

# Entropy and isotropy

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# Conservative and non conservative

- Conservative (i.e. symplectic) dynamics model systems without friction.
- Conformal symplectic dynamics model systems with friction. They may have attractors.
- The conformally symplectic dynamics (CS) on a symplectic manifold alter the symplectic form only up to a scaling factor. This class contains mechanical systems whose friction forces are proportional to the velocity

## Review of results in the conformally symplectic setting (CS)

- In dimension 2, every smooth invertible system is CS; there are many results in the dissipative setting, e.g. the study of the 2-dimensional dissipative twist maps of the annulus, that may have wild attractors, which was initiated by Birkhoff.
- In dimension at least 4, the conformal factor has to be constant (Liebermann 1958).



## Review of results in the conformally symplectic setting (CS)

- In higher dimension, several works extend KAM results for families with parameters (Calleja, Celletti, de la Llave 2013, Massetti 2019).
- Discounted Tonelli Hamiltonians were studied from the point of view of weak K.A.M. theory (Davini, Fathi, Iturriaga, Zavidovique 2016) and Aubry-Mather theory (Maro-Sorrentino 2017).

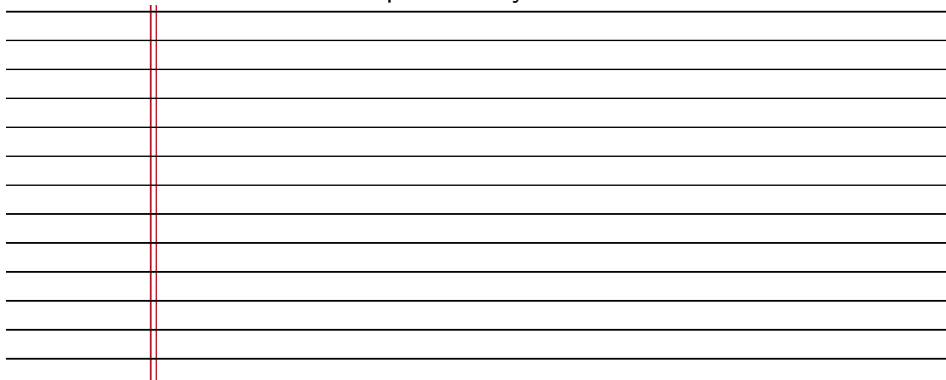


## Definitions

Let  $(\mathcal{M}^{2d}, \omega)$  be a symplectic manifold. By a *conformal symplectic dynamics*, we mean

- a diffeomorphism  $f : \mathcal{M} \rightarrow \mathcal{M}$  such that  $f^*\omega = a\omega$  with  $a > 0$ .
- or a complete vector field  $X$  such that  $L_X\omega = \alpha\omega$ , where  $L_X$  is the Lie derivative, for some real number  $\alpha$ . In this case, the flow  $(\varphi_t)$  of  $X$  is conformally symplectic and  $\varphi_t^*\omega = e^{\alpha t}\omega$ .

The case  $a < 1$  or  $\alpha < 0$  corresponds to systems with friction.



## Examples

Consider  $(\mathcal{M} = T^*Q, -d\lambda)$ , where  $T^*Q$  is the cotangent bundle of a manifold  $Q$  and  $\lambda$  is the Liouville 1-form on  $T^*Q$ .

- A continuous-time example is the Liouville vector field  $Z_\lambda$  that is defined by  $i_{Z_\lambda}(-d\lambda) = \lambda$  and a discrete-time example is  $f = \exp Z_\lambda : (q, p) \mapsto (q, ap)$ ,  $a = \frac{1}{e}$ .
- An extension of the so-called Mañé example, which exists in the conservative setting, makes possible to embed all time-continuous dynamics of a closed manifold as the restriction to the zero-section of  $T^*Q$  of a discounted Tonelli flow.

