JPM Conference

08 06 2021

QUANTITATIVE CONDITIONS FOR RIGHT-HANDEDNESS

A. FLORIO (Paris Dauphine)

joint work with U. L. HRYNIEWICZ (Aachen University)

COLLOQUE INTERNATIONAL
DE DYNAMIQUE HAMILTONIENNE
en l'honneur de
Jean-Pierre MARCO 60+1

GOAL have a quantitative criterion to assure that a dynamically convex Reeb flow is "right-handed"

- WHAT IS A "RIGHT-HANDED" FLOW?
- WHY A "RIGHT-HANDED" FLOW IS DYNAMICALLY INTERESTING?
- WHAT IS A REEB FLOW? DYNAMICALLY CONVEX?
- SOME EXAMPLES
- STATEMENT OF QUANTITATIVE CRITERION
- APPLICATIONS TO HAMILTONIAN DYNAMICS & GEODESIC FLOWS

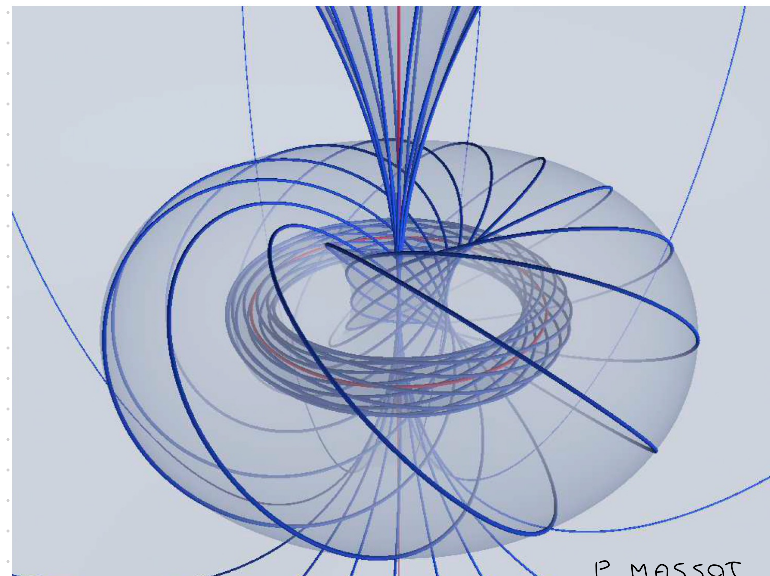
EXAMPLE / TOY MODEL. HOPF FLOW

On $\left(\mathbb{R}^4, \sum_{i=1,2} dx_i \wedge dy_i\right)$ consider Hamiltonian flow $(\phi_H^t)_{t\in\mathbb{R}}$ associated to $H : \mathbb{R}^4 \to \mathbb{R}$, $(x_1, x_2, y_1, y_2) \longmapsto x_1^2 + x_2^2 + y_1^2 + y_2^2$

$\phi_H^t|_{S^3}$ is HOPF FLOW

Every orbit is periodic of period π,

Every couple of orbits has linking number $= +1$



RIGHT-HANDED FLOW (Ghys, 2009)

Let X be a non-singular vector field on S^3 and let ϕ_t be the associated flow

Let $\mathcal{P} = \{ \phi_t\text{-invariant Borel probability measures} \}$ and

$$R = \{ (p,q) \in S^3 \times S^3 \mid p, q \text{ recurrent points } + \ \phi_{\mathbb{R}}(p) \cap \phi_{\mathbb{R}}(q) = \emptyset \}.$$

Let $\mu, \nu \in \mathcal{P}$ be <u>ergodic</u>. We distinguish 2 cases:

A) $\mu \times \nu (R) = 0$ and $\mathrm{SUPP}(\mu) \cup \mathrm{SUPP}(\nu) \subset \gamma$ periodic orbit;

B) $\mu \times \nu (R) = 1$.

