Zero noise limit for singular ODE regularized by fractional noise

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Joint work with Łukasz Mądry (Paris Dauphine)

Summary

In this talk : we consider the limit as $\varepsilon \to 0$ of

$$dX_t = sign(X_t)|X_t|^\gamma dt + \varepsilon dB_t^H, \quad X_0 = 0,$$

where B^H is a fractional Brownian motion, $\gamma < 1$, and show that it concentrates on the extremal solutions

$$x^{\pm}(t) = \pm C_\gamma t^{\frac{1}{1-\gamma}}.$$

Regularization by noise of ODE

Recall that, by the classical Cauchy-Lipschitz theory, the ODE

$$dx_t = b(x_t)dt, \quad x_0 \in \mathbb{R}^N$$

is classically well-posed for b **Lipschitz continuous**.

In contrast, in the case of SDE :

$$dX_t = b(X_t)dt + dW_t, \quad X_0 \in \mathbb{R}^N,$$

where W is a (standard) Brownian motion, it is known that b **bounded** is enough to have (strong) well-posedness for the SDE : Zvonkine '74, Veretennikov '81 (adapted solutions), Davie '07 (path-by-path uniqueness).

$$dX_t = b(X_t)dt + dW_t, \quad X_0 \in \mathbb{R}^N,$$

Idea : the map

$$T^{W;[0,t]}b : x \mapsto \int_0^t b(x + W_s)ds$$

is much more regular w.r.t. x than b.

In fact : $T^{W;[0,t]}b = b * \mu_{W;[0,t]}$ where $\mu_{W;[0,t]}$ is the occupation measure, and roughly speaking, irregularity of $W \leftrightarrow$ regularity of $\mu_{W;[0,t]}$.

Letting $\theta = X - W$, the SDE is equivalent to

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + W_s)ds$$

which is close to an ODE along $T^{W;[0,t]}b$ (assuming that θ evolves at a slower time scale than W).

Fractional Brownian motion

The above principle only requires W to have irregular paths $\rightarrow$ more general family of processes with varying degree of (irr)regularity ?

Natural candidate : $W = (W_t)_{t \geqslant 0}$ **fractional Brownian motion** (fBm) with **Hurst parameter** $H \in (0, 1)$.

- Gaussian process, stationary increments, $W_0 = 0$ and

$$\|W_t - W_s\|_{L^2(\Omega)} = |t - s|^H,$$

- sample paths are $(H - \varepsilon)$-Hölder continuous
- Representation as moving average of a standard (2-sided) BM B :

$$W_t = C_H \int_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dB_s,$$

- NOT a semimartingale, NOT a Markov process (for $H \neq \frac{1}{2}$).

Regularization of ODE by fractional noise

Consider, for W^H **fractional Brownian motion** with Hurst index $H \in (0,1)$

$$dX_t = b(X_t)dt + dW_t^H, \text{ with } b \text{ singular}$$

Early results in the scalar case by Nualart-Ouknine '02

More recently : **Catellier-Gubinelli '16** show well-posedness when

$$b \in C^\gamma, \qquad \gamma > 1 - \frac{1}{2H}$$

(Note : γ may be negative in which case $\int_0^t b(X_s)ds$ must be suitably interpreted).
Based on regularity results for the averaged field $(T^W b)(t,x) = \int_0^t b(x+W_s)ds$ and nonlinear Young integration.

Recent progress using **stochastic sewing lemma** (Lê '20), very active research area.

Selection by noise

Let b be non-Lipschitz and X^ε solve

$$dX_t = b(X_t)dt + \varepsilon dW_t, \quad X_0 \in \mathbb{R}^N,$$

what can we say about the behaviour of X^ε as $\varepsilon \to 0$?

Hope : convergence to one (or more) particular solution(s) to the ODE $\dot{x} = b(x)$, which could be interpreted as the natural "physical" solutions. (**Selection by noise**).

Difficult question in general ! In the rest of the talk : focus on scalar equations, with an isolated singularity.

