Discrete Processes – Exercise sheet n°4: Stopping times

Exercise 1. Definition of a stopping time

Let T be a random variable taking values in $\mathbb{N}\cup\{+\infty\}$, and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration. What are all the implications which hold between the following properties? For each true implication, give a proof, and for each false implication, give a counter-example.

1. $\forall n \in \mathbb{N}, \quad \{T = n\} \in \mathcal{F}_n$,
2. $\forall n \in \mathbb{N}, \quad \{T \leq n\} \in \mathcal{F}_n$,
3. $\forall n \in \mathbb{N}, \quad \{T \geq n\} \in \mathcal{F}_n$.

Definition. *(in case we have not seen it yet in the lecture)*
A stopping time relative to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a random variable T taking values in $\mathbb{N}\cup\{+\infty\}$ which satisfies the first property of Exercise 1.

Exercise 2. Operations on stopping times

If S and T are two stopping times, is it true in general that $\min(S,T)$ is also a stopping time? Same question for $\max(S,T)$, $S+T$, $S\times T$ and $S-T$. For each case, give a proof or a precise counterexample.

Exercise 3. Life expectancy ($\star$)

Let T be a stopping time relative to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. We assume that there exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, we have $\mathbb{P}(T \leq n+1 \,|\, \mathcal{F}_n) \geq \varepsilon$ almost surely. Show that $\mathbb{E}[T] < \infty$, and then deduce that T is almost surely finite

Exercise 4. The σ-field of the past

Let X be an integrable random variable and let T be a stopping time relative to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. We define the σ-field of the past until T as

$$\mathcal{F}_T \;=\; \left\{A \in \mathcal{F} \;:\; \forall\, n \in \mathbb{N},\; A \cap \{T = n\} \in \mathcal{F}_n\right\}.$$

1. Show that $\mathcal{F}_T$ is a σ-field.
2. Prove that T is $\mathcal{F}_T$-measurable.
3. Let $k \in \mathbb{N}$ and assume that $T \equiv k$ (i.e., that $T(\omega) = k$ for every $\omega \in \Omega$). Show that, in this case, $\mathcal{F}_T = \mathcal{F}_k$.
4. Prove that if $T < \infty$ almost surely and $(X_n)_{n\in\mathbb{N}}$ is a process adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, then X_T is $\mathcal{F}_T$-measurable.
5. Show that if S and T are stopping times such that $S(\omega) \leq T(\omega)$ for every $\omega \in \Omega$, then $\mathcal{F}_S \subset \mathcal{F}_T$.
6. Show that if S and T are two stopping times, then $\{S \leq T\} \in \mathcal{F}_S \cap \mathcal{F}_T$.
7. Show that if S and T are two stopping times, then $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.
8. ($\star\star$) Let $(X_n)_{n\in\mathbb{N}}$ be a process, let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be its natural filtration, assume that $\sigma((X_n)_{n\in\mathbb{N}}) = \mathcal{F}$ and let T be a stopping time relative to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that $\mathcal{F}_T = \sigma((X_{n\wedge T})_{n\in\mathbb{N}})$.
9. ($\star\star$) Does the result of Question 8 remain true if we remove the assumption that $\sigma((X_n)_{n\in\mathbb{N}}) = \mathcal{F}$? Does the result remain true if, instead of assuming that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of $(X_n)_{n\in\mathbb{N}}$, we only assume that the process $(X_n)_{n\in\mathbb{N}}$ is adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$?

Hints

- Exercice 3: first hint: recall that if X is a random variable taking values in $\mathbb{N}$, we have the equality $\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$.
- Exercice 3: second hint: to get the intuition, imagine that T is the first day when a machine breaks down, and first try to do the exercise only with your intuition, without writing maths. What does the assumption $\mathbb{P}\left(T \leq n+1 \,|\, \mathcal{F}_n\right) \geq \varepsilon$ mean in words? From this, deduce a bound on the probability, knowing that the machine still works at day n, that it also works at day $n+1$? Then, deduce a bound on the probability that the machine still works after n days. Once you got the intuition, try to write things down in formal maths.
- Exercise 4, question 2: Start by showing that for every $k \in \mathbb{N}$, we have $\{T = k\} \in \mathcal{F}_T$.
- Exercise 4, question 8: first check that for every $n \in \mathbb{N}$, we have $\{T = n\} \in \sigma((X_{n \wedge T})_{n \in \mathbb{N}})$ and then, for $A \in \mathcal{F}_T$, write

$$A \;=\; \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \big(A \cap \{T = n\}\big) \cap \{T = n\}\,.$$