Attracted by an elliptic fixed point

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IN HONOR OF JEAN-PIERRE MARCO

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David Sauzin (CNRS - IMCCE, Paris Observatory - PSL University)

Based on joint work with Bassam Fayad and Jean-Pierre Marco

P. Lochak - J.-P. Marco - D.S. 2003 *On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems*, Memoirs of the Amer. Math. Soc. **163** (2003), no. 775, viii+145 pp.

J.-P. Marco - D.S. 2003 *Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems*, Publ. Math. I.H.E.S. **96** (2003), 199–275.

J.-P. Marco - D.S. 2004 *Wandering domains and random walks in Gevrey near-integrable Hamiltonian systems*, Ergodic Theory & Dynam. Systems **24** (2004), no. 5, 1619–1666.

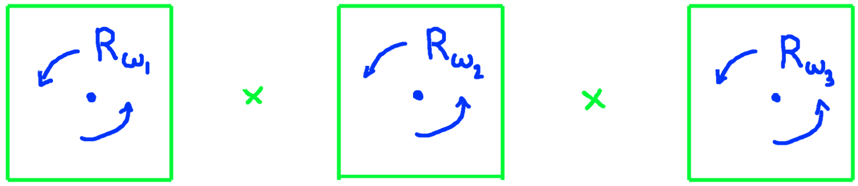
L. Lazzarini - J.-P. Marco - D.S. 2019 *Measure and capacity of wandering domains in Gevrey near-integrable exact symplectic systems*, Memoirs of the Amer. Math. Soc. **257** (2019), no. 1235, vi+110 pp.

B. Fayad - J.-P. Marco - D.S. 2020 *Attracted by an elliptic fixed point*, Asterique **416** (2020), 321–340.

Terminology: 0 fixed point of T symplectic diffeo of $\mathbb{R}^{2n}$ is elliptic of frequency vector $\omega = (\omega_1, \dots, \omega_n)$ if $DT(0)$ is conjugate to

$$S_\omega : (\mathbb{R}^2)^n \circlearrowleft, \qquad S_\omega(s_1, \dots, s_n) := (R_{\omega_1}(s_1), \dots, R_{\omega_n}(s_n)).$$

$\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$

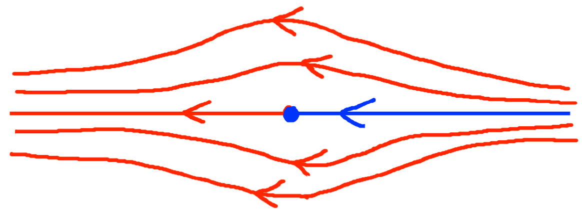


(R_β = rigid rotation around 0 in $\mathbb{R}^2$ with rotation number β)

non-resonant if moreover $k \cdot \omega \notin \mathbb{Z}$ for all $k \in \mathbb{Z}^n - \{0\}$.

Resonant case very different: e.g. time-1 map of $H(x,y) = y(x^2 + y^2)$ $\rightsquigarrow$ linear part at $(0,0)$ = Id and each point of $\mathbb{R}_{>0} \times \{0\}$ is attracted.

$$\boxed{X_H = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y}}$$
$$= -(x^2+3y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$$



Is it possible to find a symplectomorphism with a non-resonant elliptic fixed point and a non-trivial orbit converging to it in forward time?

$$T^m(z) \xrightarrow[m \to +\infty]{} 0$$

YES

[FMS20] B. Fayad, J.-P. Marco, D.S., *Asterique* **416** (2020), 321–340

https://hal.archives-ouvertes.fr/hal-01658860 or arXiv:1712.03001

Our theorem in [FMS20] gives the first such examples.

The phase space is $\mathbb{R}^6$ (or $\mathbb{R}^{2n}$ with any $n \geqslant 3$)

Our examples are C^∞ but not analytic.

Notice that, by reversing time, this provides an example of Lyapunov unstable fixed point.

(Lyapunov stability = orbits that start near 0 remain close to 0 for all forward time)

Context + historical perspective

A form of "Arnold diffusion" for the reverse dynamics...