A scalar example

Consider (for $0 < \gamma < 1$)

$$b(x) = \begin{cases} A|x|^\gamma, & x \geqslant 0 \\ -B|x|^\gamma, & x < 0. \end{cases}$$

The equation

$$dx_t = b(x_t)dt, \quad x_0 = 0$$

admits infinitely many solutions, of the form

$$x^{+,t_0} = c_{A,\gamma}(t - t_0)_+^{\frac{1}{1-\gamma}}, \text{ or } x^{-,t_0} = -c_{B,\gamma}(t - t_0)_+^{\frac{1}{1-\gamma}}$$

Theorem (Bafico-Baldi '82)

Let X^ε be the solution to $dX_t = b(X_t)dt + \varepsilon dW_t$. Then it holds that

$$\mathcal{L}_{X^\varepsilon} \to_{\varepsilon \to 0} \pi\delta_{x^{+,0}} + (1-\pi)\delta_{x^{-,0}}.$$

for some (explicit) $\pi = \pi(A, B, \gamma)$.

Their proof is based on "PDE" arguments (martingales, Markov processes)

Main result : setting

We take W fBm with Hurst index H, fix $1 > \gamma > 1 - \frac{1}{2H}$ and b such that :

$$b_{|x>0} = Ax^{\gamma}, \;\; b_{|x<0} = -B(-x)^{\gamma},$$

$$b \in C^{\gamma}(\mathbb{R}), \quad b(\lambda \cdot) = \lambda^{\gamma} b(\cdot)$$

where $A, B > 0$, and let X^{ε} solve

$$dX^{\varepsilon}_t = b(X^{\varepsilon}_t)dt + \varepsilon dW_t, \;\; X^{\varepsilon}_0 = 0.$$

Again, for $\varepsilon = 0$, we have the family of "solutions"

$$x^{+,t_0} = c_{A,\gamma}(t - t_0)_+^{\frac{1}{1-\gamma}}, \text{ or } x^{-,t_0} = -c_{B,\gamma}(t - t_0)_+^{\frac{1}{1-\gamma}}$$

Main result

Theorem (G.-Mądry)

Under the above assumptions, it holds that

$$\mathcal{L}_{X^\varepsilon} \to_{\varepsilon \to 0} \pi \delta_{x^{+,\mathbf{0}}} + (1-\pi)\delta_{x^{-,\mathbf{0}}}.$$

for some $\pi \in (0,1)$ *.*

In fact, for any $0 < \delta < 1$*, there exists* τ_ε *with*

$$\forall s \geqslant 0, \;\; X_{s+\tau_\varepsilon} \geqslant (1-\delta) x_s^{+,0} \text{ or } X_{s+\tau_\varepsilon} \leqslant (1-\delta) x_s^{-,0}$$

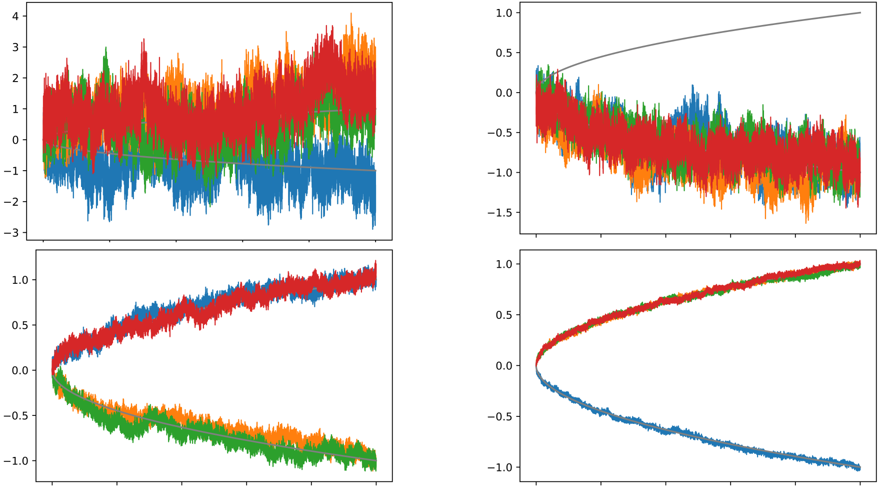
and letting $t_\varepsilon = \varepsilon^{\left(\frac{1}{1-\gamma} - H\right)^{-1}}$*, it holds that*

