Discrete Processes – Exercise sheet n°1: General probability

At the end of the sheet there are hints for some questions. Try first without looking at the hints, and look for a hint only if you are really blocked! Questions with ($\star$) are a bit more difficult, and questions marked ($\star\star$) are harder "bonus" questions, you may try them only if you get bored.

Throughout the whole sheet, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space.

Exercise 1. Modes of convergence of random variables

1. Recall the definitions of almost sure convergence, convergence in probability, convergence in distribution and convergence in L^p.
2. (a) Show that convergence in L^1 implies convergence in probability.
   (b) Show that the reverse implication is not true (that is to say, find an example of a sequence of random variables which converges in probability but not in L^1).
3. (a) Show that almost sure convergence implies convergence in probability.
   (b) Show that the reverse implication is not true.
4. Let $1 \leq p < q \leq \infty$.
   (a) Show that for every real random variable X, we have $\|X\|_p \leq \|X\|_q$. You may treat separately the case $q = \infty$.
   (b) Deduce that convergence in L^q implies convergence in L^p.
   (c) Show that the reverse implication is not true (find a counter-example for $p = 1$ and $q = 2$).
5. (a) Show that convergence in L^∞ implies almost sure convergence.
   (b) Show that the reverse implication is not true.
6. Show that if $X_n \to X$ almost surely and there exists an integrable variable Z such that $|X_n| \leq Z$ for every $n \in \mathbb{N}$, then $X_n \to X$ in L^1.
7. ($\star$) Show that if X_n converges in probability to X, then there exists a subsequence $(X_{\varphi(n)})_{n\in\mathbb{N}}$ which converges almost surely to X.
8. (a) ($\star\star$) Show that convergence in probability implies convergence in law.
   (b) Show that the reverse implication is not true.
   (c) Show that, when the limit variable X is a constant, then convergence in law implies convergence in probability.

Exercise 2.

Let $X : \Omega \to \mathbb{R}$ be a real random variable such that $X \geq 0$ almost surely. Show that if $\mathbb{E}[X] = 0$ then $X = 0$ almost surely.

Exercise 3. Characterization of cumulative distribution functions

Show that $F : \mathbb{R} \to \mathbb{R}$ is the cumulative distribution function of a real-valued random variable X if and only if it satisfies the three following conditions:

1. F is non-decreasing ;
2. F is càdlàg (right-continuous with left limits: for every $x \in \mathbb{R}$, we have $\lim_{y \to x, y > x} f(y) = f(x)$ and $\lim_{y \to x, y < x} f(y)$ exists) ;
3. We have $\lim\limits_{-\infty} F = 0$ and $\lim\limits_{+\infty} F = 1$.

Deduce from this a simple way to simulate an exponential variable with parameter 1.

Exercise 4. Around Borel-Cantelli's lemma

1. Show that independence is not equivalent to pairwise independence, i.e., construct three random variables X, Y and Z which are not independent but which are pairwise independent (that is to say, $X \perp\!\!\!\perp Y$, $Y \perp\!\!\!\perp Z$ and $X \perp\!\!\!\perp Z$).
2. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of events such that $\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) < \infty$. Show that $\mathbb{P}(\limsup A_n) = 0$, where we recall that $\limsup A_n = \cap_{n \in \mathbb{N}} \big(\cup_{k \geq n} A_k \big)$.
3. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of independent events such that $\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) = \infty$.
   (a) Show that for every $n \in \mathbb{N}$ and every $p \in \mathbb{N}$, we have
   $$P\left(\bigcap_{k \geq n} (\Omega \setminus A_k) \right) \ \leq \ \exp\left(- \sum_{k=n}^{n+p} \mathbb{P}(A_k) \right).$$
   (b) Deduce that for every $n \in \mathbb{N}$, we have $\mathbb{P}\big(\cap_{k \geq n} (\Omega \setminus A_k) \big) = 0$.
   (c) Deduce that $\mathbb{P}(\limsup A_n) = 1$.
4. Show that if $X_1, \ldots, X_n$ are pairwise independent variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ then we have the identity $\mathrm{Var}(X_1, \ldots, X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$.
5. We now consider a sequence of events $(A_n)_{n \in \mathbb{N}}$ which is such that $\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) = \infty$, but we only assume that these events are pairwise independent. Let $S_n = \sum_{k=0}^{n} 1_{A_k}$, let $m_n = \mathbb{E}[S_n]$ and let $S = \lim S_n$.
   (a) Show that $\mathrm{Var}(S_n) \leq m_n$.
   (b) Deduce from Chebychev's inequality that $\mathbb{P}(S \leq m_n/2) \leq 4/m_n$.
   (c) Conclude that $\mathbb{P}(S = \infty) = 1$, and that $\mathbb{P}(\limsup A_n) = 1$.