If a system has a non-resonant elliptic fixed point or is close to integrable (external parameter ε), then the formal perturbation theory seems to predict that the action variables cannot vary much.

In low dimension, with non-degeneracy assumptions, KAM theory makes this prediction a theorem: perpetual stability of action variables.

"Arnold diffusion" = the possibility for some orbits to have their action variables drifting away from their initial values...

$n = 1$

Linear part $(\theta, r) \mapsto (\theta + \omega, r)$ with $\omega \in \mathbb{R} - \mathbb{Q}$

Importance of the Birkhoff Normal Form
$(\tilde{\theta}, \tilde{r}) \mapsto \big(\tilde{\theta} + \omega + a_1\tilde{r} + \cdots + a_N\tilde{r}^N + O(\tilde{r}^{N+1}), \tilde{r} + O(\tilde{r}^{N+1})\big)$

Non-zero BNF ("torsion") or ω Diophantine (even without torsion)
$\implies$ accumulation by invariant quasi-periodic smooth curves and Lyapunov stability.

Anosov-Katok 1970: examples in $D \subset \mathbb{R}^2$ with Lyapunov unstability and ergodicity (BNF = 0 and ω Liouvillian), but not a single orbit converging to 0 in forward or backward time.

$n \geqslant 2$

R. Douady & P. Le Calvez 1983, R. Douady 1988: Lyapunov unstability with arbitrary non-degenerate BNF

(obtained from the existence of a sequence of points that converge to 0 and whose orbits travel along a simple resonance away from 0, not from the existence of one particular orbit.)

B. Fayad preprint 2020: examples of Lyapunov unstability with $n \geqslant 3$, analytic Hamiltonian diffeos, divergent BNF.

Our examples in [FMS20]:
– have BNF = 0
– require $n \geqslant 3$
– are not analytic but Gevrey
– rely on "Herman synchronized diffusion mechanism".

The construction of Gevrey examples of unstable Hamiltonian or exact-symplectic systems is a line of research that had started in collaboration with Michel Herman and Jean-Pierre Marco in 1999...

J.-P. Marco - D.S. 2003 *Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems*, Publ. Math. I.H.E.S. **96** (2003), 199–275.

J.-P. Marco - D.S. 2004 *Wandering domains and random walks in Gevrey near-integrable Hamiltonian systems*, Ergodic Theory & Dynam. Systems **24** (2004), no. 5, 1619–1666.

L. Lazzarini - J.-P. Marco - D.S. 2019 *Measure and capacity of wandering domains in Gevrey near-integrable exact symplectic systems*, Memoirs of the Amer. Math. Soc. **257** (2019), no. 1235, vi+110 pp.

B. Fayad - D.S. 2020 *KAM tori are no more than sticky*, Arch. Rat. Mech. Anal., **237**, 3 (2020), 1177–1211.

[FMS20] *Attracted by an elliptic fixed point*, Asterique **416** (2020), 321–340.

Central thread: Gevrey regularity + variations on "Herman synchronized diffusion mechanism"...

The Gevrey world

Given $\alpha \geqslant 1$ a real number, Gevrey-α regularity is defined by the requirement that the partial derivatives exist at all (multi)orders ℓ and are bounded by $CM^{|\ell|}|\ell|!^{\alpha}$ for some C and M (when $\alpha = 1$, this simply means analyticity).

Upon fixing a real $L > 0$ (essentially the inverse of the previous M), one can define a Banach algebra $\big(G^{\alpha,L}(\mathbb{R}^{2n}), \|\,.\,\|_{\alpha,L}\big)$.

Calculus in the Gevrey world: products of Gevrey-α functions, differentiation ("Gevrey-Cauchy inequalities"), composition of Gevrey-α maps, solving ODEs (the flow of a Gevrey-α Hamiltonian is made of Gevrey-α symplectomorphisms)...

Denote by $\mathcal{U}^{\alpha,L}$ the set of all Gevrey-(α, L) symplectic diffeomorphisms of $\mathbb{R}^6$ which fix 0 and are C^∞-tangent to Id at 0.