$$\sup_\varepsilon \mathbb{P}\left[\frac{\tau_\varepsilon}{t_\varepsilon} \geqslant \lambda\right] \leqslant \exp(-C\lambda^\kappa),$$

for some $0 < \kappa < 1$*.*

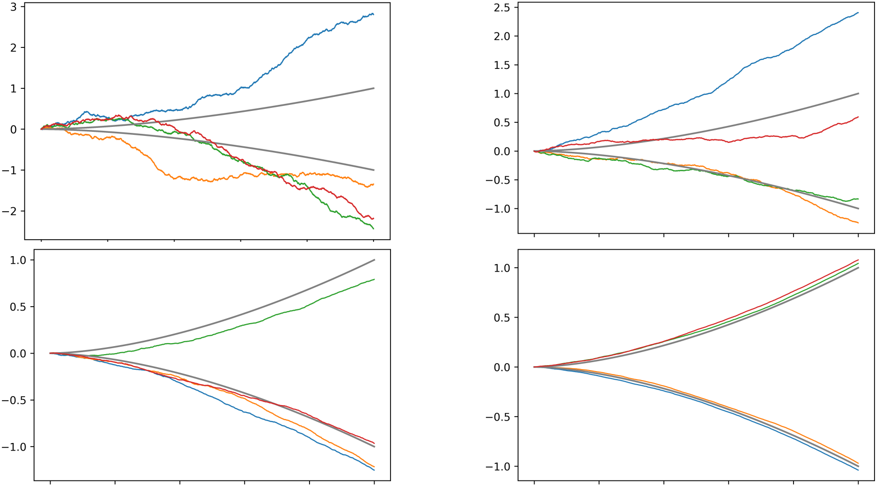
Remark : for $\gamma > 0$, the first assertion was already proven by Pilipenko-Proske '18.

Simulations



4 simulated paths of X^{ε}, $\varepsilon \in \{1, 0.3, 0.1, 0.03\}$, for $H = 0.1$, $\gamma = -1$
($t_{\varepsilon} \sim \varepsilon^{2.5}$)

Simulations



4 simulated paths of X^ε, $\varepsilon \in \{1, 0.3, 0.1, 0.03\}$, for $H = 0.7$, $\gamma = 0.4$ ($t_\varepsilon \sim \varepsilon^{1.03}$)

Ideas of proof

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \varepsilon dW_t, \quad X_0 = 0.$$

Scaling idea ("transition point") from Delarue-Flandoli '14 :
Recall that (up to constants)

$$x_t^{+,0} = t^{\frac{1}{1-\gamma}}, \qquad |\epsilon W_t| \approx \epsilon t^H.$$

Let

$$t_\varepsilon = \varepsilon^{\left(\frac{1}{1-\gamma} - H\right)^{-1}}$$

be such that these coincide, and x_ε their common value at this time.

Then :
for $t << t_\varepsilon$, $X_t \approx \varepsilon W_t$ (randomness dominates)
for $t >> t_\varepsilon$, $X_t \approx x_t^{\pm,0}$ (drift dominates)

the transition between these two regimes happens at a time of order t_ε where $|X|$ is of order x_ε.
(in fact $X^\varepsilon =^{(d)} x_\varepsilon X^1_{\cdot/t_\varepsilon}$)

Markovian proof (Delarue-Flandoli, $H = 1/2$)

Follow the following procedure :

1. Wait until X^ε hits level $\pm x_\varepsilon$, this happens at time τ_1, of order t_ε
2. Starting from $\pm x_\varepsilon$,

$$\mathbb{P}\left(\forall s \geqslant 0, \left|X^\varepsilon_{\tau_1+s}\right| \geqslant (1-\delta)x_s^{0,+}\right) \geqslant p > 0.$$

3. If the above event does not happen and fails at time σ_1, wait until time τ_2 where $|X^\varepsilon| \geqslant x_\varepsilon$
4. iterate with times τ_k, σ_k, ...

If everything were Markov, the above procedure would conclude after finitely many (independent) steps.

What about in our case ?

