# Conservative surface homeomorphisms with finitely many periodic points

Patrice Le Calvez (IMJ-PRG)

Hamiltonian Dynamical Systems in Honor of Jean-Pierre Marco

June 9, 2021

# Introduction

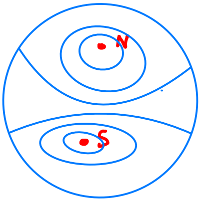
Can we characterize area preserving homeomorphisms of a closed surface that have finitely many periodic points?

Can we use this study to understand what the connected components of the complement of the closure of the periodic point set of an area preserving homeomorphism look like.

Can we extend this study to homeomorphisms with no wandering point?

# Classical examples where $\#\mathrm{per}(f) < \infty$

**On the 2 sphere:**
an irrational rotation around two fixed poles.




**On the 2 torus:**
a rotation

$$(x, y) \mapsto (x + \alpha, y + \beta), \ \ (\alpha, \beta) \notin \mathbb{Q}^2/\mathbb{Z}^2,$$

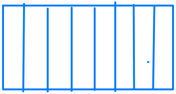
a skew product

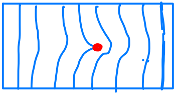
$$(x, y) \mapsto (x + y, y + \beta), \ \ \beta \notin \mathbb{Q}/\mathbb{Z}.$$

**On a surface $S_g$ of genus $\geqslant 2$:**
the time one map of a translation flow in a minimal direction.

# Simple modifications of the classical examples

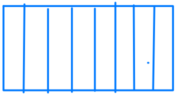
If $f$ is the time one map of a translation flow in a minimal direction on $S_g$ or a totally irrational rotation on $\mathbb{R}^2/\mathbb{Z}^2$, one can add finitely many fixed points without adding other periodic points.

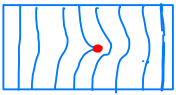




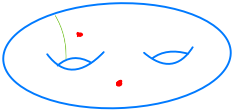
# Simple modifications of the classical examples

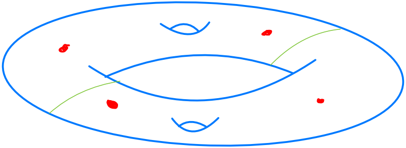
If $f$ is the time one map of a translation flow in a minimal direction on $S_g$ or a totally irrational rotation on $\mathbb{R}^2/\mathbb{Z}^2$, one can add finitely many fixed points without adding other periodic points.





In these two cases, one can consider a lift to a finite cyclic covering space and compose with a covering transformation, replacing fixed points with periodic points of same period.





# Subtle modifications of the classical examples

Using the approximation method by conjugation of Anosov-Katok we can construct pseudo versions of some of the classical examples, like irrational pseudo rotations on $\mathbb{S}^2$ or $\mathbb{R}^2/\mathbb{Z}^2$. On $\mathbb{S}^2$, it means a non wandering homeomorphism with two fixed points $N$ and $S$ such that every other point turns with an irrational angular speed around $N$ and $S$.

One can find area preserving smooth irrational pseudo rotations that are ergodic or weak-mixing (see Fayad-Saprykina for instance).

# Subtle modifications of the classical examples

Using the approximation method by conjugation of Anosov-Katok we can construct pseudo versions of some of the classical examples, like irrational pseudo rotations on $\mathbb{S}^2$ or $\mathbb{R}^2/\mathbb{Z}^2$. On $\mathbb{S}^2$, it means a non wandering homeomorphism with two fixed points $N$ and $S$ such that every other point turns with an irrational angular speed around $N$ and $S$.

One can find area preserving smooth irrational pseudo rotations that are ergodic or weak-mixing (see Fayad-Saprykina for instance).

**Main question:** Is every area preserving homeomorphism with finitely many periodic points related to one of the classical examples?

# What is known when $g \leqslant 1$ ?

**Theorem:** [Franks, L] *If* $f \in \mathrm{Homeo}_+(\mathbb{S}^2)$ *is non wandering and* $\#\mathrm{per}(f) < \infty$, *then* $f$ *is an irrational pseudo rotation.*

# What is known when $g \leqslant 1$ ?

**Theorem:** [Franks, L] *If $f \in \mathrm{Homeo}_+(\mathbb{S}^2)$ is non wandering and $\#\mathrm{per}(f) < \infty$, then $f$ is an irrational pseudo rotation.*

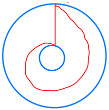
**Theorem:** [. . . , Addas-Zanata-Tal] *If $f \in \mathrm{Homeo}_+(\mathbb{R}^2/\mathbb{Z}^2)$ is non wandering and $\#\mathrm{per}(f) < \infty$, then:*

