Conley, Easton, Massera, Sullivan, Wilson and Yorke

Albert Fathi

ZoomTown, June 9, 2021

Introduction

We will assume that X is a C^k vector field, with $k \geq 1$, on the smooth connected but not necessarily compact manifold M. We will also assume that the vector field is complete, i.e. it defines a flow $\phi_t, t \in \mathbb{R}$. The flow ϕ_t is C^k on $M \times \mathbb{R}$.

We propose to give a "common" proof of three well-known theorems using a generalisation of a work of Wilson and Yorke:
F. Wesley Wilson, Jr. & James A. Yorke, *Lyapunov Functions and Isolating Blocks*, **J. Diff. Eq.** 13 (1973) 106–123.

The first theorem was obtained in:
José Luis Massera, *Contributions to stability theory*, **Annals of Math.**, 64 (1956) 182–206.

Theorem 1 (Massera's converse to Lyapunov Theorem)

Suppose that A is a non-empty compact subset of M which is invariant under the flow ϕ_t and Lyapunov asymptotically stable, then there exists a C^∞ function $g : M \to [0, +\infty[$ such that $g^{-1}(0) = A$ and $X \cdot g < 0$ on $V \setminus A$, where V is a neighborhood of A in M.

Recall that an invariant set A for the flow ϕ_t is said to be *Lyapunov asymptotically stable* if it satisfies the following two conditions:

(a) for every neighborhood V of A (in M), we can find a neighborhood V' of A such that $\bigcup_{t\geq 0} \phi_t(V') \subset V$;

(b) there exists a neighborhood V_0 of A, such that, for every $x \in V_0$, we have $d(\phi_t(x), A) \to 0$ as $t \to +\infty$.

The second Theorem is from:
DENNIS SULLIVAN, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, **Inv. math.**, 36 (1976) 225–255.

Theorem 2 (Sullivan)

Assume $A \subset M$ is a compact subset that contains no full (two-sided) orbit of the flow ϕ_t, then there exists a C^∞ function $f : M \to [0, +\infty[$ such that $X \cdot f > 0$ on A.

It is in fact a small corollary from a much deeper theory.
A simpler proof was proposed by Marc Chaperon. You can find it in the appendix of: FRANÇOIS LAUDENBACH & JEAN-CLAUDE SIKORAV, *Hamiltonian Disjunction and Limits of Lagrangian Submanifolds*, **Int. Math. Res. Notices**, 4 (1994) 161–168.

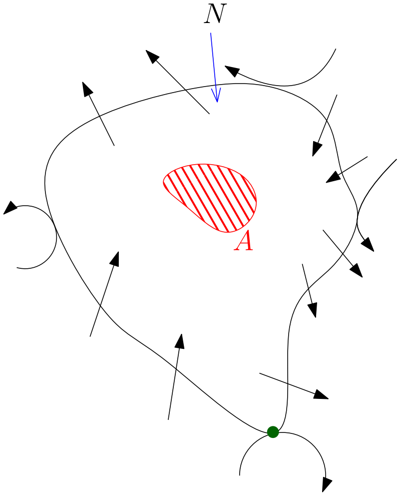
The third theorem is from: CHARLES CONLEY & ROBERT EASTON, *Isolated invariant sets and isolating blocks*, **Trans. Amer. Math. Soc.**, 158 (1983) 35–61.

Theorem 3 (Conley-Easton Isolating Block)

Suppose A is a compact subset of M which is an isolated invariant set of the flow ϕ_t, then there exists a smooth submanifold with boundary N of M which is an isolating block for A.

Recall that an invariant A of ϕ_t is said to be *isolated* if there exists a neighborhood V of A such that $A = \bigcap_{t \in \mathbb{R}} \phi_t(V)$. Such a neighborhood V is said to be an *isolating neighborhood* of A. An *isolating block* for the isolated invariant set A is a codimension 0 compact submanifold N of M, which is an isolating neighborhood of A, such that at a point $x \in \partial N$, where the vector field is tangent to ∂N, the local orbit of x lies outside N except for the point x.

This isolating block looks like



It is important to note that such tangencies of orbits to ∂N with exactly (locally) one point x of intersection with N are automatic if ∂N is more curved toward N at x than the absolute value of the curvature of the orbit at x, like at the green point.

Main Theorem

We will deduce the three theorems from the following one:

Theorem 4 (Main Theorem)

Assume C is a closed subset of M, we can find two C^∞ functions $f, g : M \to [0, +\infty[$ such that:

(i) $f^{-1}(0) = \cap_{t \geq 0} \phi_{-t}(C)$. *(Therefore* $d_x f = 0$ *for every* $x \in \cap_{t \geq 0} \phi_{-t}(C)$.*)*

(ii) $X \cdot f(x) > 0$ *for every* $x \in C \setminus \cap_{t \geq 0} \phi_{-t}(C)$.

(iii) $g^{-1}(0) = \cap_{t \geq 0} \phi_t(C)$. *(Therefore* $d_x g = 0$ *for every* $x \in \cap_{t \geq 0} \phi_t(C)$.*)*

(iv) $X \cdot g(x) < 0$ *for every* $x \in C \setminus \cap_{t \geq 0} \phi_t(C)$.

Note that $\cap_{t \geq 0} \phi_{-t}(C)$ is the set of points $x \in C$ whose whole forward orbit remains in C and $\cap_{t \geq 0} \phi_t(C)$ is the set of points $x \in C$ whose whole backward orbit remains in C.
As we said, a version of this theorem is contained in:
F. Wesley Wilson, Jr. & James A. Yorke, *Lyapunov Functions and Isolating Blocks*, **J. Diff. Eq.** 13 (1973) 106–123.