One can define a distance $\mathrm{dist}(\Phi, \Psi) = \|\Phi - \Psi\|_{\alpha,L}$ that makes it a complete metric space.

In [FMS20], we prove

THEOREM Fix $\alpha > 1$ and $L > 0$. For each $\gamma > 0$, there exist

– a non-resonant vector $\omega \in \mathbb{R}^3$,
– a point $z \in \mathbb{R}^6$,
– a diffeomorphism $\Psi \in \mathcal{U}^{\alpha,L}$ such that $\|\Psi - \mathrm{Id}\|_{\alpha,L} \leqslant \gamma$,

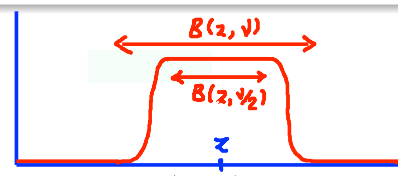
so that $T = \Psi \circ S_\omega$ satisfies $T^m(z) \underset{m \to +\infty}{\longrightarrow} 0$.

ω, z, Ψ are obtained as limits of Cauchy sequences $(\omega^{(n)})$, $(z^{(n)})$, $(\Psi^{(n)})$ that we construct inductively.

In particular, each $\omega^{(n)}$ is in $\mathbb{Q}^3$ and our "synchronized attraction scheme" for $T^{(n)} = \Psi^{(n)} \circ S_{\omega^{(n)}}$ relies on

– a fine synchronization between the three rational components of $\omega^{(n)}$
– an arrangement of the things so that $\Psi^{(n)}$ is (almost) inactive most of the time on a long portion of the $T^{(n)}$-orbit of $z^{(n)}$
– an avatar of a coupling lemma initially due to M. Herman.

$\alpha > 1 \to$ easy to construct Gevrey-α bump fcns: for any $z \in \mathbb{R}^2$ and $\nu > 0$, $\exists\ f_{z,\nu} \in G^{\alpha,L}(\mathbb{R}^2)$

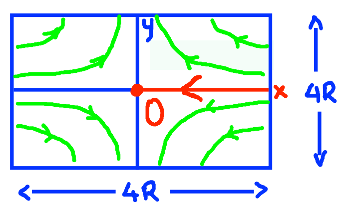


$$0 \leqslant f_{z,\nu} \leqslant 1, \qquad f_{z,\nu} \equiv 1 \text{ on } B(z,\nu/2), \qquad f_{z,\nu} \equiv 0 \text{ on } B(z,\nu)^c$$

and $\|f_{z,\nu}\|_{\alpha,L} \leqslant \exp(c\,\nu^{-\frac{1}{\alpha-1}})$ (exponentially large for small ν).

We also fix $R > 0$, pick $\eta_R \equiv 1$ on $[-2R, 2R]$, $\equiv 0$ on $[-3R, 3R]$, and set

$$g_R \colon \mathbb{R}^2 \to \mathbb{R}, \qquad g_R(x,y) := xy\,\eta_R(x)\,\eta_R(y).$$



Then we define $\Phi_{2,1,z,\nu}$, $\Phi_{1,3,z,\nu}$, $\Phi_{3,2,z,\nu}$ $\in \mathcal{U}^{\alpha,L}$ by

$$\Phi_{i,j,z,\nu} := \text{time-1 map of Hamiltonian } \exp(-c\,\nu^{-\frac{2}{\alpha-1}}) f_{z,\nu} \otimes_{i,j} g_R,$$

$$f_{z,\nu} \otimes_{i,j} g_R \colon\ s = (s_1, s_2, s_3) \mapsto f_{z,\nu}(s_i) g_R(s_j).$$

One gets $\|\Phi_{i,j,z,\nu} - \mathrm{Id}\|_{\alpha,L} \leqslant K \exp(-c\,\nu^{-\frac{1}{\alpha-1}})$ exponentially small.

The trick: $\Phi^{t\, f \otimes_{2,1} g}(s_1, s_2, s_3) = \big(\Phi^{t\, f(s_2) g}(s_1), \Phi^{t\, g(s_1) f}(s_2), s_3\big)$.