- *either $f$ is isotopic to $\mathrm{Id}$ and its* rotation set *is*
  - *a point $(\alpha, \beta) \notin \mathbb{Q}^2$,*
  - *a segment that does not meet $\mathbb{Q}^2$,*
  - *or a segment that meets $\mathbb{Q}^2$ at one endpoint;*
- *or (up to conjugacy) $f$ is isotopic to $(x, y) \mapsto (x + ky, y)$, $k \neq 0$, with a* vertical rotation set *reduced to $\beta \notin \mathbb{Q}$.*

# What happens when $g \geqslant 2$ ? Dehn twist maps

The Dehn twist is the homeomorphism $\tau$ of $\mathbb{R}/\mathbb{Z} \times [0,1]$ lifted by

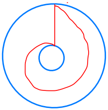
$$\tilde{\tau} : (x, y) \mapsto (x + y, y).$$



# What happens when $g \geqslant 2$ ? Dehn twist maps

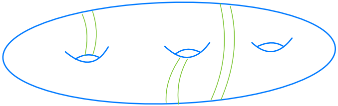
The Dehn twist is the homeomorphism $\tau$ of $\mathbb{R}/\mathbb{Z} \times [0,1]$ lifted by

$$\tilde{\tau} : (x, y) \mapsto (x + y, y).$$



A Dehn twist map is a homeomorphism $h$ of $S_g$, $g \geqslant 2$, satisfying:

- there exists a non trivial family $(A_i)_{i\in I}$ of disjoint and not homotopic essential closed annuli;
- $h(A_i) = A_i$ and $h|_{A_i}$ is conjugate to $\tau^{k_i}$, $k_i \in \mathbb{Z} \setminus \{0\}$;
- $h = \mathrm{Id}$ on $S_g \setminus \bigcup_{i\in I} A_i$.



# What happens when $g \geqslant 2$ ? The main theorem

**Theorem:** [L] *Suppose that $f \in \mathrm{Homeo}_+(S_g)$, $g \geqslant 2$, is non wandering and that $\#\mathrm{per}(f) < \infty$. Then:*

- $\mathrm{per}(f) \neq \emptyset$;
- $\exists q \geqslant 1$ *such that* $f^q$ *is isotopic to* $\mathrm{Id}$ *relative to* $\mathrm{fix}(f^q)$.

# What happens when $g \geqslant 2$ ? The main theorem

**Theorem:** [L] *Suppose that $f \in \mathrm{Homeo}_+(S_g)$, $g \geqslant 2$, is non wandering and that $\#\mathrm{per}(f) < \infty$. Then:*

- $\mathrm{per}(f) \neq \emptyset$;
- $\exists q \geqslant 1$ *such that $f^q$ is isotopic to $\mathrm{Id}$ relative to $\mathrm{fix}(f^q)$.*

**Remark**: Telling that $\exists q \geqslant 1$ such that $f^q \sim \mathrm{Id}$ means that there are no pseudo-Anosov components in the Nielsen-Thurston decomposition, but also no Dehn twist.

**Theorem:** [L] *Suppose that $f \in \mathrm{Homeo}_+(S_g)$, $g \geqslant 2$, is non wandering and that $\#\mathrm{per}(f) < \infty$. Then:*

- $\mathrm{per}(f) \neq \emptyset$;
- $\exists q \geqslant 1$ *such that* $f^q$ *is isotopic to* $\mathrm{Id}$ *relative to* $\mathrm{fix}(f^q)$.

**Remark**: Telling that $\exists q \geqslant 1$ such that $f^q \sim \mathrm{Id}$ means that there are no pseudo-Anosov components in the Nielsen-Thurston decomposition, but also no Dehn twist.

$$\left[\Omega(f) = S_g\right] + \left[f \sim \text{Dehn twist map}\right] \Longrightarrow \left[\#\mathrm{per}(f) = \infty\right].$$

$$\left[\Omega(f) = S_g\right] + \left[f \sim \mathrm{Id}\right] + \left[\#\mathrm{per}(f) < \infty\right] \Longrightarrow \underbrace{\left[\mathrm{fix}(f) \text{ is unlinked}\right]}_{\substack{\| \\ \mathrm{per}(f)}}.$$

# The theorem in the generic case

Denote $\mathrm{Diff}^r_\omega(S_g)$ the space of $\omega$-preserving diffeomorphisms of class $C^r$, where $\omega$ is a smooth area form.

