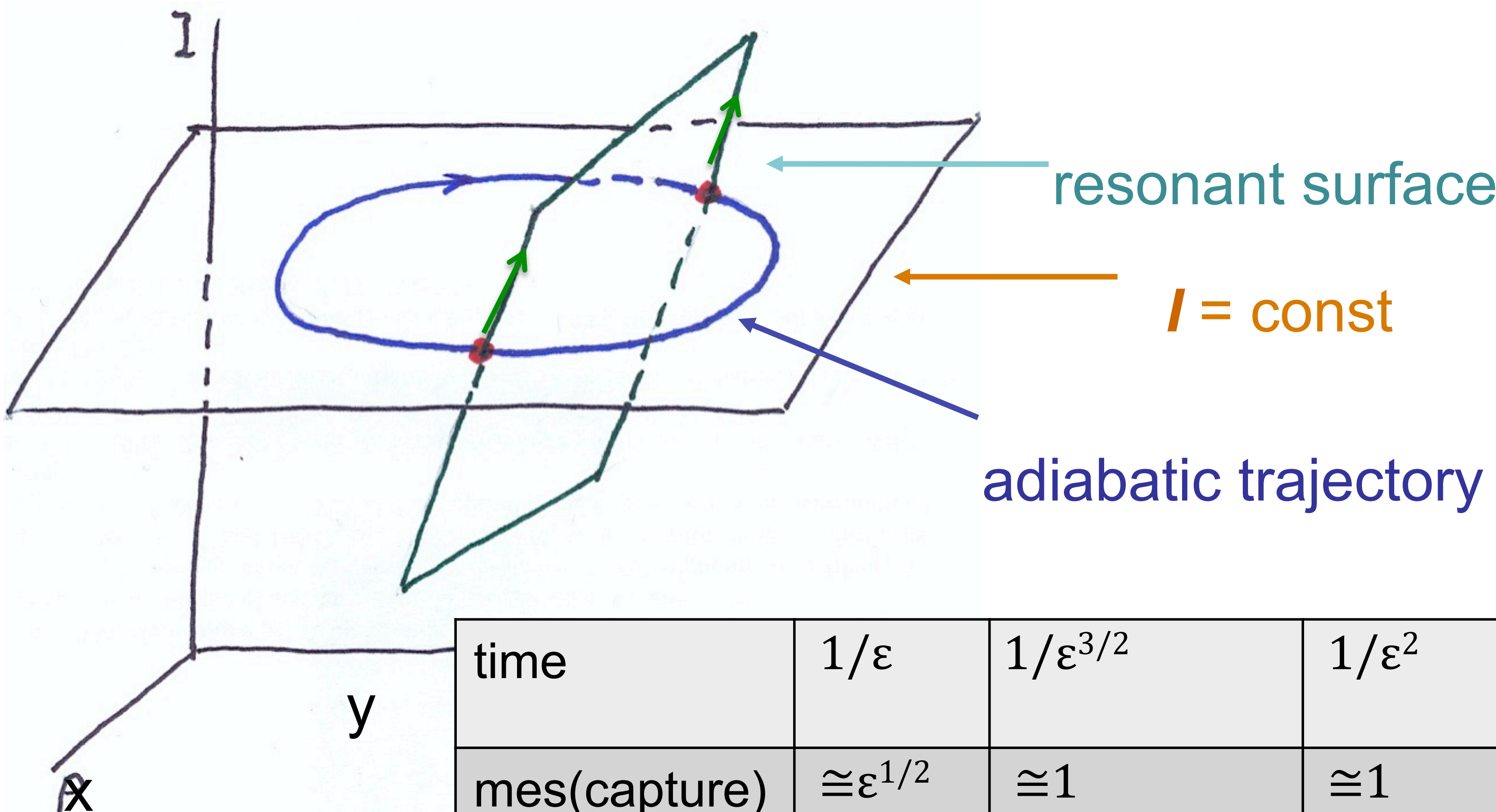
$$E(p, q, y, x) = H_0(I, y, x)$$

Unperturbed frequency:

$$\omega(I, y, x) = \frac{\partial H_0(I, y, x)}{\partial I}$$

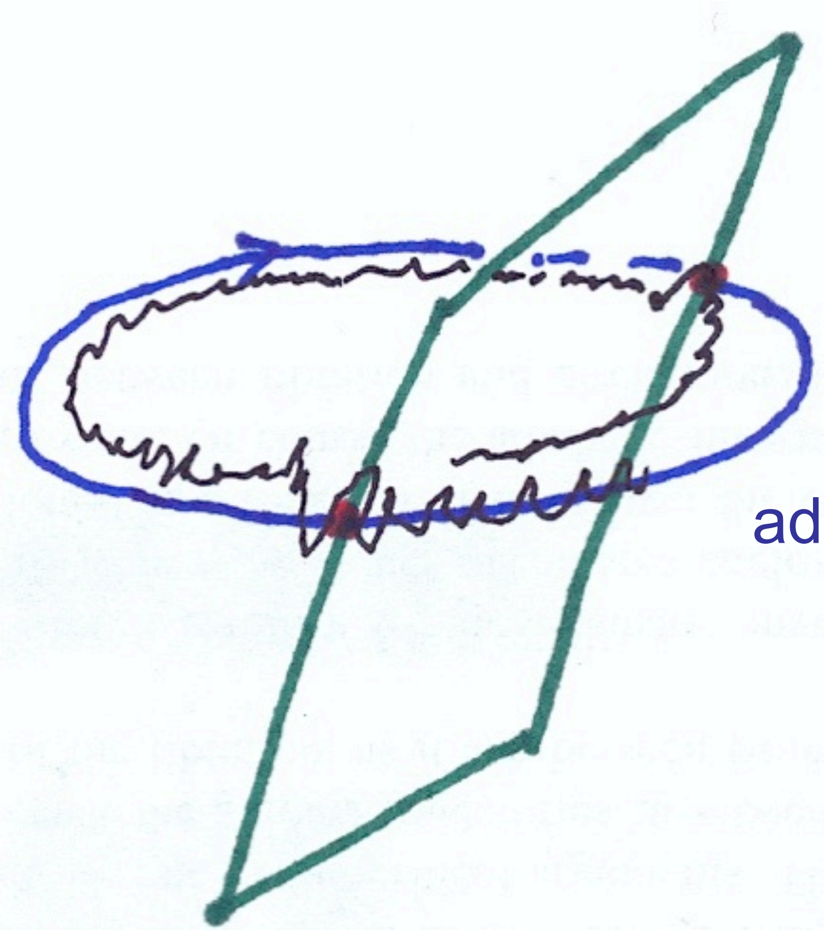
Averaging over fast phases => adiabatic approximation:

$$I = \text{const}, \quad \dot{y} = -\varepsilon \frac{\partial H_0(I, y, x)}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial H_0(I, y, x)}{\partial y}$$



time	$1/\varepsilon$	$1/\varepsilon^{3/2}$	$1/\varepsilon^2$
mes(capture)	$\cong\varepsilon^{1/2}$	$\cong 1$	$\cong 1$
average scattering	$\cong\varepsilon^{1/2}$	$\cong 1$(drift, D.Dolgopyat, 2005, 2012)	$\cong 1$
standard deviation of scattering	$\cong\varepsilon^{1/2}$	$\cong\varepsilon^{1/4}$	$\cong 1$ (diffusion)

