Université Paris-Dauphine
Master Masef
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Written exam : Evaluation des actifs financiers et arbitrage
January 10, 2023 — 2h.

Exercise 1 : Utility maximization in a one-period model

We consider a one-period financial market. The underlying probability space is $\Omega = \{\omega_1, \dots, \omega_n\}$, with $p_i := \mathbb{P}(\{\omega_i\}) > 0$ for each $i = 1, \dots, n$.

The market consists in two tradable assets : a risk-free asset with price values $B_0 = 1$ at time 0 and $B_1 = 1 + r$ at time 1, for some fixed $r > 0$, and a risky asset with price $S_0 = 1$, and $S_1 = S_1(\omega)$ at time 1.

In this market, a trading strategy (for a self-financing portfolio) is simply a number $\phi \in \mathbb{R}$, representing the number of units of the risky asset held in the portfolio between times 0 and 1.

We then denote by $X_1^{x,\phi}$ the value at time 1 of a portfolio starting with wealth x at time 0 and trading with strategy ϕ.

As usual, if X_i, $i = 0, 1$ is a stochastic process we will call $\tilde{X}$ its discounting, i.e. $\tilde{X}_0 = X_0$ and $\tilde{X}_1 = X_1/(1 + r)$.

We fix a utility function $U : \mathbb{R} \to \mathbb{R}$, which must satisfy the following properties :

$$\lim_{y \to +\infty} U(y) = +\infty, \quad U \text{ is } C^1 \text{ on } \mathbb{R}, \quad \forall y \in \mathbb{R}, U'(y) > 0.$$

Given an initial wealth x and strategy ϕ, the corresponding expected utility is given by

$$v(x, \phi) = \mathbb{E}\left[U(X_1^{x,\phi})\right]$$

and we are interested in the maximization problem

$$u(x) = \sup_{\phi \in \mathbb{R}} v(x, \phi) = \sup_{\phi \in \mathbb{R}} \mathbb{E}\left[U(X_1^{x,\phi})\right].$$

(1) For any $x \in \mathbb{R}$ and $\phi \in \mathbb{R}$, explain why

$$\forall \omega \in \omega, \quad X_1^{x,\phi}(\omega) = (x - \phi)B_1 + \phi S_1(\omega).$$

Give a similar formula for $\tilde{X}_1(\omega)$. Give an expression for $X_1^{x,\phi}$ in terms of $X_1^{0,\phi}$.

Recall that an arbitrage is an investment strategy ϕ s.t.

$$\forall \omega \in \Omega, X_1^{0,\phi}(\omega) \geq 0, \quad \text{and } \exists \omega_i \in \Omega, X_1^{0,\phi}(\omega_i) > 0.$$

(2) In this question, we assume that an arbitrage exists and we want to show that $u \equiv +\infty$.

(2a) Assume that ϕ is an arbitrage, show that there exists $1 \leq i \leq n$ such that, for any $x \in \mathbb{R}$,

$$v(x, \phi) \geq (1 - p_i)U\Big(x(1 + r)\Big) + p_i U\Big((x(1 + r) + X^{0,\phi}(\omega_i)\Big).$$

(2b) Deduce how to construct a sequence $\phi_n \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$\lim_n v(x, \phi_n) = +\infty.$$

Deduce the value of $u(x)$.

(3) In this question, we fix $x \in \mathbb{R}$, and assume that there exists an optimal ϕ^* for which $u(x) = v(x, \phi^*)$.

We also let

$$\mathcal{M}(\tilde{S}) = \left\{ \mathbb{Q} \text{ probability measure, } \mathbb{Q} \sim \mathbb{P} \text{ and } \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_1\right] = S_0 \right\}.$$

(3a) Compute $\frac{d}{d\phi} v(x, \phi)$.

(3b) Deduce that it holds that

$$\mathbb{E}\left[U'(X_1^{x,\phi^*})\,(S_1 - B_1)\right] = 0.$$

(3c) Check that letting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(X_1^{x,\phi^*})}{\mathbb{E}\left[U'(X_1^{x,\phi^*})\right]}$$

defines an element $\mathbb{Q}$ of $\mathcal{M}(\tilde{S})$.