To get a picture of the Theorem, consider the map $F : M \to \mathbb{R}^2$ defined by $F(x) = (f(x), g(x))$. Note that $X \cdot F(x) = (X \cdot f(x), X \cdot g(x))$ is in the fourth quadrant $Q_{+,-} = \{(x, y) \mid x \geq 0, y \geq 0\}$ of $\mathbb{R}^2$, for $x \in C$, and is in the interior of $Q_{+,-}$ for $x \in C \setminus (\cap_{t\geq 0}\phi_{-t}(C) \cup \cap_{t\geq 0}\phi_t(C))$.

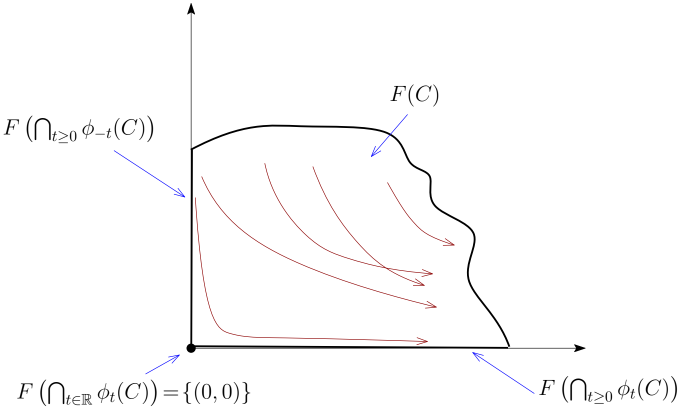


Figure: In red images by F of pieces of orbits of ϕ_t

In the work of Wilson and Yorke, the closed set C is not as general. It is an isolating neighborhood (of some invariant compact subset). They do not prove the smoothness of the functions f and g everywhere but only on the sets $C \setminus \cap_{t\geq 0}\phi_{-t}(C)$ and $C \setminus \cap_{t\geq 0}\phi_t(C)$ respectively.
They apply their work to find isolating blocks. Their proof of their version of the theorem above, like the Conley-Easton proof of existence of isolating blocks uses the exit time function from an isolating neighborhood. There are complications due to the fact that this exit function is not continuous. We go around that difficulty by not using any exit time function. In fact, by proving this more general version, with C not necessarily an isolating neighborhood, we cannot rely on exit times functions since for example C could have no interior. As often happens, the more general version has a neater proof than the particular one.
Note that an important by-product of isolating blocks is providing a "smooth" exit function defined on the interior of the isolating block (minus the subset of points whose positive orbits are contained in that interior).

We will prove the Main Theorem 4 after deducing Sullivan's Theorem 2 and Massera's converse to Lyapunov Theorem 1 from it.
We will deduce the Conley-Easton Isolating neighborhood Theorem 3 after proving the Main Theorem 4

Proof of Sullivan's Theorem 2.
By the Main Theorem 4, with $C = A$, we can find a C^∞ function f such that $X \cdot f$ is > 0 on $A \setminus \cap_{t \geq 0} \phi_{-t}(A)$. Therefore, it suffices to show that $\cap_{t \geq 0} \phi_{-t}(A)$, the set of points $x \in A$ whose whole forward orbit remains in A, is empty. This follows from the compactness of A. In fact, there existed an $x \in$ with $\phi_t(x) \in A$, for all $t \geq 0$, then, by compactness of A, the ϕ_t-omega limit set $\omega(x)$ would not be empty and would be contained entirely in A. Since $\omega(x)$ consists of full (two-sided) orbits of the flow ϕ_t, we obtain a contradiction. □

Proof of Massera's converse to Lyapunov Theorem 1.

By the fact that A is Lyapunov asymptotically stable, we can find a neighborhood V_0 of A, such that for every $x \in V_0$, we have $d(\phi_t(x), A) \to 0$ as $t \to +\infty$. Since A is compact, we can without loss of generality assume that V_0 is a compact neighborhood of A. As is well-known we will show that $A = \bigcap_{t \geq 0} \phi_t(V_0)$. Once this fact established, the Main theorem applied with $C = V_0$ provides us with a C^∞ function $g : M \to [0, +\infty[$ such that $g^{-1}(0) = \cap_{t \geq 0} \phi_t(V_0) = A$ and $X \cdot g < 0$ on $V_0 \setminus \cap_{t \geq 0} \phi_t(V_0) = V_0 \setminus A$, which proves Massera's converse to Lyapunov,.

It remains to show that $A = \bigcap_{t \geq 0} \phi_t(V_0)$. Since A is invariant by the flow ϕ_t, we have $A \subset \bigcap_{t \geq 0} \phi_t(V_0)$. Therefore, using that the closed set A is the intersection of its neighbourhoods, it suffices to show that for every neighborhood V, we can find $t_V \geq 0$ such that $\phi_{t_V}(V_0) \subset V$.

Fix a neighbohood V of A. Using again that that A is Lyapunov asymptotically stable, we can find a *open* neighborhood V' of A such that $\bigcup_{t \geq 0} \phi_t(V') \subset V$.

Since $d(\phi_t(x), A) \to 0$ as $t \to +\infty$, for every $x \in V_0$, every orbit starting in V_0 enter the neighborhood V' of A in positive time. In other words $V_0 \subset \bigcup_{s \geq 0} \phi_{-s}(V')$. By compactness of V_0, we can cover V_0 by a finite subfamily of the open sets $\phi_{-s}(V'), s \geq 0$. Hence, we can find $t_V \geq 0$ such that $V_0 \subset \bigcup_{s \in [0, t_V]} \phi_{-s}(V')$. Therefore

$$\phi_{t_V}(V_0) \subset \bigcup_{s \in [0, t_V]} \phi_{t_V - s}(V') \subset \bigcup_{t \geq 0} \phi_t(V') \subset V. \quad \square$$

Constructing smooth functions

We will need a way to construct smooth functions as infinite series. To do that, we will use an abstract Lemma on Fréchet space. Recall that a Fréchet space E is a complete Hausdorff topological vector space whose topology is defined by a countable family $p_n, n \in \mathbb{N}$ of semi-norms.