# Discounted Tonelli case

## Definition

A discounted Hamiltonian Tonelli vector field  $X$  on  $(T^*Q, \Omega = -d\lambda)$  is defined by

$$i_X \Omega = dH - \alpha \cdot \lambda$$

where

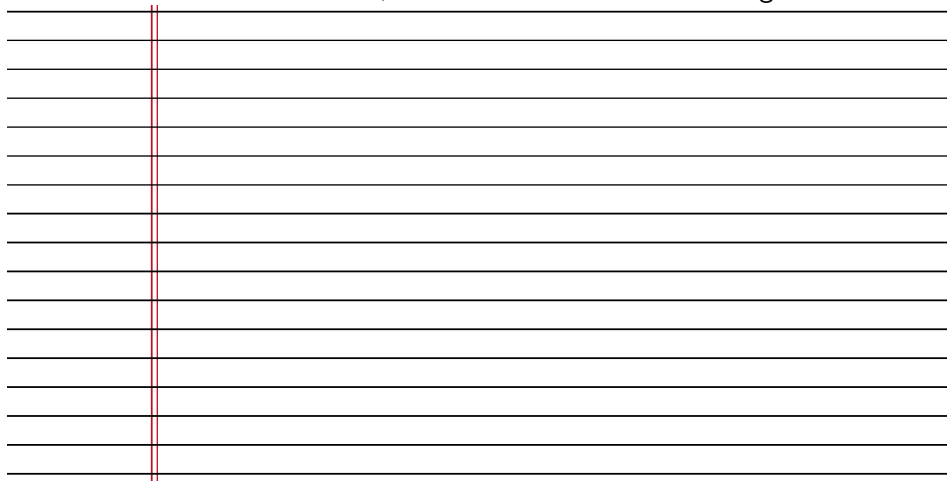
- $H : T^*Q \rightarrow \mathbb{R}$  is Tonelli, i.e.  $C^2$ , superlinear and with positive Hessian in the fiber direction;
- $\alpha > 0$ .

# Attractors

In this case, if we denote by  $(\varphi_t)$  the flow of  $X$ , we have

- $\varphi_t^* \Omega = e^{-\alpha t} \Omega$ ;
- for  $p \in T^*Q$  outside some fixed compact set,  $\frac{dH \circ \varphi_t(p)}{dt} < 0$ .

Hence there exists an attractor, and this attractor has 0 Lebesgue measure.







## Our focus: invariant submanifolds

Knowing that when existing, the attractor has zero Lebesgue measure and that Mañé example involves invariant Lagrangian submanifolds, a natural question concerns the isotropy of invariant submanifolds.

### Proposition

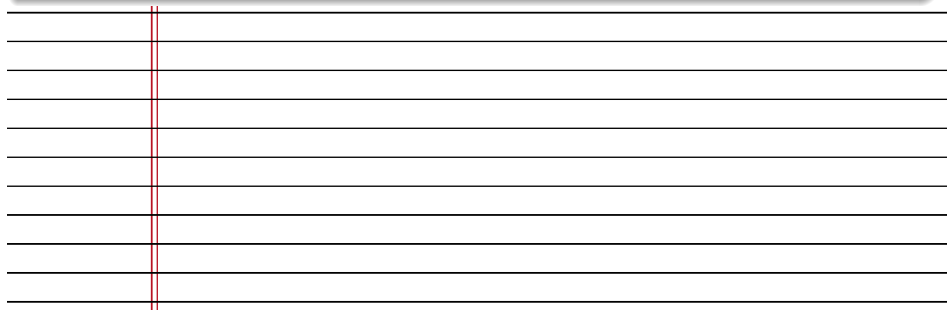
*If a closed  $C^1$  surface  $\mathcal{L}$  is invariant by a conformal and non symplectic  $C^1$  diffeomorphism of  $(\mathcal{M}, \omega)$ ,  $\mathcal{L}$  is  $\omega$ -isotropic.*