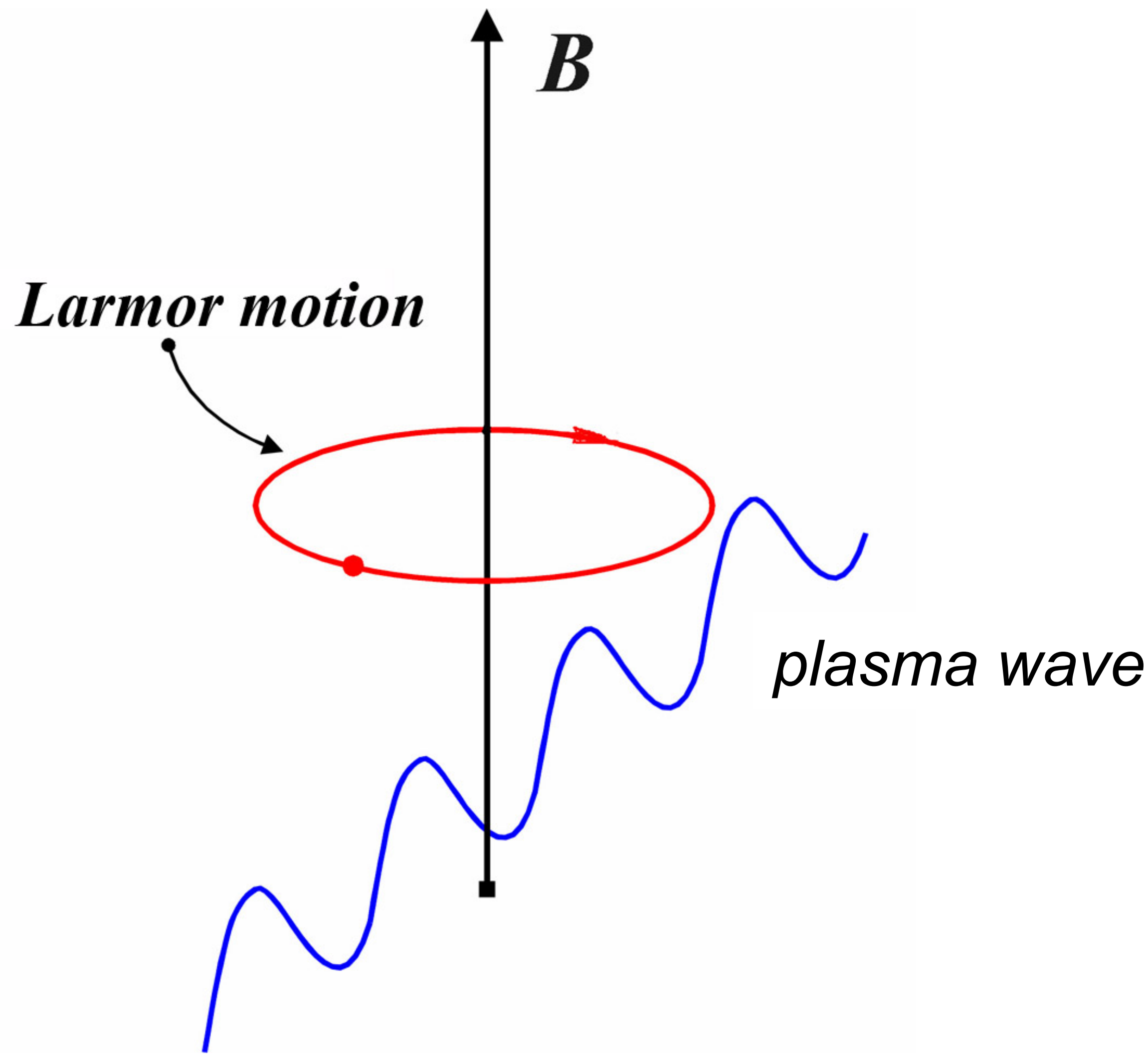
Stretching of phases is an indication of independence of results of consecutive scatterings

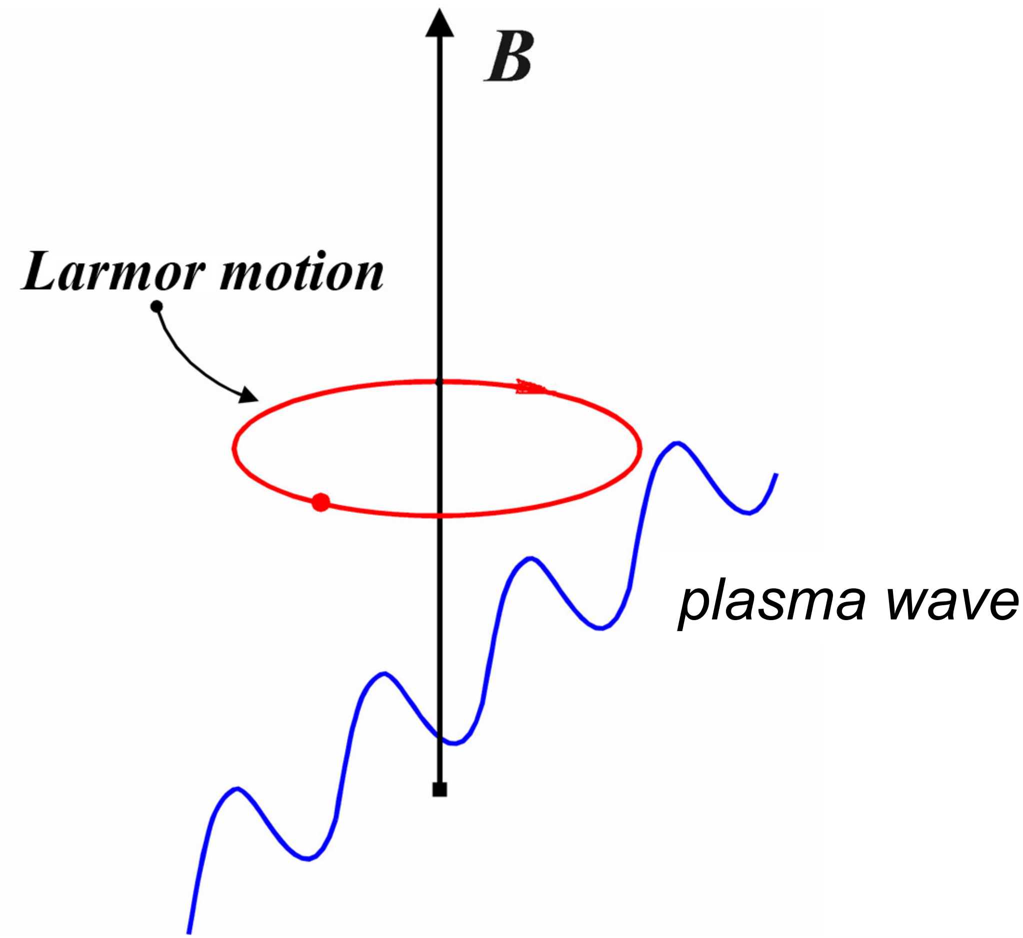


$$\Delta\tilde{\varphi} \propto \frac{1}{\varepsilon}\sqrt{\varepsilon}\Delta\varphi = \frac{1}{\sqrt{\varepsilon}}\Delta\varphi >> \Delta\varphi$$

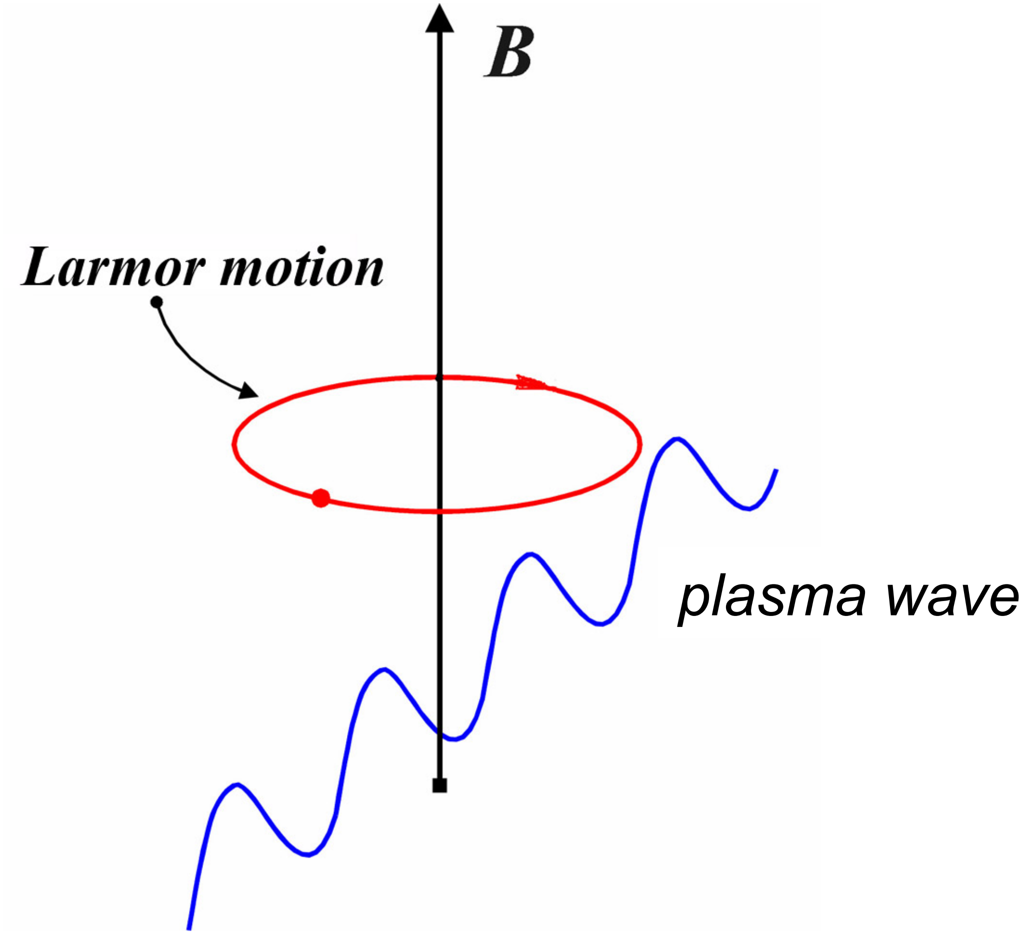
II. Example of dynamics on times >> 1/ε. Kinetic equation.

IIa. Model





We consider the case of a high frequency wave. Averaging over the phase of the wave washes out the effect of the wave. The averaged motion is just a Larmor motion.

