MÉMOIRE D'H.D.R.

DE L'UNIVERSITÉ PSL

Préparé à Université Paris-Dauphine

Quelques applications de l'approche trajectorielle en analyse stochastique

Soutenu par

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Le 11 décembre 2019

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Spécialité

Mathématiques appliquées

Université Paris-Dauphine, PSL University
Centre de Recherche en Mathématiques de la Décision

Mémoire d'habilitation à diriger des recherches

Spécialité : Mathématiques appliquées

présenté par

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Remerciements

Je suis très honoré que François Delarue, Martin Hairer et Nizar Touzi aient accepté d'être rapporteurs sur ce mémoire, et je leur suis très reconnaissant du temps qu'ils y ont consacré. Je remercie également Pierre-Louis Lions pour son soutien et son aide en tant que coordinateur. Je suis aussi reconnaissant à Pierre Cardaliaguet, Peter Friz et Huyên Pham d'avoir accepté de faire partie du jury de soutenance.

La plupart des travaux présentés dans ce mémoire ont été réalisés en collaboration avec d'autres chercheurs, et c'est ici l'occasion de les remercier pour tout ce que ces rencontres scientifiques m'ont apporté. Je remercie en particulier Peter Friz pour m'avoir initié au début de mon postdoc aux thèmes de recherche autour des trajectoires rugueuses, avec un enthousiasme et une énergie qui n'ont pas diminué lors de nos collaborations ultérieures.

Je profite également de cette page pour remercier tous mes collègues du CEREMADE. Il est indéniable que l'ambiance conviviale qui règne dans ce laboratoire en fait un endroit précieux où travailler, et les cinq années passées depuis mon arrivée ont été pour moi très enrichissantes. D'un point de vue plus pragmatique je remercie chaleureusement Daniela et Cyril pour avoir mis à ma disposition les fichiers source de leurs manuscrits.

Mes derniers remerciements vont bien sûr à Julie pour son soutien constant, et à Lison et Quentin qui mettent un enthousiasme débordant à me rappeler les vraies priorités.

Contents

Introduction

This thesis contains the results obtained since my Ph.D. defense in December 2011. It consists of the research I've pursued during my postdoc at TU Berlin between September 2012 and January 2015, and during the next four years as Maître de Conférences at CEREMADE, Paris Dauphine.

Most of this research has been focused on applications of the *pathwise approach* to stochastic analysis. Given a stochastic object of interest, namely a random variable $X = X(\omega)$ on a probability space (such as the solution to a stochastic differential equation or partial differential equation), this approach consists in identifying the correct metric space $\mathcal{M}$ on which the solution map (or Itô map) $\omega \mapsto X(\omega)$ can be factorized into

$$\omega \in \Omega \quad \mapsto \boldsymbol{\omega} \in \mathcal{M} \mapsto \mathbf{X}(\boldsymbol{\omega}) = X(\omega)$$

(the latter equality being in the a.s. sense), where the second map above is *continuous.* This allows to split the study of a stochastic problem into a probabilistic step (the first map above) and a separate deterministic (or analytic) step (the second map), which has a certain number of advantages.

As an example, consider the stochastic (Stratonovich, to simply) integral

$$X(\omega) = \int_0^1 f(B_s) \circ dB_s = \sum_{i=1}^d \int_0^1 f_i(B_s) \circ dB_s^i$$

where $B(\omega) = \omega \in \Omega = C([0,1], \mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}^d$. When $d = 1$, one simply has

$$X = F\left(B(1) - B(0)\right)$$

where F is an antiderivative of f, so that one can trivially take $\mathcal{M} = \Omega$ endowed with supremum norm. When $d \geq 2$ however, such a representation does not hold in general, and identifying the correct space is non-trivial and the object of Lyons' rough path theory : $\mathcal{M}$ is then a *non-linear* metric space.

This thesis is divided in four chapters that (except for the last one) deal with several applications of the above methodology :

- The first chapter is dedicated to fully nonlinear Stochastic Partial Differential Equations (SPDE) where the stochastic term is of Hamilton-Jacobi type. In that case, even if the noise is one-dimensional (corresponding to $d = 1$ in the above example), the Itô map is non-trivial due to the irregularity of the solution (and the fact that noise enters the equation in a non-linear way). I discuss the results of [FGLS17], establishing continuous dependence on the noise in the case where the Hamiltonian is quadratic, the results of [GG19] which concern regularizing properties of the stochastic term in such PDE, and the results of [Gas17, GGLS19] which concern finite speed of propagation properties. The latter rely on a good deterministic understanding of the Itô map, in particular on which oscillations of the noise cancel out at the PDE level, which is seen to depend on convexity properties of the Hamiltonian.