Exercise 5. σ-field generated by a random variable

1. ($\star$) Find a random variable $X : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\sigma(X) = \sigma\big(\mathcal{B}([0,1]), \{2\}\big)$.
2. ($\star\star$) For each of the following σ-fields, find a random variable $Y : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which generates it.

$$\begin{aligned}
\mathcal{E} &= \sigma\big(\{[-y,-x] \cup [x,y],\ 0 < x < y < 1\}\big), \\
\mathcal{F} &= \sigma\big(\{[-y,-x] \cup [x,y],\ 0 \leq x \leq y < 1\}\big), \\
\mathcal{G} &= \sigma\big(\{[-y,-x] \cup [x,y],\ 0 < x < y \leq 1\}\big), \\
\mathcal{H} &= \sigma\big(\{[-y,-x] \cup [x,y],\ 0 \leq x \leq y \leq 1\}\big).
\end{aligned}$$

3. Consider two random variables $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $(E, \mathcal{E})$ is any measurable space.
   (a) Show that if there exists a measurable function $g : E \to \mathbb{R}$ such that $Y = g(X)$, then the variable Y is $\sigma(X)$-measurable.
   (b) ($\star\star$) Prove the reciprocal.
   (c) ($\star\star$) Does the reciprocal also hold if Y is not real-valued?

Exercise 6. Monotone convergence, Fatou and dominated convergence

1. Prove Fatou's lemma using the monotone convergence theorem.
2. ($\star$) Deduce the dominated convergence theorem from Fatou's lemma.

Exercise 7. The Strong Law of Large Numbers

The aim of this exercise is to prove the strong law of large numbers.

1. State the weak and the strong laws of large numbers.
2. Prove the weak law of large numbers.

The goal is now to prove the strong law of large numbers. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of real integrable variables, that we assume to be centered, to simplify notation. Let $S_0 = 0$ and for every $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. We let $a > 0$ and we define $M = \sup_{n\in\mathbb{N}}(S_n - na)$. For every $n \in \mathbb{N}$, we define

$$M_n \;=\; \max_{0\leq k\leq n} (S_k - ka) \qquad \text{and} \qquad M'_n \;=\; \max_{0\leq k\leq n} (S_{k+1} - X_1 - ka)\,.$$

3. Show that for every $n \in \mathbb{N}$, the two variables M_n and M'_n have the same law.
4. ($\star$) Show that for every $n \in \mathbb{N}$, we have
$$M_{n+1} \;=\; M'_n - \min(M'_n,\, a - X_1)\,.$$
5. Deduce that for every $n \in \mathbb{N}$, we have $\mathbb{E}\big[\min(M'_n,\, a - X_1)\big] \leq 0$.
6. Deduce that $\mathbb{E}\big[\min(M',\, a - X_1)\big] \leq 0$, where $M' = \lim M'_n$.
7. Deduce from the previous question that $\mathbb{P}(M' = \infty) < 1$. You may reason by contradiction.
8. Deduce that $\mathbb{P}(M = \infty) < 1$.
9. Deduce that $\mathbb{P}(M = \infty) = 0$. You may use Kolmogorov's zero-one law: for every independent sequence of random variables $(X_n)_{n\geq 1}$, the terminal σ-field
$$\mathcal{G} \;=\; \bigcap_{n\in\mathbb{N}} \sigma(X_k,\, k \geq n)$$
is trivial, that is to say, every event in $\mathcal{G}$ has probability equal to zero or one.
10. Deduce from the previous question that $\limsup S_n/n \leq a$.
11. Conclude the proof of the strong law of large numbers. Where did we use independence? Where did we use that the variables are identically distributed?

Hints

- Exercise 1, question 2(a): Use Markov's inequality.
- Exercise 1, question 3(a): Use the Dominated Convergence Theorem.
- Exercise 1, question 4(a): When $q < \infty$, use Jensen's inequality. First try $p = 1$ and $q = 2$ if you are lost.
- Exercise 1, question 6: You may use the Dominated Convergence Theorem.
- Exercise 1, question 7: Construct an extraction φ (i.e., a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$) such that for every $n \geq 1$, we have $\mathbb{P}\big(|X_{\varphi(n)} - X| \geq \frac{1}{n}\big) \leq \frac{1}{n^2}$, and then use Borel-Cantelli's lemma.
- Exercise 1, question 8(a): First hint: you may use the result of the previous question.
- Exercise 1, question 8(a): Second hint: Take $f : \mathbb{R} \to \mathbb{R}$ measurable and bounded, and note that, since the sequence of real numbers $(\mathbb{E}[f(X_n)])_{n\in\mathbb{N}}$ is bounded, to show that it converges to $\mathbb{E}[f(X)]$, it is enough to show that for every subsequence of this sequence which converges to a certain limit ℓ, we have $\ell = \mathbb{E}[f(X)]$. To show this, you may use the result of the previous question.
- Exercise 2, first hint: Write $\{X > 0\} = \cup_{n\geq 1}\{X \geq 1/n\}$.
- Exercise 2, second hint: Use Markov's inequality.
- Exercise 3, $\Leftarrow$ implication: Consider $x = G(U)$, where $G(y) = \sup\{x \in \mathbb{R} : F(x) \leq y\}$ and U is a uniform random variable on $[0, 1]$.
- Exercise 4, question 2: Let $S = \sum_{n\in\mathbb{N}} 1_{A_n}$ and show that $\limsup A_n = \{S = \infty\}$.
- Exercise 6, question 2: If $f_n \to f$ almost surely and $|f_n| \leq g$ for every n, with g integrable, then consider the functions $h_n = 2g - |f_n - f|$.
- Exercise 7, question 2: Use characteristic functions.
- Exercise 7, question 4: Note that for every x, $y \in \mathbb{R}$, we have $x - \min(x, y) = \max(0, x - y)$.
- Exercise 7, question 6: Use the dominated convergence theorem, and do not forget to specify which integrable variable dominates.