CASE A. The measure μ, ν are POSITIVELY LINKED if rotation number of γ (in a Seifert frame) > 0

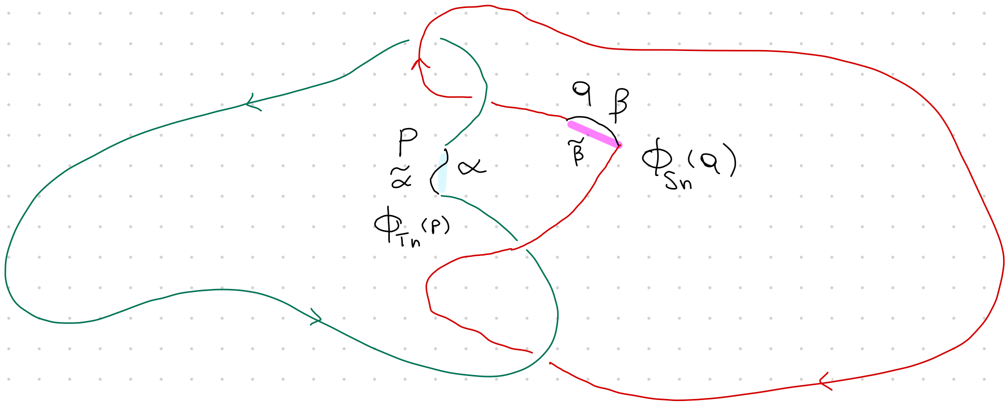
$$\left(D\phi_t(p) u \right)_{t \in [0,T]}$$

CASE B.

$(p,q) \in R$, $(T_n)_n, (S_n)_n$ $\quad T_n, S_n \to +\infty$ as $n \to +\infty$

$\phi_{T_n}(p) \to p, \ \phi_{S_n}(q) \to q$

Consider the following loops $\ell(T_n, p), \ell(S_n, q)$



α, β geodesic paths

$\tilde{\alpha}, \tilde{\beta}$ C^1 perturbations of α, β

$$\ell(T_n, p) = \phi_{[0,T_n]}(p) + \tilde{\alpha} \quad \text{and} \quad \ell(S_n, q) = \phi_{[0,S_n]}(q) + \tilde{\beta}$$

$$L(p,q) := \boxed{\inf_{(T_n)_n, (S_n)_n} \ \liminf_{n \to +\infty} \frac{\text{linking}(\ell(T_n,p), \ell(S_n,q))}{T_n S_n}}$$

The measures μ, ν are POSITIVELY LINKED if for $\mu\times\nu$- a.e. $(p,q) \in R$
$L(p,q) > 0$.

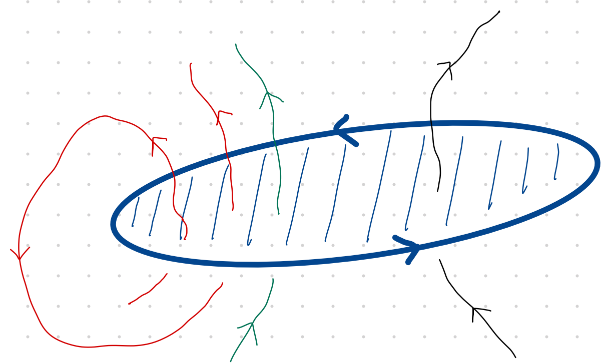
RMK. Ghys's definition is much more elegant. he introduces quadratic linking form on $\mathcal{P}\times\mathcal{P}$; nevertheless, the definitions are equivalent

DYNAMICAL INTEREST of RIGHT-HANDED FLOWS

THM (Ghys, 2009)

Let $(\phi_t)_{t\in\mathbb{R}}$ be a right-handed flow on S^3. Then **every** finite collection of periodic orbits is the boundary of a **Birkhoff section**, i.e. a compact embedded surface Σ s.t.

$\operatorname{int}\Sigma \pitchfork X$ and every orbit in $S^3 \setminus \partial\Sigma$ meets infinitely many times $\operatorname{int}\Sigma$

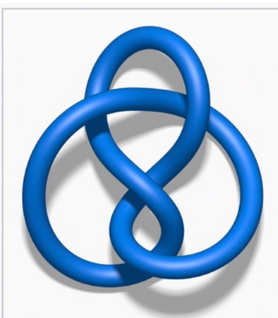


In particular...

