Weak Sobolev almost-periodic solutions for the NLS on the circle

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based on joint work with L. Biasco and M. Procesi

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Jean-Pierre's birthday in the ether 09-06-2021

Families of NLS

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} - V * \mathbf{u} + F(|\mathbf{u}|^2)\mathbf{u} = 0\,, \qquad \mathbf{u}(t,x) = \mathbf{u}(t, x+2\pi)\,,$$

- $F(y)$ is real analytic in y in a neighborhood of $y = 0$ with $f(0) = 0$
- $V* : \ell^1 \to \ell^1$ is a Fourier multiplier

$$(V * u)(x) = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}\,, \quad V = (V_j)_{j\in\mathbb{Z}} \in [-1/4, 1/4]^{\mathbb{Z}} \subset \ell^\infty(\mathbb{R})\,.$$

(substet of $\ell^1(\mathbb{C})$ $\leftrightsquigarrow$ $u(x) = \sum_j u_j e^{ijx}$ 2π-periodic x-continuous)

More precisely $(V * \mathbf{u})(t,x) := (V * \mathbf{u}(t,\cdot))(x)$ for every $t \in \mathbb{R}$.

Result: *For almost every* $V \in [-1/4, 1/4]^{\mathbb{Z}}$ *there exist infinitely many small-amplitude weak almost-periodic solutions* $\mathbf{u}$.

Definition (weak solutions)

A function $\mathbf{u} : \mathbb{R}^2 \to \mathbb{C}$ which is 2π-periodic in x and such that the map $t \mapsto \mathbf{u}(t, \cdot) \in \ell^1$ is continuous is a weak solution of NLS_V if for any smooth compactly supported function $\chi : \mathbb{R}^2 \to \mathbb{R}$ one has

$$\int_{\mathbb{R}^2} (-\mathrm{i}\chi_t + \chi_{xx})\mathbf{u} - (V * \mathbf{u} + F(|\mathbf{u}|^2)\mathbf{u})\chi \, dx \, dt = 0 \, .$$

Our target is to prove existence of solutions with very little regularity. We construct infinetely many different solutions s.t.

$$|u_j(t)| \sim \langle j \rangle^{-p}, \quad p > 1$$

for infinitely many j.

Result: *For almost every $V \in [-1/4, 1/4]^{\mathbb{Z}}$ there exist infinitely many small-amplitude weak almost-periodic solutions* $\mathbf{u}$.

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Almost-periodic: limit in the uniform topology of time quasi-periodic solutions, as $d \to \infty$

Given a vector $\omega \in \mathbb{R}^d$ of rationally independent frequencies $\omega_1, \ldots, \omega_d$ we say that $u(t,x)$ is a quasi-periodic function of frequency ω if there exists an embedding of a d-torus in the phase space:

$$\mathbb{T}^d \to \mathcal{P}, \quad \theta \mapsto U(\theta, x)$$

such that

$$u(t,x) = U(\omega t, x).$$

The linear Schrödinger

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} - V * \mathbf{u} = 0$$

Passing to the Fourier side

$$\mathbf{u} = \sum_{j\in\mathbb{Z}} \mathbf{u}_j(t) e^{\mathrm{i}jx}$$

we get

$$\mathbf{u}(t,x) = \sum_{j\in\mathbb{Z}} \mathbf{u}_j(0) e^{\mathrm{i}jx} e^{\mathrm{i}\omega_j t}, \quad \omega_j = (j^2 + V_j)$$

which is uniform limit of smooth quasi-periodic functions (provided we require some minimal decay conditions on $\mathbf{u}_j(0)$.)