Avatar of Herman's coupling lemma for $\Phi_{2,1,z,\nu}$:

(a) For $s_2 \in B(z,\nu)^c$, $\Phi_{2,1,z,\nu}(s_1,s_2,s_3) = (s_1,s_2,s_3)$.

(b) For $x_1 \in \mathbb{R}$, $\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\tilde{x}_1,0),s_2,s_3)$ with $|\tilde{x}_1| \leqslant |x_1|$.

(c) For $x_1 \in [-R,R]$, $s_2 \in B(z,\nu/2)$ and $s_3 \in \mathbb{R}^2$,

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\tilde{x}_1,0),s_2,s_3) \text{ with } |\tilde{x}_1| \leqslant \kappa|x_1|,$$

where $\kappa := 1 - \frac{1}{2}\exp(-c\,\nu^{-\frac{2}{\alpha-1}})$.

Hence, a map like $\Phi_{2,1,z_2,\nu_2}$ = time-1 map of $\exp(-c\,\nu_2^{-\frac{2}{\alpha-1}}) f_{z_2,\nu_2} \otimes_{2,1} g_R$ preserves the x_1-axis and pushes down orbits towards the origin along this axis, while keeping the other two variables frozen (item (b)). However, it is only when the 2nd variable is in $B(z_2,\nu_2)$ that $\Phi_{2,1,z_2,\nu_2}$ effectively brings down the x_1-axis towards the origin (item (c)). Moreover, if the 2nd variable is securely outside the activating ball, then $\Phi_{2,1,z_2,\nu_2}$ is completely inactive (item (a)).

Φ_{i,j,z_2,ν_2} acts as an elevator on the x_j-axis, that never goes up, and that effectively goes down when the ith variable is in the "activating" ball:

z_i may push z_j down along the x_j-axis.

The proof of the THM uses longer and longer compositions of regularly alternating 'elevators', more precisely compositions of a large number of maps of the form $\Phi_{1,3,z_1,\nu_1} \circ \Phi_{3,2,z_3,\nu_3} \circ \Phi_{2,1,z_2,\nu_2}$ (with an inductive choice of the parameters z_i and ν_i) followed by periodic rotations $S_{\omega^{(n)}}$ (with larger and larger periods):

$$T = \lim \left\{\Psi^{(n)} \circ S_{\omega^{(n)}}\right\}, \quad \Psi^{(n+1)} = \Phi_{2,1,z_2^{(n+1)},\nu_2^{(n+1)}} \circ \left(\Phi_{1,3,z_1^{(n)},\nu_1^{(n)}} \circ \Phi_{3,2,z_3^{(n)},\nu_3^{(n)}} \circ \Phi_{2,1,z_2^{(n)},\nu_2^{(n)}}\right) \circ \cdots$$

Suppose that z_2 is inside the activating ball of some elevator $\Phi_{2,1}$, which is hence actively pushing down z_1 on the x_1-axis. Suppose also that, simultaneously, some $\Phi_{3,2}$ is pushing down z_2. At some point, z_2 will exit the activating ball of $\Phi_{2,1}$, which then becomes completely inactive. The variable z_1 stops its descent and will just be rotating due to $S_{\omega^{(n)}}$. A $\Phi_{1,3}$ that is active at this height of z_1 can then be used to push down z_3. As z_3 goes down, $\Phi_{3,2}$ becomes inactive and z_2 will henceforth only rotate. This allows to introduce a new $\Phi_{2,1}$ which is active at this new height of z_2. An alternating procedure of the three types of elevators can thus be put in place.

