Investment/consumption problem in illiquid markets with regime-switching

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Abstract

We consider an illiquid financial market with different regimes modeled by a continuous-time finite-state Markov chain. The investor can trade a stock only at the discrete arrival times of a Cox process with intensity depending on the market regime. Moreover, the risky asset price is subject to liquidity shocks, which change its rate of return and volatility, and induce jumps on its dynamics. In this setting, we study the problem of an economic agent optimizing her expected utility from consumption under a non-bankruptcy constraint. By using the dynamic programming method, we provide the characterization of the value function of this stochastic control problem in terms of the unique viscosity solution to a system of integro-partial differential equations. We next focus on the popular case of CRRA utility functions, for which we can prove smoothness \(C^2\) results for the value function. As an important byproduct, this allows us to get the existence of optimal investment/consumption strategies characterized in feedback forms. We analyze a convergent numerical scheme for the resolution to our stochastic control problem, and we illustrate finally with some numerical experiments the effects of liquidity regimes in the investor’s optimal decision.

Key words : Optimal consumption, liquidity effects, regime-switching models, viscosity solutions, integro-differential system.

1 Introduction

A classical assumption in the theory of optimal portfolio/consumption choice as in Merton [21] is that assets are continuously tradable by agents. This is not always realistic in practice, and illiquid markets provide a prime example. Indeed, an important aspect of market liquidity is the time restriction on assets trading: investors cannot buy and sell them immediately, and have to wait some time before being able to unwind a position in some financial assets. In the past years, there was a significant strand of literature addressing these liquidity constraints. In [25], [20], the price process is observed continuously but the trades succeed only at the jump times of a Poisson process. Recently, the papers [23], [6], [12] relax the continuous-time price observation by considering that asset is observed only at the random trading times. In all these cited papers, the intensity of trading times is constant or deterministic. However, the market liquidity is also affected by long-term macroeconomic conditions, for example by financial crisis or political turmoil, and so the level of trading activity measured by its intensity should vary randomly over time. Moreover, liquidity breakdowns would typically induce drops on the stock price in addition to changes in its rate of return and volatility. For instance, after World War II, the Tokyo stock exchange was closed from August 1945 until May 1949 and it reopened with a loss of 95% compared to the pre-war stock prices. More recently, many emerging market bourses in Brazil, India, Thailand, etc., faced repeated halts for parts or all of a trading session in September-October 2008, during the financial crisis. More evidence of such liquidity breakdowns are reported in [7] and [19]. Let us mention that the market liquidity features addressed in this paper do not include microstructure aspects related to the size of orders and limit order book approach, largely studied in the literature over the recent years. We refer for example to the papers [2], [22], [1].

In this paper, we investigate the effects of such liquidity features on the optimal portfolio choice. We model the index of market liquidity as an observable continuous-time Markov chain with finite-state regimes, which is consistent with some cyclicality observed in financial markets. The modelisation of financial stock prices by regime-switching processes was originally proposed and justified in [13], and since then this approach has been extensively pursued in the financial litterature, see e.g. [3], [26] and the references therein.

The economic agent can trade only at the discrete arrival times of a Cox process with intensity depending on the market regimes. Moreover, the risky asset price is subject to liquidity shocks, which switch its rate of return and volatility, while inducing jumps on its dynamics. In this hybrid jump-diffusion setting with regime switching, we study the optimal investment/consumption problem over an infinite horizon under a nonbankruptcy state constraint. After some useful preliminary results we state a dynamic programming principle (DPP) which holds in our framework (due to the state constraints in two dimensions, in this case the standard continuity assumption is slightly weakened, see Remark 3.1). Then, using DPP, we state the characterization of the value function of this stochastic control problem as the unique constrained viscosity solution to a system of integro-partial differential equations. These two results (DPP and viscosity characterization) are proved in the companion paper [11]. In the particular case of CRRA utility function, we can go beyond
the viscosity properties, and prove $C^2$ regularity results for the value function in the interior of the domain. As a consequence, we show the existence of optimal strategies expressed in feedback form in terms of the derivatives of the value function. Due to the presence of state constraints, the value function is not smooth at the boundary, and so the verification theorem cannot be proved with the classical arguments of Dynkin’s formula. To overcome this technical problem, we use an ad hoc approximation procedure (see Proposition 5.2). We also provide a convergent numerical scheme for solving the system of equations characterizing our control problem, and we illustrate with some numerical results the effect of liquidity regimes in the agent’s optimal investment/consumption. We also measure the impact of continuous time observation with respect to a discrete time observation of the stock prices. Our paper contributes and extends the existing literature in several ways. First, we extend the papers [25] and [20] by considering stochastic intensity trading times and regime switching in the asset prices. For a two-state Markov chain modulating the market liquidity, and in the limiting case where the intensity in one regime goes to infinity, while the other one goes to zero, we recover the setup of [7] and [19] where an investor can trade continuously in the perfectly liquid regime but faces a threat of trading interruptions during a period of market freeze. On the other hand, regime switching models in optimal investment problems were already used in [29], [26] or [27] for continuous-time trading.

The rest of the paper is structured as follows. Section 2 describes our continuous-time market model with regime-switching liquidity, and formulates the optimization problem for the investor. In Section 3 we state some useful properties of the value function of our stochastic control problem. Section 4 is devoted to state the DPP and the analytic characterization of the value function as the unique viscosity solution to the dynamic programming equation, both proved in the companion paper [11]. The special case of CRRA utility functions is studied in Section 5: we show smoothness results for the value functions, and obtain the existence of optimal strategies via a verification theorem. Some numerical illustrations complete this last section.

2 A market model with regime-switching liquidity

Let us fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Let $I$ be a continuous-time Markov chain valued in the finite state space $I_d = \{1, \ldots, d\}$, with intensity matrix $Q = (q_{ij})$. For $i \neq j$ in $I_d$, we can associate to the jump process $I$, a Poisson process $N^j$ with intensity rate $q_{ij} \geq 0$, such that a switch from state $i$ to $j$ corresponds to a jump of $N^j$ when $I$ is in state $i$. We interpret the process $I$ as a proxy for market liquidity with states (or regimes) representing the level of liquidity activity, in the sense that the intensity of trading times varies with the regime value. This is modeled through a Cox process $(N_t)_{t \geq 0}$ with intensity $(\lambda_i)_{t \geq 0}$, where $\lambda_i > 0$ for each $i \in I_d$. For example, if $\lambda_i < \lambda_j$, this means that trading times occur more often in regime $j$ than in regime $i$. The increasing sequence of jump times $(\tau_n)_{n \geq 0}$, $\tau_0 = 0$, associated to the counting process $N$ represents the random times when an investor can trade a risky asset of price.
process $S$. Note that under these assumptions the jumps of $I$ and $N$ are a.s. disjoint.

In the liquidity regime $I_t = i$, the stock price follows the dynamics

$$dS_t = S_t(b_i dt + \sigma_i dW_t),$$

where $W$ is a standard Brownian motion independent of $(I, N)$, and $b_i \in \mathbb{R}$, $\sigma_i \geq 0$, for $i \in \mathbb{I}_d$. Moreover, at the times of transition from $I_{t^-} = i$ to $I_t = j$, the stock changes as follows:

$$S_t = S_{t^-} (1 - \gamma_{ij})$$

for a given $\gamma_{ij} \in (-\infty, 1)$, so the stock price remains strictly positive, and we may have a relative loss (if $\gamma_{ij} > 0$), or gain (if $\gamma_{ij} \leq 0$). Typically, there is a drop of the stock price after a liquidity breakdown, i.e. $\gamma_{ij} > 0$ for $\lambda_j < \lambda_i$. Overall, the risky asset is governed by a regime-switching jump-diffusion model:

$$dS_t = S_t \left( b_{I_t^-} dt + \sigma_{I_t^-} dW_t - \gamma_{I_{t^-}, I_t} dN_{I_{t^-}, I_t} \right).$$

(2.1)

**Portfolio dynamics under liquidity constraint.** We consider an agent investing and consuming in this regime-switching market. We denote by $(Y_t)$ the total amount invested in the stock, and by $(c_t)$ the consumption rate per unit of time, which is a nonnegative adapted process. Since the number of shares $Y_t / S_t$ in the stock held by the investor has to be kept constant between two trading dates $\tau_n$ and $\tau_{n+1}$, then between such trading times, the process $Y$ follows the dynamics:

$$dY_t = Y_t \frac{dS_t}{S_t}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0.$$

The trading strategy is represented by a predictable process $(\zeta_t)$ such that at a trading time $t = \tau_{n+1}$, the rebalancing on the number of shares induces a jump $\zeta_t$ in the amount invested in the stock:

$$\Delta Y_t = \zeta_t.$$

Overall, the càdlàg process $Y$ is governed by the hybrid controlled jump-diffusion process

$$dY_t = Y_t \left( b_{I_t^-} dt + \sigma_{I_t^-} dW_t - \gamma_{I_{t^-}, I_t} dN_{I_{t^-}, I_t} \right) + \zeta_t dN_t.$$

(2.2)

Assuming for simplicity a constant savings account (see Remark 2.2), i.e. zero interest rate, the amount $(X_t)$ invested in cash then follows

$$dX_t = -c_t dt - \zeta_t dN_t.$$

(2.3)

The total wealth is defined at any time $t \geq 0$, by $R_t = X_t + Y_t$, and we shall require the non-bankruptcy constraint at any trading time:

$$R_{\tau_n} \geq 0, \quad a.s. \quad \forall n \geq 0.$$

(2.4)

Actually since the asset price may become arbitrarily large or small between two trading dates, this non-bankruptcy constraint means a no-short sale constraint on both the stock and savings account, as showed by the following Lemma.
Lemma 2.1 The nonbankruptcy constraint (2.4) is formulated equivalently in the no-short sale constraint:

\[ X_t \geq 0, \quad \text{and} \quad Y_t \geq 0, \quad \forall t \geq 0. \tag{2.5} \]

This is also written equivalently in terms of the controls as:

\[ -Y_{t-} \leq \zeta_t \leq X_{t-}, \quad t \geq 0, \tag{2.6} \]

\[ \int_{t}^{\tau_{n+1}} c_s ds \leq X_t, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0. \tag{2.7} \]

Proof. By induction we can write the wealth at any trading time as

\[ R_{\tau_{n+1}} = R_{\tau_n} + Y_{\tau_n} \left( \frac{S_{\tau_{n+1}}}{S_{\tau_n}} - 1 \right) - \int_{\tau_n}^{\tau_{n+1}} c_t dt, \quad n \geq 0. \]

Since (conditionally on \( F_{\tau_n} \)) the stock price \( S_{\tau_{n+1}} \) has support in \((0, \infty)\), we see that the nonbankruptcy condition \( R_{\tau_{n+1}} \geq 0 \) is equivalent to a no-short sale constraint:

\[ 0 \leq Y_{\tau_n} \leq R_{\tau_n}, \quad n \geq 0, \tag{2.8} \]

together with the condition on the nonnegative consumption rate

\[ \int_{\tau_n}^{\tau_{n+1}} c_t dt \leq R_{\tau_n} - Y_{\tau_n} = X_{\tau_n}, \quad n \geq 0. \tag{2.9} \]

Since \( Y_{\tau_n} = Y_{\tau_n-} + \zeta_{\tau_n} \), and since \( R_{\tau_n} = R_{\tau_n-} \) a.s., the no-short sale constraint (2.8) means equivalently that (2.6) is satisfied for \( t = \tau_n \). Since \( \zeta \) is predictable, this is equivalent to (2.6) being satisfied \( d\mathbb{P} \otimes dt \) almost everywhere. Indeed, letting \( H_t = 1_{\{ \zeta < -Y_{t-} \text{ or } \zeta > X_{t-} \}} \), \( H \) is predictable, so that \( \forall t \geq 0, 0 = \mathbb{E} \left[ \sum_{\tau_n \leq t} H_{\tau_n} \right] = \mathbb{E} \left[ \int_0^t H_s \lambda_s ds \right] \), and we deduce that \( H_t = 0 \) \( d\mathbb{P} \otimes dt \) a.e. since \( \lambda_s > 0 \).