Main difficulty : Non-Markovian techniques

W^H non Markovian : what happens between and after time τ_1 is correlated, and so are the subsequent attempts...

But recall that W can be written in terms of the (Markov !) process B :

$$W_t = C_H \int_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dB_s,$$

and for $u \geqslant v \geqslant s$, with $u - v << v - s$:

$$W_u - W_v = \underbrace{\int_{-\infty}^{s} \left((u-r)^{H-\frac{1}{2}} - (v-r)^{H-\frac{1}{2}} \right) dB_r}_{\mathcal{F}_s\text{-measurable but small}}$$
$$+ \underbrace{\int_{s}^{u} \left((u-r)^{H-\frac{1}{2}} - (v-r)_+^{H-\frac{1}{2}} \right) dB_r}_{\text{independent from } \mathcal{F}_s}$$

Non-Markovian techniques

We take inspiration from works on long time behaviour of fractional SDEs, in particular Hairer '06, Panloup-Richard '20 and add :

- an "admissibility" condition on B before starting step 1 or 2, ensuring that influence of the past is small enough (otherwise : wait i.e. "Step 0")
- during step 2, we sequentially check for constraints on B on growing intervals (of size proportional to t_ε), which ensure that X stays close to $x^{0,\pm}$. (when failed : wait before restarting step 1).
- The waiting times are chosen to ensure than when we restart, the influence of the past noise (B up to "failure" is small), to have

$$\mathbb{P}\left(|X^\varepsilon| \text{ stays above } x^{+,0} \text{ after } \tau_k | \mathcal{F}_{\tau_{k-1}}\right) \geqslant p > 0.$$

- More precisely : waiting times depend on 'size of B' (in some Hölder norm) before failure, and number of attempts :

$$\rho_{k+1} - \rho_k = k^{\mu_1} + \|B\|_{[\rho_{k-1},\rho_k]}^{\mu_2}$$

Open question : optimal concentration estimates

Recall Freidlin-Wentzell large deviations : if b is smooth,

$$\mathbb{P}(X^\varepsilon \text{ not close from } x) = exp\left(-\varepsilon^{-2}\left(C + o(1)\right)\right)$$

Situation different in the singular case !
Gradinaru-Herrmann-Roynette '01 : for $H = 1/2$ and $\gamma \geqslant 0$,

$$\mathbb{P}\left(|X_1^\varepsilon| > (1+\delta)x_1^+\right) = \exp\left(-\varepsilon^{-2}\left(C + o(1)\right)\right)$$

$$\mathbb{P}\left(|X_1^\varepsilon| < (1-\delta)x_1^+\right) = \exp\left(-\varepsilon^{\frac{-2(1-\gamma)}{1+\gamma}}\left(C + o(1)\right)\right)$$

Note that the rate in the second one is t_ε^{-1}...as expected !
Q: What about $H \neq \frac{1}{2}$?
From our results, we only obtain, for some small $\kappa < 1$:

$$\mathbb{P}(|X_1^\varepsilon| < (1-\delta)x_1^+) \leqslant \exp\left(-t_\varepsilon^{-\kappa}\right)$$

Possible conjecture : the optimal rate is always

$$\varepsilon^{-2} \wedge t_\varepsilon^{-1}$$

(Related to rate of convergence to equilibrium for fractional SDE)

Conclusion

Main result : we consider the limit as $\varepsilon \to 0$ of

$$dX_t = b(X_t)dt + \varepsilon dW_t^H, \quad X_0 = 0, \text{ with } b(x) = A|x|^\gamma 1_{x>0} - B|x|^\gamma 1_{x<0}$$

where W^H is a fractional Brownian motion, $1 > \gamma > 1 - \frac{1}{2H}$, and show that it concentrates on the extremal solutions

$$x^\pm(t) = \pm C t^{\frac{1}{1-\gamma}}.$$

Some open questions :

- Optimal concentration estimates $\mathbb{P}(X_1^\varepsilon \approx 0) \approx \exp(-\varepsilon^{-?})$
- Can we say anything about the weights given to x^+ and x^- in the limit ? Dependence on H ?
- More complicated (e.g. multi-dimensional) situations ? (already difficult in the Markovian case...)