Hamiltonian flow of minimum time control affine systems

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Restricted coplanar 3 body problem :

$$\ddot{q} + \nabla V_\mu(q) - 2J\dot{q} = u, \|u\| \leq 1 \quad (1)$$

in the rotating frame (RC3BP), $V_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}$.

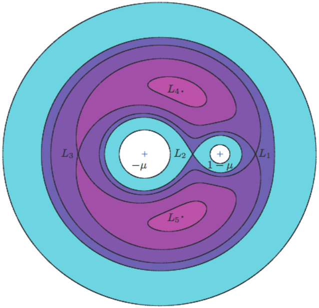

Figure 1: Hill's region and Lagrange points for the RC3BP, figure from [1].

Control affine system

→ *control affine system* :

$$\begin{cases} \dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x),\ u \in L^1_{loc}([0,t_f],B) \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{2}$$

F_i are smooth, $i = 0, 1, 2$, B is the unitary ball of $\mathbb{R}^2$.

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Remark

(1) can be written that way with $x = (q, v)$.

Geometric necessary condition : the PMP

Definition

$\forall (x,p) \in T_x^*M,\ H(x,p,u) = H_0(x,p) + u_1 H_1(x,p) + u_2 H_2(x,p)$

with $H_i = \langle p, F_i(x) \rangle,\ i = 0, 1, 2.$

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Theorem (P.M.P.)

(x,u) minimum time trajectory then there exists $p(t) \in T_{x(t)}M^* \setminus \{0\}$
- (x,p) is solution of :

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x,p,u) \\ \dot{p} = -\frac{\partial H}{\partial x}(x,p,u). \end{cases} \tag{3}$$

- $H(x(t),p(t),u(t)) = \max_{\tilde{u} \in U} H(x(t),p(t),\tilde{u})$.
- $H(x(t),p(t),u(t)) \geq 0$.

Singularities

Pros : **Autonomous** Hamiltonian system.
Cons : Dimension doubled, only necessary condition $\rightarrow$ existence of optimal control, **singularities**.

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Maximized Hamiltonian $H^*(x,p) = H_0(x,p) + \sqrt{H_1(x,p)^2 + H_2(x,p)^2}$
$u = \frac{1}{\sqrt{H_1^2+H_2^2}}(H_1, H_2)$: discontinuities of the control u are called **switchs**.

Definition (Singular locus.)

A switch is a discontinuity of the reference control.
The singular locus, or switching surface is defined by
$\Sigma = \{z = (x,p) \in T^*M,\ H_1(x,p) = H_2(x,p) = 0\}$.

Notation : $F_{ij} = [F_i, F_j]$, $H_{ij} = \{H_i, H_j\}$, $i,j = 0,1,2$.

Hypothesis

Assumption :

$$(\mathcal{A}) : \text{rank}(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) = 4, \text{ for all } x \in M.$$

Check for the RC3BP.
$\rightarrow$ Link with controllability when F_0 is *recurrent* ($\mu = 0$ or certain energy levels of the RC3BP.)

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Proposition

Any system of the form $\ddot{q} + g(q, \dot{q}) = u$ *verifies* $(\mathcal{A})$.

We will use later the following hypothesis : $(\mathcal{B}) : [F_1, F_2] = 0$.

Definition

$\Sigma = \Sigma_0 \cup \Sigma_- \cup \Sigma_+$ with :
$\Sigma_- = \{H_{12}(z)^2 < H_{02}(z)^2 + H_{01}(z)^2\}$,
$\Sigma_+ = \{H_{12}(z)^2 > H_{02}(z)^2 + H_{01}(z)^2\}$,
$\Sigma_0 = \{H_{12}(z)^2 = H_{02}(z)^2 + H_{01}(z)^2\}$.

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Remark

In a point $\bar{z} \in \Sigma$, $(\mathcal{A}) \Leftrightarrow H_{01}^2(\bar{z}) + H_{02}^2(\bar{z}) > 0$

Main result

Theorem

There exists unique solution for system (1) in a neighborhood $O_{\bar{z}}$ of $\bar{z}$, and there is at most one switch on $O_{\bar{z}}$.

- *If $\bar{z} \in \Sigma_-$: The extremal flow $z : (t, z_0) \in [0, t_f] \times O_{\bar{z}} \mapsto z(t, z_0) \in M$ is piecewise smooth, and smooth on each strata :*

$$O_{\bar{z}} = S_0 \sqcup S_1 \sqcup \Sigma$$

  - where S_1 is the codimension one submanifold of initial conditions leading to the switching surface,
  - $\Delta_0 = O_{\bar{z}} \setminus (S_1 \cup \Sigma)$.
- *If $\bar{z} \in \Sigma_+$, no extremal intersects the singular locus, and therefore, the flow is smooth on $O_{\bar{z}}$.*

Idea of the proof

Polar Blow up :

$$(H_1, H_2) = (\rho \cos\theta, \rho \sin\theta),\ (\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1.$$

$u = (\cos\theta, \sin\theta)$, and $\Sigma = \{\rho = 0\}$.