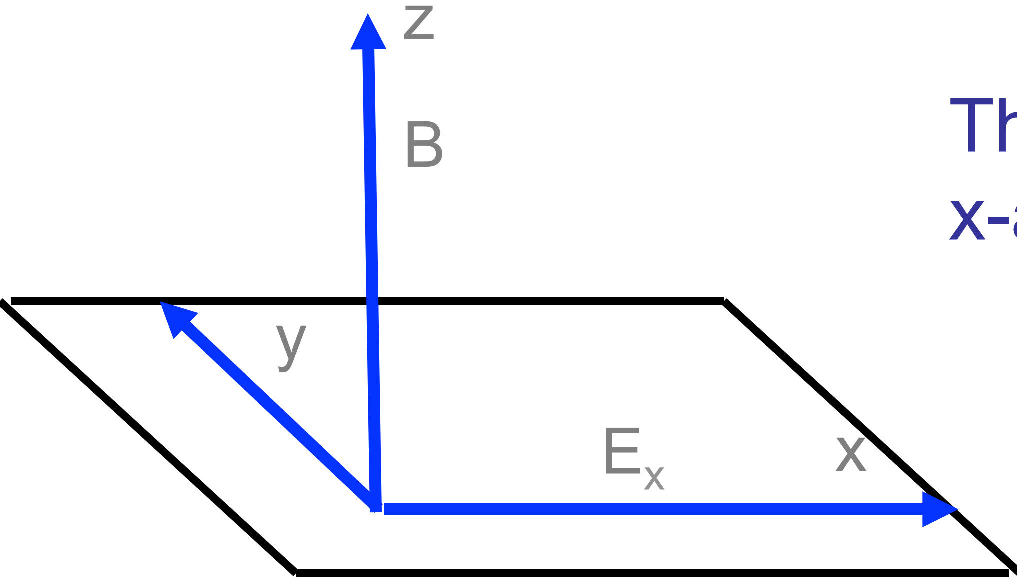


In the process of the Larmor motion the particle may approach resonance with the wave: projection of the particle’s velocity onto direction of the wave propagation is equal to the phase velocity of the wave.

Equation of motion

$$\frac{d}{dt}(m\vec{V}) = \frac{e}{c}\vec{V} \times \vec{B} + e\vec{E}$$

Electrostatic wave perpendicular to a magnetic field, non-relativistic particle



The wave propagates along x-axis

B – the magnetic field
E_x - the electric field of the wave

$$\ddot{x} = \frac{eB(x)}{mc}\dot{y} + \frac{e}{m}E_x,$$

$$\ddot{y} = -\frac{eB(x)}{mc}\dot{x}$$

$$E_x = -E_0(x)\cos(kx - \omega t).$$

$$k >> 1,\ \omega >> 1,\ v_\phi = \omega / k \cong 1,\ E_0 \cong 1$$

v_ϕ - phase velocity of the wave

The equation can be reduced to the Hamiltonian form with the Hamiltonian (assume mass *m=1*)

$$H = \frac{1}{2}p^2 + \frac{1}{2}\Omega_b^2 b(x) + \varepsilon u(x)\sin(kx - \omega t), \qquad \Omega_b = \frac{eB(0)}{c},$$

$$b(x) \cong x^2$$

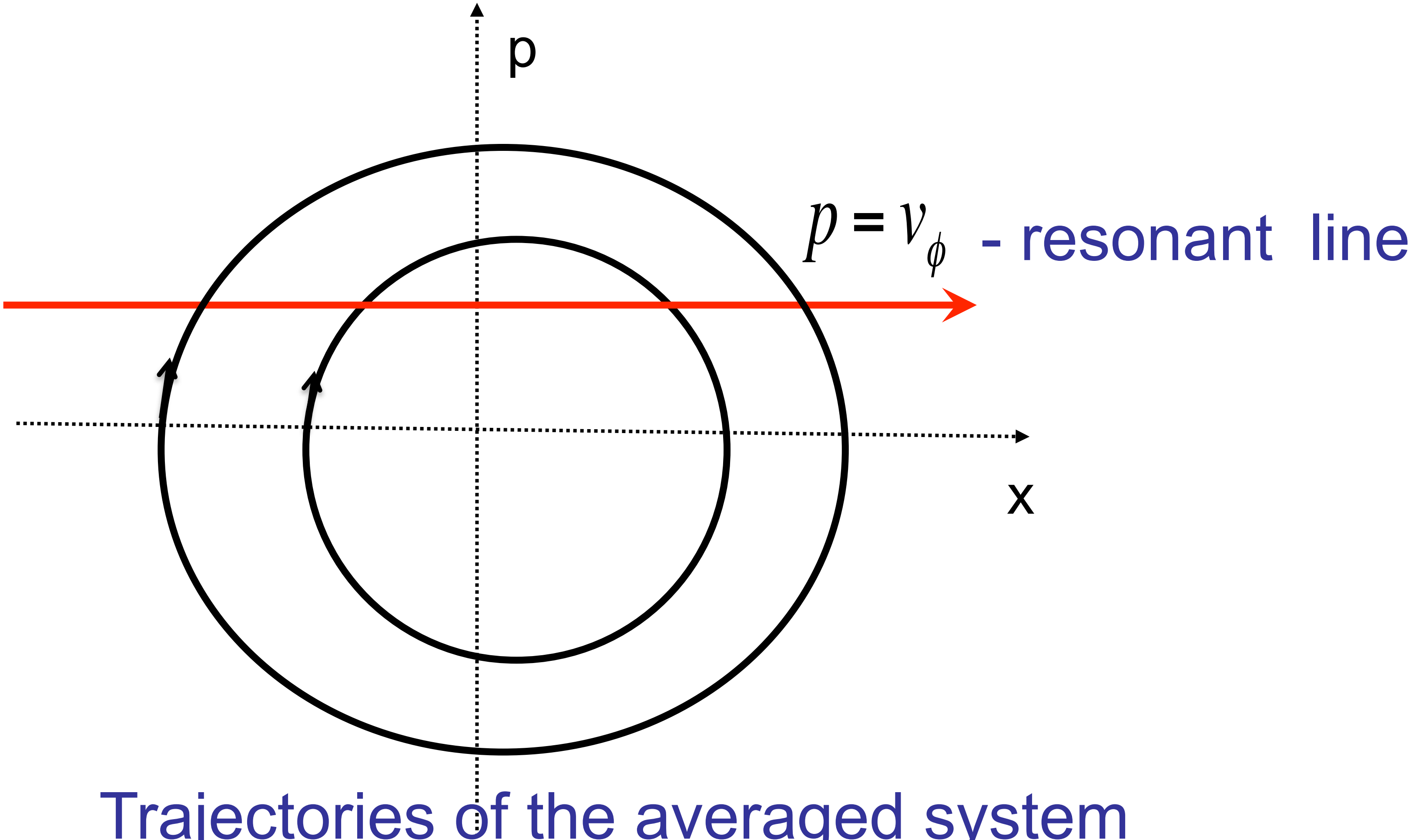
x, p - slow conjugate variables, $\varepsilon \cong 1/k << 1.$

$\phi = kx - \omega t = k(x - v_\phi t)$ - fast phase

Resonant line: $p = v_\phi$

$$\bar{H} = \frac{1}{2}p^2 + \frac{1}{2}\Omega_b^2 b(x)$$

- averaged Hamiltonian, kinetic energy of the particle



Trajectories of the averaged system

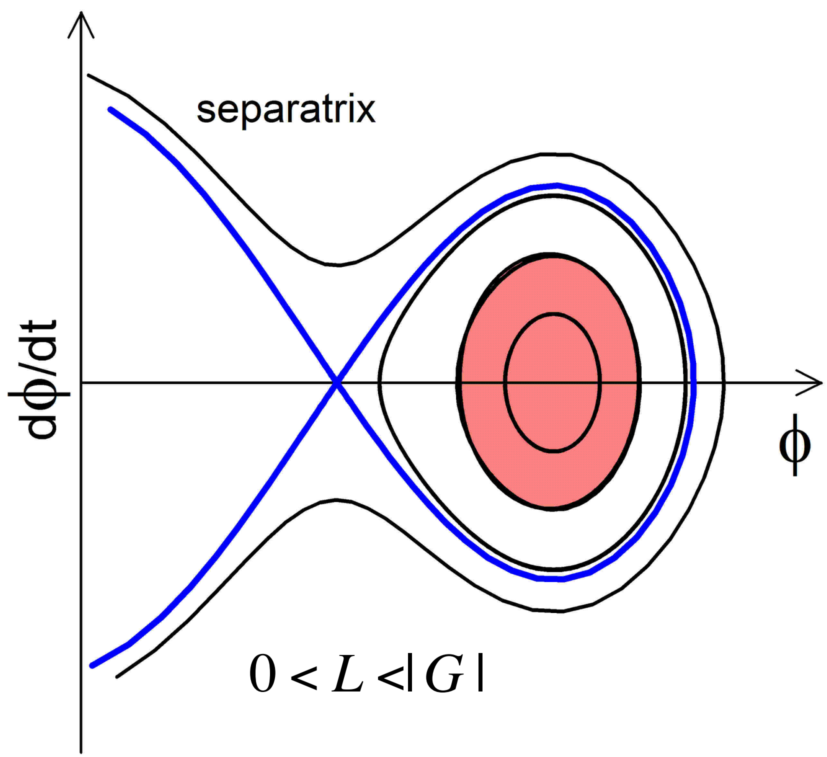
Approximate equations for motion near the resonant line:

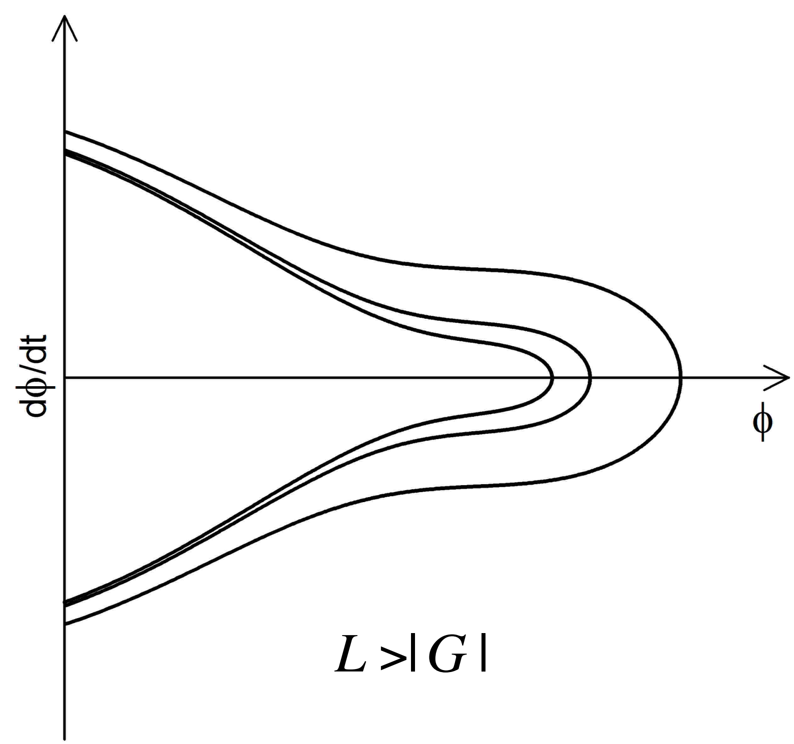
$$\ddot{\phi} + G(x)\cos\phi + L(x) = 0$$ - fast pendulum

$$\dot{x} = v_\phi$$ - slowly varying parameter

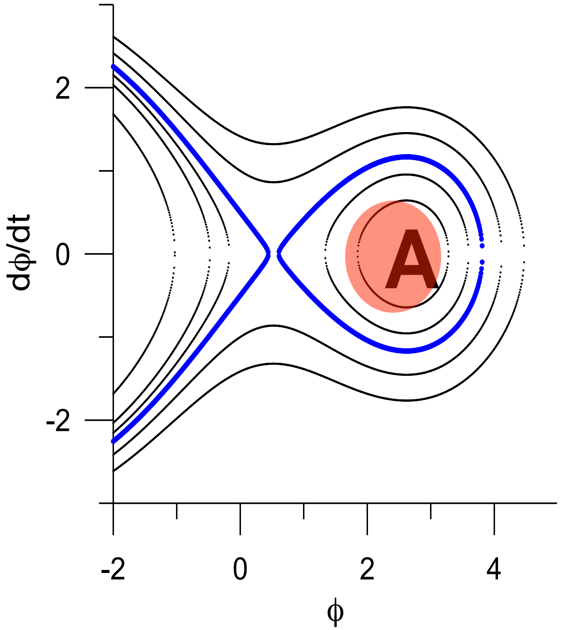
$$G(x) = \varepsilon k^2 u(x), \quad L(x) = \tfrac{1}{2} k \Omega_b^2 b'(x)$$

Phase portraits of the pendulum for frozen *x*:





IIIb. Capture and scattering



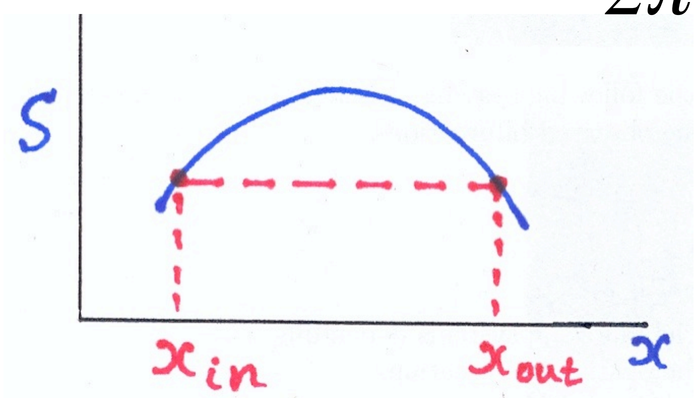
The area **A** surrounded by the trajectory is an adiabatic invariant: its value is approximately conserved in the evolution.

$S(x)$ - the area of the oscillatory domain, capture is possible for x such that $S'(x) > 0$.

Probability of capture: $\Pi = \dfrac{S'(x)v_\phi}{2\pi\,|L(x)|} \cong \dfrac{1}{\sqrt{k}}$

In-out function:

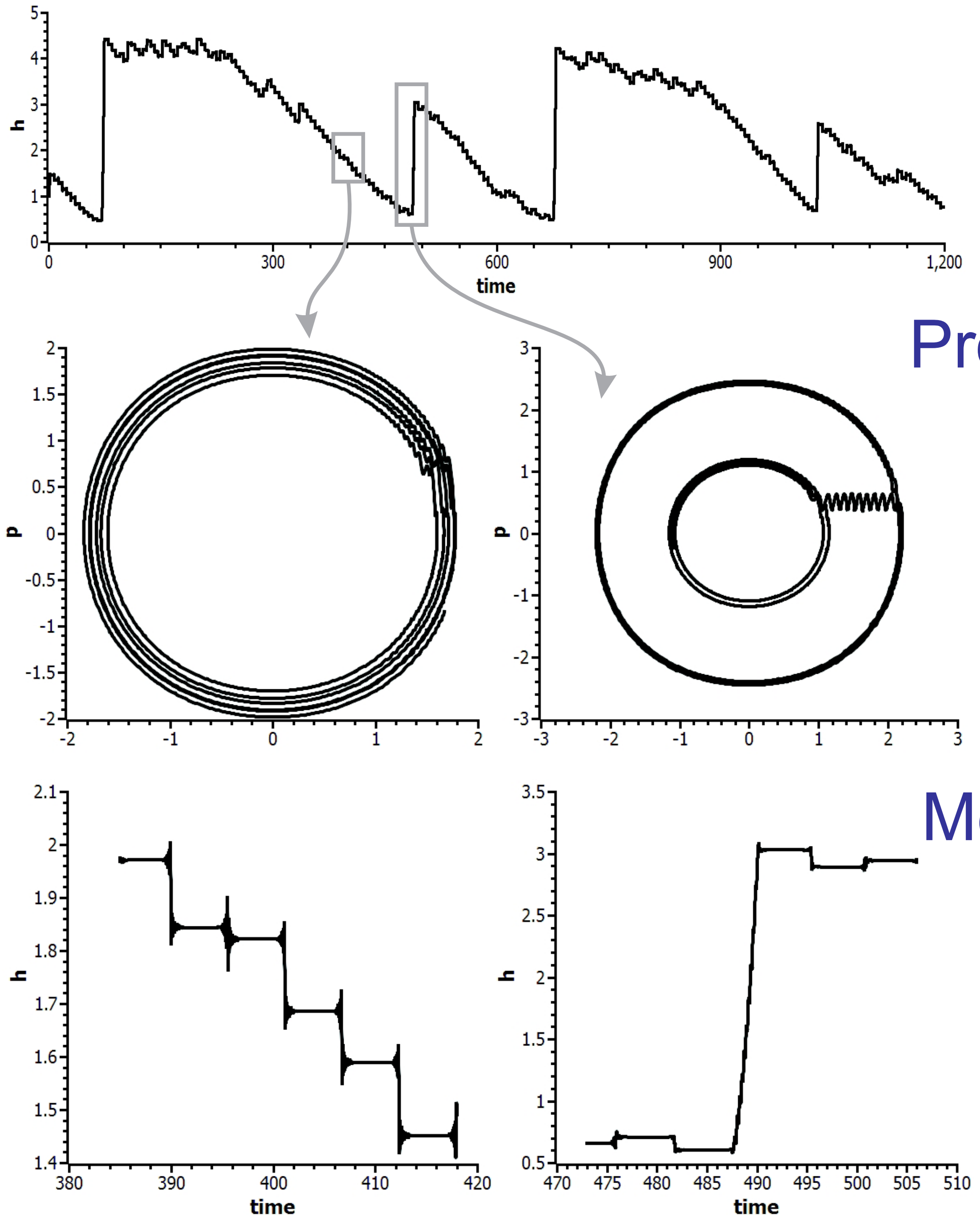
$$S(x_{out}) = S(x_{in})$$



Scattering in values of the averaged Hamiltonian: Δh

Mean value of scattering: $<\Delta h>$

$$<\Delta h> = -\frac{v_\varphi}{2\pi k}\operatorname{sgn}(L(x))S(x) \cong \frac{1}{\sqrt{k}}$$



