Université Paris-Dauphine
Master Masef
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Written exam : Evaluation des actifs financiers et arbitrage
Thursday, December 17, 2020 — 2h.

Exercise 1 : a market with transaction cost

We consider a one period discrete time financial market $(\Omega, \mathcal{F}, \mathbb{P})$, with one risky asset $S = (S_0, S_1)$ and one riskless asset of interest rate $r = 0$. Suppose that $\Omega = \{\omega_d, \omega_u\}$ with $\mathbb{P}[\{\omega\}] = \frac{1}{2}$ for each $\omega \in \Omega$, and S satisfies $S_0 = 1$, and

$$S_1(\omega_d) = 1 + d, \quad S_1(\omega_u) = 1 + u$$

for some fixed $-1 < d < u$.

We assume in addition that there exists a proportional transaction cost for trading at time $t = 1$, but no transaction cost at time $t = 0$. Concretely, at time $t = 1$, the selling price of the risky asset is $(1 - c)S_1$ and the buying price is $(1 + c)S_1$, for some constant $c \in (0, 1)$.

Then for a self-financing portfolio $V^{0,\phi}$ with initial wealth 0 and trading strategy $\phi \in \mathbb{R}$ (which is a deterministic constant in this one period setting), one has that the value of the portfolio at time 1, after liquidation, is given by

$$V_1^{0,\phi} \;=\; \phi(1-c)S_1\mathbf{1}_{\{\phi\geq 0\}} \;+\; \phi(1+c)S_1\mathbf{1}_{\{\phi<0\}} \;-\; \phi S_0.$$

We say that the no-arbitrage (NA) condition holds if

$$V_1^{0,\phi} \geq 0,\ \mathbb{P}\text{-a.s.} \quad \Longrightarrow \quad V_1^{0,\phi} = 0,\ \mathbb{P}\text{-a.s.}$$

Further, a couple $(\mathbb{Q}, \hat{S})$ is called a consistent price system (CPS) if $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, and the process $\hat{S} = (\hat{S}_0, \hat{S}_1)$ is a martingale under $\mathbb{Q}$, and $\hat{S}$ satisfies

$$\hat{S}_0 = S_0(=1), \quad \hat{S}_1(\omega) \in \big[(1-c)S_1(\omega),\ (1+c)S_1(\omega)\big], \quad \text{for all } \omega \in \Omega.$$

Denote by $\mathcal{CPS}$ the collection of all consistent price systems $(\mathbb{Q}, \hat{S})$.

(1) Assume that the set $\mathcal{CPS}$ is nonempty.

   (i) Let $(\mathbb{Q}, \hat{S}) \in \mathcal{CPS}$, prove that

$$V_1^{0,\phi} \leq \phi(\hat{S}_1 - \hat{S}_0), \quad \text{for all } \phi \in \mathbb{R}.$$

   (ii) Deduce from the above formula that the condition (NA) holds true in this case.

(2) Prove that if the condition (NA) holds true, then

$$(1-c)(1+d) < 1 < (1+c)(1+u). \qquad (*)$$

(3) (i) Let a, b be real numbers such that $a < 1 < b$. Show that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$, and a $\mathbb{Q}$-martingale M with

$$M_0 = 1, \quad M_1(\omega_d) = a, \quad M_1(\omega_u) = b.$$

(ii) Deduce that if $(*)$ holds, then the set $\mathcal{CPS}$ is not empty.

From the above results, one can conclude that the condition (NA) is equivalent to the fact that $\mathcal{CPS}$ is not empty, which is equivalent to the inequality $(*)$.

We now let $G : \Omega \to \mathbb{R}$ be the payoff of a European option, and denote by $p(G)$ its minimum super-hedging cost, i.e.

$$p(G) \ := \ \inf \big\{ p \in \mathbb{R} \ : \exists \phi \in \mathbb{R} \text{ s.t. } p + V_1^{0,\phi} \geq G, \quad \mathbb{P} - \text{a.s.} \big\}.$$

(4) Prove that

$$p(G) \geq \mathbb{E}^{\mathbb{Q}}[G], \quad \text{for all } (\mathbb{Q}, \hat{S}) \in \mathcal{CPS}.$$

(5) Let us consider a one period fictitious market defined on an enlarged space $(\overline{\Omega}, \overline{\mathbb{P}})$ with

$$\overline{\Omega} = \Omega \times \{(1-c), (1+c)\}, \text{ with } \overline{\mathbb{P}}[(\omega, \theta)] = \frac{1}{4} \text{ for all } \bar{\omega} = (\omega, \theta) \in \overline{\Omega}.$$

The interest rate $r = 0$ and the underlying risky asset $X = (X_0, X_1)$ is defined by $X_0 = S_0 = 1$ and $X_1(\omega, \theta) = \theta S_1(\omega)$ for all $(\omega, \theta) \in \overline{\Omega}$.

There is no transaction cost on the fictitious market, as the case studied during the course. Let us denote by $\bar{p}(G)$ the minimum super-hedging cost of G (G is extended to $\overline{\Omega}$ by letting $G(\omega, \theta) = G(\omega)$). Also denote by $(\overline{\text{NA}})$ the no-arbitrage condition on the fictitious market, defined as in the course.