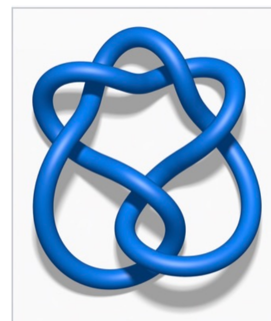
Tools for questions about entropy & cooling ;

Qualitative result : restrictions on how periodic orbits look like
(Knot types of open book decomposition)

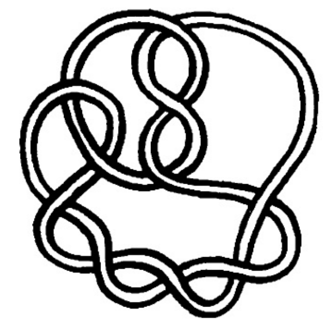
Figure-eight Knot YES



Stevedore Knot NO



10_{20}



EXAMPLES of RIGHT-HANDED FLOWS

Hopf flow, lift of geodesic flow on S^2 with round Riemannian metric, some configurations of 3B PRC problem ...

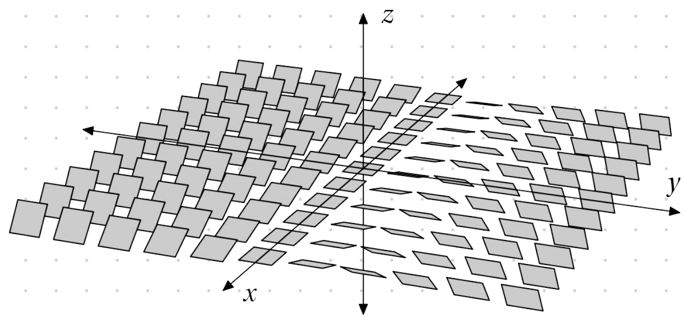
RMK. Generally speaking, right-handedness is difficult to check!

P. Dehornoy has results for geodesic flows on hyperbolic orbifolds.

REEB DYNAMICS & DYNAMICALLY CONVEX REEB FLOWS

A <u>CONTACT MANIFOLD</u> is (M^{2n+1}, ξ) where M is a $(2n+1)$-smooth mnfd and ξ is a coorientable field of hyperplanes. In particular, there exists a 1-form α s.t $\alpha \wedge (d\alpha)^n$ is a volume form and $\xi = \mathrm{Ker}(\alpha)$

EXAMPLE.



$(\mathbb{R}^3, \mathrm{Ker}(-y\,dx + dz))$

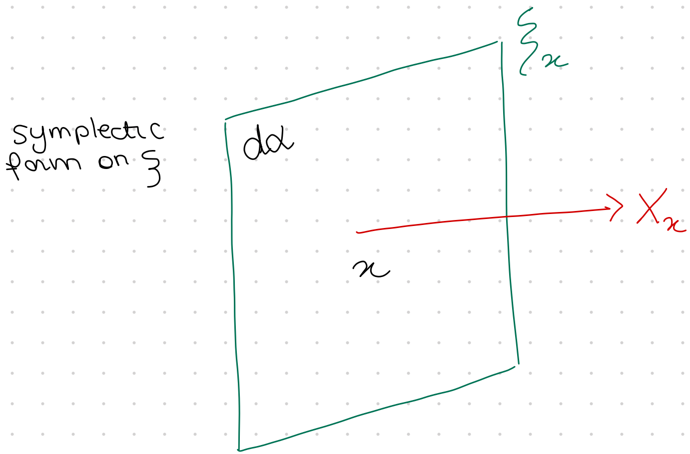
$\alpha = -y\,dx + dz$

$\xi = \langle \partial_y, \partial_x + y\partial_z \rangle$

Let X be the **Reeb vector field** associated to α, i.e.

$$i_X d\alpha = 0 \quad \text{and} \quad i_X \alpha = 1,$$

let $(\phi_t)_{t\in\mathbb{R}}$ be the **Reeb flow**, i.e. the flow generated by X.