Moreover, since \( X_t = X_{\tau_n} - \int_{\tau_n}^{t} c_s ds \) for \( \tau_n \leq t < \tau_{n+1} \), the condition (2.9) is equivalent to (2.7). By rewriting the conditions (2.8)-(2.9) as

\[ Y_{\tau_n} \geq 0, \quad X_{\tau_n} \geq 0, \quad X_{(\tau_{n+1})-} \geq 0, \quad \forall n \geq 0, \]

and observing that for \( \tau_n \leq t < \tau_{n+1} \),

\[ Y_t = \frac{S_t}{S_{\tau_n}} Y_{\tau_n}, \quad X_{\tau_n} \geq X_t \geq X_{(\tau_{n+1})-}, \]

we see that they are equivalent to (2.5).

\[ \square \]

Remark 2.1 Under the nonbankruptcy (or no-short sale constraint), the wealth \((R_t)_{t \geq 0}\) is nonnegative, and follows the dynamics:

\[ dR_t = R_{t-} Z_{t-} \left( b_{t-} dt + \sigma_{t-} dW_t - \gamma_{t-} \lambda_t dN^{I_{\tau_{n+1}}}_{t-} \right) - c_t dt, \tag{2.10} \]
where \( Z_t := \frac{Y_t}{R_t} \) valued in \([0, 1]\) is the proportion of wealth invested in the risky asset; and evolves according to the dynamics:

\[
dZ_t = Z_t(1 - Z_t) \left[ (b_{t-} - Z_t - \sigma_{t-}^2) dt + \sigma_{t-} dW_t - \frac{\gamma_{t-} - \gamma_{t-} I_t}{1 - Z_t - \gamma_{t-} I_t} dN_{I_t} \right] \\
+ \frac{\zeta_t}{R_{t-}} dN_t + Z_t c_{t-} dt,
\]

for \( t < \tau = \inf \{ t \geq 0 : R_t = 0 \} \).

Given an initial state \((i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+\), we shall denote by \( A_i(x, y) \) the set of investment/consumption control process \((\zeta, c)\) such that the corresponding process \((X, Y)\) solution to (2.2)-2.3) with a liquidity regime \( I \), and starting from \((I_0, X_0, Y_0) = (i, x, y)\), satisfy the non-bankruptcy constraint (2.5) (or equivalently (2.6)-(2.7)).

**Optimal investment/consumption problem.** The preferences of the agent are described by a utility function \( U \) which is increasing, concave, \( C^1 \) on \((0, \infty)\) with \( U(0) = 0 \), and satisfies the usual Inada conditions: \( U'(0) = \infty, \ U'(\infty) = 0 \). We assume the following growth condition on \( U \): there exist some positive constant \( K \), and \( p \in (0, 1) \) s.t.

\[
U(x) \leq K x^p, \quad x \geq 0.
\]

We denote by \( \tilde{U} \) the convex conjugate of \( U \), defined from \( \mathbb{R} \) into \([0, \infty]\) by:

\[
\tilde{U}(\ell) = \sup_{x \geq 0} [U(x) - x \ell],
\]

which satisfies under (2.12) the dual growth condition on \( \mathbb{R}_+ \):

\[
\tilde{U}(\ell) \leq \tilde{K} \ell^{-\tilde{p}}, \quad \forall \ell \geq 0, \quad \text{with} \quad \tilde{p} = \frac{p}{1-p} \quad > \quad 0,
\]

for some positive constant \( \tilde{K} \).

The agent’s objective is to maximize over portfolio/consumption strategies in the above illiquid market model the expected utility from consumption rate over an infinite horizon. We then consider, for each \( i \in \mathbb{I}_d \), the value function

\[
v_i(x, y) = \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right], \quad (x, y) \in \mathbb{R}_+^2,
\]

where \( \rho \) is a discount factor. We also introduce, for \( i \in \mathbb{I}_d \), the function

\[
\hat{v}_i(r) = \sup_{x \in [0, r]} v_i(x, r - x), \quad r \geq 0,
\]

which represents the maximal utility performance that the agent can achieve starting from an initial nonnegative wealth \( r \) and from the regime \( i \). More generally, for any locally bounded function \( w_i \) on \( \mathbb{R}_+^2 \), we associate the function \( \hat{w}_i \) defined on \( \mathbb{R}_+ \) by:

\[
\hat{w}_i(r) = \sup_{x \in [0, r]} w_i(x, r - x),
\]

so that:

\[
\hat{w}_i(x + y) = \sup_{e \in [-y, x]} w_i(x - e, y + e), \quad (x, y) \in \mathbb{R}_+^2.
\]
In the sequel, we shall often identify a \(d\)-tuple function \((w_i)_{i \in I_d}\) defined on \(\mathbb{R}_+^2\) with the function \(w\) defined on \(\mathbb{R}_+^2 \times I_d\) by \(w(x, y, i) = w_i(x, y)\).

In this paper, we focus on the analytic characterization of the value functions \(v_i\) (and so \(\hat{v}_i\), \(i \in I_d\), and on their numerical approximation.

**Remark 2.2** For simplicity we have assumed zero interest rate for the riskless asset. The case of constant \(r \neq 0\) can actually be reduced to this case, at the cost of allowing time-dependent utility of consumption. This can be seen from the identity

\[
E \int_0^\infty e^{-\rho s} U(c_s) ds = E \int_0^\infty e^{-\bar{\rho} s} \bar{U}(s, \bar{c}_s) ds,
\]

where \(\bar{\rho} = \rho - pr\), \(\bar{c}_s = e^{-rs} c_s\) and \(\bar{U}(s, \cdot) = e^{-prs} U(e^{rs} \cdot)\). Note that \(\bar{U}(s, \cdot)\) still satisfies (2.12), and in the special case of power utility \(U(c) = c^p\), one actually has \(\bar{U}(s, \cdot) = U\). It is also worth noting that in that case, one should replace \(\rho\) by \(\bar{\rho}\) in the technical assumption (3.1) below.

### 3 Some properties of the value function

We state some preliminary properties of the value functions that will be used in the next section for the PDE characterization. We first need to check that the value functions are well-defined and finite. Let us consider for any \(p > 0\), the positive constant:

\[
k(p) := \max_{i \in I_d, z \in [0,1]} \left[ pb_i z - \frac{\sigma_i^2}{2} p(1-p) z^2 + \sum_{j \neq i} q_{ij} ((1-z)\gamma_{ij}^p - 1)\right] < \infty.
\]

A natural assumption ensuring finiteness of the value function is then

\[
\rho > k(p). \tag{3.1}
\]

As will be seen in the next Lemma, such assumption simply means that, for \(t \to +\infty\), the maximal growth rate of the wealth process elevated to \(p\) (which determines the growth of the utility) is less than \(\rho\). This substantially corresponds to the well known assumption, in economic models, that the growth rate of the optimal paths must be smaller than the profit rate (see e.g. the introduction of [9]).

**Lemma 3.1** Fix some initial conditions \((i, x, y) \in I_d \times \mathbb{R}_+ \times \mathbb{R}_+\), and some \(p > 0\). Then:

1. For any admissible control \((\zeta, c) \in \mathcal{A}_i(x, y)\) associated with wealth process \(R\), the process \((e^{-k(p)t} R_t^p)_{t \geq 0}\) is a supermartingale. In particular, for \(\rho > k(p)\),
   \[
   \lim_{t \to \infty} e^{-\rho t} E[R_t^p] = 0. \tag{3.2}
   \]

2. For fixed \(T \in (0, \infty)\), the family \((R_{T \wedge \tau}^p)_{\tau, \zeta, c}\) is uniformly integrable, when \(\tau\) ranges over all stopping times, and \((\zeta, c)\) runs over \(\mathcal{A}_i(x, y)\).
Proof. (1) By Itô’s formula and (2.10), we have
\[
d(e^{-k(p)t} R_t^p) = -k(p)e^{-k(p)t} R_t^p dt + e^{-k(p)t} d(R_t^p)
\]
\[
= e^{-k(p)t} \left[ -k(p) R_t^p + p R_t^{p-1} (-c_t + b_{l_t} R_{t-} Z_{t-}) + \frac{p(p - 1)}{2} R_t^{p-2} (\sigma_{l_t} R_{t-} Z_{t-})^2 + \sum_{j \neq l_t} q_{l_t-j}(R_t^p (1 - \gamma_{l_t-j} Z_{t-})^p - R_t^{p-1}) \right] dt + dM_t,
\]
where \( M \) is a local martingale. Now, by definition of \( k(p) \), we have
\[
p R_t^{p-1} (-c_t + b_{l_t} R_{t-} Z_{t-}) + \frac{p(p - 1)}{2} R_t^{p-2} (\sigma_{l_t} R_{t-} Z_{t-})^2 + \sum_{j \neq l_t} q_{l_t-j}(R_t^p (1 - \gamma_{l_t-j} Z_{t-})^p - R_t^{p-1}) \leq -pc_t R_t^{p-1} + k(p) R_t^p
\]
\[
\leq k(p) R_t^p.
\]
Since \( R \) has countable jumps, \( R_t = R_{t-}, dt \otimes dt \) a.e., and so the drift term in \( d(e^{-k(p)t} R_t^p) \) is nonpositive. Hence \( (e^{-k(p)t} R_t^p)_{t \geq 0} \) is a local supermartingale, and since it is nonnegative, it is a true supermartingale by Fatou’s lemma. In particular, we have
\[
0 \leq e^{-pt} \mathbb{E}[R_t^p] \leq e^{-(p-k(p))t} (x + y)^p
\]
which shows (3.2).

(2) For any \( q > 1 \), we get by the supermartingale property of the process \( (e^{-k(p)t} R_t^p)_{t \geq 0} \) and the optional sampling theorem:
\[
\mathbb{E}\left[\left(R_{T \wedge \tau}^p\right)^q\right] \leq e^{k(p)\tau} (x + y)^{pq} < \infty, \quad \forall (\zeta, c) \in A_i(x, y), \tau \text{ stopping time},
\]
which proves the required uniform integrability. \( \square \)

The next proposition states a comparison result, and, as a byproduct, a growth condition for the value function.

Proposition 3.1

(1) Let \( w = (w_i)_{i \in \mathbb{I}_d} \) be a d-tuple of nonnegative functions on \( \mathbb{R}_+^2 \), twice differentiable on \( \mathbb{R}_+^2 \setminus \{(0, 0)\} \) such that
\[
\rho w_i - b_i y \frac{\partial w_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 w_i}{\partial y^2} - \sum_{j \neq i} q_{ij} [w_j(x, y(1 - \gamma_{ij})) - w_i(x, y)]
\]
\[
- \lambda_i [\tilde{w}_i(x + y) - w_i(x, y)] - \tilde{U} \left( \frac{\partial w_i}{\partial x} \right) \geq 0, \quad (3.4)
\]
for all \( i \in \mathbb{I}_d, (x, y) \in \mathbb{R}_+^2 \setminus \{(0, 0)\} \). Then, for all \( i \in \mathbb{I}_d, v_i \leq w_i \), on \( \mathbb{R}_+^2 \).