Technical intermediary result (Proposition 4.1 in [FMS20]):

Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}_+^3$ with $q_3 \mid q_1 \mid q_2$ and $\boxed{z = ((x_1,0),(x_2,0),(x_3,0))} \in B(0,R)$ with $x_1, x_2, x_3 > 0$ and $x_2 \geqslant 1/q_2$.

approximate initial condition

Then, for any $\eta > 0$, there exist

(a) $\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ such that $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$, the orbits of the translation of vector $\overline{\omega}$ on $\mathbb{T}^3$ are η-dense and $|\overline{\omega} - \omega| \leqslant \eta$;

landing point

(b) $\boxed{\bar{z} = ((\overline{x}_1,0),(\overline{x}_2,0),(\overline{x}_3,0))}$ such that $0 < \overline{x}_i \leqslant x_i/2$ and $\overline{x}_2 \geqslant 1/\overline{q}_2$;

(c) $z' \in \mathbb{R}^6$, $\hat{x}_1 \in (\overline{x}_1 + \frac{1}{\overline{q}_1^3}, x_1)$ and $N \geqslant 1$, such that $\boxed{|z' - z| \leqslant \eta}$ and the diffeomorphism

true initial condition

$$\mathcal{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\hat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}$$

satisfies

$$\mathcal{T}^N(z') = \bar{z}$$

and $|\mathcal{T}^m(z')_i| \leqslant (1+\eta)x_i$ for $m \in \{0, \dots, N\}$.

Moreover, $\overline{q}_1$, $\overline{q}_2$ and $\overline{q}_3$ can be taken arbitrarily large.

To be able to iterate the previous result, we must upgrade it by inserting a 'z-admissible' diffeomorphism...

DEFINITION Given $z = (z_1, z_2, z_3) \in \mathbb{R}^6$, we say that a diffeomorphism Φ of $\mathbb{R}^6$ is *z-admissible* if $\Phi \equiv \mathrm{Id}$ on

$$\{s \in \mathbb{R}^6 : |s_i| \leqslant \tfrac{11}{10}|z_i|, i = 1,2,3\}.$$

Iterative step with a z-admissible diffeomorphism Φ inserted (Proposition 4.3 in [FMS20]):

Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}_+^3$ with $q_3 \mid q_1 \mid q_2$ and $z = ((x_1,0),(x_2,0),(x_3,0)) \in B(0,R)$ with $x_1, x_2, x_3 > 0$ and $x_2 \geqslant 1/q_2$. Suppose $\Phi \in \mathcal{U}^{\alpha,L}$ is z-admissible and

$$T := \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi \circ S_\omega$$

satisfies $\boxed{T^M(z_0) = z}$ with $z_0 \in \mathbb{R}^6$, $M \geqslant 1$. Then, for any $\eta > 0$, there exist

(a) $\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ such that $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$, the orbits of the translation of vector $\overline{\omega}$ on $\mathbb{T}^3$ are η-dense and $|\overline{\omega} - \omega| \leqslant \eta$;

(b) $\bar{z} = ((\overline{x}_1,0),(\overline{x}_2,0),(\overline{x}_3,0))$ such that $0 < \overline{x}_i \leqslant x_i/2$ and $\overline{x}_2 \geqslant 1/\overline{q}_2$;

(c) $\bar{z}_0 \in \mathbb{R}^6$ such that $|\bar{z}_0 - z_0| \leqslant \eta$, and $\overline{M} > M$, and $\bar{\Phi} \in \mathcal{U}^{\alpha,L}$ $\bar{z}$-admissible, so that

$$\overline{T} := \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} \circ S_{\overline{\omega}}$$

satisfies $\boxed{\overline{T}^{\overline{M}}(\bar{z}_0) = \bar{z}}$ and $|\overline{T}^m(\bar{z}_0)_i| \leqslant (1+\eta)x_i$ for $m \in \{M, \dots, \overline{M}\}$.

(d) Moreover, $\|\Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} - \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi\|_{\alpha,L} \leqslant \eta$.

Proof: Apply the Technical Intermediary Result to z and get $\overline{\omega}$, $\bar{z}$, N and z' η-close to z such that $\mathcal{T}^N(z') = \bar{z}$. Let

$$\bar{\Phi} := \Phi_{1,3,\hat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi.$$

By z-admissibility we get

$$\overline{T}^m(z') = \mathcal{T}^m(z') \quad \text{for} \;\; m \in \{0, \dots, N\}.$$

Let $\overline{M} := M + N$ and $\bar{z}_0 := \overline{T}^{-M}(z')$: η-close to z_0 if we take $\overline{\omega}$ close enough to ω and the $\overline{q}_i$'s large enough.

Thank you for your attention!