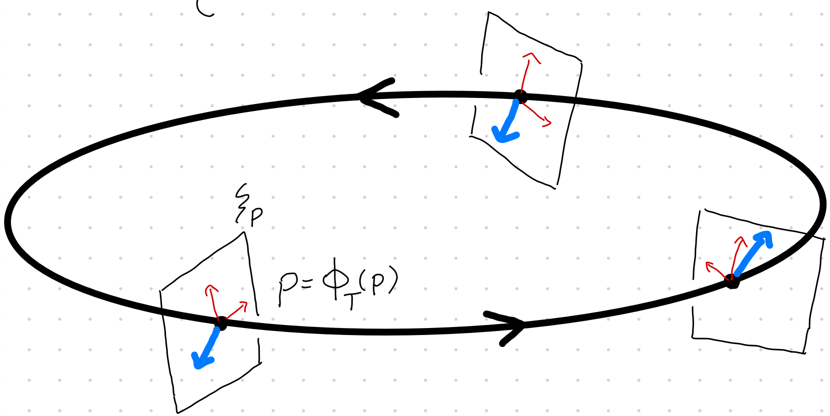


DEF (Hofer-Wysocki-Zehnder)

A Reeb flow on a homology 3-sphere (S^3) is **dynamically convex**

if <u>every periodic orbit has CONLEY-ZEHNDER index ≥ 3</u>

$$\Psi : \xi \to S^3 \times \mathbb{R}^2 \quad \text{global symplectic trivialization}$$



Once we fix Ψ, consider $v \in \xi_p$ and its orbit $\left(D\phi_t(p)v \right)_{t\in[0,T]}$

Consider the angular coordinate $t \longmapsto \Theta(t)$ of $D\phi_t(p) v$ with respect to the global trivialization Ψ

CONLEY-ZEHNDER INDEX : $2 \lfloor \Delta_{[0,T]} \Theta \rfloor + 1$

$\uparrow$

variation of a lift of Θ between 0 and T

RMK. on S^3 all global symplectic trivializations are homotopic so Conley-Zehnder index does not depend on Ψ

EXAMPLE of DYNAMICALLY CONVEX REEB FLOWS.

On $(\mathbb{R}^4, \sum_{i=1}^{2} dx_i \wedge dy_i)$ let $H: \mathbb{R}^4 \to \mathbb{R}$ be s.t. $D^2 H \geq \varepsilon\, \mathrm{Id}$ for some $\varepsilon > 0$.

Then

$$\left(\phi_H^t \big|_{H^{-1}(1)} \right)_{t \in \mathbb{R}}$$

($H^{-1}(1)$ compact energy level)

is a DYNAMICALLY CONVEX REEB FLOW

with contact form $\alpha = \frac{1}{2} \left(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \right)$

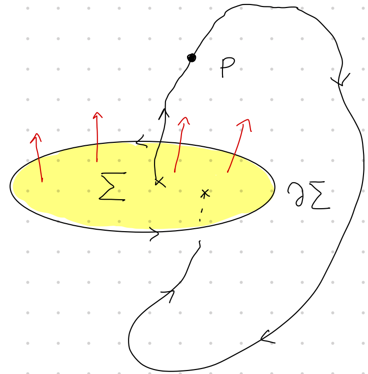
WHY DYNAMICALLY CONVEX REEB FLOWS ARE INTERESTING?