(4) We now consider an option with payoff $G = G(\omega)$ at time 1. We consider the utility maximization problem for an agent buying the option, which can be written as

$$u^G(x) := \sup_{\phi \in \mathbb{R}} v^G(x, \phi) \text{ where } v^G(x, \phi) = \mathbb{E}\left[U\left(X_1^{x,\phi} + G\right)\right].$$

$p \in \mathbb{R}$ is called an **indifference price** (for the buyer, with initial wealth x) if $u(x) = u^G(x-p)$.

Throughout this question, we take for granted the existence of optimal strategies ϕ attaining the supremum in the definitions of v and v^G, for all values of x.

(4a) Discuss the definition of indifference price.

(4b) Show that for any ϕ, $x \mapsto v^G(x, \phi)$ is strictly increasing. Deduce that u^G is strictly increasing and that, the initial wealth x being fixed, the indifference price, if it exists, is unique.

(4c) Recall the definition of a viable price. Show that if p is the indifference price, then p is a viable price.

(Hint : Show that, if p is an arbitrage for the buyer, then one can construct ϕ s.t. $v^G(x - p, \phi) > v(x, \phi^*)$, and conclude that $u^G(x - p) > u(x)$. Reason similarly for the seller).

(4d) If the market is complete, what can we say about the indifference price ?

Exercise 2 : Correlated assets

We consider a market with two risky assets, with price processes $S^i, i = 1,2$ satisfying $S_0^1 = S_0^2 = 1$ and

$$dS_t^1 = b_1 S_t^1 \, dt + \sigma_1 S_t^1 dW_t^1,$$
$$dS_t^2 = b_2 S_t^2 \, dt + \sigma_2 S_t^2 \left(\rho_t dW_t^1 + \sqrt{1-\rho_t^2} dW_t^2 \right),$$

where $b_1, b_2, \sigma_1, \sigma_2$ are strictly positive constants, W^1 and W^2 are independent Brownian motions under $\mathbb{P}$, and ρ is an adapted process with values in $[-1,1]$.

The risk-free interest rate is a fixed constant $r > 0$.

The goal of the exercise is to discuss the pricing and hedging of a European option with payoff $G = g(S_T^1, S_T^2)$ at time $T > 0$.

Part I : constant correlation

In this section, we assume that $\rho_t \equiv \rho$ is a fixed deterministic constant in $]-1,1[$.

(1) Recall how to construct a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that for $i = 1,2$,

$$dS_t^1 = r S_t^1 \, dt + \sigma_1 S_t^1 d\bar{W}_t^1, \tag{1}$$
$$dS_t^2 = r S_t^2 \, dt + \sigma_2 S_t^2 \left(\rho d\bar{W}_t^1 + \sqrt{1-\rho^2} d\bar{W}_t^2 \right),$$

where $(\bar{W}^1, \bar{W}^2)$ are independent Brownian motions under $\mathbb{Q}$.

(2) Let $p_\rho : (t, s_1, s_2) \in [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto p_\rho(t, s_1, s_2) \in \mathbb{R}$ be a smooth function which satisfies on $[0,T] \times \mathbb{R} \times \mathbb{R}$ the PDE

$$rp - \partial_t p - r\left(s_1 \partial_{s_1} p + s_2 \partial_{s_2} p\right) - \frac{1}{2}(\sigma_1)^2 s_1^2 \partial_{s_1 s_1} p - \frac{1}{2}(\sigma_2)^2 s_2^2 \partial_{s_2 s_2} p - \rho \sigma_1 \sigma_2 s_1 s_2 \partial_{s_1 s_2} p = 0,$$

as well as the terminal boundary condition

$$p(T, s_1, s_2) = g(s_1, s_2), \forall (s_1, s_2) \in \mathbb{R} \times \mathbb{R}..$$

Prove that :

(i) for all $t \in [0,T]$, $s_1, s_2 > 0$,

$$p_\rho(t, s_1, s_2) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} g((S^1)_T^{t,s_1}, (S^2)_T^{t,s_2}) \right],$$

where $((S^1)^{t,s_1}, (S^2)^{t,s_2})$ denotes the solution to (1) on $[t,T]$ with initial condition $S_t^i = s_i$, $i = 1,2$.

(ii) it holds that, $\mathbb{Q}$−a.s.,

$$e^{-rT} g(S_T^1, S_T^2) = p_\rho(0, S_0^1, S_0^2) \; + \; \sum_{i=1}^{2} \int_0^T \partial_{s_i} p_\rho(t, S_t^1, S_t^2) d\tilde{S}_t^i.$$

(iii) Recall the financial interpretation of the previous question, as it relates to the option with payoff $g(S_T^1, S_T^2)$.