(i) Prove that

$$p(G) = \bar{p}(G) \quad \text{and} \quad (\text{NA}) \Longleftrightarrow (\overline{\text{NA}}).$$

(ii) Let $\overline{\mathcal{M}}$ denotes the collection of all equivalent martingale measures (for X) on $\overline{\Omega}$, and fix $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}$. By considering $\overline{\mathbb{E}}^{\overline{\mathbb{Q}}}[X_1 | S_1]$, show that there exists $(\hat{\mathbb{Q}}, \hat{S})$ in $\mathcal{CPS}$ such that

$$\overline{\mathbb{E}}^{\overline{\mathbb{Q}}}[G] = \mathbb{E}^{\hat{\mathbb{Q}}}[G].$$

(iii) Applying the duality formula in the extended market (as seen in the course) to $\bar{p}(G)$, deduce that (if (NA) holds), one has

$$p(G) = \sup_{(\mathbb{Q}, \hat{S}) \in \mathcal{CPS}} \mathbb{E}^{\mathbb{Q}}[G].$$

Exercise 2 : Stochastic interest rate

We consider a financial market in which the dynamics of the interest rate r and of the risky asset S are given by

$$r_t = r_0 + \int_0^t a(b - r_s)ds + \int_0^t \gamma dW_s^1, \tag{1}$$

$$S_t = S_0 + \int_0^t S_s r_s ds + \int_0^t S_s \sigma_1 dW_s^1 + \int_0^t S_s \sigma_2 dW_s^2, \tag{2}$$

in which W^1 and W^2 are two independent Brownian motions under the measure $\mathbb{Q}$. Here, $a, b, r_0, S_0, \gamma, \sigma_1, \sigma_2$ are strictly positive constants.

We set $\beta_t := e^{-\int_0^t r_s ds}$ the discount factor and, as usual, $\tilde{X}_t = \beta_t X_t$ for any adapted process X.

(1) Show that

$$d\tilde{S}_t = \tilde{S}_t(\sigma_1 dW_t^1 + \sigma_2 dW_t^2).$$

(2) Using Girsanov's theorem, find a probability measure $\mathbb{Q}' \neq \mathbb{Q}$, $\mathbb{Q}' \sim \mathbb{Q}$ such that $\tilde{S}$ is a $\mathbb{Q}'$-martingale. Is the market complete if the only tradable risky asset is S ?

We now consider an option with payoff $g(S_T)$ (at time $T > 0$), where g is a continuous bounded function, and we will discuss two different ways to hedge this option.

Pricing with the wrong interest rate

We first fix $\bar{r} \in \mathbb{R}$, and we assume that we will hedge the option with the "wrong model" for which the interest rate is constant and equal to $\bar{r}$.

(3) For $0 \leq t \leq T$, let

$$\bar{p}(t, x) = \mathbb{E}^{\mathbb{Q}}\left[\, e^{-\bar{r}(T-t)} g(\bar{S}_t) |\ \bar{S}_t = x \right],$$

where $\bar{S}$ is the solution to (2) with r_t replaced by $\bar{r}$.

(a) Recall (briefly) why $\bar{p}$ should solve on $[0, T) \times \mathbb{R}_+$ the equation

$$\bar{r}\bar{p} - \partial_t \bar{p} - \bar{r} x \partial_x \bar{p} - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) x^2 \partial_{xx}^2 \bar{p} = 0.$$

(b) Using this equation, and applying Itô's formula, show that, if $\phi_t = \partial_x \bar{p}(t, S_t)$, one has

$$\bar{p}(0, S_0) + \int_0^T \phi_t d\tilde{S}_t = \beta_T g(S_T) + \int_0^T (r_t - \bar{r})\left(\bar{p}(t, S_t) - S_t \partial_x \bar{p}(t, S_t)\right) \beta_t dt.$$

(c) If an investor tries to hedge the option with payoff $g(S_T)$ using the price $\bar{p}(0, S_0)$ and the associated delta-hedging strategy, what is the hedging error ?

Market completion using a 0-coupon bond

From now on, we fix $\tau > T$ and assume that the 0-coupon bond of maturity τ is liquid and can be used as a hedging instrument. We also assume that its price at time $t < \tau$ is B_t^τ where

$$dB_t^\tau = r_t B_t^\tau dt + \gamma f^1(\tau - t) dW_t^1$$

for some function $f^1 > 0$.

(4) Recall that we consider an option of maturity $T < \tau$ and payoff $g(S_T)$ in which g is continuous and bounded. We assume that we are given a smooth function $p(t, r, x)$ such that

$$rp - \partial_t p - a(b-r)\partial_r p - rx\partial_x p - \frac{1}{2}\gamma^2 \partial_{rr} p - \frac{1}{2}x^2(\sigma_1^2 + \sigma_2^2)\partial_{xx} p - \gamma\sigma x \partial_{rx} p = 0$$

and $p(T, r, x) = g(x)$.

(a) Apply Itô's lemma to obtain the dynamics of $\tilde{P}_t = \beta_t p(t, r_t, S_t)$ in terms of the derivatives of the function p.

(b) Find two adapted processes ϕ^1 and ϕ^2 such that

$$p(0, r_0, S_0) + \int_0^T \phi_s^1 d\tilde{S}_s + \int_0^T \phi_s^2 d\tilde{B}_s^\tau = \beta_T g(S_T).$$

(c) What should be the price of the option with payoff $g(S_T)$ in this market ? What can we say on the hedging strategy of $g(S_T)$?

(d) Could we hedge perfectly this payoff $g(S_T)$ without using the 0-coupon bond ?

(e) If the payoff was instead of the form $g(B_T^\tau)$, could we hedge it without investing in S ?