Example 5 (Typical example) For $r \in \mathbb{N} \cup \{\infty\}$, the vector space $E = C^r(M, \mathbb{R})$, endowed with the topology of uniform convergence on compact subsets for all derivatives up to order, is a Fréchet space. Of course, when M is compact and r is finite, the space $E = C^r(M, \mathbb{R})$ is in fact a Banach space.

Lemma 6 *Suppose E is a Fréchet space and $x_n, n \in \mathbb{N}$ is a sequence in E, there exists a sequence $\epsilon_n > 0$ of positive numbers such that for every sequence δ_n, with $|\delta_n| \leq \epsilon_n$, the series $\sum_{n \geq 0} \delta_n x_n$ converges in E.*

The proof is standard, see for example:

Albert Fathi, *Partitions of Unity for Countable Covers*, **Amer. Math. Monthly**, 104 (1997) 720–723.

Since it is short, we give it.

Proof. The Fréchet space E has a topology defined by the countable family $p_n, n \in \mathbb{N}$ of semi-norms. We can assume $p_{\ell+1} \geq p_\ell$ (replace p_ℓ by $\sum_{i=0}^{\ell} p_i$). The fact that E is complete means that a sequence $y_n, n \in \mathbb{N}$ in E, which is Cauchy for every semi-norm p_ℓ, converges.
For every $\ell \in \mathbb{N}$, choose $\epsilon_\ell > 0$ such that

$$\epsilon_\ell p_\ell(x_\ell) \leq \frac{1}{2^\ell}.$$

If $|\delta_\ell| \leq \epsilon_\ell$, then for $n \geq \ell$, we have

$$p_\ell(\delta_n x_n) \leq p_n(\delta_n x_n) = |\delta_n| p_n(x_n) \leq \epsilon_n p_n(x_n) \leq \frac{1}{2^n}.$$

Hence

$$\sum_{n \geq \ell} p_\ell(\delta_n x_n) \leq \sum_{n \geq \ell} \frac{1}{2^n} = \frac{1}{2^{\ell-1}}$$

and the series $\sum_{n \geq 0} \delta_n x_n$ is Cauchy for every semi-norm p_ℓ. Therefore $\sum_{n \geq 0} \delta_n x_n$ converges. □

A first application of this Lemma is the well-known:

Proposition 7

If A is a closed subset of M, we can find a C^∞ function $\theta : M \to [0, +\infty[$ such that $\theta^{-1}(0) = A$.

Proof.

Set

$$V_n = \{x \in M \mid d(x, A) < \frac{1}{n}\}.$$

For each $n \geq 1$, we can find a C^∞ function $\theta_n : M \to [0, +\infty[$ such that $\theta_n | A \equiv 0$ and $\theta_n | M \setminus V_n \equiv 1$.
By Lemma 6, we can find a sequence $\epsilon_n > 0$ of positive numbers such that the series $\sum_{n \geq 1} \epsilon_n \theta_n$ converges to a C^∞ function $\theta : M \to [0, +\infty[$. We have $\theta | A \equiv 0, \theta_n \geq 0, \epsilon_n > 0$ and $\theta_n | M \setminus V_n \geq 1 > 0$, for all $n \geq 1$.Hence $\theta^{-1}(0) = A$, since $M \setminus A = \cup_{n \geq 1} M \setminus V_n$. □

Proof of the Main Theorem 4

Recall the Main Theorem

Theorem 4 (Main Theorem)

Assume C is a closed subset of M, we can find two C^∞ functions $f, g : M \to [0, +\infty[$ such that:

(i) $f^{-1}(0) = \cap_{t \geq 0} \phi_{-t}(C)$. *(Therefore $d_x f = 0$ for every $x \in \cap_{t \geq 0} \phi_{-t}(C)$.)*

(ii) $X \cdot f(x) > 0$ *for every* $x \in C \setminus \cap_{t \geq 0} \phi_{-t}(C)$.

(iii) $g^{-1}(0) = \cap_{t \geq 0} \phi_t(C)$. *(Therefore $d_x g = 0$ for every $x \in \cap_{t \geq 0} \phi_t(C)$.)*

(iv) $X \cdot g(x) < 0$ *for every* $x \in C \setminus \cap_{t \geq 0} \phi_t(C)$.

Proof.

We start with a a C^∞ function $\theta : M \to [0, +\infty[$ such that $\theta^{-1}(0) = C$, as given by Proposition 7.

For $t > 0$, we define $\theta_t : M \to [0, +\infty[$ by

$$\theta_t(x) = \int_0^t \theta(\phi_s(x))\, ds.$$

Since both θ and the flow ϕ_t are C^1, so is θ_t.
Note also that $\theta_t \geq 0$ and $(t, x) \mapsto \theta_t(0)$ is continuous on $\mathbb{R} \times M$.
Since $\theta \geq 0$ and $\theta^{-1}(0) = C$, for a given $x \in M$, we have

$$\theta_t(x) = 0 \iff x \in \bigcap_{0 \leq s \leq t} \phi_{-s}(C),$$

since

$$\begin{aligned} \theta_t(x) = 0 &\iff \theta(\phi_s(x)) = 0 \text{ for all } s \in [0, t] \\ &\iff \phi_s(x) \in C \text{ for all } s \in [0, t] \\ &\iff x \in \bigcap_{0 \leq s \leq t} \phi_{-s}(C). \end{aligned}$$