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$u = (\cos\theta, \sin\theta)$, and $\Sigma = \{\rho = 0\}$.
New time : $dt = \rho ds$ we obtain a smooth vector field

- Σ_- : 2 parabolic equilibria $\theta_\pm \rightarrow$ normally hyperbolic manifold : S_1 global stable manifold.
- Σ_+ : No equilibrium, no switch.

Theorem

The singular-regular transition is continuous, with singularities in "$z \ln z$".

Application to Kepler and the RC3BP

In this case : $(\mathcal{B})$: $[F_1, F_2] = 0 \Rightarrow H_{12} = 0$.
Previous results apply directly to the controlled RC3BP, and

$$\Sigma = \Sigma_- = \{z, H_1(z) = H_2(z) = 0, H_{02}(z)^2 + H_{01}(z)^2 > 0\}$$

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Proposition

In the controlled Kepler problem and RC3BP switching are instantaneous rotations of angle π of the control u: if t is a switching time, $u(t_-) = -u(t_+)$.

We call such switchings π-singularities.

π-singularities

We can globally bound the number of π-singularities on a time interval $[0, t_f]$.

Definition (Distance to collisions)

We define $\delta = \inf_{[0,t_f]} |q(t)|$,
$\delta_1 = \inf_{[0,t_f]} |q(t) + \mu|$,
$\delta_2 = \inf_{[0,t_f]} |q(t) - (1 - \mu)|$.
Finally note $\delta_{12}(\mu) = \frac{\delta_1 \delta_2}{((1-\mu)\delta_2^3 + \mu\delta_1^3)^{1/3}}$.

Upper bound on π-singularities

Proposition

- Keplerian case : Time interval of length $\pi\delta^{3/2}$ between two π-singularities. On a time interval $[0, t_f]$ the number of such singularities is at most $N_0 = [\frac{t_f}{\pi\delta^{3/2}}]$.

- Controlled RC3BP : Time interval of length $\pi\delta_{12}(\mu)^{3/2}$ between two π-singularities. On a time interval $[0, t_f]$ there is at most $N_\mu = [\frac{t_f}{\pi\delta_{12}(\mu)^{3/2}}]$ π-singularities.

$\rightarrow$ Sturm type estimations.

Idea of the poof.

In a fixed frame

$$\ddot{q} = -\nabla_q \Omega_\mu(t,q) + u, \|u\| \leq 1$$

with $\Omega_\mu(t,q) = \frac{1-\mu}{|q|} + \frac{\mu}{|q-\xi(t)|}$, $\xi(t) = e^{i\omega t}$.

$$\begin{aligned} H^*(t,q,v,p_q,p_v) &= p_q.v - p_v.\nabla_q \Omega_\mu(t,q) + |p_v| \\ &\Rightarrow \ddot{p}_v + \nabla_q^2 \Omega_\mu(t,q) p_v = 0. \qquad (4) \end{aligned}$$

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$$H^*(t,q,v,p_q,p_v) = p_q.v - p_v.\nabla_q \Omega_\mu(t,q) + |p_v|$$

$$\Rightarrow \ddot{p}_v + \nabla_q^2 \Omega_\mu(t,q) p_v = 0. \quad (4)$$

Set $A = \begin{pmatrix} 1 + \frac{1-\mu}{|q|^3} + \frac{\mu}{|q-\xi(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q|^3} + \frac{\mu}{|q-\xi(t)|^3} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$

$$A - \nabla_q^2 \Omega_\mu(t,q) > 0 :$$

second component of $\ddot{z} + A(t)z = 0$ gives the conclusion.

Minimum time Kepler problem

Well known : - uncontrolled Kepler problem is **integrable**
- Kepler motion with **constant force** ($u =$cte) is integrable (Charlier and St-Germain).

Minimum time Kepler problem :

$$\begin{cases} \ddot{q} + \frac{q}{\|q\|^3} = u, \|u\| \leq 1, \\ (q(0), v(0)) = (q_0, v_0), \\ (q(t_f), v(t_f)) = (q_f, v_f) \\ t_f \to \min . \end{cases} \tag{5}$$

Hamiltonian in cartesian coordinates

Note $x = (q, v) = (x_1, x_2, x_3, x_4)$:

$$H(x,p) = p_1x_3 + p_2x_4 - \frac{p_3x_1 + p_4x_2}{(x_1^2 + x_2^2)^{3/2}} + \sqrt{p_3^2 + p_4^2}, \;\; p \in T_xM^*.$$

and

$$u = \frac{1}{\sqrt{p_3^2 + p_4^2}}(p_3, p_4).$$

$\rightarrow$ **Liouville integrability** of H ?

Moralès-Ramis theorem

Theorem (Morales-Ramis)

Let us consider a Hamiltonian H analytic on a complex analytic symplectic manifold and a particular solution Γ not reduced to a point. If H is integrable in the Liouville sense with meromorphic first integrals, then the first order variational equation near Γ has a virtually Abelian Galois group over the base field of meromorphic functions on Γ.

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Theorem

The minimum time Kepler problem is not meromorphically Liouville integrable on $\mathcal{M}$.

