Entropy and isotropy

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Conservative and non conservative

- Conservative (i.e. symplectic) dynamics model systems without friction.
- Conformal symplectic dynamics model systems with friction. They may have attractors.
- The conformally symplectic dynamics (CS) on a symplectic manifold alter the symplectic form only up to a scaling factor. This class contains mechanical systems whose friction forces are proportional to the velocity

Review of results in the conformally symplectic setting (CS)

- In dimension 2, every smooth invertible system is CS; there are many results in the dissipative setting, e.g. the study of the 2-dimensional dissipative twist maps of the annulus, that may have wild attractors, which was initiated by Birkhoff.
- In dimension at least 4, the conformal factor has to be constant (Liebermann 1958).

Review of results in the conformally symplectic setting (CS)

- In higher dimension, several works extend KAM results for families with parameters (Calleja, Celletti, de la Llave 2013, Massetti 2019).
- Discounted Tonelli Hamiltonians were studied from the point of view of weak K.A.M. theory (Davini, Fathi, Iturriaga, Zavidovique 2016) and Aubry-Mather theory (Maro-Sorrentino 2017).

Definitions

Let $(\mathcal{M}^{2d}, \omega)$ be a symplectic manifold. By a *conformal symplectic dynamics*, we mean

- a diffeomorphism $f : \mathcal{M} \circlearrowleft$ such that $f^*\omega = a\,\omega$ with $a > 0$.
- or a complete vector field X such that $L_X\omega = \alpha\,\omega$, where L_X is the Lie derivative, for some real number α. In this case, the flow (φ_t) of X is conformally symplectic and $\varphi_t^*\omega = e^{\alpha t}\omega$.

The case $a < 1$ or $\alpha < 0$ corresponds to systems with friction.

Examples

Consider $(\mathcal{M} = T^*\mathcal{Q}, -d\lambda)$, where $T^*\mathcal{Q}$ is the cotangent bundle of a manifold $\mathcal{Q}$ and λ is the Liouville 1-form on $T^*\mathcal{Q}$.

- A continuous-time example is the Liouville vector field Z_λ that is defined by $i_{Z_\lambda}(-d\lambda) = \lambda$ and a discrete-time example is $f = \exp Z_\lambda : (q, p) \mapsto (q, ap)$, $a = \frac{1}{e}$.
- An extension of the so-called Mañé example, which exists in the conservative setting, makes possible to embed all time-continuous dynamics of a closed manifold as the restriction to the zero-section of $T^*\mathcal{Q}$ of a discounted Tonelli flow.

Discounted Tonelli case

Definition

A discounted Hamiltonian Tonelli vector field X on $(T^*\mathcal{Q}, \Omega = -d\lambda)$ is defined by

$$i_X\Omega = dH - \alpha.\lambda$$

where

- $H : T^*\mathcal{Q} \to \mathbb{R}$ is Tonelli, i.e. C^2, superlinear and with positive Hessian in the fiber direction;
- $\alpha > 0$.

Attractors

In this case, if we denote by (φ_t) the flow of X, we have

- $\varphi_t^*\Omega = e^{-\alpha t}\Omega$;
- for $p \in T^*\mathcal{Q}$ outside some fixed compact set, $\frac{dH\circ\varphi_t(p)}{dt} < 0$.

Hence there exists an attractor, and this attractor has 0 Lebesgue measure.

Open questions on attractors

- Does any conformal flow (resp. diffeomorphism) that is defined on the cotangent bundle of a closed manifold admit a global (compact) attractor?
- For a Tonelli flow or other dynamics, how can be the attractor (e.g. its size in any sense)? And the dynamics on it?

Our focus: invariant submanifolds

Knowing that when existing, the attractor has zero Lebesgue measure and that Mañé example involves invariant Lagrangian submanifolds, a natural question concerns the isotropy of invariant submanifolds.

Proposition

If a closed C^1 surface $\mathcal{L}$ is invariant by a conformal and non symplectic C^1 diffeomorphism of $(\mathcal{M}, \omega)$, $\mathcal{L}$ is ω-isotropic.

A non-isotropic example

Proposition

There exists a conformal vector field X on a 4-dimensional symplectic manifold $(\mathcal{M}, \omega)$,with a 3-dimensional invariant submanifold $\mathcal{L}$ (hence $\mathcal{L}$ is not isotropic).
Moreover, the submanifold $\mathcal{L}$ is the global attractor for the flow (φ_t) of X, $(\varphi_{t|\mathcal{L}})$ is conjugated to the suspension of an Anosov automorphism of $\mathbb{T}^2$ with 2-dimensional stable and unstable foliations, and $(\varphi_{t|\mathcal{L}})$ is transitive with entropy equal to $|\alpha|$ where $L_X\omega = \alpha\omega$.