Once the nonlinearity is plugged in...
for most choices of V existence of infinitely many almost-periodic solutions with finite (actually very low) regularity both in time and space, under appropriate arithmetic conditions on the ω_j

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around almost-periodic solutions

Instead of looking at the solutions, let us consider its support i.e. the infinite torus $\mathbb{T}^{\mathbb{Z}} = \mathbb{R}^{\mathbb{Z}}/2\pi\mathbb{Z}^{\mathbb{Z}}$

$$\mathbb{T}^{\mathbb{Z}} \to \ell^1(\mathbb{C}) \quad \varphi = (\varphi_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} |\mathbf{u}_j(0)| e^{\mathrm{i}\varphi_j + \mathrm{i}jx}$$

We endow $\mathbb{T}^{\mathbb{N}}$ with a Banach manifold structure, based on ℓ^∞, in the usual manner

$$\mathrm{dist}(\theta, \varphi) = \sup_{j\in\mathbb{Z}} |\theta_j - \varphi_j|_{2\pi}$$

The natural expected solution of the nonlinear problem is of the form:

$$f(\varphi, x) = \sum_{\substack{\ell\in\mathbb{Z}^{\mathbb{Z}}:|\ell|_1<\infty \\ j\in\mathbb{Z}}} \widehat{f}(\ell, j) e^{\mathrm{i}\ell\cdot\varphi + \mathrm{i}jx}$$

We require some decay on the Fourier coefficients

- The regularity in x depends on the $\mathtt{a}(j)$;
- f is analytic in each angle φ_j in the strip $|\mathrm{Im}(\varphi_j)| \le \mathtt{p}_j$.

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$\varphi = (\varphi_1, \varphi_2, \dots)$ are infinitely many angles, in principle the same holds for $\ell = \ell_1, \dots$

but the condition $|\ell|_1 < \infty$ implies that ℓ has finite support!

Hence in each sum $\ell \cdot \varphi$ is a finite sum

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towards finite regularity solutions for NLS

NLS as an infinite dimensional Hamiltonian system:

$$\mathrm{i}u_t + u_{xx} - V * u + F(|u|^2)u = 0 \rightsquigarrow H = \sum_{j\in\mathbb{Z}} \omega_j |u_j|^2 + P(F,u)$$

One wishes to fix some positive sequence $(I_j)_{j\in\mathcal{S}} := (|u_j(0)|^2)_{j\in\mathcal{S}}$ and prove that, up to an analytic symplectic change of variables of the phase space, the torus

$$\mathcal{T}_I = \left\{ u \in \mathcal{P} : |u_j|^2 = I_j,\, j \in \mathcal{S}, \quad |u_j|^2 = 0,\, j \in \mathcal{S}^c \right\},$$

is an **invariant** torus supporting almost-periodic solutions of frequency ω, under Diophantine conditions.

key point: choice of the phase space.
more regularity $\rightsquigarrow$ weaker Diophantine conditions $\rightsquigarrow$ easier result

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- Quasi-periodic solutions have been widely studied (starting from '90), by KAM theory for PDE's (Kuksin-Wayne-Pöschel) and by the Craig-Wayne-Bourgain method (newton like scheme + multiscale analysis)
- While there are many results on quasi-periodic solutions also in the infinite dimensional context, very few is known about the almost-periodic ones:

All on NLS with external parameters & very high (at least Gevrey) regularity both in time and space!

For the quasi-periodic case one can set V=0 and imitate the "finite dimensional KAM" (cf. Kuksin-Poschel 96 for ex, in the NLS):

- Under a non degenerate twist condition on the non-linearity, after one step of BNF
- Itroduce Action-Angle and use some torsion property in the usual manner to modulate the frequencies and linearize the dynamics on the invariant torus

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Problems:

Problem: in these results quasi-periodic solutions with d frequencies have size $\varepsilon_d \to 0$ as $d \to \infty$.

Goal: prove existence and linear stability of quasi-periodic functions with d frequencies with a strategy and smallness condition uniform in the dimension of the torus d!

...at least in the case with external parameters!