Moreover, for every $x \in M$, we have

$$X \cdot \theta_t(x) = \theta(\phi_t(x)) - \theta(x).$$

In fact, to prove $X \cdot \theta_t(x) = \theta(\phi_t(x)) - \theta(x)$, we compute

$$\begin{aligned}
X \cdot \theta_t(x) &= \lim_{\epsilon \to 0} \frac{1}{s}[\theta_t(\phi_\epsilon(x)) - \theta(x)] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_0^t \theta(\phi_s(\phi_\epsilon(x)))\, ds - \int_0^t \theta(\phi_s(x))\, ds \right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_0^t \theta(\phi_{s+\epsilon}(x))\, ds - \int_0^t \theta(\phi_s(x))\, ds \right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_\epsilon^{t+\epsilon} \theta(\phi_s(x))\, ds - \int_0^t \theta(\phi_s(x))\, ds \right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_t^{t+\epsilon} \theta(\phi_s(x))\, ds - \int_0^\epsilon \theta(\phi_s(x))\, ds \right] \\
&= \theta(\phi_t(x)) - \theta(x).
\end{aligned}$$

Hence, since $\theta | C \equiv 0$, for all $x \in C$, we obtain that

$$X \cdot \theta_t(x) = \theta(\phi_t(x)) \geq 0 \text{ and } X \cdot \theta_t(x) = 0 \iff \theta(\phi_t(x)) = 0,$$

therefore

$$\forall x \in C, X \cdot \theta_t(x) = 0 \iff x \in \phi_{-t}(C).$$

Choose a sequence $t_n \in [0, \infty[$ dense in $[0, \infty[$. By Lemma 6, we can find a sequence $\epsilon_n > 0$ such that the series $\sum_{n=0}^{\infty} \epsilon_n \theta_{t_n}$ converges in the C^1 compact open topology to a C^1 function $\hat{f} : M \to [0, +\infty[$. Since the convergence is C^1, we also have

$$X \cdot \hat{f} = \sum_{n=0}^{\infty} \epsilon_n X \cdot \theta_{t_n}.$$

Since $\epsilon_n > 0$ and $\theta_{t_n} \geq 0$, we have $\hat{f}(x) = 0 \iff \theta_{t_n}(x) = 0$ for all n. By density of t_n in $[0, +\infty[$ and the continuity of $(t, x) \mapsto \theta_t(x)$, we obtain $\hat{f}(x) = 0 \iff \theta_t(x) = 0$, for all $t \geq 0$. But $\theta_t(x) = 0 \iff x \in \bigcap_{0 \leq s \leq t} \phi_{-s}(C)$, as we already observed. Hence

$$\hat{f}(x) = 0 \iff \theta_t(x) = 0 \text{ for all } t \geq 0 \iff x \in \bigcap_{t \geq 0} \phi_{-t}(C).$$

This precisely shows that $\hat{f}$ satisfies condition (i) of the Main Theorem 4.

Since $X \cdot \hat{f} = \sum_{n=0}^{\infty} \epsilon_n X \cdot \theta_{t_n}$ and, as we showed, $X \cdot \theta_t(x) = \theta(\phi_t(x)) \geq 0$ for $x \in C$, we obtain

$$X \cdot \hat{f}(x) \geq 0, \text{ for all } x \in C.$$

By $X \cdot \hat{f}(x) = \sum_{n=0}^{\infty} \epsilon_n X \cdot \theta_{t_n}(x)$ and $X \cdot \theta_t(x) = \theta(\phi_t(x)) \geq 0$ for $x \in C$, we obtain

$$\begin{aligned} \forall x \in C, X \cdot \hat{f}(x) = 0 &\iff X \cdot \theta_{t_n}(x) = 0 \text{ for all } n \\ &\iff \theta(\phi_{t_n}(x)) = 0 \text{ for all } n. \end{aligned}$$

Since t_n is dense in $[0, +\infty[$, we obtain

$$\forall x \in C, X \cdot \hat{f}(x) = 0 \iff \theta_t(x) = \theta(\phi_t(x)) = 0, \text{ for all } t \geq 0.$$

But we already obtained
$\theta_t(x) = 0$ for all $t \geq 0 \iff x \in \bigcap_{t \geq 0} \phi_{-t}(C)$, hence

$$\forall x \in C, X \cdot \hat{f}(x) = 0 \iff x \in \bigcap_{t \geq 0} \phi_{-t}(C).$$

Taken together with $X \cdot \hat{f}(x) \geq 0$, for all $x \in C$, we obtain

$$\forall x \in C \setminus \bigcap_{t \geq 0} \phi_{-t}(C), X \cdot \hat{f}(x) > 0.$$

This means that $\hat{f}$ satisfies condition (ii) of the Main Theorem 4. The function $\hat{f}$ is only C^1, we now proceed to show how to obtain from it a C^∞ function.

We know that $\hat{f}$ is C^1, with $\hat{f} > 0$ on $M \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$ and $X \cdot \hat{f} > 0$ on $C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$. By the density of the C^∞ functions in the C^1 Whitney topology, we can find $\tilde{f} : M \setminus \bigcap_{t\geq 0} \phi_{-t}(C) \to]0, +\infty[$ such that

$$|\tilde{f}(x) - \hat{f}(x)| < \hat{f}(x), \text{ for all } x \in M \setminus \bigcap_{t\geq 0} \phi_{-t}(C) \quad (1)$$

and

$$|X \cdot \tilde{f}(x) - X \cdot \hat{f}(x)| < X \cdot \hat{f}(x), \text{ for all } x \in C \setminus \bigcap_{t\geq 0} \phi_{-t}(C). \quad (2)$$

From (1), we can extend $\tilde{f}$ to a continuous function $\tilde{f} : M \to (0, +\infty[$ such that $\tilde{f}^{-1}(0) = \bigcap_{t\geq 0} \phi_{-t}(C)$. Note that by (2), we have $X \cdot \tilde{f} > 0$ on $C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$.
We should still smooth $\tilde{f}$ on $C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$.
For each integer $n \geq 1$, we choose a C^∞ non-decreasing function $\rho_n : [0, +\infty[\to [0, +\infty[$ such that

$$\rho_n^{-1}(0) = [0, 1/n] \text{ and } \rho_n'(t) > 0, \text{ for all } t > 1/n.$$

For every $n \geq 1$, we define the function $f_n : M \to [0, +\infty[$ by

$$f_n = \rho_n \circ \tilde{f}.$$

Since $\rho_n = 0$ on $[0, 1/n]$ and $\tilde{f}$ is C$^\infty$ outside $\tilde{f}^{-1}(0)$, the function f_n is C^∞ on M.