# A non-isotropic example

## Proposition

*There exists a conformal vector field  $X$  on a 4-dimensional symplectic manifold  $(\mathcal{M}, \omega)$ , with a 3-dimensional invariant submanifold  $\mathcal{L}$  (hence  $\mathcal{L}$  is not isotropic).*

*Moreover, the submanifold  $\mathcal{L}$  is the global attractor for the flow  $(\varphi_t)$  of  $X$ ,  $(\varphi_t|_{\mathcal{L}})$  is conjugated to the suspension of an Anosov automorphism of  $\mathbb{T}^2$  with 2-dimensional stable and unstable foliations, and  $(\varphi_t|_{\mathcal{L}})$  is transitive with entropy equal to  $|\alpha|$  where  $L_X\omega = \alpha\omega$ .*



## Some remarks and a question

- In the later example the invariant submanifold is an hypersurface of a manifold  $N \times \mathbb{R}$ .
- For a Tonelli flow of a cotangent bundle, the attractor cannot separate.
- **Question.** Is it possible that a Tonelli flow has an invariant closed submanifold that is not isotropic?



## Yomdin theory

Let  $\mathcal{L}$  be a compact Riemannian  $C^r$  manifold,  $\mathcal{S} \subset \mathcal{L}$  be a compact  $C^r$  submanifold of dimension  $s$  and  $f$  be a  $C^r$  self-map of  $\mathcal{L}$  ( $r \geq 1$ ). Denote by  $\text{vol}$  the  $s$ -dimensional Riemannian volume. Define the logarithmic volume growth of  $f|_{\mathcal{S}}$  as

$$\text{logvol}(f|_{\mathcal{S}}) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log |\text{vol}(f^n(\mathcal{S}))|,$$

and

$$\text{rad}(Df) = \limsup_{n \rightarrow +\infty} \|Df^n\|^{1/n}, \quad \|Df\| = \sup_x \|Df_x\|.$$

### Theorem (Yomdin 1987)

$$\text{logvol}(f|_{\mathcal{S}}) \leq \text{ent}(f) + \log^+(\text{rad}(Df)^{s/r}).$$

*In particular, if  $\mathcal{L}$ ,  $\mathcal{S}$  and  $f$  are smooth,*

$$\text{logvol}(f|_{\mathcal{S}}) \leq \text{ent}(f).$$

# A result of Yomdin theorem

## Proposition

Let  $f$  be a conformal diffeomorphism of a symplectic manifold  $(\mathcal{M}, \omega)$ , i.e. such that  $f^*\omega = a\omega$  with  $a \in ]0, 1[$ . Let  $\mathcal{L}$  be an  $\ell$ -dimensional invariant closed submanifold. Assume one of the following hypothesis.

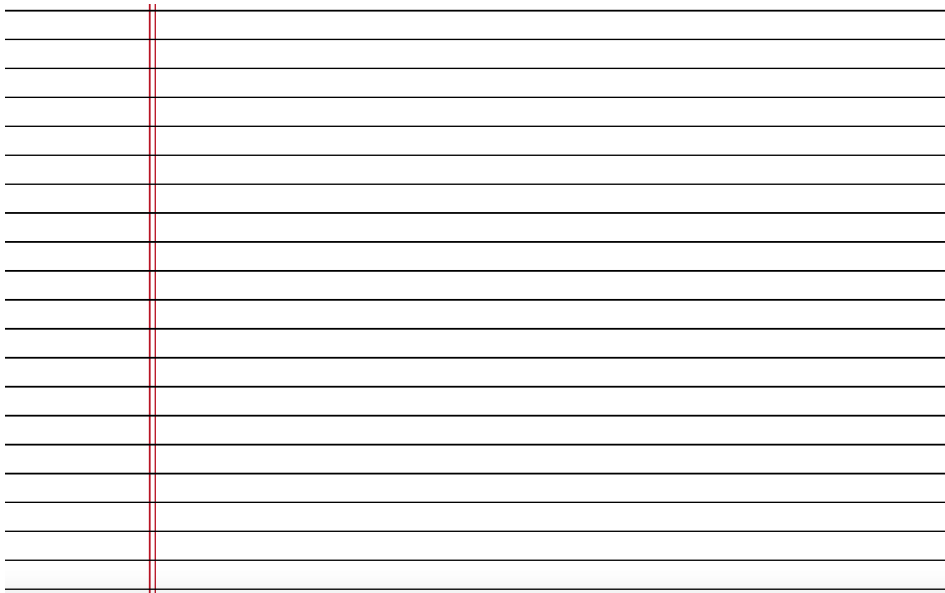
- 1 The diffeomorphism  $f$  is smooth,  $\mathcal{L}$  is smooth and

