Université Paris-Dauphine — Year: 2026-2027
M1 Mathématiques et applications — Discrete processes

Partiel – 29/10/2026 – Subject in English – duration: 2 hours

Documents and electronic devices are forbidden. You can write in English or in French. The quality of the writing will be taken into account.

Most questions are independent of each other, and in each exercise there are questions close to the lecture. Thus, if you are stuck, do not hesitate to pass on to the following question, assuming the results if needed. In all what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Exercise 1. (2 points) Let X be an integrable random variable, and let $\mathcal{G}$ and $\mathcal{H}$ be two sub-σ-field of $\mathcal{F}$, with $\mathcal{G} \subset \mathcal{H}$. Show that

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\,\big|\,\mathcal{H}\big] \;=\; \mathbb{E}\big[\mathbb{E}[X|\mathcal{H}]\,\big|\,\mathcal{G}\big] \;=\; \mathbb{E}[X|\mathcal{G}]\,.$$

Exercise 2. (2 points) Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field of $\mathcal{F}$ and let $X \in L^2(\mathcal{F})$. Show (without using the notions of conditional expectation or projection) that for every $Y \in L^2(\mathcal{G})$, we have the equivalence

$$\|X - Y\|_2 \;=\; \inf_{Z \in L^2(\mathcal{G})} \|X - Z\|_2 \qquad \Longleftrightarrow \qquad (X - Y) \;\perp\; L^2(\mathcal{G})\,.$$

Exercise 3. (1 point) State and prove the conditional Fatou's lemma. You may use the conditional monotone convergence theorem without reproving it.

Exercise 4. (3 points) Let X be a square integrable random variable, and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a non-increasing sequence of sub-σ-fields of $\mathcal{F}$, that is to say, $\mathcal{G}_{n+1} \subset \mathcal{G}_n$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we let $Y_n = \mathbb{E}[X \,|\, \mathcal{G}_n]$, and $Z_n = Y_{n+1} - Y_n$.

1. Show that the variables $(Z_n)_{n\in\mathbb{N}}$ are orthogonal in L^2.
2. Deduce that the series $\sum_{n\in\mathbb{N}} Z_n$ converges in L^2.
3. Let $\mathcal{G}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$. Show that $Y_n \to \mathbb{E}[X \,|\, \mathcal{G}_\infty]$ in L^2.

Exercise 5. (2 points) Let X and Y be two real random variables. Show that X and Y are independent if and only if, for every bounded and measurable function $f : \mathbb{R} \to \mathbb{R}$ we have $\mathbb{E}[f(X) \,|\, Y] = \mathbb{E}[f(X)]$.

Exercise 6. (10 points) Let X and Y be two independent exponential variables with parameters a and b, respectively. One may think for example of X and Y being the waiting time of the next bus on two different lines, which do not necessarily have the same frequency. We let $M = \min(X, Y)$, the arrival time of the first bus. One of the goals of the exercise is to study the expectation of X knowing M.

1. Compute $\mathbb{E}[X + Y \,|\, X]$.
2. Compute $\mathbb{P}(X < Y \,|\, X)$.
3. Compute $\mathbb{E}\big[X \,\big|\, \sigma(\{X < Y\})\big]$.
4. What is the law of the variable M?
5. For this question, we assume that $a = b = 1$.
   (a) Compute $\mathbb{E}[X \,|\, X + Y]$.
   (b) Show that the variable $\Delta = \max(X, Y) - M$ is independent of M, and determine its law.
   (c) Deduce that $\mathbb{E}[X + Y \,|\, M] = 2M + 1$.
   (d) Deduce that $\mathbb{E}[X \,|\, M] = M + \frac{1}{2}$.
6. We go back to the general case of any parameters a and b. Let X', Y', Z and B be independent variables, with X', Y' and Z following exponential distributions with respective parameters a, b and $a + b$, and B a Bernoulli variable with parameter $b/(a + b)$. We define $X'' = Z + BX'$ and $Y'' = Z + (1 - B)Y'$. Show that the couple (X'', Y'') has the same law as (X, Y).
7. Deduce $\mathbb{E}[X \,|\, M]$.
8. Bonus question: what is the conditional distribution of X knowing M?