Discrete Processes – Exercise sheet n°2: Conditional expectation

At the end of the sheet there are hints for some questions. Try first without looking at the hints, and look for a hint only if you are really blocked! Questions with ($\star$) are a bit more difficult.

Throughout the whole sheet, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space.

Exercise 1.

Let X and Y be two square-integrable variables and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Show that

$$\mathbb{E}\big[X\,\mathbb{E}[Y|\mathcal{G}]\big] \;=\; \mathbb{E}\big[Y\,\mathbb{E}[X|\mathcal{G}]\big]\,.$$

Exercise 2.

Let X and Y be two independent Poisson variables of parameters λ and μ, respectively.

1. What is the distribution of $X+Y$?
2. Compute $\mathbb{E}[X \mid X+Y]$.

Exercise 3.

In all what follows, $\mathcal{G}$, $\mathcal{G}_1, \mathcal{G}_2$ are sub-σ-algebras of $\mathcal{F}$, the random variables X, $(X_n)_{n\in\mathbb{N}}$ are in $L^1(\mathcal{F})$ and the properties are true almost surely.

1. Prove the positivity property of conditional expectation: if $X \geq 0$ a.s., then $\mathbb{E}[X|\mathcal{G}] \geq 0$ a.s.
2. Show that $\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big] = \mathbb{E}[X]$.
3. Show that if X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$.
4. ($\star$) Prove the conditional version of the monotone convergence theorem: let $(X_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of non-negative random variables which converges almost surely to a random variable X, then
$$\mathbb{E}[X|\mathcal{G}] \;=\; \lim_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]\,,$$
that is to say, the sequence of variables $(\mathbb{E}[X_n|\mathcal{G}])_{n\in\mathbb{N}}$ converges almost surely to the variable $\mathbb{E}[X|\mathcal{G}]$. You may use the usual monotone convergence theorem, without reproving it.
5. Show that if $\mathcal{G}_2 \subset \mathcal{G}_1$ then
$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}_1]\,\big|\,\mathcal{G}_2\big] \;=\; \mathbb{E}[X|\mathcal{G}_2] \qquad \text{and} \qquad \mathbb{E}\big[\mathbb{E}[X|\mathcal{G}_2]\,\big|\,\mathcal{G}_1\big] \;=\; \mathbb{E}[X|\mathcal{G}_2]\,.$$
6. Prove that if X is independent of $\mathcal{G}$, then
$$\mathbb{E}[X|\mathcal{G}] \;=\; \mathbb{E}[X]\,.$$

Exercise 4.

1. Let X, Y be two independent uniform variables on $[0,1]$. Compute $\mathbb{E}[X|XY]$.

2. Let $X \sim \mathcal{N}(0,1)$. Compute $\mathbb{E}[X^2|X]$ and $\mathbb{E}[X|X^2]$.
3. Let X and Y be two independent uniform variables on $[-\pi/2,\, \pi/2]$. Compute

$$\mathbb{E}[\sin X|\cos X], \quad \mathbb{E}[X|e^X], \quad \mathbb{E}[\cos X|\sin X], \quad \text{and} \quad \mathbb{E}[\sin X|\cos(X+2Y)]\,.$$

Exercise 5.

($\star$) Let X be an exponential variable with parameter 1 and let $t > 0$. We define $Y = \min(X,t)$ and $Z = \max(X,t)$. Compute $\mathbb{E}[X|Y]$ and $\mathbb{E}[X|Z]$.

Exercise 6.

Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables, and let $S = X_1 + \cdots + X_n$.

1. Compute $\mathbb{E}[S|X_1]$.
2. ($\star$) Compute $\mathbb{E}[X_1|S]$.

Exercise 7.

($\star$) Let (X,Y) be a Gaussian vector with mean m and covariance matrix Σ given by

$$m = \begin{pmatrix} m_X \\ m_Y \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix}.$$

Find two constants $a, b \in \mathbb{R}$ (which depend on m and Σ) such that $\mathbb{E}[Y|X] = aX + b$.

Hints

- Exercise 2, question 2: use the formula for conditional expectation in the case of two discrete random variables.
- Exercise 4, question 1: call $Z = XY$ and first compute the joint density of the couple (X,Z), then use the formula for conditional expectations of variables with a joint density.
- Exercise 5: consider any function $g : \mathbb{R} \to \mathbb{R}$ measurable and bounded and try, from the expression $\mathbb{E}[Xg(V)]$, to obtain an expression of the form $\mathbb{E}[\phi(V)g(V)]$, for a certain measurable function ϕ, to be found.
- Exercise 6, question 2: first guess the result, using that the n variables play symmetric roles. Then, show that the answer that you guessed satisfies the three assumptions of the definition of conditional expectation.
- Exercise 7: first hint: start by determining a and b assuming that they exist, and then show that $Y - (aX + b)$ is independent of X.
- Exercise 7: second hint: recall that the components of a Gaussian vector are independent if and only if their covariance is zero.