By Lemma 6, we can find a sequence $\epsilon_n > 0$ such that both series $\sum_{n=1}^\infty \epsilon_n f_n$ and $\sum_{n=1}^\infty \epsilon_n \rho_n$ converge respectively in the C^∞ compact open topology to the C^∞ functions $f : M \to [0, +\infty[$ and $\rho : M \to [0, +\infty[$.

We now check that the C^∞ function f satisfies both properties (i) and (ii) of the Main Theorem 4.

To check (i), namely $f(x) = 0 \iff x \in \bigcap_{t \geq 0} \phi_{-t}(C)$, we note that, for $x \in M$, we have

$$\begin{aligned} f(x) = 0 &\iff \sum_{n=1}^{\infty} \epsilon_n f_n(x) = 0 \\ &\iff f_n(x) = \rho_n(\tilde{f}(x)) = 0, \text{ for all } n \geq 1 \\ &\iff \tilde{f}(x) \in [0, 1/n], \text{ for all } n \geq 1 \iff \tilde{f}(x) = 0 \\ &\iff x \in \bigcap_{t \geq 0} \phi_{-t}(C). \end{aligned}$$

To check (ii), namely $X \cdot f > 0$ on $C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$, we first note that

$$f = \sum_{n=1}^{\infty} \epsilon_n f_n = \sum_{n=1}^{\infty} \epsilon_n \rho_n \circ \tilde{f} = \left(\sum_{n=1}^{\infty} \epsilon_n \rho_n\right) \circ \tilde{f} = \rho \circ \tilde{f}.$$

Using that ρ is C^∞ on $[0, +\infty[$ and $\tilde{f}$ is C^∞ on $M \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$, we have

$$X \cdot f(x) = \rho'(\tilde{f}(x)) X \cdot \tilde{f}(x), \text{ for all } x \in M \setminus \bigcap_{t\geq 0} \phi_{-t}(C).$$

But $\rho' = \sum_{n=1}^{\infty} \epsilon_n \rho'_n > 0$ on $]0, +\infty[$, since $\epsilon_n > 0$ and $\rho'_n > 0$ on $]1/n, +\infty[$, and $\tilde{f}, X \cdot \tilde{f}$ are both > 0 on $C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$. Hence $X \cdot f(x) > 0$, for all $x \in C \setminus \bigcap_{t\geq 0} \phi_{-t}(C)$.

To obtain the function g satisfying properties (iii) and (iv) of Main Theorem 4, we consider the flow $\check{\phi}_t = \phi_{-t}$, whose vector field is $-X$. By what we showed above, we can find a C$^\infty$ function$g : M \to [0, +\infty[$ such that $g^{-1}(0) = \cap_{t\geq 0} \check{\phi}_{-t}(C)$ and $-X \cdot g(x) > 0$ for every $x \in C \setminus \cap_{t\geq 0} \check{\phi}_{-t}(C)$. This finishes the proof of the Main Theorem since $\cap_{t\geq 0} \check{\phi}_{-t}(C) = \cap_{t\geq 0} \phi_t(C)$. □

Isolated Invariant Set and Isolating Block

Again we suppose that the flow ϕ_t on M is given by the C^r vector field (with $r \geq 1$).

Recall that an invariant set A of the flow ϕ_t is said to be *isolated* if we can find a neighborhood V of A such that $A = \bigcap_{t\in\mathbb{R}} \phi_t(V)$.

Such a V is called an *isolating neighborhood* for A.

We give more precisely the definition of an isolating block.

An *isolating block* for the isolated invariant subset A is a smooth codimension 0 submanifold N, of class C$^\infty$, with boundary ∂N of M, which is an isolating neighborhood for A , such that

$$\partial N^t = \{x \in \partial N \mid X(x) \text{ is tangent to } \partial N\},$$

is a codimension 1 submanifold (without boundary) of ∂N, of class C^r, which is the common boundary in ∂N of the closure of the two open subsets

$$\partial N^o = \{x \in \partial N \mid \; X(x) \text{ points out of } N\}$$

$$\partial N^i = \{x \in \partial N \; X(x) \text{ points in } N\}$$

and for every $x \in \partial N^t$, we can find $\epsilon > 0$ such that $\phi_s(x) \notin N$ for $0 < |s| < \epsilon$.