(3) (i) Using Itô's formula, obtain an explicit expression for $(S^1)_T^{t,s_1}$ and $(S^2)_T^{t,s_2}$ in terms of $T-t$, r, ρ, s_i, $\bar{W}_T^i - \bar{W}_t^i$, σ_i $(i=1,2)$.

Deduce an expression for $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g\left((S^1)_T^{t,s_1},(S^2)_T^{t,s_2})\right)\right]$ involving two independent standard Gaussian variables Z_1 and Z_2.

(ii) We now assume that the payoff is of the form $g(S_T^1, S_T^2) = h(S_T^1 - S_T^2)$, where h is a C^2 convex function. Using the representation from question (3)(i) and the previous question, deduce that $\partial_{s_1 s_2} p_\rho \leq 0$.

(4) We now consider the case $\rho = 1$. Show that in that case, if (b_1, σ_1) and (b_2, σ_2) are not proportional, one can construct an arbitrage.

Part II: Hedging with the wrong correlation

In this section, we go back to the case of a general correlation, which we assume not known a priori : we only know that, under a measure $\mathbb{Q} \sim \mathbb{P}$,

$$dS_t^1 = rS_t^1\, dt + \sigma_1 S_t^1 d\bar{W}_t^1,$$

$$dS_t^2 = rS_t^2\, dt + \sigma_2 S_t^2 \left(\rho_t d\bar{W}_t^1 + \sqrt{1-\rho^2} d\bar{W}_t^2\right),$$

where (ρ_t) is an adapted process, and $\bar{W}^i$, $i=1,2$ are independent $\mathbb{Q}$-Brownian motions. We assume that, it further holds that,

$$\underline{\rho} \leq \rho_t \leq \overline{\rho}, \quad \forall t \in [0,T], \mathbb{Q}\text{-a.s.}$$

where $-1 < \underline{\rho} < \overline{\rho} < 1$ are some fixed constants.

We now fix $\tilde{\rho} \in]-1,1[$, and assume that we sell and hedge the option according to the model with constant correlation parameter $\tilde{\rho}$. To be more precise, this means that we initially sell the option at price $P_0 = p_{\tilde{\rho}}(0, S_0^1, S_0^2)$, and then invest in the risky assets with a strategy $(\phi_t^1, \phi_t^2) = (\partial_{s_1} p_{\tilde{\rho}}(t, S_t^1, S_t^2), \partial_{s_2} p_{\tilde{\rho}}(t, S_t^1, S_t^2))$, where $p_{\tilde{\rho}}$ is the function considered in Part I.

Since the correlation ρ_t is not assumed constant, we will make a hedging error. We now aim at computing this error.

(1) Apply Itô's formula, and the equation satisfied by $p_{\tilde{\rho}}$, to obtain that, a.s.,

$$e^{-rT} g(S_T^1, S_T^2) = p_{\tilde{\rho}}(0, S_0) + \frac{1}{2}\int_0^T e^{-rt}\left(\rho_t - \tilde{\rho}\right)\sigma_1\sigma_2 S_t^1 S_t^2 \partial_{s_1 s_2} p_{\tilde{\sigma}}(t, S_t^1, S^2) dt$$
$$+ \int_0^T e^{-rt}\partial_{s_1} p_{\tilde{\rho}}(t, S_t^1, S_t^2)\sigma_1 d\bar{W}_t^1 + \int_0^T e^{-rt}\partial_{s_2} p_{\tilde{\rho}}(t, S_t^1, S_t^2)\left(\rho_t d\bar{W}_t^1 + \sqrt{1-\rho_t^2} d\bar{W}_t^2\right).$$

(2) Deduce an expression for the (discounted) hedging error, defined as

$$\tilde{\mathcal{E}} := \left(P_0 + \sum_{i=1}^2 \int_0^T \phi_t^i d\tilde{S}_t^i\right) - e^{-rT} g(S_T^1, S_T^2).$$

(3) We assume in this question that $g(s_1, s_2) = h(s_1 - s_2)$ where h is a C^2 convex function. Using the result of (4) (ii) in Part I, deduce a choice of $\tilde{\rho}$ such that $\tilde{\mathcal{E}} \geq 0$ (namely, the considered strategy is super-hedging).