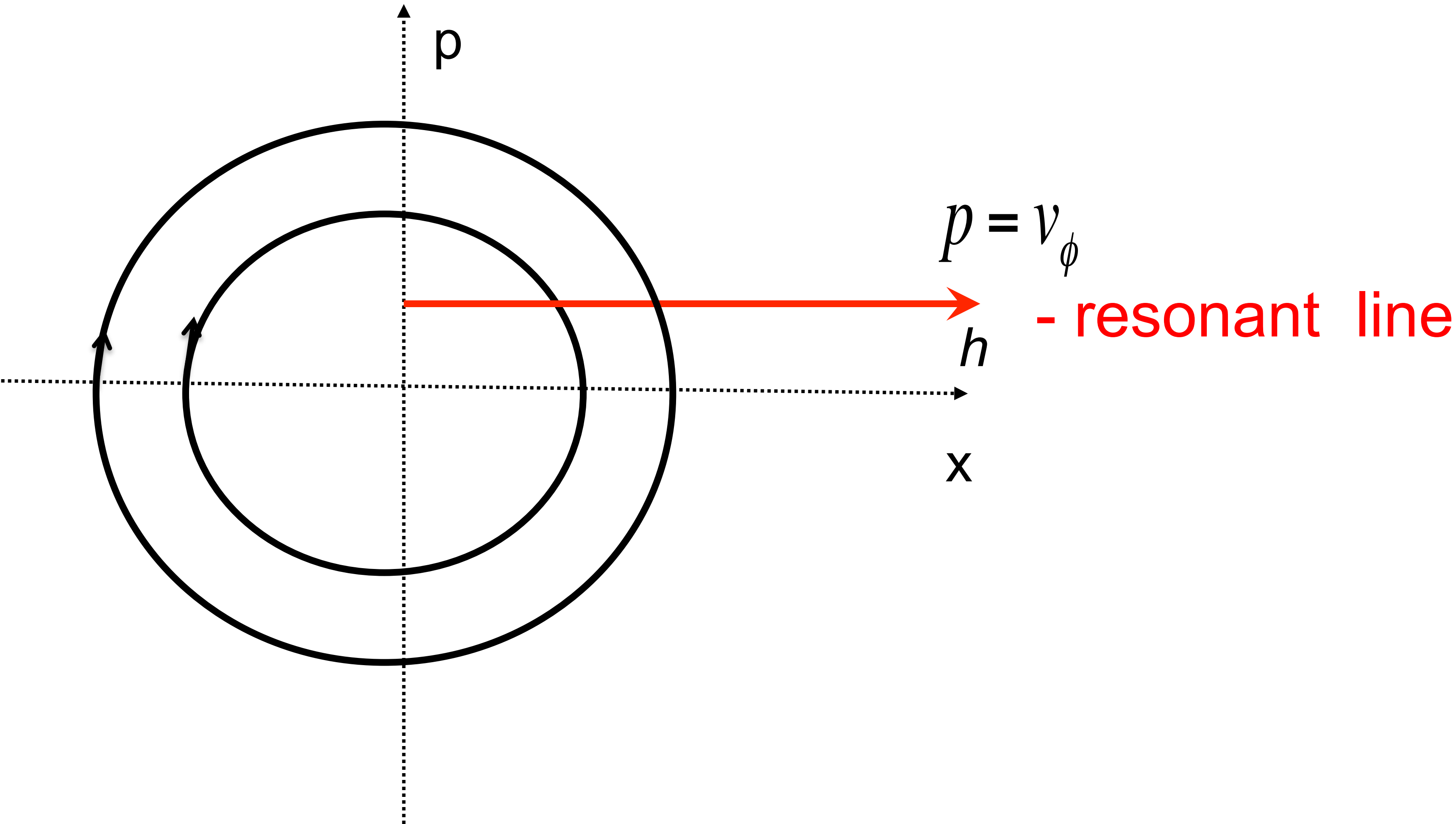
Probability of capture

$$\Pi = \frac{S'(x) v_\phi}{2\pi |L(x)|}$$

Mean value of scattering

$$< \Delta h > = -\frac{v_\varphi}{2\pi k} \operatorname{sgn}(L(x)) S(x)$$

Assume that the particles interact with the wave only for *x>0* (i.e. *u(x) =0 for x<0*). So we can consider only part of the resonant line with *x>0.* We use *h* as a coordinate instead of *x.*



Denote:

S(h) – the area of the oscillatory domain,

Π*(h)* –probability of capture into resonance

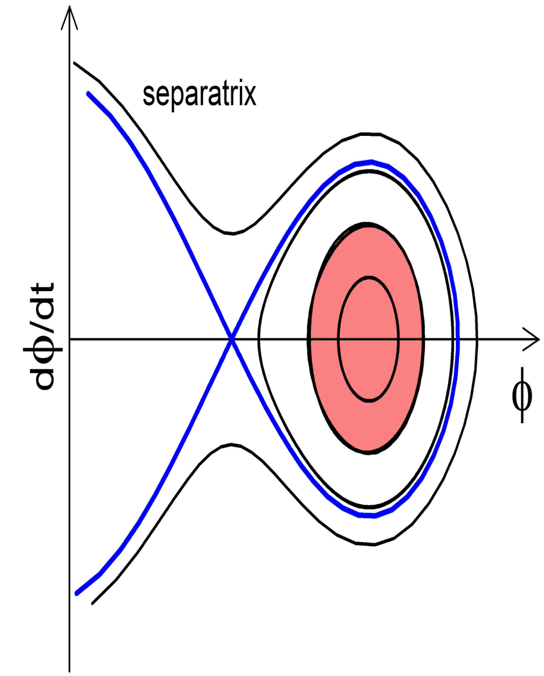
Δh(h, ξ) – change of h for one passage through resonance (ξ characterises the phase for this passage, it is considered as a random value with the uniform distribution on [0,1]),

$<\Delta h>$ - mean value of Δh

$<(\Delta h)^2>$ - mean value of $(\Delta h)^2$

T(h) - *the period of the averaged motion*

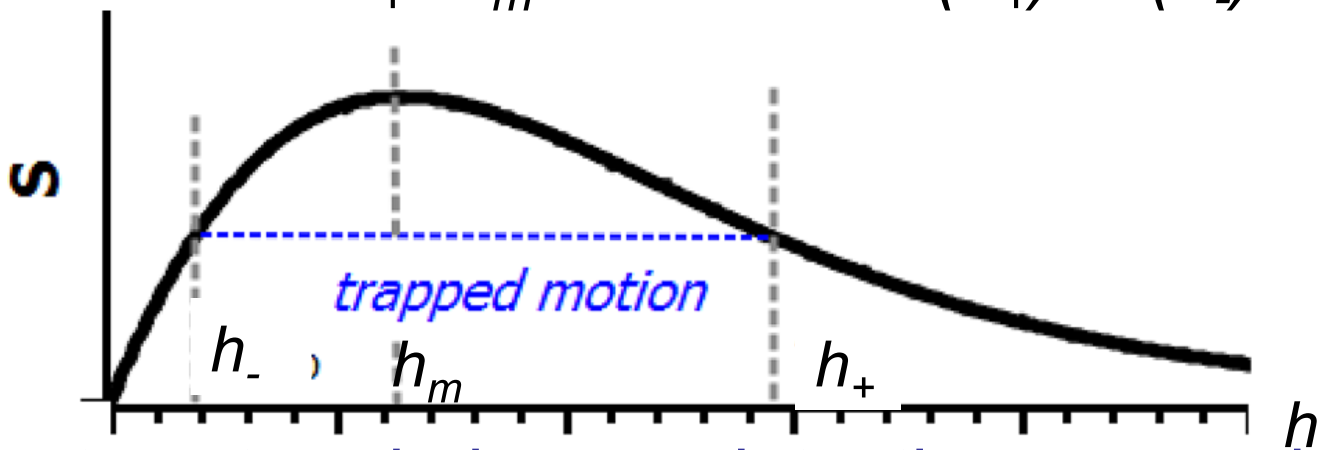
$$V(h)=<\Delta h>/T(h),\ D(h)=<(\Delta h)^2>/T(h)$$



$$\Pi(h) = \frac{v_\phi dS/dh}{2\pi k} \text{ (for d}S/dh > 0), \quad <\Delta h> = -\frac{v_\phi S}{2\pi k}$$