$$\text{ent}(f|_{\mathcal{L}}) < -\log(a);$$

- 2 The diffeomorphism  $f$  is  $C^r$ ,  $\mathcal{L}$  is  $C^r$  for some  $r \geq 1$  and

$$\text{ent}(f|_{\mathcal{L}}) + \log^+ \left( \text{Rad}(Df|_{\mathcal{L}}^{-1})^{\ell/r} \right) < -\log(a).$$

Then  $\mathcal{L}$  is  $\omega$ -isotropic.



# An improvement of regularity hypothesis in a particular case

Let

- $\mathcal{M}^{(n)}$  be a compact Riemannian manifold
- $\mathcal{F}$  be a foliation induced by a subbundle  $F$  of  $T\mathcal{M}$  of rank  $p \leq n - 1$
- $\Omega$  be an  $(n - p)$ -form on  $\mathcal{M}$  which induces a volume on submanifolds transverse to  $\mathcal{F}$
- $f$  be a  $C^1$ -diffeomorphism of  $\mathcal{M}$  preserving  $\mathcal{F}$  and such that

$$f^*\Omega = \beta \Omega$$

for some  $\beta > 1$ .

## Theorem

*The topological entropy of  $f$  satisfies*

$$\text{ent } f \geq \ln \beta.$$



## An improvement of regularity hypothesis in a particular case

Assume that  $\omega$  is a pre-symplectic form of (even) rank  $2\ell \geq 2$  and

$$f^*\omega = \alpha\omega, \quad \alpha > 1.$$

The kernel of  $\omega$  is a uniquely integrable subbundle  $F$  of even corank  $2\ell$ .

### Corollary

*The topological entropy of  $f$  satisfies*

$$\text{ent } f \geq \frac{\text{rank}(\omega)}{2} \ln \alpha.$$

# An improvement of regularity hypothesis in a particular case

## Corollary

Let  $f : \mathcal{M} \rightarrow \mathcal{M}$  be a  $C^2$  conformal symplectic diffeomorphism such that  $f^*\omega = a\omega$ . Suppose that  $\mathcal{N}$  is an invariant  $C^2$  submanifold such that  $\omega|_{\mathcal{N}}$  has constant rank and

$$\text{ent } f|_{\mathcal{N}} < \frac{\text{rank}(\omega|_{\mathcal{N}})}{2} \ln a.$$

Then  $\mathcal{N}$  is isotropic. This result applies e.g. to an invariant submanifold on which the dynamics is minimal (every orbit is dense) and the entropy vanishes.

## Statement to be proved

We assume that  $\mathcal{M}^{(n)}$  is a compact Riemannian manifold with distance  $d$ , that  $\Omega$  is a  $(n - p)$ -form on  $\mathcal{M}$  that induces a volume form transversally to the  $p$  foliation  $\mathcal{F}$  whose tangent subspace is the kernel of  $\Omega$ . We assume that  $f$  be a  $C^1$ -diffeomorphism of  $\mathcal{M}$  preserving  $\mathcal{F}$  and such that

$$f^*\Omega = \beta \Omega$$

for some  $\beta > 1$ .

Then the topological entropy of  $f$  satisfies

$$\text{ent } f \geq \ln \beta.$$



