Discrete Processes – Exercise sheet n°5: Martingales

Exercise 1.

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that $(X_n)_{n\in\mathbb{N}}$ is also a martingale with respect to its natural filtration.

Exercise 2.

Let $(\Delta_n)_{n\geq 1}$ be a sequence of square integrable random variables (not necessarily independent, nor identically distributed). We define $X_0 = 0$ and, for every $n \geq 1$, $X_n = \Delta_1 + \cdots + \Delta_n$. We assume that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

1. Show that for every $n \geq 1$, we have $\mathbb{E}[\Delta_n] = 0$.
2. Show that for every $n \geq 1$, we have $\mathbb{E}[X_n^2] = \mathbb{E}[\Delta_1^2] + \cdots + \mathbb{E}[\Delta_n^2]$.

Exercise 3. Martingales vs. tartingales

A real process $(M_n)_{n\in\mathbb{N}}$ is said to be a tartingale if, for every $n \in \mathbb{N}$, the variable M_n is integrable and satisfies $\mathbb{E}[M_{n+1}|M_n] = M_n$ (tartingale is not a standard terminology, it is only invented for this exercise).

1. Recall the definition of a martingale.
2. Show that if $(M_n)_{n\in\mathbb{N}}$ is a martingale, then it is also a tartingale.
3. We consider the following example. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. variables with

$$\mathbb{P}(X_1 = 1) \;=\; \mathbb{P}(X_1 = -1) \;=\; \frac{1}{2}\,.$$

We then define a process $(M_n)_{n\in\mathbb{N}}$ by letting $M_0 = 0$ and, for every $n \in \mathbb{N}$,

$$M_{n+1} \;=\; \begin{cases} M_n + X_{n+1} & \text{if } M_n \neq 0\,, \\ X_1 & \text{if } M_n = 0\,. \end{cases} \tag{1}$$

In other words, $(M_n)_{n\in\mathbb{N}}$ is a random walk, except that at each time the walk comes back to 0, the next step is a copy of the first step, instead of being a new random step. We write $(\mathcal{F}_n)_{n\in\mathbb{N}}$ for the canonical filtration associated with the process $(M_n)_{n\in\mathbb{N}}$. Show that, for every $n \geq 1$,

$$\mathbb{E}\big[M_{n+1}\,|\,\mathcal{F}_n\big] \;=\; \begin{cases} M_n & \text{if } M_n \neq 0\,, \\ X_1 & \text{if } M_n = 0\,. \end{cases}$$

4. Is the process $(M_n)_{n\in\mathbb{N}}$ a martingale?
5. Show that the process $(M_n)_{n\in\mathbb{N}}$ is a tartingale.
6. ($\star$) Bonus question: prove that if a process $(M_n)_{n\in\mathbb{N}}$ taking values in $\mathbb{Z}$ is both a tartingale and a Markov chain, then it is a martingale. You may use, as you prefer, one of the two following definitions of a Markov chain with values in $\mathbb{Z}$ (no other knowledge about Markov chains is necessary):

• A process $(M_n)_{n\in\mathbb{N}}$ valued in $\mathbb{Z}$ is a Markov chain if and only if there exists a sequence of independent real random variables $(X_n)_{n\geq 1}$, independent of M_0, and measurable functions $f_n : \mathbb{Z}\times\mathbb{R}\to\mathbb{Z}$ such that $M_{n+1} = f_n(M_n,\, X_{n+1})$ for every $n\in\mathbb{N}$.

• A process $(M_n)_{n\in\mathbb{N}}$ valued in $\mathbb{Z}$ is a Markov chain if and only if there exist functions $Q_n : \mathbb{Z}^2\to\mathbb{R}$ such that, for every $n\in\mathbb{N}$, and every $x_0,\,\ldots,\,x_{n+1}\in\mathbb{Z}$ such that $\mathbb{P}(M_0 = x_0,\,\ldots,\,M_n = x_n) > 0$, it holds

$$\begin{aligned}\mathbb{P}\big(M_{n+1} = x_{n+1}\,\big|\,M_0 = x_0,\,M_1 = x_1,\,\ldots,\,M_n = x_n\big) &= \mathbb{P}\big(M_{n+1} = x_{n+1}\,\big|\,M_n = x_n\big)\\ &= Q_n(x_n,\,x_{n+1})\,.\end{aligned}$$

Hints

- Exercise 3, question 3: write (1) in one formula, with a sum of two terms with indicators, and then use the properties of conditional expectation.