THM (HOFER-WYSOCKI-ZEHNDER, '98)

$\boxed{\text{EVERY DYNAMICALLY CONVEX REEB FLOW HAS A DISK-LIKE BIRKHOFF SECTION.}}$

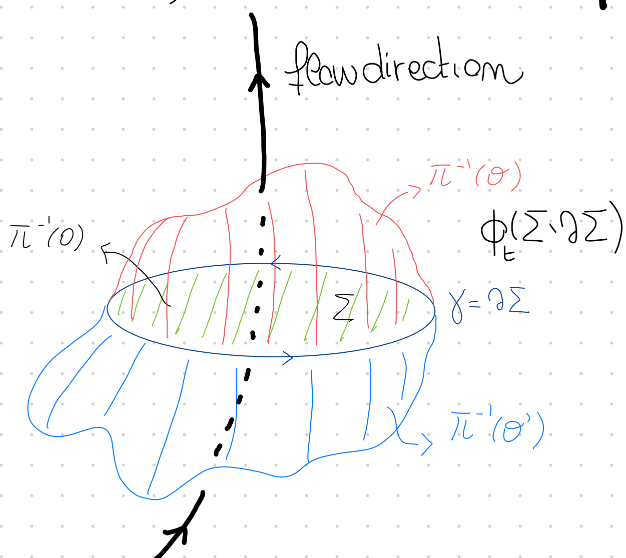
... that is: $\exists$ embedded disk Σ such that

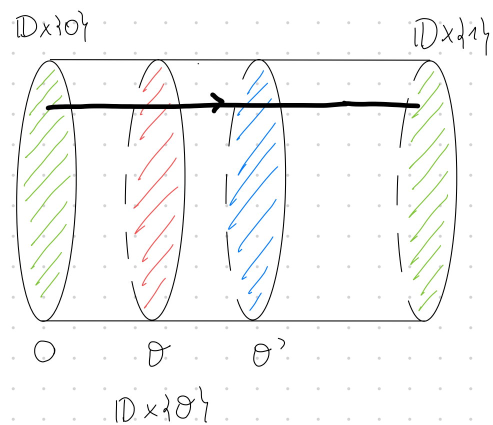
1) $\partial\Sigma$ is a periodic orbit
2) $\mathrm{int}\,\Sigma \pitchfork X$
3) every orbit in $S^3 \setminus \partial\Sigma$ meets $\mathrm{int}\,\Sigma$ infinitely many times



disk-like Birkhoff section

In particular, we have an **open book decomposition**:





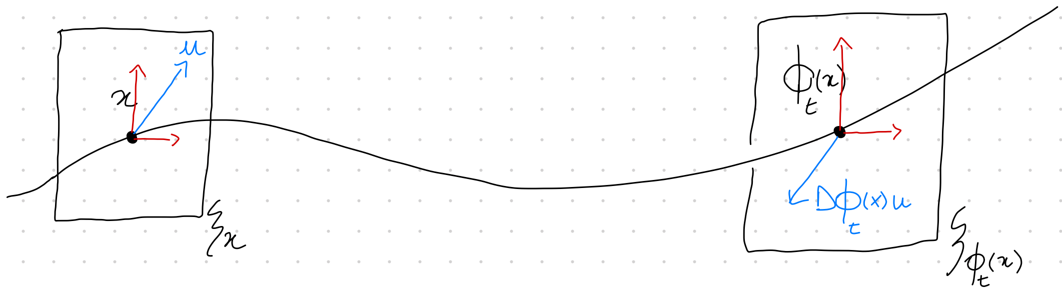
QUANTITATIVE CRITERION FOR RIGHT-HANDEDNESS

Let γ_0 be an unknotted periodic orbit with self-linking number $= -1$.
Then, by HRYNIEWICZ (2014) it bounds a disk-like Birkhoff section Σ.

$$\boxed{\kappa(\gamma_0) := \liminf_{T \to +\infty} \inf_{(x,u) \in \xi} \frac{\Theta(T,x,u) - \Theta(0,x,u)}{\operatorname{linking}(K(T,x,\Sigma), \gamma_0)}}$$

