Conference in honour of Jean Pierre M

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Kolmogorov's non-degeneracy at simple resonances:
a non perturbative problem
(joint with LUCA BIASCO since 2014)

$$H(y,x) = h(y) + \varepsilon f(y,x) \quad , \quad dy \wedge dx$$

$$(y,x) \in B \times \mathbb{T}^n$$

region in $\mathbb{R}^n$

Classical (1950-1960) If h K-non-degenerate ($\det h'' \neq 0$ in B) and $\varepsilon < \varepsilon_0 \Rightarrow$ $B \times \mathbb{T}^n$ is filled by KAM primary tori (continuation of integrable tori) up to a set of meas $\underline{\sqrt{\varepsilon}}$.

Big problem: what is the generic dynamics in $\mathcal{P}^c$??

We know many things.

- $\exists$ of lower d'l tori (elliptic/hyp.)
- exp stability of actions (Nekhoroshev)
- $\exists$ Arnold diffusion.

Structure of simple resonances

$k \in \mathbb{Z}^d \setminus \{0\}$, m.c.d. $(k_j) = 1$

$$R_k = \{ y \in B \mid h'(y) \cdot k = 0 \text{ but } h'(y) \cdot \ell \neq 0 \text{ if } \ell \notin \langle k \rangle = k\mathbb{Z}^n \}$$

Observation 1

simple resonances are related to a *fund* nearly-integrable structure

Fix $\alpha, \bar{K} > 0$ $\longleftarrow$ Fourier cut-off, $k \in \mathbb{Z}^d$, m.c.d. $(k_j) = 1$, $|k| \leq \bar{K}$

α ← size of small divisors

$\exists \ \Phi_k : \mathcal{N}_{\alpha, k} \times \mathbb{T}^d \circlearrowleft$ s.t.

$\mathcal{N}_{\alpha,k}$ ← α-neighb of R_k (exact simple resonance)

$$H \circ \Phi_k = h(y) + \varepsilon \left(g_0(y) + (\pi_{\langle k \rangle} f)(y, k \cdot x) + G(y, k \cdot x) + f_k(y, x) \right)$$

where

$|g_0|$ and $\|G\| \leq \frac{c}{\bar{K}^b}$

$\|f_k\| \leq e^{-c\bar{K}}$

$$(\pi_k f)(\theta) = \sum_{j \in \mathbb{Z}^n} f_{jk} \, e^{ij\theta}$$

θ ← $k \cdot x$

$$\boxed{R = \frac{1}{\varepsilon^a}}\ , \left(|\log \varepsilon|^2 , \quad \|f\|_* = \varepsilon^{|\log \varepsilon|} \right)$$

$$\alpha \sim \sqrt{\varepsilon}\, K$$

Disregard the (quasi) exponentially small terms f_* you get

$$H_* = h(y) + \varepsilon\big(g_0(y) + F(y,\theta) + G(y,\theta) \big)$$

small π small, θ and $k \cdot x$

Arnold's 1964 example

$$h = \frac{y_1^2}{2} + \frac{y_2^2}{2} + y_3 \ , \quad F(x_1) = \cos x_1 - 1$$

$$f_* = \mu (\cos x_1 - 1) \cdot (\sin x_2 + \cos x_3)$$

Observation 2 (Arnold–Kozlov–Neishtadt)

$\mathcal{P}\, h(\ldots)^c$ there is an $O(\varepsilon)$-zone

which NON PERTURBATIVE

$n = 2 \qquad \frac{y_1^2 + y_2^2}{2} + \varepsilon f(x_1, x_2)$

interaction of two single resonances

which a double resonance

$\ldots$

$y \to \sqrt{\varepsilon}\,\bar{y}$, rescale the Hamilton. div by ε

$\to$ equivalent Hamiltonian

$$\frac{\bar{y}_1^2 + \bar{y}_2^2}{2} + f(x_1, x_2)$$

from now on $\boxed{H = \frac{|y|^2}{2} + \varepsilon f(x)}$

Class of generic potentials f

$\mathbb{P} = \bigcup_{s>0} \mathbb{P}_s$, $\mathbb{P}_s \subseteq \mathbb{B}_s := \{ f : \mathbb{T}^n \to \mathbb{R}^n \mid \|f\|_s = \sup_k |f_k| e^{|k|s} < \infty \}$

$\mathbb{P}_s$ is the set of f's s.t.