**Theorem:** [L-Sambarino] *There exists an "explicit" residual set $\mathcal{G} \subset \mathrm{Diff}^r_\omega(S)$ (for the $C^r$ topology) such that if $f \in \mathcal{G}$ is isotopic to a Dehn twist map, there exist hyperbolic fixed points and these points have homoclinic intersections.*

# The theorem in the generic case

Denote $\mathrm{Diff}^r_\omega(S_g)$ the space of $\omega$-preserving diffeomorphisms of class $C^r$, where $\omega$ is a smooth area form.

**Theorem:** [L-Sambarino] *There exists an "explicit" residual set $\mathcal{G} \subset \mathrm{Diff}^r_\omega(S)$ (for the $C^r$ topology) such that if $f \in \mathcal{G}$ is isotopic to a Dehn twist map, there exist hyperbolic fixed points and these points have homoclinic intersections.*

Outline of the proof:

(1) Lefschetz formula $\Longrightarrow \#\mathrm{fix}_h(f) \geqslant 2g-2$;

(2) stable and unstable branches are dense if $\nexists$ homoclinic intersection (due to results of Mather and Koropecki-L-Nassiri);

(3) this is not compatible with the twist condition.

# Outline of the proof of the general theorem

To prove

$$\Big[\Omega(f) = S_g\Big] + \Big[f \sim \text{Dehn twist map}\Big] \Longrightarrow \Big[\#\text{per}(f) = \infty\Big].$$

we first use results proven in the generic case, then replace arguments involving stable and unstable branches, with extensions of results of [L-Tal] stated for homeomorphisms isotopic to $\mathrm{Id}$:

- a fixed point theorem;
- a forcing lemma on free lines.

# Outline of the proof of the general theorem

To prove

$$\Big[\Omega(f) = S_g\Big] + \Big[f \sim \text{Dehn twist map}\Big] \Longrightarrow \Big[\#\text{per}(f) = \infty\Big].$$

we first use results proven in the generic case, then replace arguments involving stable and unstable branches, with extensions of results of [L-Tal] stated for homeomorphisms isotopic to $\mathrm{Id}$:

- a fixed point theorem;
- a forcing lemma on free lines.

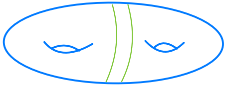
Curiously, the proof of

$$\Big[\Omega(f) = S_g\Big] + \Big[f \sim \mathrm{Id}\Big] + \Big[\#\text{per}(f) < \infty\Big] \Longrightarrow \Big[\text{fix}(f) \text{ is unlinked}\Big].$$

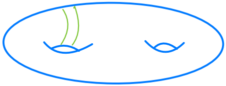
is very similar

# The simplest case: $g = 2$, $I = \{i\}$, $k_i = 1$

We suppose that $g = 2$ and that $f \sim h$, where $h$ is a Dehn twist map with a unique twisted annulus $A$ and with a simple twist.



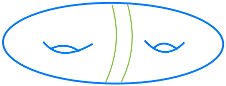
$A$ is separating



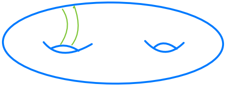
$A$ is not separating

## The simplest case: $g = 2$, $I = \{i\}$, $k_i = 1$

We suppose that $g = 2$ and that $f \sim h$, where $h$ is a Dehn twist map with a unique twisted annulus $A$ and with a simple twist.



$A$ is separating



$A$ is not separating

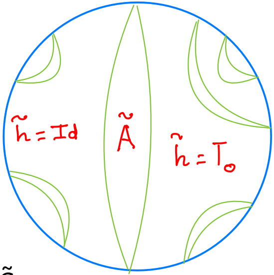
$\tilde\pi : \tilde S \to S_g$ is the universal covering

$\tilde A$ is a lift of $A$

$T \in \mathrm{Aut}(\tilde\pi)$ stabilizes $\tilde A$

$\tilde h$ is the "simplest" lift of $h$ fixing $\tilde A$

$\tilde f$ is the associated lift of $f$

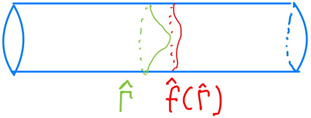



**Remark:** $\tilde f$ and $\tilde h$ extend to $\partial\tilde S$ and $\tilde f = \tilde h$ on $\partial\tilde S$.

# The annular lift. Application to the separating case

$\tilde{f}$ commutes with $T$ and lifts a map $\hat{f}$ of $\hat{S} = \tilde{S}/T$ that satisfies a boundary twist condition. So, by Poincaré-Birkhoff Theorem:

1. *either $\hat{f}$ (and $f$) have infinitely many periodic points;*
2. *or there exists an essential loop $\hat{\Gamma} \subset \hat{S}$, such that $\hat{f}(\hat{\Gamma}) \cap \hat{\Gamma} = \emptyset$.*



# The annular lift. Application to the separating case

$\tilde{f}$ commutes with $T$ and lifts a map $\hat{f}$ of $\hat{S} = \tilde{S}/T$ that satisfies a boundary twist condition. So, by Poincaré-Birkhoff Theorem:

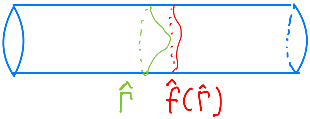
1. *either $\hat{f}$ (and $f$) have infinitely many periodic points;*
2. *or there exists an essential loop $\hat{\Gamma} \subset \hat{S}$, such that $\hat{f}(\hat{\Gamma}) \cap \hat{\Gamma} = \emptyset$.*

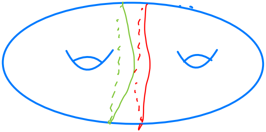
**Theorem:** [L-Sambarino] *(2) never occurs if A is separating.*

*Proof:* Easy in the two following situations:

$\hat{\Gamma} \cup \hat{f}(\hat{\Gamma})$ is included in an annulus projecting injectively in $S_g$;

$f$ is a $C^1$ diffeomorphism that preserves an area form. □

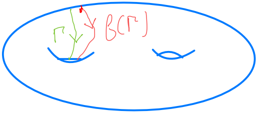


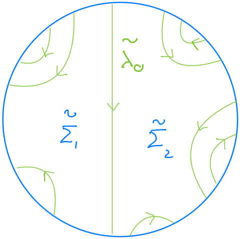



# The non separating case (I)

Taking a finite covering if necessary, one can suppose that $\hat{\Gamma}$ projects onto a simple loop $\Gamma \subset S_g$ such that $f(\Gamma) \cap \Gamma = \emptyset$.

Denote $\tilde{\lambda}_0 \subset \tilde{S}$ the lift of $\Gamma$ that is invariant by $T$ and $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$, the components of $\tilde{\pi}^{-1}(S \setminus \Gamma)$ that are bounded by $\tilde{\lambda}_0$.




It is sufficient to prove the following:

**Proposition:** *There exists a sequence* $(T_m)_{m \geqslant m_0}$ *in* $\mathrm{Aut}(\tilde{\pi})$ *s. t.:*

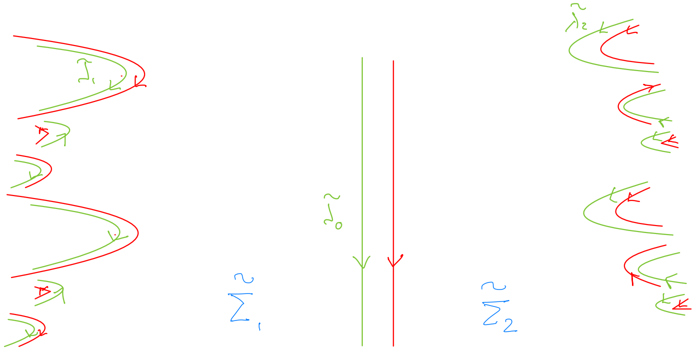
- $T_m$ *sends* $\tilde{\Sigma}_1$ *onto* $\tilde{\Sigma}_2$*;*
- $\tilde{f}^m \circ T_m^{-1}$ *has a fixed point.*

# The non separating case (II)

**Lemma:** *There exist* $\tilde\lambda_1 \in \tilde\Lambda_{1,l}$, $\tilde\lambda_2 \in \tilde\Lambda_{2,r}$ *and* $n_0 \geqslant 0$ *such that*

$$0 < p < q \Rightarrow \tilde f^{n_0+q}(\tilde\lambda_1) \cap \tilde T^p(\tilde\lambda_2) \neq \emptyset,$$

*where* $\tilde\Lambda_{i,l}$ *(resp.* $\tilde\Lambda_{i,r}$*) is the set of lifts of* $\Gamma$ *bounding* $\tilde\Sigma_i$ *on its left (resp. right).*