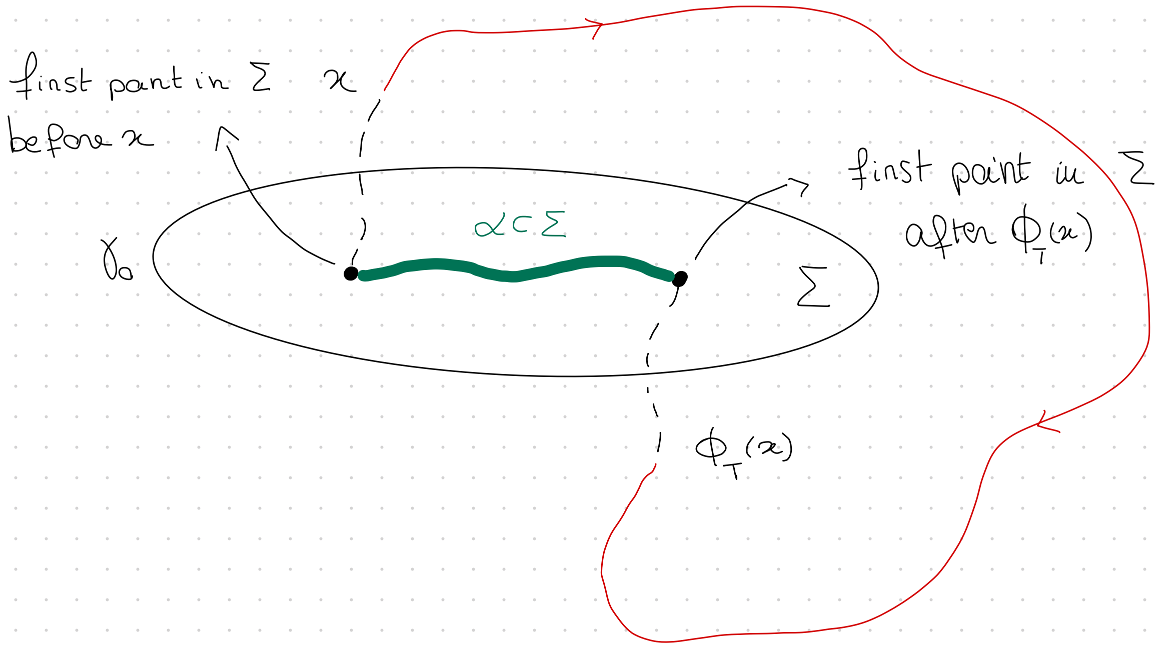
where ...

for $x \in S^3$, $u \in \xi_x \setminus 0$, consider the ANGULAR COORDINATE $\Theta(t, x, u)$ of $D\phi_t(x)u \in \xi_{\phi_t(x)}$ (wrt global symplectic trivialization)



$\Theta(T, x, u) - \Theta(0, x, u)$ is the variation (between 0 and T) of the lift of $t \mapsto \Theta(t, x, u)$

- For $x \in S^3$ we consider linking$(K(T, x; \Sigma), \gamma_\alpha)$ where $K(T, x, \Sigma)$ is a special loop containing $\phi_{[0,T]}(x)$:



The loop $K(T, x; \Sigma)$

RMK: the quantity $K(\gamma_0)$ do not depend on Σ nor on trivialization

THM A (F-HRYNIEWICZ)

Let $(\phi_t)_{t\in\mathbb{R}}$ be a dynamically convex Reeb flow on S^3.

Let γ_0 be an unknotted periodic orbit with self-linking number -1.

If $\quad \kappa(\gamma_0) > 2\pi$

then $\quad (\phi_t)_{t\in\mathbb{R}}$ is **right-handed**

COROLLARY (STRICTLY CONVEX HAMILTONIANS)

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a STRICTLY CONVEX HAMILTONIAN, $D^2 H \geq \varepsilon \, \mathrm{Id}$.

