Reflected SDE and pathwise stochastic analysis

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Reflected SDE

Given domain $D \subset \mathbb{R}^N$ and driving signal X, takes the form

$$dY_t = f(Y_t)dX_t + dK_t, \quad Y_t \in D \quad \forall t \geqslant 0, \quad dK_t = 1_{\{Y_t \in \partial D\}} n(Y_t)|dK_t|$$

where $n(y)$ is the inner normal at $y \in \partial D$.

Natural object when considering processes constrained to remaining in a given subdomain. Many applications (queueing theory, finance,...). Classical objects in stochastic analysis when X is a semimartingale. (Skorokhod '61, Tanaka '79, Lions-Sznitman '84,...)

Q : can we apply rough path theory to these equations ? In particular, can we study these equations for non-semimartingale X such as fBm ?

Classical well-posedness

If D is convex, it holds that $n(y) \cdot (y - y') \leqslant 0$ for $y \in \partial D, y' \in D$.

This implies that, given two solutions Y, Y' to $dY = f(Y)dX_t + dK_t$

if $dX_t = dt$, $d|Y - Y'|^2 \leqslant C|Y - Y'|^2 dt$ (also for $dY_t = f(Y_t)dt + dX_t$ (additive noise))

if $dX_t = dB_t$ Brownian motion, $d\,\mathbb{E}\,|Y - Y'|^2 \leqslant C\,\mathbb{E}\,|Y - Y'|^2 dt$ from which uniqueness follows by Gronwall's lemma.

Formally, reflection corresponds to the singular drift equation

$$dY_t = f(Y_t)dX_t - \nabla\Phi(Y_t)dt,$$

where Φ is the convex indicator function of D

$$\Phi(y) = +\infty 1_{\{y \notin D\}}.$$

(singular but **monotone** drift).

- Is this compatible with rough path type solutions ? Recall for $dY = f(Y)dX$, $X \in C^\alpha$

$$X_t - X_s = f(Y_s)(X_t - X_s) + o(|t-s|) \ (\alpha > \tfrac{1}{2}, \text{ Young case})$$

$$X_t - X_s = f(Y_s)(X_t - X_s) + (f'f)(Y_s)\left(\int_s^t X_{s,r} \otimes dX_r\right) + o(|t-s|)$$

$$(\tfrac{1}{2} \geqslant \alpha > \tfrac{1}{3}, \text{ rough case}).$$

A priori, different type of argument
(''almost equalities'' vs. ''inequalities (in expectation)'')

Some positive results

$$dY_t = f(Y_t)dX_t + dK_t, \quad Y_t \in D, dK_t = 1_{\{Y_t \in \partial D\}} n(Y_t)|dK_t|$$

- Aida ('15,'16) : **Existence** holds for X α-Hölder rough path ($\alpha > \frac{1}{3}$), and some regularity assumptions on D (similar to classical theory.
- In $1d$ ($D = \mathbb{R}_+$), one can show that uniqueness holds. (Deya-Gubinelli-Hofmanova-Tindel '19, Richard-Tanré-Torrès '21, Allan-Liu-Prömel '22).

What about uniqueness in the multi-dimensional case ?

Non-uniqueness for reflected RDE

In general, uniqueness will not hold for rough driving signals :

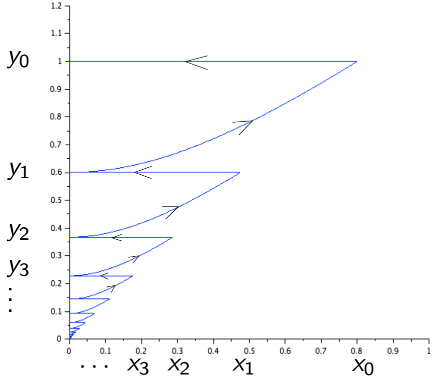
Theorem (G. *AIHP '21*)

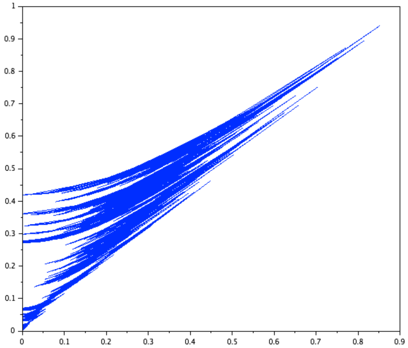
The equation

$$\begin{cases} dx_t & = y_t d\beta_t - dt + dk_t, x_t \geqslant 0, dk_t \geqslant 0,\ x_t dk_t = 0, \\ dy_t & = x_t d\beta_t. \end{cases}$$

admits infinitely many solutions with initial condition $(0,0)$*, if* β *is :*
(1) a fractional Brownian motion with Hurst index $H < \frac{1}{2}$*.*
(2) a well-chosen deterministic path, which can have modulus of continuity $\omega(r) = r^{1/2}|\log(r)|^{\gamma}$*,* $\gamma > \frac{1}{2}$*.*

$$\begin{cases} dx_t &= y_t d\beta_t - dt + dk_t, x_t \geqslant 0, dk_t \geqslant 0,\ x_t dk_t = 0, \\ dy_t &= x_t d\beta_t. \end{cases}$$