(2) Assume (2.12) and (3.1). There then exists some positive constant \( C \) s.t.
\[
v_i(x, y) \leq C(x + y)^p, \quad \forall (i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+^2.
\]
Proof. (1) First notice that for \((x, y) = (0, 0)\), the only admissible control in \(A_i(x, y)\) is the zero control \(\zeta = 0, c = 0\), so that \(v_i(0,0) = 0\). Now, fix \((x, y) \in \mathbb{R}_+^2 \setminus \{(0,0)\}, i \in \mathbb{R}_d\), and consider an arbitrary admissible control \((\zeta, c) \in A_i(x, y)\). By Itô’s formula applied to \(e^{-pt}w(X_t, Y_t, I_t)\), we get:

\[
d[e^{-pt}w(X_t, Y_t, I_t)] = e^{-pt} \left[ -\rho w - ct \frac{\partial w}{\partial x} + b_{1_-, Y_t^-} \frac{\partial w}{\partial y} + \frac{1}{2} \sigma_{1_-, Y_t^-} \frac{\partial^2 w}{\partial y^2} \right. \\
\left. + \sum_{j \notin I_{-}} q_{1_{-}, j} [w(X_{t^-}, Y_{t^-} - \gamma_{1_{-}, j}, j) - w(X_{t^-}, Y_{t^-}, I_{-})] - \lambda_{1_{-}} [w(X_{t^-} - \zeta_{t}, Y_{t^-} + \zeta_{t}, I_{-}) - w(X_{t^-}, Y_{t^-}, I_{-})] \\
+ e^{-pt} \sum_{j \notin I_{-}} \left[ w(X_{t^-}, Y_{t^-} - \gamma_{1_{-}, j}, j) - w(X_{t^-}, Y_{t^-}, I_{-}) \right] (dN_{I_{-}}(t) - q_{1_{-}, j} dt) \right] \\
+ e^{-pt} \left[ w(X_{t^-} - \zeta_{t}, Y_{t^-} + \zeta_{t}, I_{-}) - w(X_{t^-}, Y_{t^-}, I_{-}) \right] (dN_t - \lambda_{1_{-}} dt). \tag{3.6}
\]

Denote by \(\nu = \inf\{t \geq 0 : (X_t, Y_t) = (0,0)\}\), and consider the sequence of bounded stopping times \(\nu_n = \inf\{t \geq 0 : X_t + Y_t \geq n \text{ or } X_t + Y_t \leq 1/n\} \wedge n, n \geq 1\). Then, \(\nu_n \nearrow \nu\) a.s. when \(n\) goes to infinity, and \(c_t = 0, X_t = Y_t = 0\) for \(t \geq \nu\), and so

\[
\mathbb{E} \left[ \int_{0}^{\infty} e^{-pt} U(c_t) dt \right] = \mathbb{E} \left[ \int_{0}^{\nu} e^{-pt} U(c_t) dt \right]. \tag{3.7}
\]

From Itô’s formula (3.6) between time \(t = 0\) and \(t = \nu_n\), and observing that the integrands of the local martingale parts are bounded for \(t \leq \nu_n\), we obtain after taking expectation:

\[
w(x, y, i) = \mathbb{E} \left[ e^{-pt_n} w(X_{\nu_n}, Y_{\nu_n}, I_{\nu_n}) \right. \\
\left. + \int_{0}^{\nu_n} e^{-pt} \left( \rho w + ct \frac{\partial w}{\partial x} - b_{1_-, Y_t^-} \frac{\partial w}{\partial y} - \frac{1}{2} \sigma_{1_-, Y_t^-} \frac{\partial^2 w}{\partial y^2} \right. \right. \\
\left. \left. - \sum_{j \notin I_{-}} q_{1_{-}, j} [w(X_{t^-}, Y_{t^-} - \gamma_{1_{-}, j}, j) - w(X_{t^-}, Y_{t^-}, I_{-})] \\
- \lambda_{1_{-}} [w(X_{t^-} - \zeta_{t}, Y_{t^-} + \zeta_{t}, I_{-}) - w(X_{t^-}, Y_{t^-}, I_{-})] \right) dt \right] \\
\geq \mathbb{E} \left[ e^{-pt_n} w(X_{\nu_n}, Y_{\nu_n}, I_{\nu_n}) + \int_{0}^{\nu_n} e^{-pt} U(c_t) dt \right] \geq \mathbb{E} \left[ \int_{0}^{\nu_n} e^{-pt} U(c_t) dt \right],
\]

where we used (3.4), and the nonnegativity of \(w\). By sending \(n\) to infinity with Fatou’s lemma, and (3.7), we obtain the required inequality: \(w_t \geq v_t\) since \((c, \zeta)\) are arbitrary.
(2) Consider the function \( w_i(x, y) = C(x + y)^p \). Then, for \((x, y) \in \mathbb{R}_+^2 \setminus \{(0, 0)\} \), and denoting by \( z = y/(x + y) \in [0, 1) \), a straightforward calculation shows that

\[
\rho w_i - b_i y \frac{\partial w_i}{\partial y} - \frac{1}{2} \sigma_i^2 y \frac{\partial^2 w_i}{\partial y^2} - \sum_{j \neq i} q_{ij} [w_j(x, y(1 - \gamma_{ij})) - w_i(x, y)]
- \lambda_i [\hat{\vartheta}_i(x + y) - w_i(x, y)] - \tilde{U} \left( \frac{\partial w_i}{\partial x} \right)
= C(x + y)^p \left[ \rho - pb_i z + \frac{\sigma^2}{2} p(1 - p) z^2 - \sum_{j \neq i} q_{ij} ((1 - z\gamma_{ij})^p - 1) \right] - \tilde{U} ((x + y)^{p - 1} pC)
\geq (x + y)^p \left( C - k(p) - \tilde{K}(pC) - \frac{p}{1 - \rho} \right)
\]  (3.8)

by (2.13). Hence, for \( \rho > k(p) \), and for \( C \) sufficiently large, the r.h.s. of (3.8) is nonnegative, and we conclude by using the comparison result in assertion 1).

In the sequel, we shall assume the standing condition (3.1) so that the value functions are well-defined and satisfy the growth condition (3.5). We now prove continuity properties of the value functions.

**Proposition 3.2** The value functions \( v_i, i \in \mathbb{N}_d \), are concave, nondecreasing in both variables, and continuous on \( \mathbb{R}_+^2 \). This implies also that \( \hat{v}_i, i \in \mathbb{N}_d \), are nondecreasing, concave and continuous on \( \mathbb{R}_+ \). Moreover, we have the boundary conditions for \( v_i, i \in \mathbb{N}_d \), on \( \{0\} \times \mathbb{R}_+ \):

\[
v_i(0, y) = \begin{cases} 0, & i f \ y = 0 \\
\mathbb{E} \left[ e^{-\rho \tau_{i_1}} \hat{\vartheta}_i \left( y \frac{S_{\tau_1}}{S_0} \right) \right], & i f \ y > 0.
\end{cases}
\]  (3.9)

Here \( I \) denotes the continuous-time Markov chain \( I \) starting from \( i \) at time 0, and \( \tau_1 \) is the first jump time of the process \( N \), i.e. the first positive trading time.

**Proof.** Fix some \((x, y, i) \in \mathbb{R}_+^2 \times \mathbb{N}_d, \delta_1 \geq 0, \delta_2 \geq 0, \) and take an admissible control \((\zeta, c) \in \mathcal{A}_i(x, y)\). Denote by \( R \) and \( R' \) the wealth processes associated to \((\zeta, c)\), starting from initial state \((x, y, i)\) and \((x + \delta_1, y + \delta_2, i)\). We thus have \( R' = R + \delta_1 + \delta_2 S/S_0 \). This implies that \((\zeta, c)\) is also an admissible control for \((x + \delta_1, y + \delta_2, i)\), which shows clearly the nondecreasing monotonicity of \( v_i \) in \( x \) and \( y \), and thus also the nondecreasing monotonicity of \( \hat{v}_i \) by its very definition.

The concavity of \( v_i \) in \((x, y)\) follows from the linearity of the admissibility constraints in \( X, Y, \zeta, c \), and the concavity of \( U \). This also implies the concavity of \( \hat{v}_i(r) \) by its definition.

Since \( v_i \) is concave, it is continuous on the interior of its domain \( \mathbb{R}_+^2 \). From (3.5), and since \( v_i \) is nonnegative, we see that \( v_i \) is continuous on \((x_0, y_0) = (0, 0)\) with \( v_i(0, 0) = 0 \). Then, \( \hat{v}_i \) is continuous on \( \mathbb{R}_+ \) with \( \hat{v}_i(0) = 0 \). It remains to prove the continuity of \( v_i \) at \((x_0, y_0)\) when \( x_0 = 0 \) or \( y_0 = 0 \). We shall rely on the following implication of the dynamic programming principle (see Proposition 4.1 below):

\[
v_i(x, y) = \sup_{c \in \mathcal{C}(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_i \left( R_{\tau_1} \right) \right]
= \sup_{c \in \mathcal{C}(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_i \left( x - \int_0^{\tau_1} c_t dt + y \frac{S_{\tau_1}}{S_0} \right) \right], \forall (x, y) \in \mathbb{R}_+^2,
\]  (3.10)
where $C(x)$ denotes the set of nonnegative adapted processes $(c_t)$ s.t. $\int_0^{\tau_1} c_t dt \leq x$ a.s. (i) We first consider the case $x_0 = 0$ (and $y_0 > 0$).

In this case, the constraint on consumption $c$ in $C(x_0)$ means that $c_t = 0, t \leq \tau_1$, so that (3.10) implies (3.9). Now, since $v_1$ is nondecreasing in $x$, we have: $v_1(x,y) \geq v_1(0,y)$. Moreover, by concavity and thus continuity of $v_1(0, \cdot)$, we have: $\lim_{y \to y_0} v_1(0,y) = v_1(0, y_0)$. This implies that $\lim_{(x,y) \to (0, y_0)} v_1(x,y) \geq v_1(0, y_0)$. The proof of the converse inequality requires more technical arguments. For any $x, y \geq 0$, we have:

$$v_1(x,y) = \sup_{c \in C(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho s} U(c_s) ds + e^{-\rho \tau_1} \hat{v}_{\tau_1} (x - \int_0^{\tau_1} c_s ds + y \frac{S_{\tau_1}}{S_0}) \right]$$

$$\leq \sup_{c \in C(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho s} U(c_s) ds \right] + \mathbb{E} \left[ e^{-\rho \tau_1} \hat{v}_{\tau_1} \left( x + y \frac{S_{\tau_1}}{S_0} \right) \right]$$

$$=: E_1(x) + E_2(x,y).$$

(3.11)

Now, by Jensen’s inequality, and since $U$ is concave, we have:

$$\int_0^\infty U \left( c_s \mathbb{1}_{\{s \leq \tau_1\}} \right) \rho e^{-\rho s} ds \leq U \left( \int_0^\infty c_s \mathbb{1}_{\{s \leq \tau_1\}} \rho e^{-\rho s} ds \right),$$

and thus:

$$\int_0^{\tau_1} e^{-\rho s} U(c_s) ds \leq \frac{U(\rho x)}{\rho}, \quad \text{a.s. } \forall c \in C(x),$$

(3.12)

by using the fact that $\int_0^{\tau_1} c_t dt \leq x$ a.s. By continuity of $U$ in 0 with $U(0) = 0$, this shows that $E_1(x)$ converges to zero when $x$ goes to $x_0 = 0$. Next, by continuity of $\hat{v}_1$, we have: $\hat{v}_{\tau_1} (x + y \frac{S_{\tau_1}}{S_0}) \to \hat{v}_{\tau_1} (y_0 \frac{S_{\tau_1}}{S_0})$ a.s. when $(x,y) \to (0, y_0)$. Let us check that this convergence is dominated. Indeed from (3.5), there is some positive constant $C$ s.t.

$$\hat{v}_{\tau_1} \left( x + y \frac{S_{\tau_1}}{S_0} \right) \leq C \left( x + y \frac{S_{\tau_1}}{S_0} \right)^p \leq C (x+y)^p \left( 1 \vee \left( \frac{S_{\tau_1}}{S_0} \right) \right).$$