# The non separating case (III)

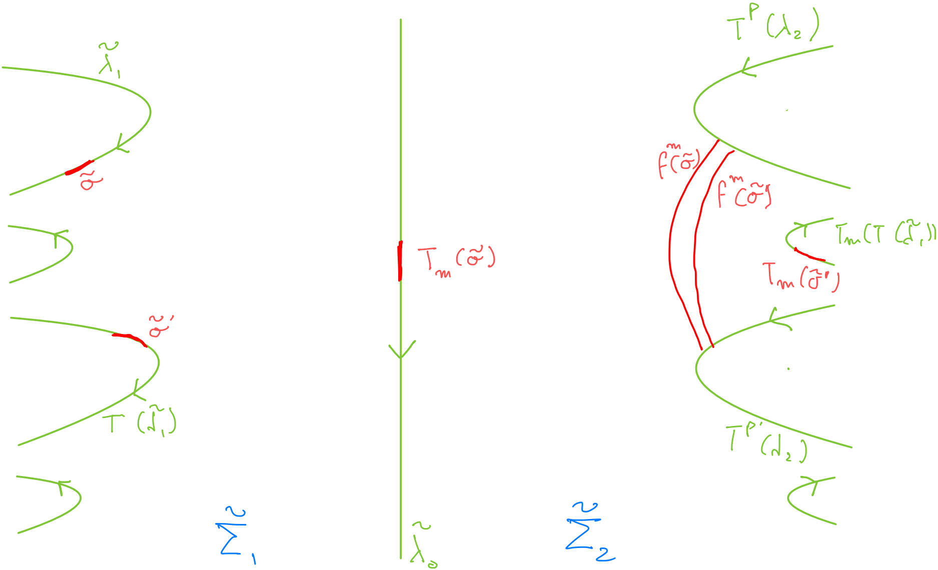
.

**Lemma:** *If $m \geqslant 0$ is large enough, there exist*

- *integers $p < p'$;*
- *segments $\tilde{\sigma} \subset \tilde{\lambda}_1$ and $\tilde{\sigma}' \subset T(\tilde{\lambda}_1)$ such that $\tilde{f}^m(\tilde{\sigma})$ and $\tilde{f}^m(\tilde{\sigma}')$*
  - *join $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$,*
  - *are on the right of $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$,*
  - *are on the left of $\tilde{\lambda}_0$;*
- *an automorphism $T_m$ such that:*
  - *$T_m(\tilde{\lambda}_1) = \tilde{\lambda}_0$,*
  - *$T_m(T(\tilde{\lambda}_1))$ is between $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$,*
  - *$T_m(\tilde{\lambda}_0)$ is different from $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$.*

By an index argument, the lemma implies that $\tilde{g} = \tilde{f}^m \circ T_m^{-1}$ has a fixed point.

# The case non homological to zero (IV)



# The case non homological to zero (V)

.

The reasons why $\tilde{g}$ has a fixed point are the following:

- $T_m(T(\tilde{\lambda}_1))$, $T^p(\tilde{\lambda}_2)$, $\tilde{\lambda}_0$, $T^{p'}(\tilde{\lambda}_2)$ are cyclically ordered;
- $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$ are free lines of $\tilde{g}$;
- $T_m(\tilde{\sigma}) \subset \tilde{\lambda}_0$ and $\tilde{g}(T_m(\tilde{\sigma})) \cap \tilde{\lambda}_0 = \emptyset$;
- $T_m(\tilde{\sigma}') \subset T_m(T(\tilde{\lambda}_1))$ and $\tilde{g}(T_m(\tilde{\sigma}')) \cap T_m(T(\tilde{\lambda}_1)) = \emptyset$;
- $\tilde{g}(T_m(\tilde{\sigma}))$ and $\tilde{g}(T_m(\sigma'))$ join $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$.

# The case non homological to zero (V)

.

The reasons why $\tilde{g}$ has a fixed point are the following:

- $T_m(T(\tilde{\lambda}_1))$, $T^p(\tilde{\lambda}_2)$, $\tilde{\lambda}_0$, $T^{p'}(\tilde{\lambda}_2)$ are cyclically ordered;
- $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$ are free lines of $\tilde{g}$;
- $T_m(\tilde{\sigma}) \subset \tilde{\lambda}_0$ and $\tilde{g}(T_m(\tilde{\sigma})) \cap \tilde{\lambda}_0 = \emptyset$;
- $T_m(\tilde{\sigma}') \subset T_m(T(\tilde{\lambda}_1))$ and $\tilde{g}(T_m(\tilde{\sigma}')) \cap T_m(T(\tilde{\lambda}_1)) = \emptyset$;
- $\tilde{g}(T_m(\tilde{\sigma}))$ and $\tilde{g}(T_m(\sigma'))$ join $T^p(\tilde{\lambda}_2)$ and $T^{p'}(\tilde{\lambda}_2)$.