Meromophic hamiltonian

Extension in complex domain :

$$\mathcal{M} = \{(x, p, r) \in \mathbb{C}^8 \times \mathbb{C}^{*2},\ r_1^2 = x_1^2 + x_2^2,\ \ r_2^2 = p_3^2 + p_4^2\}.$$

Rational expression :

$$H(x, p, r) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{r_1^3} + r_2$$

H is meromophic on $\mathcal{M}$.
4 **degrees of freedom** $\rightarrow$ 4 first integrals.

Reduced system

Stable subspace (collisions) :

$$S = \{x_2 = x_4 = p_2 = p_4 = 0, r_1 = x_1, r_2 = p_3\} \cap \mathcal{M}$$

On S, $H_{|S}(x,p) = p_1 x_3 - \frac{p_3}{x_1^2} + p_3$

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Remark

First integral : $C = \frac{1}{2}x_3^2 + x_1 - \frac{1}{x_1}$

Equations of motion :

$$\begin{cases} \ddot{x}_1 = -1 - \frac{1}{{x_1}^2}, \\ \ddot{p}_3 - 2\frac{p_3}{x_1^3} = 0. \end{cases}$$

Particular solution Γ

Time $s = x_1(t)$, $' = \frac{d}{ds}$:

$$2\left(C + \frac{1}{x_1} - x_1\right)p_3''(x_1) - \left(1 + \frac{1}{x_1^2}\right)p_3'(x_1) - \frac{2}{x_1^3}p_3(x_1) = 0.$$

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$\rightarrow$ A solution is :

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}.$$

Then, on the level $C = 2i$: $p_3 = \frac{x_1 - i}{\sqrt{x_1}}$, and

$$x_3 = \sqrt{2}\frac{x_1 - i}{\sqrt{x_1}}, p_1 = -\frac{x_1^2 + 1}{\sqrt{2}x_1^2}.$$

Variational equation

Definition (Variational equation along Γ)

$\dot{Z}(t) = A(t)Z(t)$ with $A(t) = \mathrm{Jac}(J_4\nabla H)(\Gamma(t))$

$\rightarrow$ In time x_1 : $Z'(x_1(t)) = \frac{1}{x_3(t)}A(x_1(t))Z(x_1(t))$

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$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

Since the **Picard-Vessiot** field is generated by all the components of the solutions, the Picard Vessiot field K generated by the sub-equation $X' = A_3 X$ is smaller : $\mathrm{Gal}(A) \supset \mathrm{Gal}(A_3)$, we focus on A_3.

Cyclic vector method

Cyclic vector method gives :

$$X_1^{(4)} + \frac{2(3i - 5x_1)}{x_1(i - x_1)} X_1^{(3)} + \frac{(-3x_1 + i)(-29x_1 + 23i)}{4(x_1 - i)^2 x_1^2} X_1'' - \frac{(i - 3x_1)(7x_1 + i)}{4(x_1 - i)^2 x_1^3} X_1' + \frac{3x_1 + i}{4(x_1 - 1)^3 x_1^4} X_1 = 0 \quad (6)$$

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Solutions contain a primitive of
$x_1 \mapsto {}_2F_1\left(\frac{5}{2}-\frac{\sqrt{3}}{2}i, \frac{1}{2}+\frac{\sqrt{3}}{2}i, 1+\sqrt{3}i, 1+ix_1\right)$

Conclusion

Picard Vessiot field is a differential field : it contains the function aswell.
The (rational) Galois group of ${}_2F_1(\frac{5}{2}-\frac{\sqrt{3}}{2}i, \frac{1}{2}+\frac{\sqrt{3}}{2}i, 1+\sqrt{3}i, 1+x_1 i)$ over $\mathbb{C}(x_1)$ is $SL_2(\mathbb{C})$: **Not virtually abelian** (and not even solvable).
In this case, **meromophic** and **rational** Galois groups are equal since the linear equation is Fuschian (Schlesinger density theorem).

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The (rational) Galois group of ${}_2F_1(\frac{5}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, 1 + \sqrt{3}i, 1 + x_1 i)$ over $\mathbb{C}(x_1)$ is $SL_2(\mathbb{C})$: **Not virtually abelian** (and not even solvable).
In this case, **meromophic** and **rational** Galois groups are equal since the linear equation is Fuschian (Schlesinger density theorem).
$\rightarrow$ Enough real first integrals with **natural frontiers** could exist. (Symmetry of the Hamiltonian ?)

References :

[1] J.-B. Caillau, Daoud, B. Minimum time control of the restricted three-body problem SIAM J. Control Optim. 50 (2012), no. 6, 3178-3202.
[2] J. Moralès, J.-P. Ramis, Galoisian obstructions to integrab ility of Hamiltonian systems I and II Methods and Applications of A nalysis 8 (2001) 33-95 et 97-111
[3] J. Pesin Lectures on partial hyperbolicity and stable ergodicity Zurich Lectures in Advanced Mathematics, EMS, 2004.
[4] On the extremal flow of some affine control system, J.-B. Caillau, M.O. (in preparation)
[5] Agrachev, A. and Biolo, C., Switching in Time-Optimal Problem: the 3D Case with 2D Control, Journal of Dynamical and Control Systems, 2016.