

Discrete Processes – Exercise sheet n°3: Complements on conditional expectation

Exercise 1. A counter-example to the tower property

By considering for example $\Omega = \{a, b, c\}$, show that, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are any two sub- σ -fields of $\mathcal{F}$, in general $\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2]$ is not almost surely equal to $\mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]|\mathcal{G}_1]$.

Exercise 2. Independence and conditional independence

We say that two variables X and Y are conditionally independent knowing a σ -field $\mathcal{G}$ is for every bounded and measurable functions f and g we have $\mathbb{E}[f(X)g(Y)|\mathcal{G}] = \mathbb{E}[f(X)|\mathcal{G}] \times \mathbb{E}[g(Y)|\mathcal{G}]$. The goal of this exercise is to show that there is no implication between independence and conditional independence, in either direction.

1. Let X and Y be two independent Bernoulli variables with respective parameters $p, q \in (0, 1)$. Let $\mathcal{G} = \sigma(\{X + Y > 0\})$.
 - (a) Compute $\mathbb{E}[X|\mathcal{G}]$ and $\mathbb{E}[Y|\mathcal{G}]$.
 - (b) Are the random variables $\mathbb{E}[X|\mathcal{G}]$ and $\mathbb{E}[Y|\mathcal{G}]$ independent?
 - (c) Are the variables X and Y conditionally independent knowing $\mathcal{G}$?
2. (★) Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two variables X and Y and a sub- σ -field $\mathcal{G}$ of $\mathcal{F}$ such that X and Y are conditionally independent knowing $\mathcal{G}$ but are not independent.

Exercise 3. Two implications and two counter-examples

1. Let X and Y be real random variables such that X, Y and XY are integrable. Show the following implications:
 - (a) X, Y independent $\Rightarrow \mathbb{E}[X|Y] = \mathbb{E}[X]$.
 - (b) $\mathbb{E}[X|Y] = \mathbb{E}[X] \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

2. We want to show that the reciprocal of the implication (a) is not true in general. Let X be a standard Gaussian variable, and let $Y = |X|$.

- (a) Show that $\mathbb{E}[X|Y] = \mathbb{E}[X]$.
- (b) Are the variables X and Y independent?

3. (★) Show that the reciprocal of the implication (b) is not true in general, i.e., find a couple of variables (X, Y) with X, Y and XY integrable and such that $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$ but $\mathbb{E}[X|Y]$ is not almost surely equal to $\mathbb{E}[X]$.

Exercise 4. The L^2 space is complete

The aim of this exercise is to show that $L^2(\Omega, \mathcal{F}, P)$, endowed with the L^2 norm, is complete, i.e., that Cauchy sequences converge.

Let $(X_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in L^2 , i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $\|X_n - X_m\|_2 \leq \varepsilon$.

We want to show that there exists a random variable $X \in L^2$ such that $X_n \rightarrow X$ for the L^2 norm.

1. (★) Construct a subsequence $(X_{\varphi(n)})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, we have $\|X_{\varphi(n+1)} - X_{\varphi(n)}\|_2 \leq 2^{-n}$.
2. We write $\hat{X}_n = X_{\varphi(n)}$ and we consider, for every $n \in \mathbb{N}$, the variable $Y_n = \sum_{k=1}^n |\hat{X}_k - \hat{X}_{k-1}|$.
 - (a) What ensures that these variables converge almost surely to a certain random variable Y , taking values in $[0, +\infty]$?
 - (b) Show that for every $n \in \mathbb{N}$, we have $\|Y_n\|_2 \leq 2$.
 - (c) Deduce that $Y \in L^2$.
 - (d) Deduce that $Y < \infty$ almost surely.
3. Let $N = \{Y = \infty\}$.