The Skorokhod map

Given a $\mathbb{R}^N$-valued path W, there exists a unique Z s.t.

$$Z = W + K, \quad Z_t \in D \forall t \geqslant 0, dK = 1_{\{Z_t \in \partial D\}} n(Z_t) d|K|_t.$$

The map $\Gamma : W \mapsto Z$ is called the **Skorokhod map**.
Reflected SDE are then equivalently rewritten as fixed points :

$$Y = \Gamma\left(y_0 + \int_0^{\cdot} f(Y_t) dX_t \right).$$

Very simple form when $D = \mathbb{R}_+$ (Skorokhod '61)

$$K_t = -\left(\inf_{0 \leqslant s \leqslant t} W_s \right) \wedge 0.$$

If $D = D_1 \times \ldots \times D_k$, $\Gamma_D = (\Gamma_{D_1}, \ldots, \Gamma_{D_k})$.
For general D, no such simple expression.

A Lipschitz continuity result in p-variation

Theorem (Falkowski-Slominski ('15,'22))

Let $D = \Pi_{i=1}^{N}[a_i, b_i]$ (with $-\infty \leqslant a_i < b_i \leqslant +\infty$). Then

$$\|\Gamma(Y) - \Gamma(Y')\|_{p-var} \leqslant C\left(|Y_0 - Y_0'| + \|Y - Y'\|_{p-var}\right).$$

Combined with Young integration theory, this allows them to prove well-posedness of reflected equations :

Theorem

With D as above, B^H fBm with $H > \frac{1}{2}$, the reflected equation

$$dY_t = f(Y_t)dB_t^H + dK_t, \ldots$$

is well-posed.

(Remark : they also obtain well-posedness for mixed SDE $dY_t = f^1(Y_t)dB_t^H + f^2(Y_t)dB_t + dK_t$, where B is classical BM).

A regularization by noise result

Theorem (G. - Mądry *arXiv* '22)

Let $D = \Pi_{i=1}^N [a_i, b_i]$, $0 < H < \frac{1}{2}$ *and* $b \in C^\alpha$, *with* $\alpha > 1 - \frac{1}{2H}$. *Then well-posedness holds for*

$$Y_t = b(Y_t)dt + dB_t^H + dK_t.$$

The argument is an extension of Catellier-Gubinelli ('16), which uses nonlinear Young integration, and relies on regularity of the averaged field

$$T^{B^H} b : (t, x) \mapsto \int_0^t b(x + B_s^H) ds$$

wich has much better regularity in x than C^α.
Letting $\theta = Y - B^H$, the equation is equivalent to

$$\theta = \Gamma \left(\theta + \int_0^\cdot T^{B^H} b(\theta_s, ds) + B^H \right) - B^H$$

We combine with the (Falkowski-Slominski) Lipschitz property of Γ to conclude.

- Actually, we need to obtain regularity of $T^{\Gamma(B^H)}b$. We show that if K is adapted and in C^{q-var} with $\frac{1}{q} > H + \frac{1}{2}$, then with b as above,

$$T^{B^H+K}b \in C_t^{(2-\epsilon)-var} C_x^1$$

  This is obtained via the **stochastic sewing lemma**.

- We also have some result for $H > \frac{1}{2}$, in which case we need $b \in C^\alpha$ with $\alpha > 2 - \frac{1}{H}$ (probably not optimal).

- The argument is not specific to reflection, also works e.g. for equations of the form

$$dY_t = f(Y_t)dt + dB_t^H + \alpha \max_{0 \leqslant s \leqslant t} Y_s + \beta \min_{0 \leqslant s \leqslant t} Y_s.$$

Some open questions (Young case)

- For (non-smooth) convex domains D and/or oblique reflection direction, is Γ_D Lipschitz in p-variation ?
  (Note $p = \infty$ is known but nontrivial, cf Dupuis-Ishii '91)
- More specific to the regularization by noise / singular drift equations : better results for $H > \frac{1}{2}$?
- Can we understand reflection in a stochastic way ? e.g. by considering properties of the map

$$(s, t, x) \mapsto K_{s,t}(x + B_{s,\cdot})$$

Some open questions (rough case)

We have seen that in general, there is no uniqueness for reflected RDE. Can we find criteria to restore uniqueness ?

- Does uniqueness hold for sufficiently regular X, e.g. 2-variation, or for ψ-variation with $\psi(r) = r^2 \log\log(1/r)$? (This is the regularity of **Markovian** rough paths, cf Friz-Victoir chap. XVI).
- Does uniqueness hold if the driving vector field / noise are **non-degenerate** e.g. X fBm and $f(y) \cdot n(y) \neq 0$ for $y \in \partial D$?
- In general, does uniqueness hold for a.e. initial condition, is there a **unique flow** for the RRDE ?