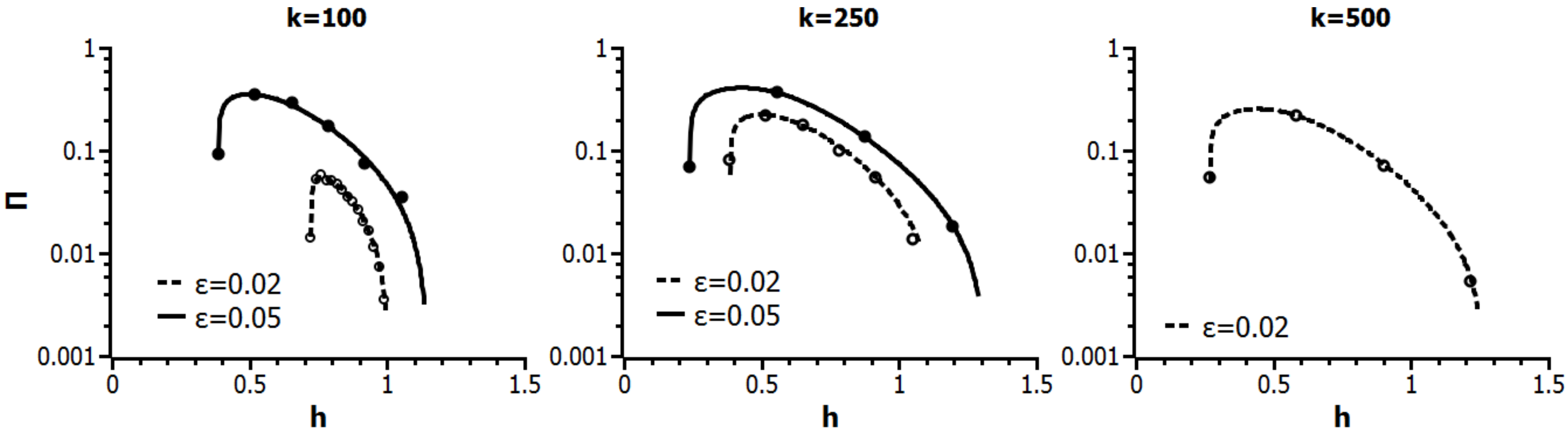
$$\Rightarrow \ \Pi = -d<\Delta h>/dh \text{ (for d}S/dh > 0)$$

We assume that the function S has a unique maximum at $h=h_m$ Thus, phase points captured at $h_-<h_m$ fly to the right along resonant line and escape from the resonance at $h_+<h_m$ such *that* $S(h_+)=S(h_-)$.

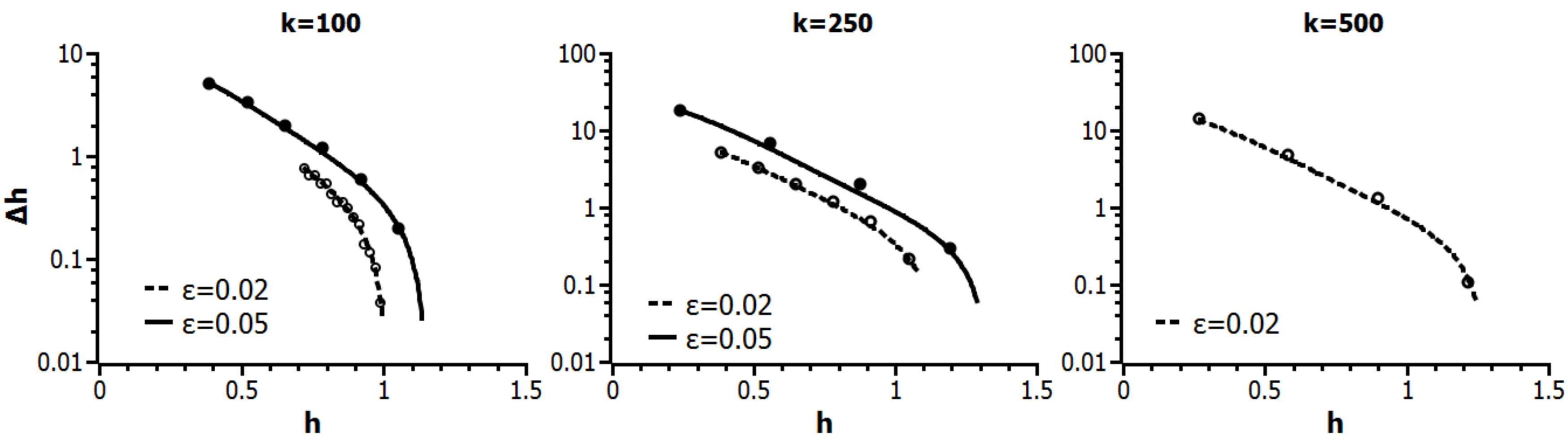


For not captured phase points the energy drifts slowly to the left with the velocity $V(h)=<\Delta h>/T(h)$.

Numerical check for capture

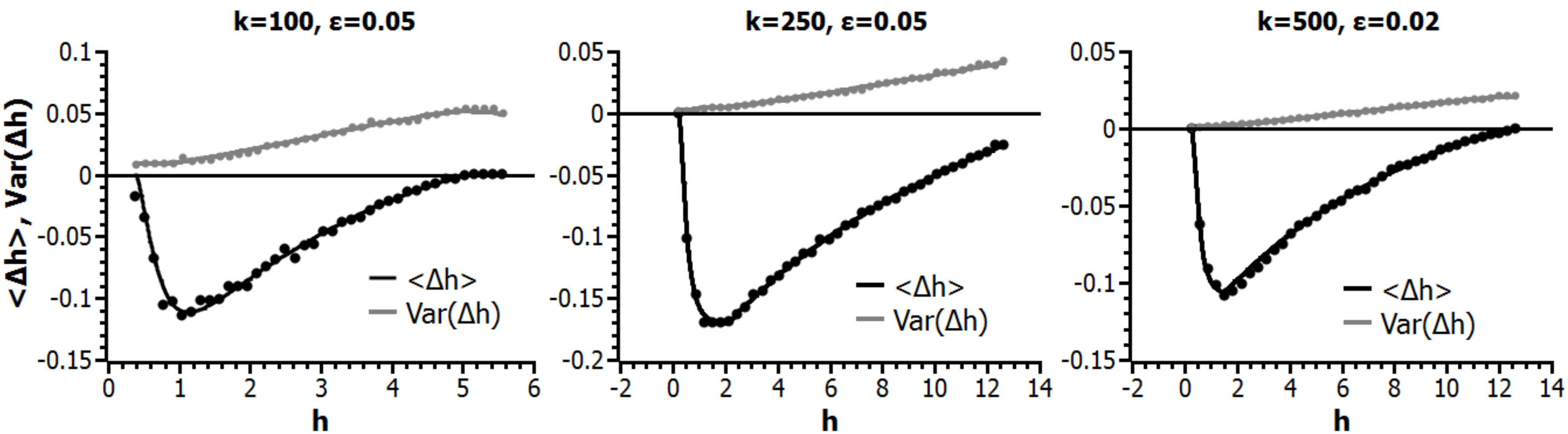


. 3: Probability of trapping: numerical (dots) and analytical (curves) results. System parameters are: $\Omega_b = 1$, $v_\phi = 0.5$, $\tilde{b}_0 = 0.1$.



4: Energies gained by trapped particles: numerical (dots) and analytical (curves) results. System parameters are: $\Omega_b = 1$, $v_\phi = 0.5$, $\tilde{b}_0 = 0.1$.