Moreover,

$$\mathbb{E} \left[ e^{-\rho \tau_1} \left( \frac{S_{\tau_1}}{S_0} \right)^p \right] \leq \int_0^\infty \lambda_i e^{-\int_0^t \lambda_l e^{-\rho t} \left( \frac{S_t}{S_0} \right)^p dt} dt \leq \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty e^{-\rho t} \left( \frac{S_t}{S_0} \right)^p dt,$$

and so

$$\mathbb{E} \left[ e^{-\rho \tau_1} \left( \frac{S_{\tau_1}}{S_0} \right)^p \right] \leq \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty \mathbb{E} \left[ e^{-\rho t} \left( \frac{S_t}{S_0} \right)^p \right] dt \leq \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty e^{-(\rho-k(p)) t} dt < \infty,$$

where we used in the second inequality the supermartingale property in Lemma 3.1 (and, more precisely, equation (3.3)) for $x = 0, y = 1, c \equiv \zeta \equiv 0$. One can then apply the dominated convergence theorem to $E_2(x,y)$, to deduce that $E_2(x,y)$ converges to $\mathbb{E} \left[ e^{-\rho \tau_1} \hat{v}_{\tau_1} (y_0 \frac{S_{\tau_1}}{S_0}) \right]$ when $(x,y) \to (0, y_0)$. This, together with (3.9), (3.11), proves that $\limsup_{(x,y) \to (0, y_0)} v_1(x,y) \leq v_1(0, y_0)$, and thus the continuity of $v_1$ at $(0, y_0)$. 11
We consider the case $y_0 = 0$ (and $x_0 > 0$).
Similarly, as in the first case, from the nondecreasing and continuity properties of $v_i(.,0)$, we have: $\liminf_{(x,y) \to (x_0,0)} v_i(x,y) \geq v_i(x_0,0)$. Conversely, for any $x \geq 0$, and $c \in C(x)$, let us consider the stopping time $\tau_c = \inf \{ t \geq 0 : \int_0^t c_s ds = x_0 \}$. Then, the nonnegative adapted process $c'$ defined by: $c'_t = c_1\{ t \leq \tau_c \wedge \tau_1 \}$, lies obviously in $C(x_0)$. Furthermore,

$$\int_0^{\tau_1} e^{-\rho s} U(c_s) ds = \int_0^{\tau_c \wedge \tau_1} e^{-\rho s} U(c'_s) ds + \int_{\tau_c \wedge \tau_1}^{\tau_1} e^{-\rho s} U(c_s) ds \leq \int_0^{\tau_1} e^{-\rho s} U(c'_s) ds + \frac{U(\rho (x-x_0)_+)}{\rho},$$

(3.13)

by the same Jensen’s arguments as in (3.12). Also note that for all $y \geq 0$,

$$\hat{v}_{\tau_1} \left( x - \int_0^{\tau_1} c'_t dt + y \frac{S_{\tau_1}}{S_0} \right) \leq \hat{v}_{\tau_1} \left( x_0 - \int_0^{\tau_1} c'_t dt + (x-x_0)_+ + y \frac{S_{\tau_1}}{S_0} \right) \leq \hat{v}_{\tau_1} \left( x_0 - \int_0^{\tau_1} c'_t dt \right) + \hat{v}_{\tau_1} \left( (x-x_0)_+ + y \frac{S_{\tau_1}}{S_0} \right),$$

(3.14)

where we have used the fact that $\hat{v}_i$ is nondecreasing, and subadditive (as a concave function with $\hat{v}_i(0) \geq 0$). By adding the two inequalities (3.13)-(3.14), and taking expectation, we obtain from (3.10):

$$v_i(x,y) \leq v_i(x_0,0) + \frac{U(\rho (x-x_0)_+)}{\rho} + E \left[ e^{-\rho \tau_1} \hat{v}_{\tau_1} \left( (x-x_0)_+ + y \frac{S_{\tau_1}}{S_0} \right) \right],$$

and by the same domination arguments as in the first case, this shows that

$$\limsup_{(x,y) \to (x_0,0)} v_i(x,y) \leq v_i(x_0,0),$$

which ends the proof.

**Remark 3.1** The above proof of continuity of the value functions at the boundary by means of the dynamic programming principle is somehow different from other similar proofs that one can find e.g. in [8, 23, 29]. Indeed in such problems the proof of dynamic programming principle is done (or referred to) in two parts: the “easy” one ($\leq$) which does not require continuity of the value function, and the “difficult” one ($\geq$) which requires the continuity of the value function up to the boundary. The proof of continuity at the boundary in such cases uses only the “easy” inequality. In our case, due to the specific boundary condition of our problem, the “easy” inequality is not enough to prove the continuity at the boundary. We need also the “hard” inequality. For this reason we prove, in the companion paper [11], a proof of the dynamic programming principle in our case that, in the “hard” inequality part, uses the continuity of $v_i$ in the interior and the continuity of its restriction to the boundary (which are both implied by the concavity and by the growth condition (3.5)).
Remark 3.2 For simplicity we have restricted our study to the case where $U$ is defined on the positive half-line $\mathbb{R}_+$. With some work, our results can be extended to the case $U(0) = -\infty$, assuming $U(c) \geq -Kc^q$, for some $K \geq 0$, $q < 0$. In that case (assuming $\rho > 0$), $v_i(x, y) > -\infty$ whenever $x > 0$, $y \geq 0$, while $v_i(0^+, y) = -\infty$ for all $y$. The proof of such results follows substantially standard arguments. For similar results with similar methods one can see e.g [9], Section 4 or [15], Section 11.

The following technical lemma will be needed later on in the feedback characterization of optimal controls (see the proof of Lemma 5.1).

Lemma 3.2 There exists some positive constant $C > 0$ s.t. for any $(x, y, i) \in (0, \infty) \times \mathbb{R}_+ \times \mathbb{I}_d$,

$$\frac{\partial v_i}{\partial x}(x^+, y) := \lim_{\delta \downarrow 0} \frac{v_i(x + \delta, y) - v_i(x, y)}{\delta} \geq C U'(2x). \quad (3.15)$$

Proof. Fix some $x > 0$, $y \geq 0$, and set $x_1 = x + \delta$ for $\delta > 0$. For any $(\zeta, c) \in \mathcal{A}_i(x, y)$ with associated cash/amount in shares $(X, Y)$, notice that $(\tilde{\zeta}, \tilde{c}) := (\zeta, c + \delta 1_{[0,1]}(0))$ is admissible for $(x_1, y)$. Indeed, the associated cash amount satisfies

$$\tilde{X}_t = X_t + (x_1 - x) - \int_0^t \delta 1_{[0,1]}(s) ds \geq X_t \geq 0,$$

while the amount in cash $\tilde{Y}_t = Y_t \geq 0$ since $\zeta$ is unchanged. Thus, $(\tilde{\zeta}, \tilde{c}) \in \mathcal{A}_i(x_1, y)$, and we have

$$v_i(x_1, y) \geq \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(\tilde{c}_t) dt \right]$$

$$= \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right] + \mathbb{E} \left[ \int_0^{1\wedge \tau_1} e^{-\rho t} \left( U(c_t + \delta) - U(c_t) \right) dt \right]. \quad (3.16)$$

Now, by concavity of $U$: $U(\delta c_t + \delta) - U(c_t) \geq \delta U'(c_t + \delta)$, and

$$\int_0^{1\wedge \tau_1} e^{-\rho t} \left( U(c_t + \delta) - U(c_t) \right) dt \geq \int_0^{1\wedge \tau_1} e^{-\rho t} \delta U'(c_t + \delta) dt$$

$$\geq \delta e^{-\rho(1\wedge \tau_1)} \int_0^{1\wedge \tau_1} U'(c_t + \delta) dt$$

$$\geq \delta e^{-\rho(1\wedge \tau_1)} U'(2x + \delta) \int_0^{1\wedge \tau_1} 1_{\{c_t < 2x\}} dt. \quad (3.17)$$

Moreover,

$$2x \int_0^{1\wedge \tau_1} 1_{\{c_t \geq 2x\}} dt \leq \int_0^{1\wedge \tau_1} c_t dt \leq x,$$

since $(\zeta, c)$ is admissible for $(x, y)$, so that

$$\int_0^{1\wedge \tau_1} 1_{\{c_t < 2x\}} dt \geq (1 \wedge \tau_1) - \left( \frac{1}{2} \wedge \tau_1 \right) \geq \frac{1}{2} 1_{\{\tau_1 \geq 1\}}. \quad (3.18)$$

13
By combining (3.17) and (3.18), and taking the expectation, we get
\[
\mathbb{E} \left[ \int_0^{1 \wedge \tau_1} e^{-\rho t} (U(c_t + \delta) - U(c_t)) dt \right] \geq \delta U'(2x + \delta) \mathbb{E} \left[ e^{-\rho (1 \wedge \tau_1)} \frac{1}{2} \mathbb{1}_{\{\tau_1 \geq 1\}} \right].
\]

By taking the supremum over \((\zeta, c)\) in (3.16), we thus obtain with the above inequality
\[
v_i(x + \delta, y) \geq v_i(x, y) + \delta U'(2x + \delta) \mathbb{E} \left[ e^{-\rho (1 \wedge \tau_1)} \frac{1}{2} \mathbb{1}_{\{\tau_1 \geq 1\}} \right].
\]

Finally, by choosing \(C = \mathbb{E} \left[ e^{-\rho (1 \wedge \tau_1)} \frac{1}{2} \mathbb{1}_{\{\tau_1 \geq 1\}} \right] > 0\), and letting \(\delta\) go to 0, we obtain the required inequality (3.15). \(\square\)

4 Dynamic programming and viscosity characterization

In this section, we provide an analytic characterization of the value functions \(v_i, \ i \in \mathbb{I}_d\), to our control problem (2.14), by relying on the dynamic programming principle, which is shown to hold and formulated as:

**Proposition 4.1 (Dynamic programming principle)** For all \((x, y, i) \in \mathbb{R}_+^2 \times \mathbb{I}_d\), and any stopping time \(\tau\), we have
\[
v_i(x, y) = \sup_{(\zeta, c) \in \mathcal{A}_i(x, y)} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} v_{\tau_1}(X_\tau, Y_\tau) \right]. \tag{4.1}
\]

**Proof.** See [11]. \(\square\)

The associated dynamic programming system (also called Hamilton-Jacobi-Bellman or HJB system) for \(v_i, \ i \in \mathbb{I}_d\), is written as
\[
\rho v_i - b_i y \frac{\partial v_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2} - \tilde{U} \left( \frac{\partial v_i}{\partial x} \right) - \sum_{j \neq i} q_{ij} \left[ v_j(x, y(1 - \gamma_{ij})) - v_i(x, y) \right] - \lambda_i \left[ \hat{v}_i(x + y) - v_i(x, y) \right] = 0, \quad (x, y) \in (0, \infty) \times \mathbb{R}_+, \ i \in \mathbb{I}_d,
\]

together with the boundary condition (3.9) on \(\{0\} \times \mathbb{R}_+\) for \(v_i, \ i \in \mathbb{I}_d\). Notice that, arguing as one does for the deduction of the HJB system above, the boundary condition (3.9) may also be written as:
\[
\rho v_i(0, \cdot) - b_i y \frac{\partial v_i}{\partial y}(0, \cdot) - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2}(0, \cdot) - \sum_{j \neq i} q_{ij} \left[ v_j(0, y(1 - \gamma_{ij})) - v_i(0, y) \right] - \lambda_i \left[ \hat{v}_i(y) - v_i(0, y) \right] = 0, \quad y > 0, \ i \in \mathbb{I}_d. \tag{4.3}
\]

Notice that in this boundary condition the term \(\tilde{U} \left( \frac{\partial v_i}{\partial x} \right)\) has disappeared. This implicitly comes from the fact that, on the boundary \(\{x = 0\}\) the only admissible consumption rate
is $c = 0$; this is in contrast to the boundary $\{ y = 0 \}$ which does not induce any constraints on the policy (and as a consequence (4.2) also holds on $(0, \infty) \times \{0\}$). We will say more on this in studying the case of CRRA utility function in Section 5.1.