Again an isolating block looks like

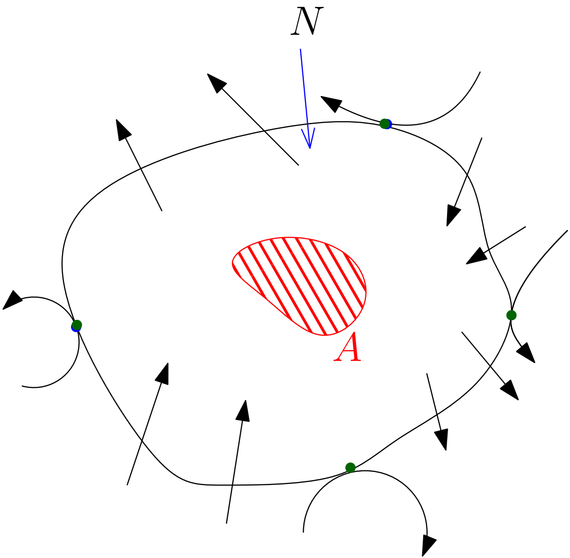


Figure: N^t is represented by the four green points

Note that if V is an isolating neighborhood of the compact invariant set A, we can always cut it down to a codimension 0 compact submanifold with boundary N of M, which is still a neighborhood of A. Of course N is also an isolating neighborhood of A. The challenge is to make

$$\partial N^t = \{x \in \partial N \mid X(x) \text{ is tangent to } \partial N\}$$

a codimension 1 submanifold of ∂N, which is the common boundary in ∂N of the closures of

$$\partial N^o = \{x \in \partial N \mid \ X(x) \text{ points out of } N\}$$
$$\partial N^i = \{x \in \partial N \ X(x) \text{ points in } N\}.$$

together with the existence for every $x \in \partial N^t$ of an $\epsilon > 0$ such that $\phi_s(x) \notin N$ for $0 < |s| < \epsilon$.
We now give a criterion to achieve the challenge.

Lemma 8 *Suppose that the smooth codimension* 0 *submanifold, with boundary* N *of* M *is an isolating neighborhood of the invariant set* A. *If for every* $x_0 \in N^t$, *we can find a* C^∞ *function* $\theta : V \to \mathbb{R}$, *where* V *is an open neighborhood* V *of* x_0, *and a regular value* c *of* f *such that* $N \cap V = \theta^{-1}(]-\infty, c])$ *and* $X \cdot (X \cdot \theta)(x_0) > 0$, *then* N *is an isolating block for* A.

Proof. Since c is a regular value $\partial N \cap V = \theta^{-1}(c)$, we have

$$\partial N \cap V = \theta^{-1}(c) = \{x \in V \cap \partial N \mid \theta(x) = c\},$$

and

$$\partial N^t \cap V = \{x \in V \cap \partial N \mid X \cdot \theta(x) = 0\}.$$

Since $X \cdot (X \cdot \theta)(x_0) > 0$, cutting down the neighborhood V of x_0, we can assume that $X \cdot (X \cdot \theta) > 0$ on all of V. This implies that 0 is a regular value of $\mu = X \cdot \theta | (V \cap \partial N)$, the restriction of the C^r function $X \cdot \theta$ to $V \cap N$.

In fact, if $x \in V \cap \partial N$ is such that $\mu(x) = 0$, then $x \in \partial N^t$ and $X(x) \in T_x\partial N = \ker d_x\theta$. But $d_x\mu(X(x)) = X \cdot (X \cdot \theta)(x) > 0$, which implies $d_x\mu \neq 0$. Therefore $\partial N^t = \mu^{-1}(0)$ is a C^r submanifold of ∂N (since μ is C^r) and the two-sides in $\partial N \cap V$ of the submanifold $\partial N^t \cap V = \mu^{-1}(0)$ are

$$\{x \in V \cap \partial N \,|\, X \cdot \theta(x) > 0\} \subset \partial N^o \text{and } \{x \in V \cap \partial N \,|\, X \cdot \theta(x) < 0\} \subset \partial N^i.$$

It remains to show that for every $x \in \partial N^t \cap V$, we can find $\epsilon > 0$ such that $\phi_s(x) \notin N$ for $0 < |s| < \epsilon$.
In fact, if $x \in \partial N^t \cap V$, the function $s \mapsto \theta(\phi_s(x))$ satisfies

$$\frac{d}{ds}\theta(\phi_s(x))_{s=0} = X \cdot \theta(x) = 0 \text{ and } \frac{d^2}{ds^2}\theta(\phi_s(x))_{s=0} = X \cdot (X \cdot \theta)(x) > 0.$$

Since $\theta(\phi_0(x)) = \theta(x) = c$, we can find $\epsilon > 0$ such that $\phi_s(x) \in V$ and $\theta(\phi_s(x)) > c$, for all $0 < |s| < \epsilon$. Therefore $\phi_s(x) \notin N$ for $0 < |s| < \epsilon$, because $N \cap V = \theta^{-1}(]-\infty, c])$.
This finishes the proof of the criterion Lemma 8. □

Finding the Isolating Block

Suppose now that A is an isolated invariant set, call V a compact neighborhood of A such that $A = \bigcap_{t\in\mathbb{R}} \phi_t(V)$. We proceed to show that A has an isolating block contained in V.

We can apply the main Theorem to the compact set $C = V$ to obtain two C^∞ functions $f, g : M \to [0, +\infty[$ such that:

(i) $f^{-1}(0) = \cap_{t\geq 0}\phi_{-t}(V)$ and $X \cdot f(x) > 0$ for every $x \in V \setminus \cap_{t\geq 0}\phi_{-t}(V)$.

(ii) $g^{-1}(0) = \cap_{t\geq 0}\phi_t(V)$ and $X \cdot g(x) < 0$ for every $x \in V \setminus \cap_{t\geq 0}\phi_t(V)$.

As before, we define the C^∞ function $F : M \to \mathbb{R}^2$ by

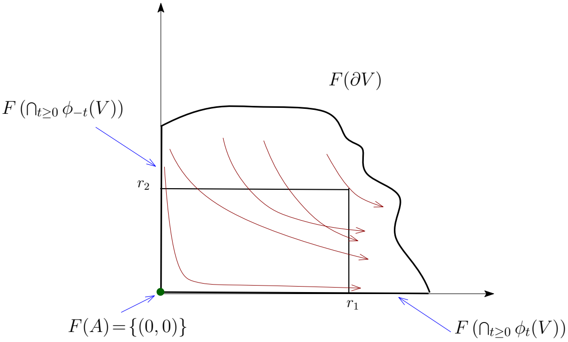
$$F(x) = (f(x), g(x)).$$

Note that $F(M) \subset [0, +\infty[\times[0, +\infty[$ and $F^{-1}(0, 0) = A$ from the first parts of (i) and (ii).