Pöschel ('02) considered (Dirichlet b.c.)
$\mathrm{i}u_t + u_{xx} - V(x)u + \text{smooth NL} = 0$
Result: for most choices of $V(x)$ in L^2 one can constuct a sequence of invariant tori converging to an almost periodic solution

$$\mathbb{T}^n \to \mathbb{T}^{n+1} \to \mathbb{T}^{n+2} \to \cdots \to \mathbb{T}^{\mathbb{N}}$$

at each step one introduces action-angle variables to parameterize $\mathbb{T}^n$ and then constructs $\mathbb{T}^{n+1}$

He needs the actions $I_j \to 0$ superexponentially to get the infinite torus! $|u_j| \to 0$ super-exponentially

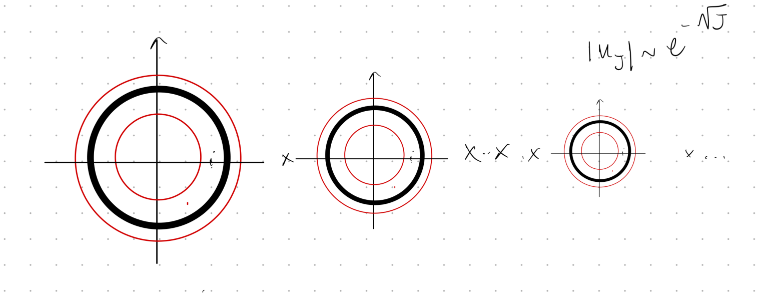
Bourgain('04) studied $\mathrm{i}u_t + u_{xx} - V * u + |u|^4 u = 0$
Result: for most choices of $V \in (-1,1)^{\mathbb{Z}}$ there exists at least one almost-periodic solution

$$|u_j| \sim r e^{-s\sqrt{j}}$$

He proved the persistence of an almost-periodic torus in **one shot**:
no approximate finite dim. tori, no action-angle variables

∞-dim torus as product of circles and requiring lower bounds on all the "actions":

$$\frac{r}{2}e^{-s\sqrt{\langle j\rangle}} < |u_j^{(0)}| < re^{-s\sqrt{\langle j\rangle}}$$



This means that there is a neighborhood $u^{(0)}$ made all of maximal tori uniformly bounded away from the singularities $u_j^{(0)} = 0$ indeed on his approximately invariant tori action-angle variables would be well defined (in ∞ dimension this is not trivial)

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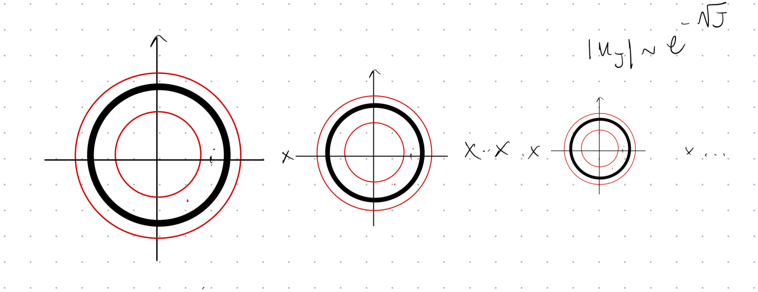

Remark: The set of actions of Bourgain's case has zero measure w.r.t the probability measure on $B_r(\mathfrak{g}_s)$

$$\mathfrak{g}_s := \{u = (u_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{C}) : \quad \sup_{j\in\mathbb{Z}} |u_j| e^{s\sqrt{\langle j\rangle}} < \infty\}$$

Biasco-M.-Procesi(19): inspired by Bourgain's idea
- remove the lower bound and generalize the strategy dealing with any choice of the action and x-dependent non-linearity
- scheme and smallness assumptions uniform in dimension
- Within the same scheme: existence and linear stability for almost-periodic and quasi-periodic Gevrey solutions

Construction of a flexible method based on

- a functional setting of Banach scales with good properties of norms (monotonicity, closeness w.r.t. Poisson brackets etc.)
- decomposition of the problem of persistence of the invariant torus in two steps:

  1) prove a general normal form with counter-terms in order to modulate the frequency (containing the hard analysis!)