In our context, the notion of viscosity solution to the non local second-order system (4.2) is defined as follows.

**Definition 4.1** (i) A $d$-tuple $w = (w_i)_{i \in I_d}$ of continuous functions on $\mathbb{R}^2_+$ is a viscosity supersolution (resp. subsolution) to the non local second-order system (4.2) if

$$
\rho \varphi_i(\bar{x}, \bar{y}) - b_i \bar{y} \frac{\partial \varphi_i}{\partial y}(\bar{x}, \bar{y}) - \frac{1}{2} \sigma_i^2 \bar{y}^2 \frac{\partial^2 \varphi_i}{\partial y^2}(\bar{x}, \bar{y}) - \tilde{U} \left( \frac{\partial \varphi_i}{\partial x}(\bar{x}, \bar{y}) \right) - \sum_{j \neq i} q_{ij} \left[ \varphi_j(\bar{x}, \bar{y}(1 - \gamma_{ij})) - \varphi_i(\bar{x}, \bar{y}) \right] - \lambda_i \left[ \varphi_i(\bar{x} + \bar{y}) - \varphi_i(\bar{x}, \bar{y}) \right] \geq (\text{resp.} \leq) 0,
$$

for all $d$-tuple $\varphi = (\varphi_i)_{i \in I_d}$ of $C^2$ functions on $\mathbb{R}^2_+$, and any $(\bar{x}, \bar{y}, i) \in (0, \infty) \times \mathbb{R}_+ \times I_d$, such that $w_i(\bar{x}, \bar{y}) = \varphi_i(\bar{x}, \bar{y})$, and $w \geq (\text{resp.} \leq) \varphi$ on $\mathbb{R}^2_+ \times I_d$.

(ii) A $d$-tuple $w = (w_i)_{i \in I_d}$ of continuous functions on $\mathbb{R}^2_+$ is a viscosity solution to (4.2) if it is both a viscosity supersolution and subsolution to (4.2).

The main result of this section is to provide an analytic characterization of the value functions in terms of viscosity solutions to the dynamic programming system.

**Theorem 4.1** The value function $v = (v_i)_{i \in I_d}$ is the unique viscosity solution to (4.2) satisfying the boundary condition (3.9), and the growth condition (3.5).

**Proof.** The proof of viscosity property follows as usual from the dynamic programming principle. The uniqueness and comparison result for viscosity solutions is proved by rather standard arguments, up to some specificities related to the non local terms and state constraints induced by our hybrid jump-diffusion control problem. The details are written in the companion paper [11].

5 The case of CRRA utility

In this section, we consider the case where the utility function is of CRRA type in the form:

$$
U(x) = \frac{x^p}{p}, \quad x > 0, \text{ for some } p \in (0, 1).
$$

We shall exploit the homogeneity property of the CRRA utility function, and go beyond the viscosity characterization of the value function in order to prove some regularity results, and provide an explicit characterization of the optimal control through a verification theorem. We next propose an iterative numerical scheme to compute the value functions and optimal strategies, and illustrate with some tests the impact of our illiquidity features.
5.1 Regularity results and verification theorem

For any \((i, x, y) \in \mathbb{I}_d \times \mathbb{R}^2_+\), \((\zeta, c) \in \mathcal{A}(x, y)\) with associated state process \((X, Y)\), we notice from the dynamics (2.3)-(2.2) that for any \(k \geq 0\), the state \((kX, kY)\) is associated to the control \((k\zeta, kc)\). Thus, for \(k > 0\), we have \((\zeta, c) \in \mathcal{A}_i(x, y)\) iff \((k\zeta, kc) \in \mathcal{A}(kx, ky)\), and so from the homogeneity property of the power utility function \(U\) in (5.1), we have:

\[
v_i(kx, ky) = k^p v_i(x, y), \quad \forall (i, x, y) \in \mathbb{I}_d \times \mathbb{R}^2_+, \quad k \in \mathbb{R}_.\tag{5.2}
\]

Let us now consider the change of variables:

\[(x, y) \in \mathbb{R}^2_+ \setminus \{(0, 0)\} \quad \mapsto \quad (r = x + y, z = \frac{y}{x+y}) \in (0, \infty) \times [0, 1].\]

Then, from (5.2), we have \(v_i(x, y) = v_i(r(1-z), rz) = r^p v_i(1-z, z)\), and we can separate the value function \(v_i\) into:

\[
v_i(x, y) = U(x+y)\varphi_i\left(\frac{y}{x+y}\right), \quad \forall (i, x, y) \in \mathbb{I}_d \times (\mathbb{R}^2_+ \setminus \{(0, 0)\}) \tag{5.3}
\]

where \(\varphi_i(z) = p v_i(1-z, z)\) is a continuous function on \([0, 1]\). By substituting this transformation for \(v_i\) into the dynamic programming equation (4.2) and the boundary condition (4.3), and after some straightforward calculations, we see that \(\varphi = (\varphi_i)_{i \in \mathbb{I}_d}\) should solve the system of (nonlocal) ordinary differential equations (ODEs):

\[
\begin{align*}
(p - pb_i z + \frac{1}{2} p(1-p)\sigma_i^2 z^2)\varphi_i - (1-p)\left(\varphi_i - \frac{z}{p} \varphi_i'\right)^{-\frac{p}{1-p}} \\
- z(1-z)(b_i - z(1-p)\sigma_i^2)\varphi_i' - \frac{1}{2} z^2(1-z)^2 \sigma_i^2 \varphi_i'' \\
- \sum_{j \neq i} q_{ij} \left[(1-z\gamma_{ij})p \varphi_j\left(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\right) - \varphi_i(z)\right] \\
- \lambda_i \sup_{\pi \in [0, 1]} \left[\varphi_i(\pi) - \varphi_i(z)\right] = 0, \quad z \in [0, 1], \quad i \in \mathbb{I}_d,
\end{align*}
\]

together with the boundary condition for \(z = 1:\)

\[
(p - pb_i + \frac{1}{2} p(1-p)\sigma_i^2)\varphi_i(1) \\
- \sum_{j \neq i} q_{ij} [(1-\gamma_{ij})p \varphi_j(1) - \varphi_i(1)] - \lambda_i \sup_{\pi \in [0, 1]} \left[\varphi_i(\pi) - \varphi_i(1)\right] = 0, \quad i \in \mathbb{I}_d. \tag{5.5}
\]

The following boundary condition for \(z = 0\), obtained formally by taking \(z = 0\) in (5.4),

\[
\begin{align*}
\rho \varphi_i(0) - (1-p)(\varphi_i(0))^{-\frac{p}{1-p}} \\
- \sum_{j \neq i} q_{ij} [\varphi_j(0) - \varphi_i(0)] - \lambda_i \sup_{\pi \in [0, 1]} \left[\varphi_i(\pi) - \varphi_i(0)\right] = 0, \quad i \in \mathbb{I}_d,
\end{align*}
\]

is proved rigorously in the below Proposition.
Proposition 5.1 The d-tuple $\varphi = (\varphi_i)_{i \in \mathbb{I}_d}$ is concave on $[0,1]$, $C^2$ on $(0,1)$. We further have
\[
\lim_{z \to 0} z\varphi_i'(z) = 0, \quad \lim_{z \to 0} z^2 \varphi_i''(z) = 0, \quad \lim_{z \to 1} (1 - z)\varphi_i'(z) = 0, \quad \lim_{z \to 1} (1 - z)^2 \varphi_i''(z) = 0, \quad \lim_{z \to 1} \varphi_i'(z) = -\infty,
\]
and $\varphi$ is the unique bounded classical solution of (5.4) on $(0,1)$, with boundary conditions (5.5)-(5.6).

Proof. Since $\varphi_i(z) = p v_i(1 - z, z)$, and by concavity of $v_i(.,.)$ in both variables, it is clear that $\varphi_i$ is concave on $[0,1]$. From the viscosity property of $v_i$ in Theorem 4.1, and the change of variables (5.3), this implies that $\varphi$ is the unique bounded viscosity solution to (5.4) on $[0,1]$, satisfying the boundary condition (5.5). Now, recalling that $q_{ii} = -\sum_{j \neq i} q_{ij}$, we observe that the system (5.4) can be written as:
\[
(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2} p(1 - p)\sigma_i^2 z^2)\varphi_i - z(1 - z)(b_i - z(1 - p)\sigma_i^2)\varphi_i'
- \frac{1}{2} z^2(1 - z)^2 \sigma_i^2 \varphi_i'' - (1 - p)\varphi_i - \frac{z}{p} \varphi_i'
= \sum_{j \neq i} q_{ij} \left[ (1 - z\gamma_{ij}) \varphi_j \left( \frac{z(1 - \gamma_{ij})}{1 - z\gamma_{ij}} \right) \right] + \lambda_i \sup_{\pi \in [0,1]} \varphi_i(\pi), \quad z \in (0,1), \ i \in \mathbb{I}_d. \quad (5.12)
\]
Let us fix some $i \in \mathbb{I}_d$, and an arbitrary compact $[a, b] \subset (0,1)$. By standard results, see e.g. [5], we know that the second-order ODE:
\[
(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2} p(1 - p)\sigma_i^2 z^2)w_i - z(1 - z)(b_i - z(1 - p)\sigma_i^2)w_i'
- \frac{1}{2} z^2(1 - z)^2 \sigma_i^2 w_i'' - (1 - p)(w_i - \frac{z}{p} w_i')^{-\frac{p}{p-1}}
= \sum_{j \neq i} q_{ij} \left[ (1 - z\gamma_{ij}) \varphi_j \left( \frac{z(1 - \gamma_{ij})}{1 - z\gamma_{ij}} \right) \right] + \lambda_i \sup_{\pi \in [0,1]} \varphi_i(\pi) \quad (5.13)
\]
has a unique viscosity solution $w_i$ satisfying $w_i(a) = \varphi_i(a)$, $w_i(b) = \varphi_i(b)$, and that this solution $w_i$ is twice differentiable on $[a, b]$ since the second term $z(1 - z)\sigma_i^2$ is uniformly elliptic on $[a, b]$, see [18]. Since $\varphi_i$ is a viscosity solution to (5.13) by (5.12), we deduce by uniqueness that $\varphi_i = w_i$ on $[a, b]$. Since $a, b$ are arbitrary, this means that $\varphi$ is $C^2$ on $(0,1)$. By concavity of $\varphi_i$, we have for all $z \in (0,1)$,
\[
\frac{\varphi_i(1) - \varphi_i(z)}{1 - z} \leq \varphi_i'(z) \leq \frac{\varphi_i(z) - \varphi_i(0)}{z}.
\]
Letting $z \to 0$ and $z \to 1$, and by continuity of $\varphi_i$, we obtain (5.7) and (5.9).
Now letting $z$ go to 0 in (5.4), we obtain $\lim_{z \to 0} z^2 \varphi''(z) = l$ for some finite $l \leq 0$. If $l < 0$, $z^2 \varphi''(z) \leq -\frac{l}{2}$ whenever $z \leq \eta$, for some $\eta > 0$. By writing that

$$z(\varphi'_i(z) - \varphi'_i(\eta)) = z \int_\eta^z \varphi''(u)du \geq -\frac{l}{2}z \int_\eta^\eta \frac{du}{u^2} = \frac{l}{2} \left( \frac{1}{\eta} - \frac{1}{z} \right),$$

and sending $z \to 0$, we get $\liminf_{z \to 0} z\varphi'_i(z) \geq -l/2$, which contradicts (5.7). Thus $l = 0$, and the boundary condition (5.6) follows by letting $z \to 0$ in (5.4). In the same way, letting $z \to 1$ in (5.4) and comparing with (5.5), we have

$$\lim_{z \to 1} \frac{1}{2} (1 - z)^2 \varphi''(z) = (\varphi_i(1) - \varphi_i'(1-))^{-\frac{p}{1-p}} \in [0, \infty].$$

(5.9) implies that this limit is 0, and we obtain (5.10) and (5.11).