Numerical check for scattering

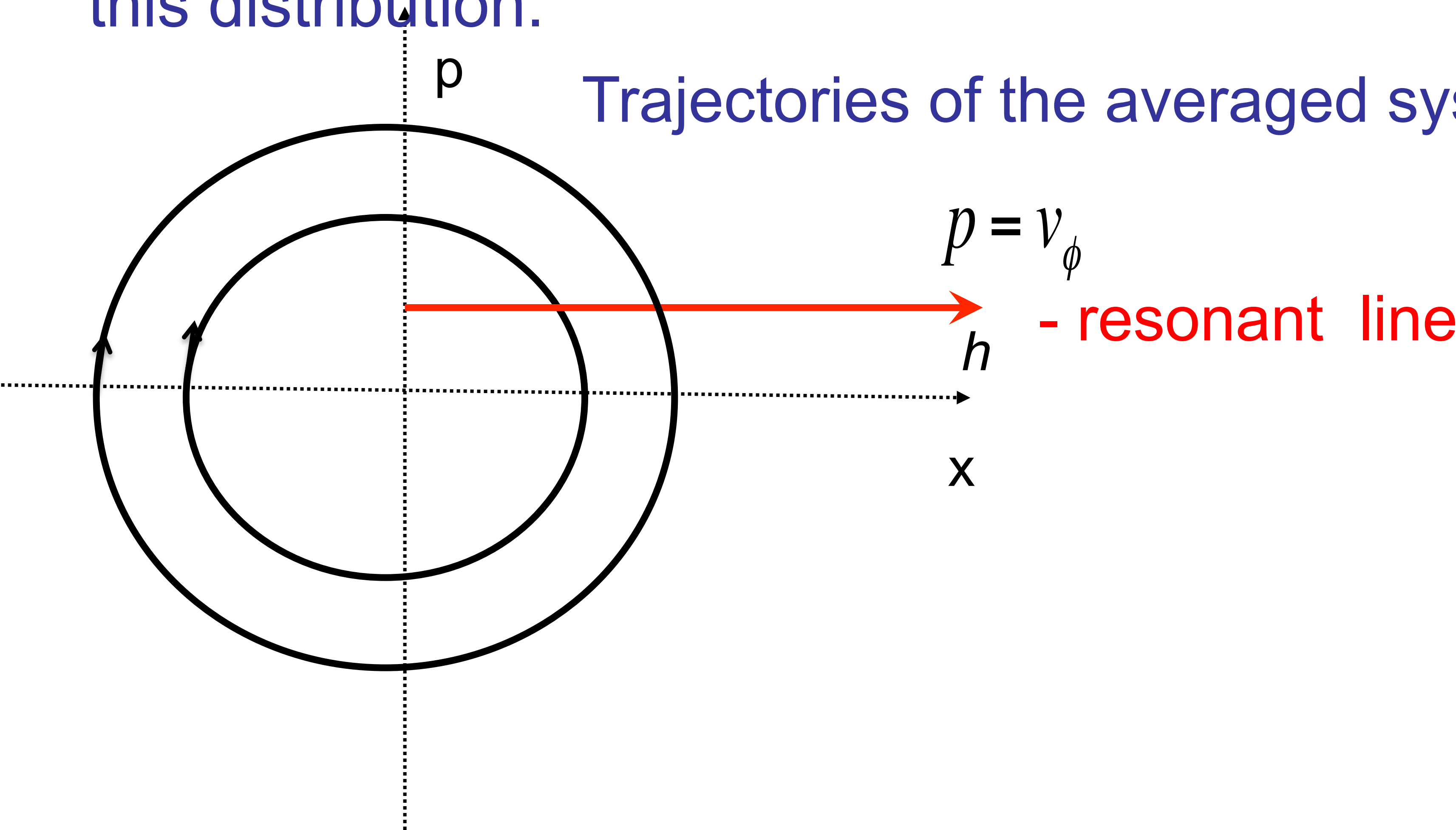


7: Energies gained by scattered particles: numerical (dots) and analytical (curves) results. System parameters are: $\Omega_b = 1$, $v_\phi = 0.5$, $\tilde{b}_0 = 0.1$.

IIIc. Kinetic equation

Let $f(t,h)$ be the distribution function of particles, where $h+v_\phi^2/2$ is the averaged Hamiltonian. We would like to describe approximately evolution of this distribution.

Trajectories of the averaged system



Kinetic equation:

$$\frac{\partial f}{\partial t} = L_s f + L_c f,$$

where operators L_s and L_c are related to scattering and capture/escape, respectively.

Scattering part has a standard form

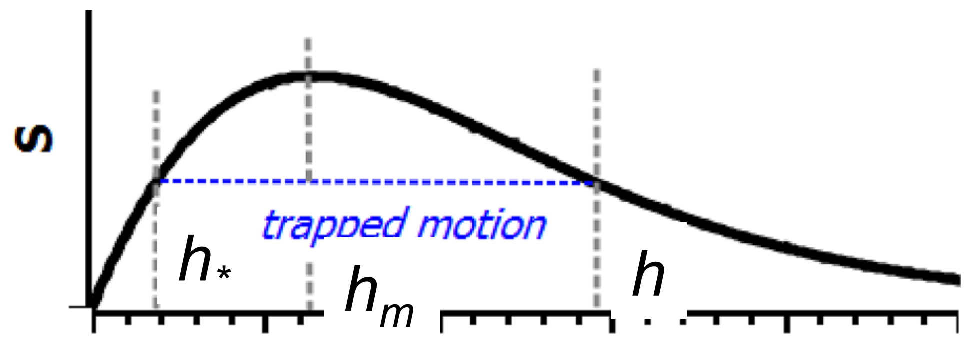
$$L_s f = -\frac{\partial (Vf)}{\partial h} + \frac{1}{2}\frac{\partial}{\partial h}\left(D \frac{\partial f}{\partial h} \right) + L_{sm} f.$$

Here $L_{sm} f$ is a small ($\approx D$) additional drift term which appear because V is calculated in the principal order in $1/k^{1/2}$. This term is omitted in formulas below.

The capture/escape part has different forms for $h<h_m$ (capture) and $h>h_m$ (escape)

For the capture, $h<h_m$:

$$L_c f = -\frac{\Pi f}{T} = \frac{1}{T}\frac{d<\Delta h>}{dh} f$$



For the escape, $h>h_m$:

Denote $\Pi_* = \Pi(h_*)$,

$$T_* = T(h_*),\ f_*(h,t) = f(h_*,t).$$

Then

$$L_c f = \frac{\Pi_* f_*}{T_*}\left|\frac{dh_*}{dh}\right| = -\frac{\Pi(h_*)}{T_*}\frac{dh_*}{dh} f_* = -\frac{\Pi(h_*)}{T_*}\frac{dS(h)/dh}{dS(h_*)/dh_*} f_*$$

$$= -\frac{v_\phi}{T_*}\frac{dS(h_*)/dh_*}{2\pi k}\frac{dS(h)/dh}{dS(h_*)/dh_*} f_* = -\frac{v_\phi}{T_*}\frac{dS(h)/dh}{2\pi k} f_* = \frac{d<\Delta h>}{dh}\frac{f_*}{T_*}$$

Finally, the kinetic equation takes the following form

For $h \le h_m$:

$$\frac{\partial f}{\partial t} = -V\frac{\partial f}{\partial h} + \frac{1}{T}\frac{\partial T}{\partial h}Vf + \frac{1}{2}\frac{\partial}{\partial h}\left(D\frac{\partial f}{\partial h}\right)$$

For $h > h_m$:

$$\frac{\partial f}{\partial t} = -V\frac{\partial f}{\partial h} - \frac{\partial V}{\partial h}\left(f - \frac{T}{T_*}f_*\right) + \frac{1}{T_*}\frac{\partial T}{\partial h}Vf_* + \frac{1}{2}\frac{\partial}{\partial h}\left(D\frac{\partial f}{\partial h}\right)$$

Here $V = \frac{<\Delta h>}{T}, \quad D = \frac{<(\Delta h)^2>}{T}$

One can rewrite this equation using action of the averaged system *I* instead of *h*. Denote corresponding values $\tilde{f}, \tilde{V}, \tilde{D}.$ We have

$$f = \frac{\tilde{f}T}{2\pi}, \; V = \frac{2\pi\tilde{V}}{T}, \; D = \frac{4\pi^2\tilde{D}}{T^2}$$

(we use here that $\frac{\partial I}{\partial h} = \frac{1}{2\pi}T$).

The kinetic equation takes the following form (tildes are omitted):

For $h \le h_m$:

$$\frac{\partial f}{\partial t} = -V\frac{\partial f}{\partial I} + \frac{1}{2}\frac{\partial}{\partial I}\left(D\frac{\partial f}{\partial I}\right)$$

For $h > h_m$:

$$\frac{\partial f}{\partial t} = -V\frac{\partial f}{\partial I} - \frac{\partial V}{\partial I}\left(f - f_*\right) + \frac{1}{2}\frac{\partial}{\partial I}\left(D\frac{\partial f}{\partial I}\right)$$