  2) *elimination* of the counter-terms using external (or internal) parameters and convenient non-degeneracies assumption, via the implicit function theorem

  (cf. Arnold, Moser, Herman,Rüssmann, Féjoz...)

The result in Gevrey regularity

Following Bourgain's strategy we fix as phase space:

$$\mathtt{g}_s(\mathbb{C}) = \left\{ v \in \ell^\infty(\mathbb{C}) : \quad |v|_s := \sup_{j\in\mathbb{Z}} |v_j| \langle j\rangle^2 e^{s\langle j\rangle^\theta} < \infty \right\},$$

with $s > 0$. We define

$$\mathtt{R} := \left\{ \omega = (\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}, \quad \sup_j |\omega_j - j^2| < 1/2 \right\}. \qquad (1)$$

Isomorphic to $[-1/2, 1/2]^{\mathbb{Z}}$.
We endow $\mathtt{R}$ with the probability measure μ induced by the product measure on $[-1/2, 1/2]^{\mathbb{Z}}$.
We say that $\omega \in \mathtt{R}$ is γ-Diophantine if

$$\omega \in \mathtt{D}_\gamma := \left\{ \omega \in \mathtt{R} \, : \, |\omega \cdot \ell| > \gamma \prod_{n\in\mathbb{Z}} \frac{1}{(1 + |\ell_n|^2 \langle n\rangle^2)} , \quad \forall \ell \in \mathbb{Z}^{\mathbb{Z}} : |\ell| < \infty \right\}.$$

$$\text{NB.} \qquad \omega \cdot \ell = \sum_{n=j_{\min}}^{j_{\max}} \omega_n \ell_n$$

Diophantine frequencies are *typical* in $\mathtt{R}$!

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Diophantine frequencies are *typical* in $\mathtt{R}$!

Nice Hamiltonians are of the form $N = \sum_j \omega_j |u_j|^2 + O((|u|^2 - I)^2)$

$\mathcal{T}_I = \left\{ u \in B_r(\mathtt{g}_s) : |u_j|^2 = I_j \right\}$ is an ω-almost-periodic invariant torus for N!

Theorem ('a la Herman, Biasco, M., Procesi)

Let ω be Diophantine and N^0 be a Hamiltonian possessing an invariant ω-almost-periodic torus. If H is sufficiently close to N_0, then

- $\exists!$ *simplectic diffeomorphism* $\Phi : B_r(\mathtt{g}_s) \to B_{r^0}(\mathtt{g}_s), \quad r < r^0, \, s > s^0$
- $\exists!$ *counter term* $\Lambda = \sum_j \lambda_j(|u_j|^2 - I_j), \quad (\lambda_j) \in \ell_\infty$
- $\exists!$ *Hamiltonian N with an invariant ω-almost-periodic torus*

such that

$$H = N \circ \Phi^{-1} + \Lambda$$

$$(\textit{equiv. } (H - \Lambda) \circ \Phi = N)$$

Rmk: Since the Hamiltonian H depend on $(V_j)_{j \in \mathbb{Z}} \subset \ell_\infty$ and Λ smoothly depend on them, one can solve $\Lambda(V_j, \omega) = 0$ by direct application of an implicit function theorem in a Banach space and get the desired dynamical conjugacy: $H = N \circ \Phi^{-1}$

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Result in Gevrey regularity

Theorem (Biasco, M., Procesi)

For any γ-Diophantine frequency $\omega \in \mathtt{R}$.
For any $\sqrt{I} := u^{(0)} \in \mathtt{g}_s$ sufficiently small (say $|u_j^{(0)}|\langle j\rangle^2 e^{s\langle j\rangle^\theta} \le r \ll \sqrt{\gamma}$)

There exists $V = V(u^{(0)}, \omega) \in \ell_\infty$ and a change of variables $\Phi : \bar{B}_r(\mathtt{g}_s) \to \bar{B}_r(\mathtt{g}_s)$ such that

$$\mathcal{T}_I := \{u \in \mathtt{g}_s : |u_j| = |u_j^{(0)}| \quad \forall j\}$$

is an invariant torus for $H_V \circ \Phi$ on which the dynamics is $\theta \to \theta + \omega t$.