**Remark 5.1** From (5.3) and the above Proposition, we deduce that the value functions $v_i$, $i \in \mathbb{I}_d$, are $C^2$ on $(0, \infty) \times (0, \infty)$, and so are solutions to the dynamic programming system (4.2) on $(0, \infty) \times (0, \infty)$ in classical sense.

**Remark 5.2** With some straightforward modifications, the viscosity results of Section 4 could be extended to more general regime-switching diffusions (assuming e.g. Lipschitz coefficients). However, the dimension reduction in the case of power utility which allowed us to prove the above regularity, and will make the numerical resolution easier, is specific to our (regime-switching) Black Scholes dynamics.

We now provide an explicit construction of the optimal investment/consumption strategies in feedback form in terms of the smooth solution $\varphi$ to (5.4)-(5.6). We start with the following Lemma.

**Lemma 5.1** For any $i \in \mathbb{I}_d$, let us define:

$$c^*(i, z) = \begin{cases} (\varphi_i(z) - \frac{\pi^*}{p} \varphi'_i(z))^{\frac{1}{1-p}} & \text{when } 0 < z < 1 \\ (\varphi_i(0))^{\frac{1}{1-p}} & \text{when } z = 0 \\ 0 & \text{when } z = 1 \end{cases}$$

$$\pi^*(i) \in \arg \max_{\pi \in [0,1]} \varphi_i(\pi).$$

Then for each $i \in \mathbb{I}_d$, $c^*(i, \cdot)$ is continuous on $[0, 1]$, $C^1$ on $(0, 1)$, and given any initial conditions $(r, z) \in \mathbb{I}_d \times \mathbb{R}_+ \times [0, 1]$, there exists a solution $(\hat{R}_t, \hat{Z}_t)_{t \geq 0}$ valued in $\mathbb{R}_+ \times [0, 1]$ to the SDE:

$$d\hat{R}_t = \hat{R}_t - \hat{Z}_t - \left( b_{1, \cdots, i} dt + \sigma_{1, \cdots, i} dW_t - \gamma_{1, \cdots, i} \frac{dN_t^{l, i}}{1 - \hat{Z}_t} \right) - \hat{R}_t c^*(I_{t-}, \hat{Z}_t) dt,$$

$$d\hat{Z}_t = \hat{Z}_t - (1 - \hat{Z}_t) \left( [b_{1, \cdots, i} - \hat{Z}_t \sigma_{1, \cdots, i}] dt + \sigma_{1, \cdots, i} dW_t - \frac{\gamma_{1, \cdots, i}}{1 - \hat{Z}_t} \frac{dN_t^{l, i}}{1 - \hat{Z}_t} \right)$$

$$\quad + (\pi^*(I_{t-}) - \hat{Z}_t) dN_t + \hat{Z}_t c^*(I_{t-}, \hat{Z}_t) dt.$$ (5.15)

Moreover, if $r > 0$, then $\hat{R}_t > 0$, a.s. for all $t \geq 0$. 

18
Proof. First notice that Lemma 3.2, written in terms of the variables \((r, z)\), is formulated equivalently as
\[
\varphi_1(z) - \frac{z}{p} \varphi_1'(z) \geq C 2^{p-1}(1-z)^{p-1}, \quad z \in (0, 1).
\]
This implies that \(c^*(i, \cdot)\) is well-defined on \((0, 1)\), and \(C^1\) since \(\varphi\) is \(C^2\). The continuity of \(c^*(i, \cdot)\) at 0 and 1 comes from (5.7) and (5.11).

Let us show the existence of a solution \(Z\) to the SDE (5.15). We start by the existence of a solution for \(t < \tau_1\) (recall that \((\tau_n)\) is the sequence of jump times of \(N\)). In the case where \(z = 1\) (resp. \(z = 0\)), then \(Z_t \equiv 1\) (resp. \(Z_t \equiv 0\)) is clearly a solution on \([0, \tau_1]\).

Consider now the case where \(z \in (0, 1)\). From the local Lipschitz property of \(z \mapsto z c^*(i, z)\), and recalling that \(\gamma_{ij} < 1\), we know, adapting e.g. the result of Theorem 38, page 303 of [24], that there exists a solution to
\[
\begin{aligned}
d\hat{Z}_t &= \hat{Z}_{t-} (1 - \hat{Z}_{t-}) \{ (b_{I_{t-}} - \hat{Z}_{t-} \sigma_{I_{t-}}^2) dt + \sigma_{I_{t-}} dW_t - \frac{\gamma_{I_{t-}, I_{t-}}}{1 - \hat{Z}_{t-} \gamma_{I_{t-}, I_{t-}}} dN_{I_{t-}, I_{t-}} \} \\
&\quad + \hat{Z}_t c^*(I_{t-}, \hat{Z}_{t-}) dt,
\end{aligned}
(5.16)
\]
which is valued in \([0, 1]\) up to time \(t < \tau_1 := \tau_1 \wedge \left( \lim_{\varepsilon \to 0} \inf \{ t \geq 0|\hat{Z}_t (1 - \hat{Z}_t) \leq \varepsilon \} \right)\). By noting that \(\hat{Z}_t \geq \hat{Z}_t^0\), where
\[
\hat{Z}_t^0 = \frac{z S_{I_{t-}}}{S_{I_{t-}}} + (1 - z), \quad t \geq 0,
\]
is the solution to (5.16) without the consumption term, and since \(S\) is locally bounded away from 0, we have \(\lim_{t \to \tau_1} \hat{Z}_t = 1\) on \(\{ \tau_1 < \tau_1 \}\). By extending \(\hat{Z}_t \equiv 1\) on \([\tau_1, \tau_1]\), we obtain actually a solution on \([0, \tau_1]\). Then at \(\tau_1\), by taking \(\hat{Z}_{\tau_1} = \pi^*(I_{\tau_1-})\), we obtain a solution to (5.15) valued in \([0, 1]\) on \([0, \tau_1]\). Next, we obtain similarly a solution to (5.15) on \([\tau_1, \tau_2]\) starting from \(\hat{Z}_{\tau_1}\). Finally, since \(\tau_n \not\to \infty\), a.s., by pasting we obtain a solution to (5.15) for \(t \in \mathbb{R}_+\).

Given a solution \(\hat{Z}\) to (5.15), the solution \(\hat{R}\) to (5.14) starting from \(r\) at time 0 is determined by the stochastic exponential:
\[
\hat{R}_t = r \cdot \mathcal{E} \left( \int_0^t \hat{Z}_{s-} \left( b_{I_{s-}} ds + \sigma_{I_{s-}} dW_s - \gamma_{I_{s-}, I_{s-}} dN_{I_{s-}, I_{s-}} \right) - c^*(I_{s-}, \hat{Z}_{s-}) dt \right)_t.
\]
Since \(-\hat{Z}_{t-} \gamma_{I_{t-}, I_{t-}} > -1\), we see that \(R_t > 0, \ t \geq 0\), whenever \(r > 0\), while \(R \equiv 0\) if \(r = 0\). \(\Box\)

**Proposition 5.2** Given some initial conditions \((i, x, y) \in \mathbb{I}_d \times (\mathbb{R}_+^2 \setminus \{(0, 0)\})\), let us consider the pair of processes \((\hat{\zeta}, \hat{c})\) defined by:
\[
\hat{\zeta}_t = \hat{R}_t (\pi^*(I_{t-}) - \hat{Z}_{t-}) \tag{5.17}
\]
\[
\hat{c}_t = \hat{R}_t c^*(I_{t-}, \hat{Z}_{t-}) \tag{5.18}
\]
where the functions \((e^*, \pi^*)\) are defined in Lemma 5.1, and \((\hat{R}, \hat{Z})\) are solutions to (5.14)-(5.15), starting from \(r = x + y, z = y/(x+y)\), with \(I\) starting from \(i\). Then, \((\hat{\zeta}, \hat{c})\) is an optimal investment/consumption strategy in \(A_i(x, y)\), with associated state process \((\hat{X}, \hat{Y}) = (\hat{R}(1 - \hat{Z}), \hat{R}\hat{Z})\), for \(v_i(x, y) = U(r)\phi_r(z)\).

**Proof.** For such choice of \((\hat{\zeta}, \hat{c})\), the dynamics of \((\hat{R}, \hat{Z})\) evolve according to (2.10)-(2.11) with a feedback control \((\hat{\zeta}, \hat{c})\), and thus correspond (via Itô’s formula) to a state process \((\hat{X}, \hat{Y}) = (\hat{R}(1 - \hat{Z}), \hat{R}\hat{Z})\) governed by (2.2)-(2.3), starting from \((x, y)\), and satisfying the nonbankruptcy constraint (2.5). Thus, \((\hat{\zeta}, \hat{c}) \in A_i(x, y)\). Moreover, since \(r = x + y > 0\), this implies that \(\hat{R} > 0\), and so \((\hat{X}, \hat{Y})\) lies in \(\mathbb{R}^2_+ \setminus \{(0,0)\}\).

As in the proof of the standard verification theorem, we would like to apply Itô’s formula to the function \(e^{-rt}v(\hat{X}_t, \hat{Y}_t, I_t)\) (denoting by \(v(x, y, i) = v_i(x, y) = U(x+y)\phi_r(y/(x+y))\)). However this is not immediately possible since the process \((\hat{X}_t, \hat{Y}_t)\) may reach the boundary of \(\mathbb{R}^2_+\) where the derivatives of \(v\) do not have classical sense. To overcome this problem, we approximate the function \(\phi_i\) (and so \(v(x, y, i)\)) as follows. We define, for every \(\epsilon > 0\) a function \(\phi^\epsilon = (\phi_i^\epsilon)_{i \in \mathbb{I}} \in C^2([0, 1], \mathbb{R}^d)\) as in the proof of Theorem 4.24 in [8], such that

- \(\phi_i^\epsilon = \phi_i\) on \([\epsilon, 1 - \epsilon]\),
- \(\phi_i^\epsilon \rightarrow \phi_i\) uniformly on \([0, 1]\) as \(\epsilon \rightarrow 0\),
- \(z(1 - z)(\phi_i^\epsilon)' \rightarrow z(1 - z)\phi_i'\) uniformly on \([0, 1]\) as \(\epsilon \rightarrow 0\),
- \(z^2(1 - z)^2(\phi_i^\epsilon)'' \rightarrow z^2(1 - z)^2\phi_i''\) uniformly on \([0, 1]\) as \(\epsilon \rightarrow 0\).