Some remarks and a question

- In the later example the invariant submanifold is an hypersurface of a manifold $N \times \mathbb{R}$.
- For a Tonelli flow of a cotangent bundle, the attractor cannot separate.
- Question. Is it possible that a Tonelli flow has an invariant closed submanifold that is not isotropic?

Yomdin theory

Let $\mathcal{L}$ be a a compact Riemannian C^r manifold, $\mathcal{S} \subset \mathcal{L}$ be a compact C^r submanifold of dimension s and f be a C^r self-map of $\mathcal{L}$ ($r \geqslant 1$). Denote by vol the s-dimensional Riemannian volume. Define the logarithmic volume growth of $f_{|\mathcal{S}}$ as

$$\mathrm{logvol}\,(f_{|\mathcal{S}}) = \limsup_{n\to+\infty} \frac{1}{n} \log \left|\mathrm{vol}\,(f^n(\mathcal{S})\right|,$$

and

$$\mathrm{rad}\,(Df) = \limsup_{n\to+\infty} \|Df^n\|^{1/n}, \quad \|Df\| = \sup_x \|Df_x\|.$$

Theorem (Yomdin 1987)

$$\mathrm{logvol}\,(f_{|\mathcal{S}}) \leqslant \mathrm{ent}\,(f) + \log^+(\mathrm{rad}\,(Df)^{s/r}).$$

In particular, if $\mathcal{L}$, $\mathcal{S}$ and f are smooth,

$$\mathrm{logvol}\,(f_{|\mathcal{S}}) \leqslant \mathrm{ent}\,(f).$$

A result of Yomdin theorem

Proposition

Let f be a conformal diffeomorphism of a symplectic manifold $(\mathcal{M}, \omega)$, i.e. such that $f^*\omega = a\,\omega$ with $a \in]0,1[$. Let $\mathcal{L}$ be an ℓ-dimensional invariant closed submanifold. Assume one of the following hypothesis.

1. *The diffeomorphism f is smooth, $\mathcal{L}$ is smooth and*

$$\mathrm{ent}\,(f_{|\mathcal{L}}) < -\log(a);$$

2. *The diffeomorphism f is C^r, $\mathcal{L}$ is C^r for some $r \geqslant 1$ and*

$$\mathrm{ent}\,(f_{|\mathcal{L}}) + \log^+ \left(\mathrm{Rad}(Df_{|\mathcal{L}}^{-1})^{\ell/r}\right) < -\log(a).$$

Then $\mathcal{L}$ is ω-isotropic.

An improvement of regularity hypothesis in a particular case

Let

- $\mathcal{M}^{(n)}$ be a compact Riemannian manifold
- $\mathcal{F}$ be a foliation induced by a subbundle F of $T\mathcal{M}$ of rank $p \leqslant n-1$
- Ω be an $(n-p)$-form on $\mathcal{M}$ which induces a volume on submanifolds transverse to $\mathcal{F}$
- f be a C^1-diffeomorphism of $\mathcal{M}$ preserving $\mathcal{F}$ and such that

$$f^*\Omega = \beta\,\Omega$$

  for some $\beta > 1$.

Theorem

The topological entropy of f satisfies

$$\mathrm{ent}\, f \geqslant \ln \beta.$$

An improvement of regularity hypothesis in a particular case

Assume that ω is a pre-symplectic form of (even) rank $2\ell \geqslant 2$ and

$$f^*\omega = \alpha\,\omega, \quad \alpha > 1.$$

The kernel of ω is a uniquely integrable subbundle F of even corank 2ℓ.

Corollary

The topological entropy of f satisfies

$$\mathrm{ent}\, f \geqslant \frac{\mathrm{rank}\,(\omega)}{2} \ln \alpha.$$

An improvement of regularity hypothesis in a particular case

Corollary

Let $f : \mathcal{M} \circlearrowleft$ be a C^2 conformal symplectic diffeomorphism such that $f^*\omega = a\omega$. Suppose that $\mathcal{N}$ is an invariant C^2 submanifold such that $\omega_{|\mathcal{N}}$ has constant rank and

$$\mathrm{ent}\, f_{|\mathcal{N}} < \frac{\mathrm{rank}\,(\omega_{|\mathcal{N}})}{2} \ln a.$$

Then $\mathcal{N}$ is isotropic. This result applies e.g. to an invariant submanifold on which the dynamics is minimal (every orbit is dense) and the entropy vanishes.

Statement to be proved

We assume that $\mathcal{M}^{(n)}$ is a compact Riemannian manifold with distance d, that Ω is a $(n-p)$-form on $\mathcal{M}$ that induces a volume form transversally to the p foliation $\mathcal{F}$ whose tangent subspace is the kernel of Ω. We assume that f be a C^1-diffeomorphism of $\mathcal{M}$ preserving $\mathcal{F}$ and such that

$$f^*\Omega = \beta\,\Omega$$

for some $\beta > 1$.
Then the topological entropy of f satisfies

$$\mathrm{ent}\, f \geqslant \ln \beta.$$