Consider $\left(\phi_t^H|_{H^{-1}(1)}\right)_{t\in\mathbb{R}}$ and Σ corresponding disk-like Birkhoff section

If

$$\varepsilon \, \tau_{\inf}(\Sigma) > \pi$$

where $\tau_{\inf}(\Sigma) = \inf\{\tau(x)$ returning time of x on $\Sigma\}$

then

$$\left(\phi_t^H|_{H^{-1}(1)}\right)_{t\in\mathbb{R}} \text{ is right-handed}$$

Almost direct from previous abstract criterion + Lemma of Grotta-Ragazzo & SaLomão

THM B (Geodesic flows on S^2)

Let g be a Riemannian metric on S^2. If

$$\delta = \frac{\min K}{\max K} > 0.712$$

where K is the Gaussian curvature, then the geodesic flow on S^2 lifts to a right-handed flow on S^3

RMK. There exist Riemannian metrics on S^2 such that the geodesic flow does not lift to a right-handed one (P. Dehornoy, private communication).

Q. What is the optimal pinching constant δ ?

SOME IDEAS of the proof of THM B

Let g be a Riemannian metric on S^2 that is **δ-PINCHED**:

$$\underline{\min K \geq \delta \max K}$$

(for simplicity, assume $\max K = 1$)

- Consider the geodesic flow associated to (S^2, g): $(\phi_t)_{t\in\mathbb{R}}$ is defined on $T^1_g S^2 = \{ v \mid g(v,v) = 1 \}$.

  $\pi : TS \to S$

  1-form: $\alpha_v \cdot z := g(v, d\pi\, z)$ $\qquad v \in TS,\ z \in T_v TS$

  α induces on $T^1_g S$ a CONTACT FORM

  geodesic flow is the corresponding Reeb flow

We can double lift $(\phi_t)_{t\in\mathbb{R}}$ to a flow $(\hat{\phi}_t)_{t\in\mathbb{R}}$ in S^3

$$D_g : S^3 \longrightarrow T_g^1 S^2$$

and

$$D_g^* \alpha = f_g \lambda_0 \qquad \text{where } \lambda_0 = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)\Big|_{S^3}$$

$f_g : S^3 \to \mathbb{R} \setminus \{0\}$ smooth function

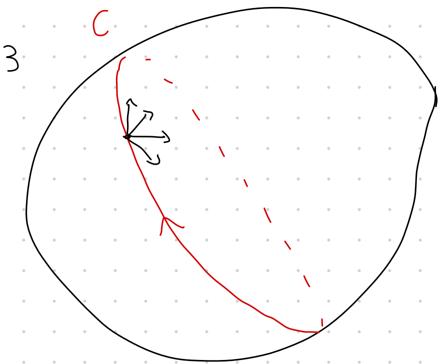
THM (Harris-Paternain, 2008)

If (S^3, g) is δ-pinched with $\delta > 1/4$, then the Reeb flow of $D_g^* \alpha$ is dynamically convex

$c : \mathbb{R}/L\mathbb{Z} \to S^2$ unit speed embedding (L = length of c)

γ_c is double cover of $s \mapsto (c(s), \dot{c}(s))$ in S^3

$\hat{\gamma}_c$ is double cover of $s \mapsto (c(-s), -\dot{c}(-s))$ in S^3

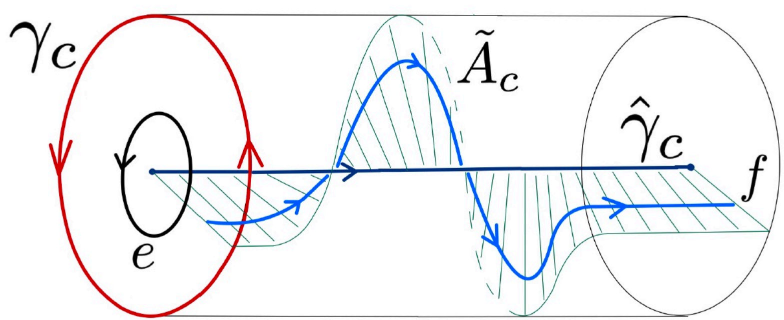


Birkhoff annulus A_c

$(s,\theta) \longmapsto (c(s), \cos\theta\, \dot{c}(s) + \sin\theta\, \dot{c}(s)^{\perp})$

$\tilde{A}_c$ is the lift of A_c on S^3

In S^3 blow up γ_c:



We want to verify the abstract quantitative criterion:

$$\liminf_{T\to+\infty} \inf_{x,u} \frac{\Delta \Theta(T;x,u)}{\text{linking}(K(T,x;\Sigma),\hat{\gamma}_c)} > 2\pi \ ?$$

$$\frac{\Delta \Theta(T;x,u)}{\text{linking}(K(T,x;\Sigma),\hat{\gamma}_c)} = \frac{\Delta \Theta(T;x,u)}{\text{int}(K(T,x,\Sigma),\tilde{A}_c)} \cdot \frac{\text{int}(K(T,x,\Sigma),\tilde{A}_c)}{\text{linking}(K(T,x;\Sigma),\hat{\gamma}_c)}$$

$$\Delta \Theta(T;x,u) \geq T \cdot \dot{\Theta} \underset{\substack{\text{HRYNIEWICZ - SALOMÃO} \\ 2013}}{\geq} T \cdot \min K = T\delta$$

$$\text{int}(K(T,x,\Sigma),\tilde{A}_c) \lesssim T/\tau_{min}$$

τ_{MIN} = infimum of return time function to $\tilde{A}_c$

So

$$\boxed{\frac{\Delta \Theta(T; x, u)}{\mathrm{int}\left(K(T,x,\Sigma), \widetilde{A}_c\right)} \gtrsim \delta \, \tau_{min}}$$

$$\frac{\mathrm{int}\left(K(T,x,\Sigma), \widetilde{A}_c\right)}{\mathrm{linking}\left(K(T,x;\Sigma), \hat{\gamma}_c\right)} = \frac{M(T,x) + N(T,x)}{M(T,x)}$$

since $\partial \widetilde{A}_c = \gamma_c \cup \hat{\gamma}_c$ so

$$\mathrm{int}\left(K(T,x,\Sigma), \widetilde{A}_c\right) = \mathrm{linking}\left(K(T,x;\Sigma), \hat{\gamma}_c\right) + \mathrm{linking}\left(K(T,x;\Sigma), \gamma_c\right)$$

<u>Lemma</u>: $M(T,x) \approx \dfrac{p_1 \circ \Psi^{M+N}(s,\theta) - s}{2L}$ $\qquad \Psi$ lift of first return map on $\widetilde{A}_c$

Tools $\to$ Klingenberg's thm + Corollary of Toponogov's thm

Lemma: If $\delta > 4/9$ $\quad p_1 \circ \Psi(s,\theta) - s \in \left(L + 2\pi\left(1 - \frac{1}{\sqrt{\delta}}\right), L + 2\pi\left(\frac{1}{\sqrt{\delta}} - 1\right) \right)$

So
$$\boxed{\frac{M(T,x) + N(T,x)}{M(T,x)} \gtrsim \frac{2L}{L + 2\pi\left(\frac{1}{\sqrt{\delta}} - 1\right)}}$$

Hence

$$\frac{\Delta \Theta(T,x,u)}{\operatorname{linking}(K(T,x,\Sigma), \hat{\gamma}_c)} \gtrsim \delta T_{min} \frac{2L}{L + 2\pi\left(\frac{1}{\sqrt{\delta}} - 1\right)}$$

Toponogov's thm $\Rightarrow$ $\tau_{min} \geq 2\pi \left(2 - \frac{1}{\sqrt{\delta}} \right)$

Hence

$$\delta \; 2\pi \left(2 - \frac{1}{\sqrt{\delta}} \right) \frac{2L}{L + 2\pi\left(\frac{1}{\sqrt{\delta}} - 1\right)} > 2\pi$$

$$\Updownarrow$$

$$2\delta(2\sqrt{\delta} - 1) - 1 > 0 \quad \leftarrow \text{unique real root}$$

$\Rightarrow$ if $\delta > 0.712\ldots$, then the abstract criterion is satisfied

$\Downarrow$

the flow is right-handed !

$\square$

Thank you!

Joyeux anniversaire, Jean-Pierre!