- If all the $|u_j^{(0)}| > 0$ the we have a maximal torus
- If all the $|u_j^{(0)}| = 0$ except a finite number we have a quasi-periodic solution
- In between we have infinite dimensional elliptic tori

NLS- Sobolev regularity

We construct solutions with finite regularity for

$$\mathbf{i}\mathbf{u}_t + \mathbf{u}_{xx} - V * \mathbf{u} + F(|\mathbf{u}|^2)\mathbf{u} = 0$$

by considering special lower dimensional tori.
Example: consider the following (infinite) subset of $\mathbb{Z}$

$$\mathcal{S} := 2^{\mathbb{N}} \equiv \left\{2^h, \quad h \in \mathbb{N}\right\}, \quad \mathbb{Z} = \mathcal{S} \cup \mathcal{S}^c \tag{2}$$

We know that NLS_V has Gevrey solutions mostly supported on $\mathcal{S}$ of frequency $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$:

$$\mathtt{D}_{\gamma,\mathcal{S}} := \left\{\omega \in \mathtt{R} : |\omega \cdot \ell| > \gamma \prod_{n\in\mathbb{Z}} \frac{1}{(1+|\ell_n|^2\langle n\rangle^2)}, \quad \sum_{j\in\mathcal{S}^c} |\ell_j| \le 2\right\}.$$

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NLS- Sobolev regularity

By using the special structure of $\mathcal{S}$ (and momentum conservation) we impose much stronger Diophantine conditions:

$$\widehat{\mathtt{D}}_{\gamma,\mathcal{S}} := \left\{ \omega \in \mathtt{R} \,:\, |\omega\cdot\ell| > \gamma \prod_{n\in\mathcal{S}} \frac{1}{(1+|\ell_n|^2\langle \log_2 n\rangle^2)}, \quad \sum_{j\in\mathcal{S}^c} |\ell_j| \le 2 \right\}.$$

This allows us to prove existence of finite regularity solutions mostly supported on $\mathcal{S}$ for the translation invariant NLS.

$$\mathtt{w}_p(\mathbb{C}) = \left\{ v \in \ell^1(\mathbb{C}) : \quad |v|_s := \sup_{j\in\mathbb{Z}} |v_j| \langle j \rangle^p < \infty \right\}, \qquad \langle j \rangle := \max(1, |j|)$$

Fix $\mathcal{S} = 2^{\mathbb{N}}$, for any $0 < \gamma \ll 1$, for any $p > 1$, for all $r < r^*(\gamma, p)$ and every $\sqrt{I} \in \bar{B}_r(\mathtt{w}_p)$ with $I_j = 0$ for $j \notin \mathcal{S}$

Theorem (Biasco, M.,Procesi; Sobolev case)

There exist a positive measure Cantor-like set

$$\mathcal{C} \subset \{\nu \in \mathbb{R}^{\mathcal{S}} : \quad |\nu_j - j^2| \le 1/2\}$$

such that for all $\nu \in \mathcal{C}$ there exists a potential $V \in [-1/2, 1/2]^{\mathcal{S}}$ and a change of variables $\Phi : \bar{B}_r(\mathtt{w}_p) \to \bar{B}_r(\mathtt{w}_p)$ such that

$$\mathcal{T}_I := \{u \in \mathtt{w}_p : |u_j|^2 = I_j \quad \forall j \in \mathcal{S}\,, \quad u_j = 0\,, \quad j \in \mathcal{S}^c\}$$

is an elliptic KAM torus of frequency α for $H_V \circ \Phi$.
Finally V depends on ν in a Lipschitz way.