(P1) $\lim_{|k|\to\infty} |f_k| e^{|k|s} |k|^n = 0$

$\left(|f_k| \sim e^{-|k|s} \right)$

(P2) $F = \pi_{\langle k \rangle} f$ $\quad$ (m.c.d. $k_j = 1$)

is Morse : $F'(\theta_0) = 0 \Rightarrow F''(\theta_0) \neq 0$.

H_k in a.a. coordinates is

K - non degenerate ?

Thm. $\exists\, c > 1 \ \forall \mu$: $\operatorname{meas}\{(I,J) \in B_\lambda \times D \mid |\det \partial^2 E| < \mu\}$

$$\leq c\, \mu^{\frac{1}{c}}$$

Sketch of proof.

$$\Delta(I,J) = (q_k + \partial_J^2 E)\, \partial_I^2 E - (\partial_J^2 E)^2$$

Analytic prop. of $E(I,J)$

(1) $|\partial_I^2 E| > \dfrac{\varepsilon}{\lambda(\log \lambda)}$

(2) $|\partial_{IJ}^2 E|, |\partial_J^2 E| \leq \text{Const}\, \dfrac{\varepsilon}{\lambda}$ small $\varepsilon^{\frac{1}{2}}$

(3) $(I,J) \mapsto E$ is holomorphic $(\mathbb{C} - (-\infty, 0]) \times D$

$\forall\, \lambda_* > 0 \ \exists\, \eta_*, c > 0 \ \forall\, \mu \ \forall\, \eta < \eta_*$

$\forall\, \lambda > \lambda_*$

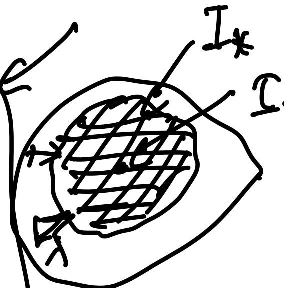
$I \to \Delta(I,J)$ is holom. ($|\partial_I E| \to \infty$)

not constant on $\mathbb{C}_0$

$\exists\, m \mid$

$$\Longleftrightarrow \inf_{[0, I_*]} \max_{1 \leq j \leq m} |\partial_I^j E| \geq \beta > 0$$

f



[Calculus lemma: If $f \in C^{m+1}(a,b)$

$$\inf_{(a,b)} \max_{1\le j\le n} |D^j f| \ge \varepsilon$$

$$\operatorname{meas}\{x \in (a,b) \mid |f(x)| \le \mu\} \le c\,\frac{\mu^a}{\varepsilon^b}$$

$$a = \frac{1}{n(n+1)} \qquad b = a + \frac{1}{2}$$

$$\Delta = c_2\, \partial_I^2 E + O(\eta) \qquad \eta \text{ small respect to } \varepsilon.$$

$$I = I(E) = \frac{1}{2\pi} \oint p\,dq$$

(4)

$$I(E_{sep} - \lambda; J) = \Phi(\lambda; J) + \Psi(\lambda, J)\, \lambda \log \lambda$$

Φ, Ψ holom. in $|\lambda| \le r_0$

$$\Psi(0, J) = \Psi_0 \ne 0$$

$$\Delta(\lambda) = \underbrace{\left(\lambda^2 (\partial_E f)^6\right)}_{\text{don't care it}} \Delta\left(I(E_c - \lambda), J\right)$$

comp 1 $\quad \hat{\Delta} = \Psi_0\, \lambda (\log \lambda)^3 + \mathcal{O}(0,2) + \mathcal{O}(2,4)$

$$\mathcal{O}(k,h) := \lambda^k \sum_{j=0}^{h} a_j(\lambda) (\log \lambda)^j$$

holom

$n=2$

$L := \lambda \partial_\lambda \quad , \mathcal{L} = L^3 (\partial_\lambda L^3)$

comp. 2

$\mathcal{L}[\hat{\Delta}] = 6 \psi_0 + O(\lambda, 4)$

$O(\lambda)$

$\Rightarrow \hat{\Delta}$ is C^7 - non deg.