Numerical check for the kinetic eqation

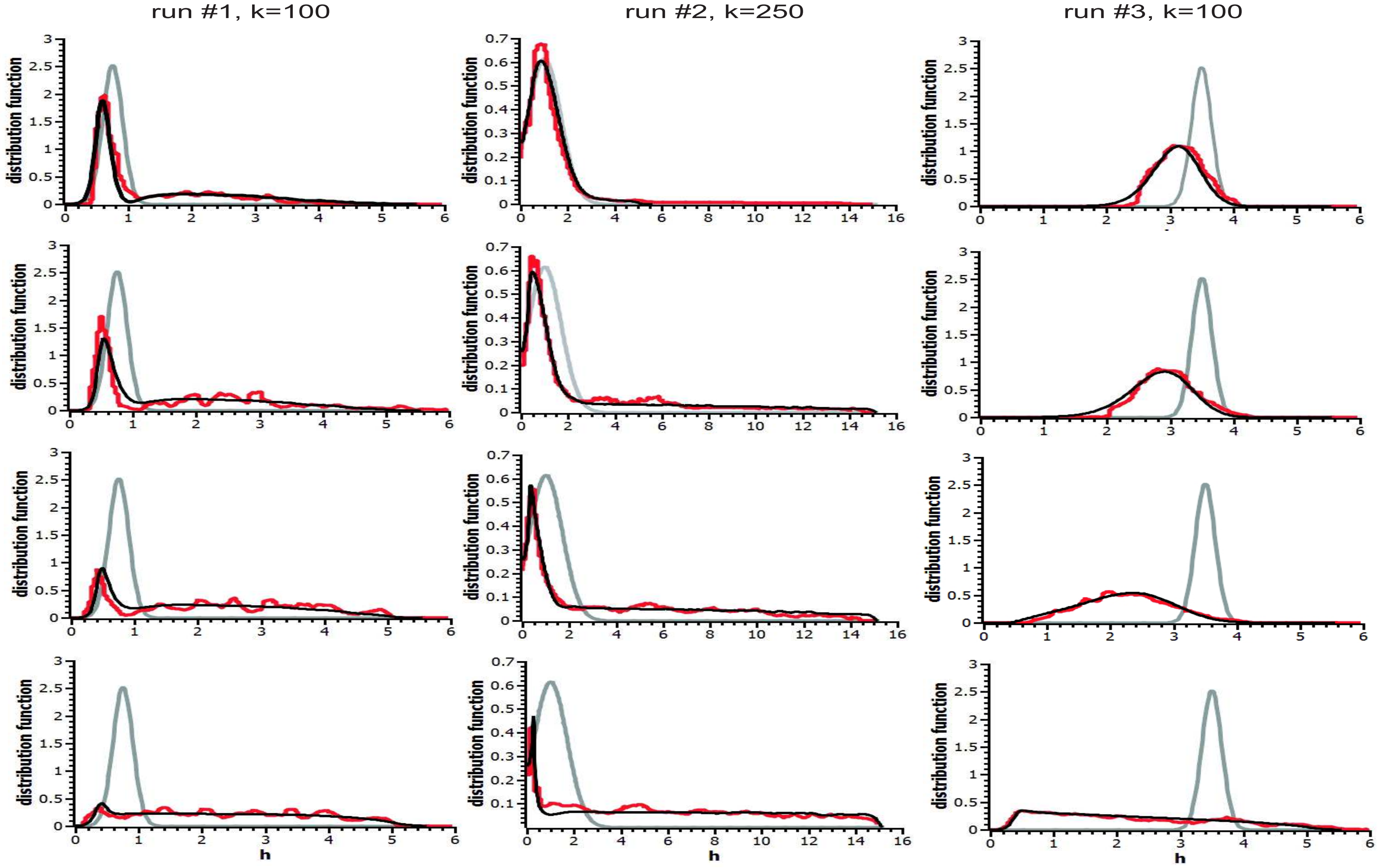
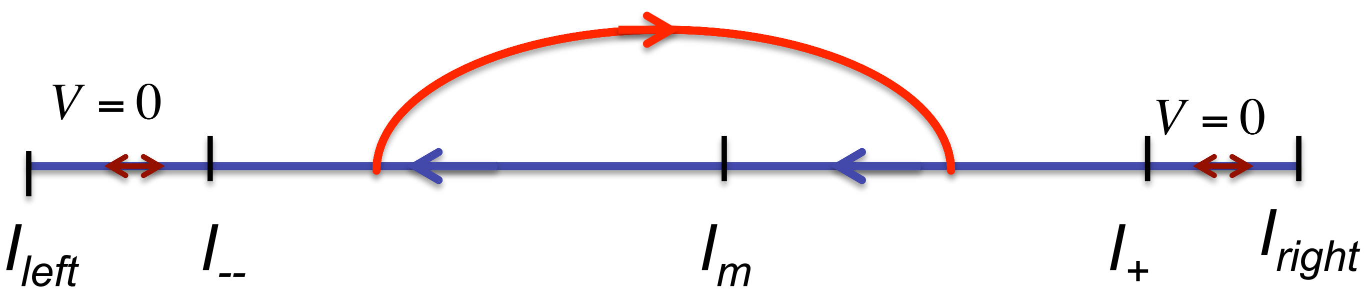


FIG. 9: Particle distributions obtained as a solution of Eq. (19) are shown in black, whereas results of test particle simulations are shown in red. The initial distribution is shown in grey. Time evolution from top to bottom. System parameters are: $\Omega_b = 1$, $v_\phi = 0.5$, $\tilde{b}_0 = 0.1$, $\varepsilon = 0.05$. Four time moments are displayed: $t\Omega_b = \{1, 3, 5, 10\}$ for runs #1 & #2 and $t\Omega_b = \{3, 5, 10, 25\}$ for run #3.

IIId. Some properties of the kinetic equation (work in progress)

$$I_{left} < I \leq I_m : \quad \frac{\partial f}{\partial t} = -V(I)\frac{\partial f}{\partial I} + \frac{1}{2}\frac{\partial}{\partial I}\left(D(I)\frac{\partial f}{\partial I}\right),$$

$$I_m \leq I < I_{right} : \quad \frac{\partial f}{\partial t} = -V(I)\frac{\partial f}{\partial I} - \frac{\partial V(I)}{\partial I}\left(f - f_*\right) + \frac{1}{2}\frac{\partial}{\partial I}\left(D(I)\frac{\partial f}{\partial I}\right).$$

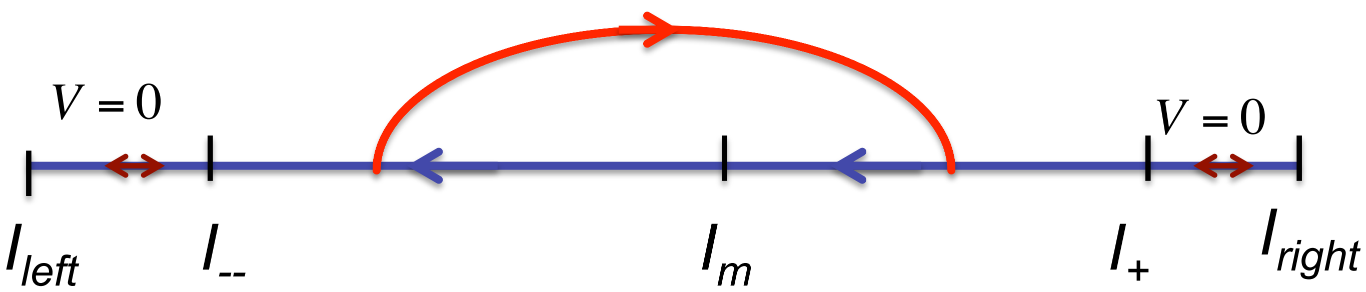


$$V(I) \propto \frac{1}{\sqrt{k}}\,|\,I - I_{-,+}\,|^{5/4}, \quad D(I) \propto \frac{1}{k}\,|\,I - I_{left,\,right}\,|^{2}\,.$$

Number of particles is conserved: $\int_{I_{left}}^{I_{right}} f(t,I)\,dI = \text{const.}$

$$D(I) \propto \frac{1}{k} \, | I - I_{left,\,right} |^2 \; .$$

The only smooth on (I_{left}, I_{right}) stationary solutions are $f = \text{const.}$

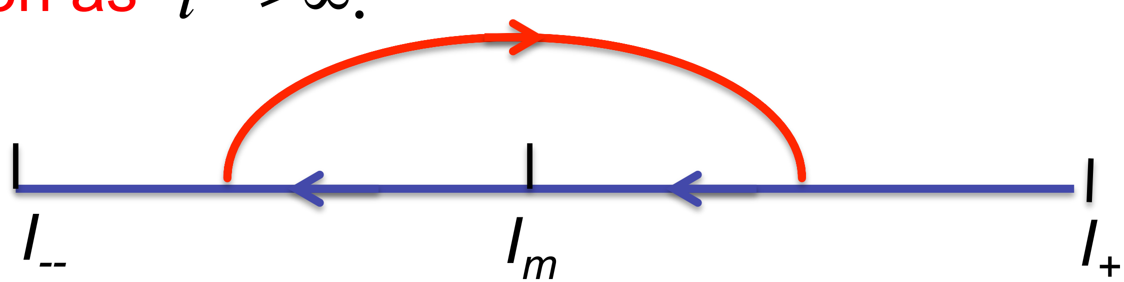


Neglect diffusion term as $\mathrm{V} \propto \frac{1}{\sqrt{k}},\ D(I) \propto \frac{1}{k},\ k >> 1.$ Then

$$I_- < I \le I_m: \quad \frac{\partial f}{\partial t} = -V(I)\frac{\partial f}{\partial I},$$

$$I_m \le I < I_+: \quad \frac{\partial f}{\partial t} = -V(I)\frac{\partial f}{\partial I} - \frac{\partial V(I)}{\partial I}\left(f - f_*\right).$$

The first equation is a linear first order PDE.
After solving the first equation, the second equation takes the form of a linear first order non-homogeneous PDE.
Thus one can write explicit formulas for the general solution.
All smooth solutions of Cauchy problem tend to the uniform distribution as $t \to \infty$.



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