Now we can apply Dynkin’s formula to the function \(v^\epsilon(x, y, i) = U(x+y)\phi_i^\epsilon(y/(x+y))\) calculated on the process \((\hat{X}, \hat{Y}, I)\) between time 0 and \(\nu_n \wedge T\), where \(\nu_n = \inf\{t \geq 0 : \hat{X}_t + \hat{Y}_t \geq n\}\):

\[
v^\epsilon(x, y, i) = \mathbb{E}
\left[
    e^{-r(\nu_n \wedge T)}v^\epsilon(\hat{X}_{\nu_n \wedge T}, \hat{Y}_{\nu_n \wedge T}, I_{\nu_n \wedge T})
    + \int_0^{\nu_n \wedge T} e^{-rt}
    \left(
        \rho v^\epsilon + \hat{c}_t \frac{\partial v^\epsilon}{\partial x} - b_{I_{t-}} \hat{Y}_t - \frac{1}{2} \sigma_{I_{t-}}^2 \hat{Y}_t \frac{\partial^2 v^\epsilon}{\partial y^2}
        - \sum_{j \neq i} q_{I_{t-}} [v^\epsilon(\hat{X}_{t-}, \hat{Y}_{t-}(1 - \gamma_{I_{t-}})) - v^\epsilon(\hat{X}_{t-}, \hat{Y}_{t-}, I_{I_{t-}})]
    \right)
    dt
\right]
\]

(5.19)

We denote by \(\hat{\zeta}(i, r, z) = r(\pi^*(i) - z), \hat{c}(i, r, z) = r\pi^*(i, z)\), and define \(g^\epsilon\) on \((\mathbb{R}^2_+ \setminus \{(0,0)\}) \times \mathbb{I}_d\) by

\[
    \rho v^\epsilon_i - b_y \frac{\partial v^\epsilon_i}{\partial y} - \frac{1}{2} \sigma_y^2 \frac{\partial^2 v^\epsilon_i}{\partial y^2} + \hat{c}(i, x + y, \frac{y}{x+y}) \frac{\partial v^\epsilon_i}{\partial x} - U(\hat{c}(i, x + y, \frac{y}{x+y}))
    - \sum_{j \neq i} q_{ij} [v^\epsilon_j(x,y(1 - \gamma_{ij})) - v^\epsilon_j(x, y)]
    - \lambda_i v^\epsilon_i \left(\hat{\zeta}(i, x + y, \frac{y}{x+y}) + \hat{c}(i, x + y, \frac{y}{x+y})\right) - v^\epsilon_i(x, y)
    =: g^\epsilon_i(x, y).
\]

20
so that from (5.19):

\[ v^\varepsilon(i, x, y) = \mathbb{E}\left[ e^{-\rho(\nu_n \wedge T)}v^\varepsilon(\hat{X}_{\nu_n \wedge T}, \hat{Y}_{\nu_n \wedge T}, I_{\nu_n \wedge T}) + \int_{\nu_n \wedge T}^{T} e^{-\rho t}(U(\hat{c}_t) + g^\varepsilon(\hat{X}_t, \hat{Y}_t, I_t))dt \right]. \] (5.20)

Notice that the properties of \( \varphi^\varepsilon \) imply:

- \( v^\varepsilon_i = v_i \) on \( \{ \varepsilon \leq \frac{y}{x+y} \leq 1 - \varepsilon \} \),
- \( v^\varepsilon_i \to v_i \) uniformly on bounded subsets of \( \mathbb{R}^2_+ \),
- \( \hat{c}(i, x+y, \frac{y}{x+y}) \frac{\partial v^\varepsilon_i}{\partial x} \to \begin{cases} \hat{c}(i, x+y, \frac{y}{x+y}) \frac{\partial v_i}{\partial x}, & x > 0 \\ 0, & x = 0 \end{cases} \) uniformly on bounded subsets of \( \mathbb{R}^2_+ \),
- \( y \frac{\partial v^\varepsilon_i}{\partial y} \to \begin{cases} y \frac{\partial v_i}{\partial y}, & y > 0 \\ 0, & y = 0 \end{cases} \) uniformly on bounded subsets of \( \mathbb{R}^2_+ \),
- \( y^2 \frac{\partial^2 v^\varepsilon_i}{\partial y^2} \to \begin{cases} y^2 \frac{\partial^2 v_i}{\partial y^2}, & y > 0 \\ 0, & y = 0 \end{cases} \) uniformly on bounded subsets of \( \mathbb{R}^2_+ \).

The details can be found in [10]. Since \( v \) is a classical solution of (4.2) on \( (0, \infty) \times (0, \infty) \), this implies that \( g^\varepsilon \) converges to 0 uniformly on bounded subsets of \( \mathbb{R}^2_+ \) when \( \varepsilon \) goes to 0. We then obtain by letting \( \varepsilon \to 0 \) in (5.20):

\[ v(x, y, i) = \mathbb{E}\left[ \int_{0}^{\infty} e^{-\rho t}U(\hat{c}_t)dt \right], \]

From the growth condition (3.5) we get

\[ \mathbb{E}\left[ e^{-\rho(\nu_n \wedge T)}v(\hat{X}_{\nu_n \wedge T}, \hat{Y}_{\nu_n \wedge T}, I_{\nu_n \wedge T}) \right] \leq CE\left[ e^{-\rho(\nu_n \wedge T)}R^p_{\nu_n \wedge T} \right]. \]

So, using Lemma 3.1, sending \( n \) to infinity, and then \( T \) to infinity, we get

\[ \lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[ e^{-\rho(\nu_n \wedge T)}v(\hat{X}_{\nu_n \wedge T}, \hat{Y}_{\nu_n \wedge T}, I_{\nu_n \wedge T}) \right] = 0. \]

Applying monotone convergence theorem to the second term in the r.h.s. of (5.20), we then obtain

\[ v_i(x, y) = \mathbb{E}\left[ \int_{0}^{\infty} e^{-\rho t}U(\hat{c}_t)dt \right], \]

which proves the optimality of \( (\hat{\zeta}, \hat{c}) \). \( \square \)
5.2 Numerical analysis

We focus on the numerical resolution of the system of ODEs (5.4)-(5.6) satisfied by \((\varphi_i)_{i \in \mathbb{I}_d}\), and rewritten for all \(i \in \mathbb{I}_d\) as:

\[
(\rho - q_{ii} + \lambda_i - p b_i z + \frac{1}{2} p (1 - p) \sigma_i^2 z^2) \varphi_i - z (1 - z) (b_i - z (1 - p) \sigma_i^2) \varphi_i' = - \frac{1}{2} z^2 (1 - z)^2 \sigma_i^2 \varphi_i'' - (1 - p) (\varphi_i - \frac{z}{p} \varphi_i') - \frac{r_i}{\pi} \\
= \sum_{j \neq i} q_{ij} \left[ (1 - z \gamma_{ij}) p \varphi_j \left( \frac{z (1 - \gamma_{ij})}{1 - z \gamma_{ij}} \right) \right] + \lambda_i \sup_{\pi \in [0,1]} \varphi_i (\pi), \quad z \in (0, 1),
\]

\[
(\rho - q_{ii} + \lambda_i) \varphi_i (0) - (1 - p) \varphi_i (0) - \frac{r_i}{\pi} = \sum_{j \neq i} q_{ij} \varphi_j (0) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i (\pi),
\]

\[
(\rho - q_{ii} + \lambda_i - p b_i + \frac{1}{2} p (1 - p) \sigma_i^2) \varphi_i (1) = \sum_{j \neq i} q_{ij} (1 - \gamma_{ij}) p \varphi_j (1) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i (\pi).
\]

We shall adopt an iterative method to solve this system of integro-ODEs: starting with \(\varphi^0 = (\varphi^0_i)_{i \in \mathbb{I}_d} = 0\), we solve \(\varphi^{n+1} = (\varphi^{n+1}_i)_{i \in \mathbb{I}_d}\) as the (classical) solution to the local ODEs where the non local terms are calculated from \((\varphi^n_i)\):

\[
(\rho - q_{ii} + \lambda_i - p b_i z + \frac{1}{2} p (1 - p) \sigma_i^2 z^2) \varphi^n_{i+1} - z (1 - z) (b_i - z (1 - p) \sigma_i^2) (\varphi^{n+1}_i)' = - \frac{1}{2} z^2 (1 - z)^2 \sigma_i^2 (\varphi^{n+1}_i)'' - (1 - p) (\varphi^{n+1}_i - \frac{z}{p} (\varphi^{n+1}_i)) - \frac{r_i}{\pi} \\
= \sum_{j \neq i} q_{ij} \left[ (1 - z \gamma_{ij}) p \varphi^n_j \left( \frac{z (1 - \gamma_{ij})}{1 - z \gamma_{ij}} \right) \right] + \lambda_i \sup_{\pi \in [0,1]} \varphi^n_i (\pi),
\]

with boundary conditions

\[
(\rho - q_{ii} + \lambda_i) \varphi^{n+1}_i (0) - (1 - p) \varphi^{n+1}_i (0) - \frac{r_i}{\pi} = \sum_{j \neq i} q_{ij} \varphi^n_j (0) + \lambda_i \sup_{\pi \in [0,1]} \varphi^n_i (\pi),
\]

\[
(\rho - q_{ii} + \lambda_i - p b_i + \frac{1}{2} p (1 - p) \sigma_i^2) \varphi^{n+1}_i (1) = \sum_{j \neq i} q_{ij} (1 - \gamma_{ij}) p \varphi^n_j (1) + \lambda_i \sup_{\pi \in [0,1]} \varphi^n_i (\pi).
\]

Let us denote by:

\[
v^n_i (x, y) = \begin{cases} U (x + y) \varphi^n_i \left( \frac{y}{x+y} \right), & \text{for } (i, x, y) \in \mathbb{I}_d \times (\mathbb{R}_+^2 \setminus \{(0, 0)\}) \\ 0, & \text{for } i \in \mathbb{I}_d, \ (x, y) = (0, 0). \end{cases}
\]

A straightforward calculation shows that \(v^n = (v^n_i)_{i \in \mathbb{I}_d}\) are solutions to the iterative local PDEs:

\[
(\rho - q_{ii} + \lambda_i) v^{n+1}_i - b_i y \frac{\partial v^{n+1}_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v^{n+1}_i}{\partial y^2} - \tilde{U} \left( \frac{\partial v^{n+1}_i}{\partial x} \right) = \sum_{j \neq i} q_{ij} v^n_j (x, y (1 - \gamma_{ij})), \quad (x, y) \in (0, \infty) \times \mathbb{R}_+, \ i \in \mathbb{I}_d, \ (5.21)
\]
Proof. Denoting by \( \iota \) \( \theta \) easier since there are only local terms in this case). Since we already know that to (5.21), satisfying boundary condition (5.22) and growth condition (3.5) (it is actually \( w \) Proposition 4.1) we have the following Dynamic Programming Principle for the \( \mathcal{P} \) (with a similar proof to Proposition 4.1) we have the following Dynamic Programming Principle for the \( w \)

\[
\begin{align*}
    (\rho - q_i + \lambda_i) v_{i+1}(0, \cdot) - b_i y \frac{\partial v_{i+1}(0, \cdot)}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_{i+1}(0, \cdot)}{\partial y^2} &= \\
    = \sum_{j \neq i} q_{ij} v_j^n(0, y(1 - \gamma_{ij})) + \lambda_i \bar{v}_i^n(y), & y > 0, \ i \in \mathbb{I}_d.
\end{align*}
\]

(5.22)

We then have the stochastic control representation for \( v^n \) (and so for \( \varphi^n \)).

Proposition 5.3 For all \( n \geq 0 \), we have

\[
v^n_i(x, y) = \sup_{(\zeta, c) \in \mathcal{A}_i(x, y)} \mathbb{E} \left[ \int_0^{\theta_n} e^{-\rho t} U(c_t) dt \right], \quad (i, x, y) \in \mathbb{I}_d \times \mathbb{R}^2_+,
\]

(5.23)

where the sequence of random times \( (\theta_n)_{n \geq 0} \) are defined by induction from \( \theta_0 = 0 \), and:

\[
\theta_{n+1} = \inf \left\{ t > \theta_n : \Delta N_t \neq 0 \text{ or } \Delta N_t^{\tau_i - \tau_i} \neq 0 \right\},
\]

i.e. \( \theta_n \) is the \( n \)-th time where we have either a change of regime or a trading time.

Proof. Denoting by \( w^n_i(x, y) \) the r.h.s. of (5.23), we need to show that \( w^n_i = v^n_i \). First (with a similar proof to Proposition 4.1) we have the following Dynamic Programming Principle for the \( w^n \): for each finite stopping time \( \tau \),

\[
\begin{align*}
    w^{n+1}_i(x, y) &= \sup_{(\zeta, c) \in \mathcal{A}_i(x, y)} \mathbb{E} \left[ \int_0^{\tau \wedge \theta_1} e^{-\rho t} U(c_t) dt + 1_{\{\tau \geq \theta_1\}} e^{-\rho \theta_1} w^n_{I_{\theta_1}}(X_{\theta_1}, Y_{\theta_1}) \right. \\
    &\quad \quad \quad \left. + 1_{\{\tau < \theta_1\}} e^{-\rho \tau} w^{n+1}_{I_{\tau}}(X_{\tau}, Y_{\tau}) \right].
\end{align*}
\]

(5.24)

The only difference with the statement of Proposition 4.1 is the fact that when \( \tau \geq \theta_1 \), we substitute \( w^{n+1} \) with \( w^n \) since there are only \( n \) stopping times remaining before consumption is stopped due to the finiteness of the horizon in the definition of \( w^n \).