- (a) Show that for every fixed $\omega \in \Omega \setminus N$, the following series converges:

$$\sum_{k \geq 1} (\hat{X}_k(\omega) - \hat{X}_{k-1}(\omega)).$$

- (b) Deduce that for every fixed $\omega \in \Omega \setminus N$, the real sequence $\hat{X}_n(\omega)$ converges to some real finite limit, which we call $X(\omega)$.
- (c) $(\star)$ For every $\omega \in N$, we define $X(\omega) = 0$. Show that this map $X : \Omega \rightarrow \mathbb{R}$ is measurable.
- (d) Conclude that $\hat{X}_n \rightarrow X$ almost surely.
4. We now want to show that this sequence $(\hat{X}_n)_{n \in \mathbb{N}}$ converges to X also in L^2 . Recall that, in general, almost sure convergence does not imply L^2 convergence.
- (a) Show that there exists $Z \in L^2$ such that $|\hat{X}_n| \leq Z$ for all n .
- (b) Deduce that $\hat{X}_n \rightarrow X$ in L^2 .
5. $(\star)$ Show the following general result, which is enough to conclude the proof: in any metric space (E, d) , if a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ admits a subsequence which converges, then the sequence $(x_n)_{n \in \mathbb{N}}$ converges.

Exercise 5. True or false?

Determine if the following assertions are true or false, justifying each answer with a proof or a counter-example.

- For every probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and every $X \in L^1(\mathcal{F})$, if $\mathcal{G} \subset \mathcal{F}$ is a trivial sub- σ -algebra of $\mathcal{F}$ (that is to say, if for every event $A \in \mathcal{G}$ we have $\mathbb{P}(A) \in \{0, 1\}$), then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$ a.s.
- For every $1 \leq p < \infty$, for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for every sub- σ -field $\mathcal{G} \subset \mathcal{F}$, the map $\Phi : (L^p(\mathcal{F}), \|\cdot\|_p) \rightarrow (L^p(\mathcal{G}), \|\cdot\|_p)$ defined by $\Phi(X) = \mathbb{E}[X|\mathcal{G}]$ is continuous.
- $(\star\star)$ For every probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and every $X \in L^1(\mathcal{F})$, if $\mathcal{G} \subset \mathcal{F}$ is a sub- σ -field of $\mathcal{F}$ then $\mathbb{E}[X|\mathcal{G}]$ is the variable in $L^1(\mathcal{G})$ which minimizes the L^1 distance to X .

4. $(\star\star)$ For every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if $X : \Omega \rightarrow [0, 1]$ is a random variable taking values in $[0, 1]$ and $\mathcal{G} \subset \mathcal{F}$ is a sub- σ -field of $\mathcal{F}$ such that $\mathbb{E}[X|\mathcal{G}] \in \{0, 1\}$ almost surely, then $X \in \{0, 1\}$ almost surely.

Hints

- Exercise 1: Recalling the tower property, note that you need to consider two σ -fields which are not included one into the other.
- Exercise 2, question 1(a): Write explicitly what is $\mathcal{G}$.
- Exercise 2, question 1(b): Compare the expectation of the product with the product of the expectations.
- Exercise 3, question 2(a): Compute $\mathbb{E}[Xf(Y)]$ for f a bounded and measurable function, and use that the law of X is symmetric, i.e., X has the same law as $-X$.
- Exercise 3, question 2(b): Recall that two real-valued variables X and Y are independent if and only if for every Borel sets A, B , we have $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \times \mathbb{P}(Y \in B)$. So you should either show that this is true for every A and B , or find a counter-example for one particular choice of A and B .
- Exercise 3, question 3(b): Look for an example where $\mathbb{E}[Y] = 0$.
- Exercise 4, question 2(b): You may use the triangular inequality for the L^2 norm.
- Exercise 4, question 2(c): You may use the monotone convergence theorem.
- Exercise 4, question 3(a): Show that this series is absolutely convergent.
- Exercise 4, question 4(a): Take $Z = |\hat{X}_0| + Y$.
- Exercise 4, question 4(b): Use the dominated convergence theorem. What is the integrable variable which dominates?