If we choose I appropriately then the solution has finite regularity.

Finite regularity solutions for NLS

Theorem (Biasco-Massetti-P. 20)

For any $p > 1$ and for most choices of $V \in \ell^\infty$ there exist infinitely many almost periodic solutions

$$u(t,x) = \sum_j \hat{u}_j(t) e^{\mathrm{i} jx}, \qquad |u|_p := \sup_j |\hat{u}_j| \langle j \rangle^p \ll 1 .$$

here the frequency is $\sim j^2$

Such solutions are approximately supported on sparse subsets of $\mathbb{Z}$.

- Since the condition $|u|_p := \sup_j |\hat{u}_j| \langle j \rangle^p$ only implies that $u \in H^{1/2}(\mathbb{T})$ for all $t \rightsquigarrow$ solutions only in a weak-sense: $u \notin C^2$ in x, $u \notin C^1$ in t
- What is the "minimal regularity"?
- What is the role of the sparse set $\mathcal{S}$?

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Basic strategy

- Work on the Hamiltonian (for the NLS $H = \sum_j (j^2 + V_j)|u_j|^2 + P$)
- Fix a sparse subset $\mathcal{S}$ satisfying appropriate separation conditions.
- Look for invariant infinite dimensional tori $\mathtt{i} : \mathbb{T}^{\mathcal{S}} \to \mathcal{T}_I \subset \mathtt{w}_p, \quad \varphi = (\varphi_j)_{j\in\mathcal{S}} \mapsto \mathtt{i}(\varphi)$, with

$$\mathtt{i}_j(\varphi) := \sqrt{I_j} e^{\mathrm{i}\varphi_j} \text{ for } j \in \mathcal{S}\,,\ \mathtt{i}_j(\varphi) := 0 \text{ otherwise}\,,$$

- Show that such tori are the support of almost-periodic weak solutions by showing that $\mathbf{u}(t,x) := \Phi(\mathtt{i}(\nu t), x)$ are uniform limit of smooth quasi-periodic functions.
- A careful control on the parameter dependence (V, ν, I) is needed: continuity w.r.t. product topology & Lipschitz w.r.t. ℓ^∞ for measure estimates and implicit fct thm respectively
  in ∞ dim Lipschitz w.r.t. $\ell^\infty \not\Rightarrow$ measurability w.r.t. product!
- Infinitely many choices of $\mathcal{S}$ lead to infinitely many different solutions

around regularity...

- the map $\mathfrak{i} : (\varphi_j)_{j\in\mathcal{S}} \mapsto \sqrt{I_j} e^{\mathrm{i}\varphi_j} \in \mathtt{w}_p$, is analytic (provided that we endow $\mathbb{T}^{\mathcal{S}}$ with the ℓ^∞-topology) BUT this does not imply that $\mathbf{u}$ is analytic in time or space !

Here

the map $t \mapsto \nu t \in \mathbb{T}^{\mathcal{S}}$ is not even continuous & the regularity of $t \mapsto \mathfrak{i}(\nu t)$ depends on the choice of I_j

EX: our solutions u is a slight deformation of

$$V(\varphi, x) = \sum_{j\in\mathcal{S}} \frac{1}{\langle j\rangle^p} e^{\mathrm{i}jx+\mathrm{i}\varphi_j} \quad \rightsquigarrow v(t,x) = \sum_{j\in\mathcal{S}} \frac{1}{\langle j\rangle^p} e^{\mathrm{i}jx+\mathrm{i}j^2 t} \quad p > 1$$

V is analytic in φ but v is not even C^1 in time.

- if $p < 2$ they are not classical solutions !
  (we construct $v(t,\cdot) \in \mathtt{w}_p$ but not in $\mathtt{w}_{p'}$ $\forall p' > p$)

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Thanks !