Discrete Processes – Exercise sheet n°6: More martingales

Exercise 1. Notion of process

Imagine that someone asks you whether the process $(M_n)_{n\in\mathbb{N}}$ defined by

$$\forall n \in \mathbb{N} \qquad \mathbb{P}(M_n = 2^n) \;=\; \frac{1}{2^n} \;=\; 1 - \mathbb{P}(M_n = 0) \tag{1}$$

is a martingale or not.

1. What is the problem with this question?
2. Construct a process $(M_n)_{n\in\mathbb{N}}$ which satisfies (1) and which is a martingale (prove it).
3. Construct a process $(M_n)_{n\in\mathbb{N}}$ which satisfies (1) and which is not a martingale (prove it).

Exercise 2. A strange couple

This exercise has nothing to do with martingales, it is a revision about conditional expectation.

1. In this question, X and Y are two real integrable random variables, and we assume that (X, Y) and (Y, X) have the same law.
   (a) Show that $\mathbb{E}[X \,|\, X+Y] = \mathbb{E}[Y \,|\, X+Y]$.
   (b) Deduce the value of $\mathbb{E}[X \,|\, X+Y]$.
2. In the following questions, we consider X a variable which is uniformly distributed in $\{0,1,2\}$ and we let

$$Y \;=\; X+1 \text{ mod. } 3 \;=\; \begin{cases} 1 & \text{if } X = 0\,, \\ 2 & \text{if } X = 1\,, \\ 0 & \text{if } X = 2\,. \end{cases}$$

   Find a function $f : \{1,2,3\} \to \{0,1,2\}$ such that $X = f(X+Y)$.
3. Deduce $\mathbb{E}[X \,|\, X+Y]$.
4. Deduce $\mathbb{E}[Y \,|\, X+Y]$.
5. What do you notice? Do X and Y have the same law? Do the couples (X,Y) and (Y,X) have the same law? Is it true that for any two real integrable variables X and Y, if X and Y have the same law, then $E[X|X+Y] = \mathbb{E}[Y|X+Y]$?

Exercise 3. Galton-Watson process

Let $(X_{n,k})_{n\geq 0,\, k\geq 1}$ be a collection of i.i.d. random variables taking values in $\mathbb{N}$, with mean m. We assume that $0 < m < \infty$, and we consider the process $(Z_n)_{n\in\mathbb{N}}$ defined by $Z_0 = 1$ and, for all $n \in \mathbb{N}$,

$$Z_{n+1} \;=\; \sum_{k=1}^{Z_n} X_{n,k}\,,$$

with the convention $\sum_{k=1}^{0} \cdots = 0$. We can think of Z_n as the size of a population at time n, in which each individual number k, for $1 \leq k \leq Z_n$, is replaced at time $n+1$ by a random number $X_{n,k}$ of children. The reference filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of $(Z_n)_{n\in\mathbb{N}}$.

1. Show that for every $n \in \mathbb{N}$, we have $\mathbb{E}[Z_{n+1} \,|\, \mathcal{F}_n] = mZ_n$.
2. Deduce that for every $n \geq 1$, the variable Z_n is integrable.
3. Find a deterministic sequence $(c_n)_{n\in\mathbb{N}}$ such that the process $(Z'_n)_{n\in\mathbb{N}}$ defined by $Z'_n = c_n Z_n$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$

Exercise 4. THE martingale

Originally the term *martingale* referred to the strategy consisting, in a series of coin flip games, to double the bet as long as the player looses and to stop at the first success. The goal of this exercise is to analyze this strategy.

Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of Rademacher variables (i.e. $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$). We consider the process $(M_n)_{n\in\mathbb{N}}$ defined by $M_0 = 0$ and, for every $n \geq 1$,

$$M_n = \sum_{k=1}^{n} 2^{k-1} X_k \,,$$

and we let $T = \inf\{n \in \mathbb{N} : X_n = 1\}$ (with the usual convention $\inf \varnothing = +\infty$).

1. Show that $(M_n)_{n\in\mathbb{N}}$ is a martingale and that T is a stopping time (with respect to the same filtration).
2. Deduce the value of $\mathbb{E}[M_{n\wedge T}]$ for every $n \in \mathbb{N}$.
3. Show that T is almost surely finite. What is the value of M_T? Comment.
4. Show that $\mathbb{E}[M_{T-1}] = -\infty$. Comment.

Exercise 5. Azuma-Hoeffding's inequality

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. Rademacher variables. Let L be an integer with $L \geq 1$, and let $D : \mathbb{Z} \to \{1, \ldots, L\}$ be a function. Let us consider the process $(M_n)_{n\in\mathbb{N}}$ defined by $M_0 = 0$ and, for every $n \in \mathbb{N}$,

$$M_{n+1} \;=\; M_n + D(M_n) X_{n+1} \,.$$

The reference filtration is $(\mathcal{F}_n)_{n\in\mathbb{N}}$ defined by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for every $n \geq 1$, and $\mathcal{F}_0 = \{\varnothing, \Omega\}$

1. Show that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
2. Show that for every $n \in \mathbb{N}$, we have
$$\mathbb{E}\big[M_n^2\big] \;=\; \sum_{k=0}^{n-1} \mathbb{E}\big[D(M_k)^2\big] \,.$$
3. Deduce that for every $n \in \mathbb{N}$, we have $n \leq \mathbb{E}[M_n^2] \leq nL^2$.
4. Let $\lambda > 0$. Show that, for every $n \in \mathbb{N}$,
$$\mathbb{E}\big[e^{\lambda M_{n+1}} \,\big|\, \mathcal{F}_n\big] \;=\; e^{\lambda M_n} \cosh\big(\lambda D(M_n)\big) \,.$$
5. Using the bound $\cosh(x) \leq e^{x^2/2}$, conclude that
$$\mathbb{E}\big[e^{\lambda M_n}\big] \;\leq\; \exp\left(\frac{n\lambda^2 L^2}{2}\right) .$$
6. Show the Azuma-Hoeffding inequality: for all $a > 0$ and $n \geq 1$,
$$\mathbb{P}\big(M_n \geq a\big) \;\leq\; \exp\left(-\frac{a^2}{2nL^2}\right) .$$
7. Deduce that for all $a > 0$ and $n \geq 1$, we have
$$\mathbb{P}\big(|M_n| \geq a\big) \;\leq\; 2\exp\left(-\frac{a^2}{2nL^2}\right) .$$
8. Deduce that $M_n/n \to 0$ almost surely.
9. ($\star$) Repeat the last three questions replacing $|M_n|$ with $\max_{0\leq k\leq n} |M_k|$.

Hints

- Exercise 2, question 2: simply consider the three cases to determine what f must be.
- Exercise 5, question 6: use that $\mathbb{P}(M_n \geq a) = \mathbb{P}(e^{\lambda M_n} \geq e^{\lambda a})$ and optimize over $\lambda > 0$.