Hence, we can find $r_0 > 0$ such that $F(\partial V) \cap [0, r_0] \times [0, r_0] = \emptyset$.

From (i) and (ii), the restriction $F|V$ is transversal to every horizontal $\{r\} \times \mathbb{R}$ and every vertical $\mathbb{R} \times \{r\}$, with $r \neq 0$.

Since F is C$^\infty$, we can apply Sard's theorem to find $(r_1, r_2) \in]0, r_0[\times]0, r_0[$, which is a regular value of F.



Since F is C$^\infty$, transversal to both $\{r_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{r_2\}$ and (r_1, r_2) is a regular value of F, the intersection $\hat{N} = V \cap F^{-1}([0, r_1] \times [0, r_2]) = V \cap F^{-1}(] - \infty, r_1] \times (] - \infty, r_2])$ is a compact C^∞ manifold with corners contained in the interior of V and containing A in its interior. The corner of $\hat{N}$ is $V \cap F^{-1}(r_1, r_2)$.

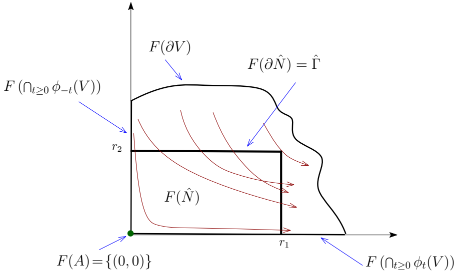
The boundary $\partial\hat{N}$ of $\hat{N}$ is the union of the two submanifolds with boundaries $\partial\hat{N}^i = V \cap F^{-1}([0, r_1] \times \{r_2\})$ and $\partial\hat{N}^o = V \cap F^{-1}(\{r_1\} \times [0, r_2])$. Therefore, we can consider that $\hat{N}$ is an isolating block. However we want an isolating block which has no corners.

To obtain an isolating block which has no corners we replace the corner of $\hat{N}$ on the inside with a quarter of a circle of small radius tangent to the sides.

More precisely, setting

$$\hat{R} = [0, r_1] \times [0, r_2] \text{ and } \Gamma = [0, r_1] \times \{r_2\} \cup \{r_1\} \times [0, r_2],$$

we have $\hat{N} = V \cap F^{-1}(\hat{R})$ and $\partial\hat{N} = V \cap F^{-1}(\hat{\Gamma})$.

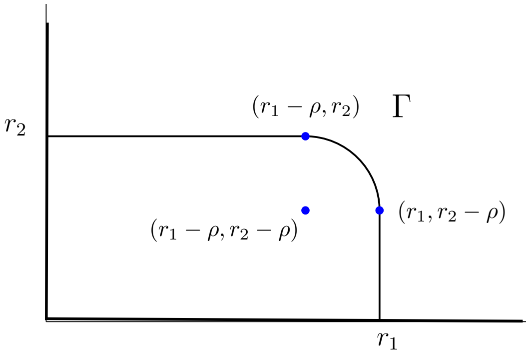


We change the rectangle $\hat{R}$ into a region R by changing the part $\hat{\Gamma}$ to Γ where the corner is replaced by a quarter circle.
More precisely we replace $\hat{\Gamma}$ by

$$\Gamma = [0, r_1 - \rho] \times \{r_2\} \cup C_{++} \cup \{r_1\} \times [0, r_2 - \rho],$$

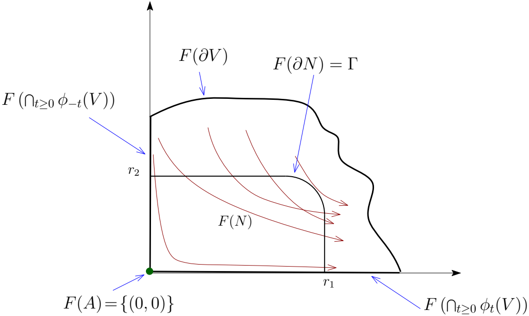
where C_{++} is the quarter circle

$$C_{++} = (r_1 - \rho, r_2 - \rho) + \{(x, y) \mid x^2 + y^2 = \rho^2, x \geq 0, y \geq 0\}.$$



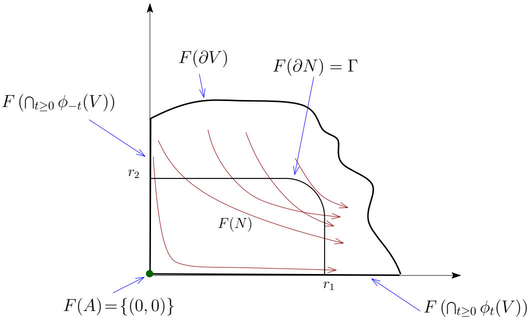
Note that $\hat{\Gamma}$ is a piecewise C$^\infty$ curve and Γ is C^1.

The rectangle $\hat{R} = [0, r_1] \times [0, r_2]$ is replaced by the region R bounded by Γ and $[0, r_1] \times \{0\} \cup \{0\} \times [0, r_2]$. We take $N = V \cap F^{-1}(R)$. Its boundary is $\partial N = V \cap F^{-1}(\Gamma)$. Of course, since Γ is only C^1, the boundary N is only C^1.