By using (5.24), we can show as in Theorem 4.1 that \( w^n \) is the unique viscosity solution to (5.21), satisfying boundary condition (5.22) and growth condition (3.5) (it is actually easier since there are only local terms in this case). Since we already know that \( v^n \) is such a solution, it follows that \( w^n = v^n \).

As a consequence, we obtain the following convergence result for the sequence \((v^n)_n\).

Proposition 5.4 The sequence \((v^n)_n\) converges increasingly to \( v \), and there exists some positive constants \( C \) and \( \delta < 1 \) s.t.

\[
0 \leq v_i - v^n_i \leq C\delta^n (x + y)^p, \quad \forall (i, x, y) \in \mathbb{I}_d \times \mathbb{R}^2_+.
\]

(5.25)

Proof. First let us show that

\[
\delta := \sup_{(\zeta, c) \in \mathcal{A}_i(x, y), \ (x, y) \in \mathbb{R}^2_+ : x + y = 1} \mathbb{E} \left[ e^{-\rho \theta_1} R^p_{\theta_1} \right] < 1.
\]

(5.26)
By writing that $e^{-pt} R_t^p = D_t L_t$, where $(L_t)_t = (e^{-k(p)t} R_t^p)_t$ is a nonnegative supermartingale by Lemma 3.1, and $(D_t)_t = (e^{-(p-k(p))t})_t$ is a decreasing process, we see that $(e^{-pt} R_t^p)_t$ is also a nonnegative supermartingale for all $(\zeta, c) \in A_i(x, y)$, and so:

$$
\mathbb{E} \left[ e^{-\rho \theta_1} R_{\theta_1}^p \right] \leq \mathbb{E} \left[ e^{-\rho (\theta_1 \wedge 1)} R_{\theta_1 \wedge 1}^p \right] = \mathbb{E} \left[ e^{-(\rho-k(p))(\theta_1 \wedge 1)} e^{-k(p)(\theta_1 \wedge 1)} R_{\theta_1 \wedge 1}^p \right].
$$

Now, since $e^{-(\rho-k(p))(\theta_1 \wedge 1)} < 1$ a.s., $\mathbb{E} \left[ e^{-k(p)(\theta_1 \wedge 1)} R_{\theta_1 \wedge 1}^p \right] \leq 1$, for all $(\zeta, c) \in A_i(x, y)$ with $x + y = 1$ (recall the supermartingale property of $(e^{-k(p)t} R_t^p)_t$), and by using also the uniform integrability of the family $(e^{-k(p)(\theta_1 \wedge 1)} R_{\theta_1 \wedge 1}^p)_{\zeta, c}$ from Lemma 3.1, we obtain the relation (5.26).

The nondecreasing property of the sequence $(v_i^n)_n$ follows immediately from the representation (5.23), and we have: $v_i^n \leq v_i^{n+1} \leq v$ for all $n \geq 0$. Moreover, the dynamic programming principle (5.24) applied to $\tau = \theta_1$ gives:

$$v_i^{n+1}(x, y) = \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ \int_{\theta_1}^{\theta_1} e^{-pt} U(c_t) dt + e^{-p \theta_1} v_i^n (X_{\theta_1}, Y_{\theta_1}) \right] \quad (5.27)$$

Let us show (5.25) by induction on $n$. The case $n = 0$ is simply the growth condition (3.5) since $v^0 = 0$. Assume now that (5.25) holds true at step $n$. From the dynamic programming principle (4.1) and (5.27) for $v$ and $v^{n+1}$, we then have:

$$v_i^{n+1}(x, y) \geq v_i(x, y) - \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ e^{-p \theta_1} (v_i(\theta_1) - v_i^n (X_{\theta_1}, Y_{\theta_1})) \right]
\geq v_i(x, y) - \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ e^{-p \theta_1} C \delta^n R_{\theta_1}^p \right]
= v_i(x, y) - C \delta^{n+1} (x + y)^p,$$

by definition of $\delta$. This proves the required inequality at step $n + 1$, and ends the proof. \( \square \)

In the next section, we solve the local ODEs for $\varphi^n$ with Newton’s method by a finite-difference scheme (see section 3.2 in [17]).

### 5.3 Numerical illustrations

#### 5.3.1 Single-regime case

In this paragraph, we consider the case where there is only one regime ($d = 1$). In this case, our model is similar to the one studied in [23], with the key difference that in their model, the investor only observes the stock price at the trading times, so that the consumption process is piecewise-deterministic. We want to compare our results with [23], and take the same values for our parameters: $p = 0.5$, $\rho = 0.2$, $b = 0.4$, $\sigma = 1$.

Let us recall from [23] the reason behind this choice of parameters (which are not very realistic for a typical financial asset): to allow meaningful comparison to the Merton (liquid) problem, the optimal Merton investment proportion should be in $[0, 1]$, while the liquid
value function should be significantly higher than the value function corresponding to the consumption problem without trading. These two constraints correspond to a high risk-return market. In the next subsection (multi-regime case), the choice of parameters will also follow from this reasoning.

Defining the cost of liquidity as the extra amount needed to have the same utility as in the Merton case: , we compare the results in our model and in the discrete observation model in [23]. The results in Table 1 indicate that the impact of the lack of continuous observation is quite large, and more important than the constraint of only being able to trade at discrete times.

\[
\begin{array}{|c|c|c|}
\hline
\lambda & \text{Discrete observation} & \text{Continuous observation} \\
\hline
1 & 0.275 & 0.153 \\
5 & 0.121 & 0.015 \\
40 & 0.054 & 0.001 \\
\hline
\end{array}
\]

Table 1: Cost of liquidity as a function of \( \lambda \).

In Figure 1 we have plotted the graph of \( \varphi(z) \) (actually \( \varphi^n(z) \) for \( n \) large) and of the optimal consumption rate \( c^*(z) \) for different values of \( \lambda \). Notice how the value function, the optimal proportion and the optimal consumption rate converge to the Merton values when \( \lambda \) increases. In Table 2 we compare the results given by our finite difference scheme for the optimal investment proportion \( z^* \), to the asymptotic results (when the intensity \( \lambda \) goes to infinity) contained in Rogers and Zane [25]. Let us observe that while the asymptotic value gives good result for large value of \( \lambda \), it is not so good for small value of intensity.

We observe that the optimal investment proportion is increasing with \( \lambda \). When \( z \) is close to 1 i.e. the cash proportion in the portfolio is small, the investor faces the risk of “having nothing more to consume” and the further away the next trading date is the smaller the consumption rate should be, i.e. \( c^* \) is increasing in \( \lambda \). When \( z \) is far from 1 it is the opposite: when \( \lambda \) is smaller the investor will not be able to invest optimally to maximize future income and should consume more quickly.

### 5.3.2 Two regimes

In this paragraph, we consider the case of \( d = 2 \) regimes. We assume that the asset price is continuous, i.e. \( \gamma_{12} = \gamma_{21} = 0 \). In this case, the value functions and optimal strategies

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>FD</th>
<th>RZ (1st order)</th>
<th>RZ (2nd order)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.539</td>
<td>0.704</td>
<td>0.637</td>
</tr>
<tr>
<td>3</td>
<td>0.696</td>
<td>0.768</td>
<td>0.761</td>
</tr>
<tr>
<td>5</td>
<td>0.744</td>
<td>0.781</td>
<td>0.778</td>
</tr>
<tr>
<td>10</td>
<td>0.786</td>
<td>0.790</td>
<td>0.790</td>
</tr>
</tbody>
</table>

Table 2: Numerical approximations to \( z^* \) obtained by finite difference (FD) scheme and Rogers & Zane (RZ) asymptotics.
for the continuous trading (Merton) problem are explicit, see [26]: \( v_{i,M}(r) = \frac{r^p}{p} \varphi_{i,M} \) where \((\varphi_{i,M})_{i=1,2}\) is the only positive solution to the equations:

\[
\left( \rho - q_{ii} - \frac{b_i^2 p}{2\sigma_i^2(1-p)} \right) \varphi_{i,M} - (1-p) \varphi_{i,M}^{-\frac{1}{p}} = q_{ij} \varphi_{j,M}, \quad i, j \in \{1, 2\}, \ i \neq j.
\]

The optimal proportion invested in the asset \( \pi^*_{i,M} = \frac{b_i}{(1-p)\sigma_i^2} \) is the same as in the single-regime case, and the optimal consumption rate is \( c^*_{i,M} = (\varphi_{i,M})^{-\frac{1}{p}} \). We take for values of the parameters

\[
p = 0.5, \\
q_{12} = q_{21} = 1, \\
b_1 = b_2 = 0.4, \\
\sigma_1 = 1, \quad \sigma_2 = 2,
\]

i.e. the difference between the two market regimes is the volatility of the asset. In Figure 2, we plot the value function and optimal consumption for each of the two regimes in this market, for various values of the liquidity parameters \((\lambda_1, \lambda_2)\). As in the single-regime case, when the liquidity increases, \( \varphi \) and \( c^* \) converge to the Merton value. A subject for future research would be to study the asymptotic analysis when \( \lambda \) goes to infinity, in the spirit of [25], of the value function, optimal consumption and investment proportion, when the number of regimes is greater than 1.

Note that while in the single regime-case the optimal investment proportion is usually increasing with the liquidity parameter \( \lambda \), in the presence of several regimes there does not appear to be a simple similar effect, as can be seen for instance in the upper-right panel of Figure 2.
To quantify the impact of regime-switching on the investor, it is also interesting to compare the cost of liquidity with the single-regime case, see Tables 3 and 4. We observe that, for equivalent trading intensity, the cost of liquidity is higher in the regime-switching case. This is economically intuitive: in each regime the optimal investment proportion is different, so that the investor needs to rebalance his portfolio more often (at every change of regime).

<table>
<thead>
<tr>
<th>(λ₁, λ₂)</th>
<th>( P₁(1) )</th>
<th>( P₂(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>0.257</td>
<td>0.224</td>
</tr>
<tr>
<td>(5,5)</td>
<td>0.112</td>
<td>0.103</td>
</tr>
<tr>
<td>(10,10)</td>
<td>0.069</td>
<td>0.064</td>
</tr>
</tbody>
</table>

Table 3: Cost of liquidity \( P_i(1) \) as a function of \((λ₁, λ₂)\).

<table>
<thead>
<tr>
<th>( λ )</th>
<th>( P₁(1) )</th>
<th>( P₂(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.153</td>
<td>0.087</td>
</tr>
<tr>
<td>5</td>
<td>0.015</td>
<td>0.042</td>
</tr>
<tr>
<td>10</td>
<td>0.004</td>
<td>0.024</td>
</tr>
</tbody>
</table>

Table 4: Cost of liquidity \( P_i(1) \) for the single-regime case.

\[
(λ₁, λ₂) = (1, 1) \quad (λ₁, λ₂) = (10, 1) \\
(λ₁, λ₂) = (10, 10) \quad \text{Merton}
\]

Figure 2: \( ϕ_i \) and \( c^*_i \) for different values of \((λ₁, λ₂)\).
6 Conclusion

In this paper we proposed a simple model of an illiquid market with regime-switching, in which the investor may only trade at discrete times corresponding to the arrival times of a Cox process. In this context, we studied an investment/consumption problem over an infinite horizon. In the general case, we proved that the value function for this problem is characterized as the unique viscosity solution to the HJB equation (which is a system of integro-PDEs). In the case of power utility, we proved the regularity of our value function and we were able to characterize the optimal policies. Finally we have presented some numerical results in this special case.
References