We now note that the images of the orbit of ϕ_t are transversal to the part $[0, r_1 - \rho] \times \{r_2\} \cup \{r_1\} \times [0, r_2 - \rho]$ of $\Gamma = [0, r_1 - \rho] \times \{r_2\} \cup C_{++} \cup \{r_1\} \times [0, r_2 - \rho]$. Hence the only tangency points of the vector field X with $\partial N = F^{-1}(\Gamma)$ will be coming from the quarter circle C_{++} away from its end points.

Therefore, if we approximate the curve Γ in the C^1 by a smooth C^∞ curve, changing Γ only in the neighborhood of the end points of C_{++}, we will not change the tangency set. This allow us to have to just check that N is an isolated block for ρ small.



Since the curvature of the circle is $1/\rho$ (in absolute value), taking ρ small enough, the curvature of the circle will be strictly bigger than the curvature of any image by F of an orbit. This implies that the orbits tangent to ∂N will have images under F locally outside R (except for the tangency) point.

Hence the orbit of a point $x \in \partial N^t$ must be locally outside N except for x itself. That should convince you that N is an isolating block.

The rigorous argument is based on the fact that we can apply Lemma 8 for the radius ρ small enough we can apply.
Since (r_1, r_2) is a regular value of $F : V \to \mathbb{R}^2$ and V is compact, we can find $\delta > 0$ such that the square $S = [r_1 - \delta, r_1 + \delta] \times [r_2 - \delta, r_2 + \delta]$ is contained in $]0, +\infty[\times]0, +\infty[$ and every point in the square $S = [r_1 - \delta, r_1 + \delta] \times [r_2 - \delta, r_2 + \delta]$ is a regular value of $F : V \to \mathbb{R}^2$. Since $S \subset]0, +\infty[\times]0, +\infty[$, imply that $X \cdot F(x) = T_x F(X(x))$ never vanishes for x in the compact subset $V \cap F^{-1}(S)$. Therefore

$$\kappa = \inf_{x \in V \cap F^{-1}(S)} \|X \cdot F(x)\| > 0,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Moreover, since F is C^∞ and X is at least C^1, the function $X \cdot (X \cdot F)$ is C^1 on the compact subset V, therefore

$$K = \sup_{\in V} \|X \cdot (X \cdot F)(x)\| < +\infty.$$

With

$$\kappa = \inf_{x \in V \cap F^{-1}(S)} \|X \cdot F(x)\| > 0$$

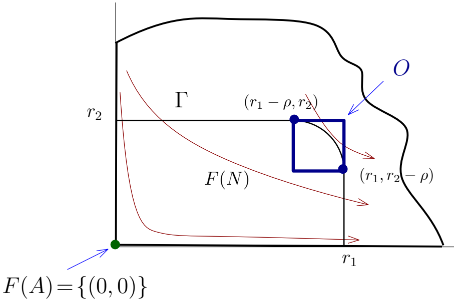
and

$$K = \sup_{\in V} \|X \cdot (X \cdot F)(x)\| < +\infty,$$

Choose now $\rho > 0$ such that $\rho < \delta$ and

$$\kappa^2 - \rho\sqrt{2}K > 0.$$

Note that $O =]r_1 - \delta, r_1[\times]r_2 - \delta, r_2[$ is a neighborhood of $C_{++} \setminus \{(r_1 - \rho, r_2), (r_1, r_2 - \rho))\}$, since $(r_1 - \rho, r_2), (r_1, r_2 - \rho)$ are the end point of C_{++}.



Therefore $W = F^{-1}(O)$ is an open neighborhood of ∂N^t. Define $\theta : F^{-1}(O) \cap V \to \mathbb{R}$ by

$$\theta(x) = \|F(x) - (r_1 - \rho, r_1 - \rho)\|^2.$$

Note that

$$\partial N \cap O \cap V = V \cap F^{-1}(C_{++} \setminus \{(r_1 - \rho, r_2), (r_1, r_2 - \rho))\}) = \theta^{-1}(\rho^2).$$

Moreover, since every point in
$O \subset S = [r_1 - \delta, r_1 + \delta] \times [r_2 - \delta, r_2 + \delta]$ is a regular value of $F : V \to \mathbb{R}^2$, the positive number ρ^2 is a regular value of θ.
Therefore, by Lemma 8, to finish proving that N is an isolating neighbourhood, it suffices to show we need to show that $X \cdot (X \cdot \theta) > 0$ on $F^{-1}(O) \cap V$.
We have

$$X \cdot \theta(x) = 2\langle F(x) - (r_1 - \rho, r_2 - \rho), X \cdot F(x)\rangle$$

and

$$X \cdot (X \cdot \theta)(x) =$$
$$= 2\left(\langle X \cdot F(x), X \cdot F(x)\rangle + \langle F(x) - (r_1 - \rho, r_2 - \rho), X \cdot (X \cdot F)(x)\rangle\right)$$

Note that $F(x)$ and $(r_1 - \rho, r_2 - \rho)$ are both in $[r_1 - \delta, r_1] \times [r_2 - \delta, r_2]$, which is of Euclidean diameter $\rho\sqrt{2}$, hence

$$
\begin{aligned}
& X \cdot (X \cdot \theta)(x) = \\
&= 2 \left(\langle X \cdot F(x), X \cdot F(x) \rangle + \langle F(x) - (r_1 - \rho, r_2 - \rho), X \cdot (X \cdot F)(x) \rangle \right) \\
&\geq 2 \left(\|X \cdot F(x)\|^2 - \|F(x) - (r_1 - \rho, r_2 - \rho)\| \|X \cdot (X \cdot F)(x)\| \right) \\
&\geq 2 \left(\kappa^2 - \rho\sqrt{2}K \right) > 0,
\end{aligned}
$$

where the last strict inequality follows from the fact that ρ small enough to satisfy the definition $\kappa^2 - \rho\sqrt{2}K > 0$.