- The second chapter deals with applications of Malliavin calculus to so-called singular SPDE. The solution theory for these equations relies on Hairer's regularity structures, a multi-parameter generalization of rough path theory. In regularity structures, the elements of the space $\mathcal{M}$ are the so-called models, which give meaning to a certain number of polynomial functions of the noise. Interestingly, applying Malliavin calculus relies on understanding further analytical properties of the Itô map, namely its differentiability in the Cameron-Martin directions. This then allows to obtain information on the laws of the solutions, such as absolute continuity of finite-dimensional projections with respect to Lebesgue measure. I describe in this chapter the results obtained in [CFG17] and [GL19] on two such equations, namely the generalized 2d Parabolic Anderson Model and the 3d Stochastic Quantization equation.

- The third chapter is concerned with the analysis of rough volatility models in finance. This class of models, where the volatility has sample paths which are rougher than that of a Brownian motion, has been recently observed to reproduce remarkably well historical and pricing market data. I describe the results of [BFG$^+$19], where we remarked that studying these models via pathwise methods (and precisely : the theory of regularity structures) had a number of advantages. I then present the results of [FGP18] where this methodology allowed us in particular to prove precise large deviation estimates. Finally, I discuss the results of the note [Gas19] on the martingale property in the rough Bergomi model.

- The fourth chapter is thematically separate from the others (it does not rely on pathwise considerations), and is concerned with the study of the Skorokhod Embedding Problem (SEP). The latter consists in finding stopping times such that the law of a Brownian motion at this time coincides with a prescribed measure. The results of this chapter concern a solution to the SEP discovered by Root, where the stopping time is a hitting time of a barrier by the time-space process. I describe the results of [GOdR15], where we identify this barrier as the free boundary to a PDE, those of [GMO15] where we compute this barrier with an integral equation, and finally the results recently obtained in [GOZ19] where we extend the free boundary representation to the case of a rather general Markov process.

Every chapter contains a brief introduction, the motivation, contextualization and description of the obtained results, and some open problems and perspectives.

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Chapter 1

Stochastic Hamilton-Jacobi equations

1.1 Introduction

In this chapter we consider evolution equations of the general type

$$\begin{cases} \mathrm{d}u = F(D^2u, Du, u, x, t)\mathrm{d}t + \sum_{i=1}^{d} H_i\left(Du, u, x\right)\mathrm{d}\xi^i(t) \;\; \text{in} \;\; \mathbb{R}^N \times (0, T], \\ u = u_0 \text{ on } \mathbb{R}^N \times \{0\}; \end{cases} \tag{1.1}$$

Here the unknown is a scalar function $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and $\xi \in C([0, T], \mathbb{R}^d)$ is a noise term (in typical applications one may take $\xi = B(\omega)$ be a Brownian path, and the equation should then be understood in Stratonovich sense).

Several technical assumptions have to be made on F and H but the most important assumption is that F is degenerate elliptic, namely

$$F \text{ is nondecreasing in its first argument,} \tag{1.2}$$

which formally guarantees that solutions of (1.1) should satisfy the maximum principle.

Equations of the form (1.1) have been introduced by Lions and Souganidis in a series of notes [87, 88, 90, 91] under the name "Fully nonlinear SPDEs". They have shown that suitable modifications of the deterministic theory of viscosity solutions allow to study these equations.

An important example of application of equation (1.1) is given by the stochastic motion of hypersurfaces. Indeed, assume that an oriented hypersurface Γ embedded in $\mathbb{R}^N$ moves according to normal velocity at each point $x \in \Gamma$

$$V = f(x, \vec{n}, D\vec{n}) + \sum_{i=1}^{d} g_i(x, \vec{n})\dot{B}^i, \tag{1.3}$$

where $\vec{n}$ is the (outer) normal of Γ (at the point x), and B^i are independent Brownian motions. The level set approach [103] associates to this geometric motion a PDE of the form (1.1). More precisely, given f and g there exist F and H such that if u is a solution to (1.1) above, formal computations yield that

$$\Gamma(t) := \{x, \;\; u(t,x) = 0\}$$

evolves according to (1.3). As an (important) example, when f is the mean curvature operator and $d = 1$, $g \equiv 1$, the equation for u is

$$\mathrm{d}u = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) \mathrm{d}t + |Du| \circ \mathrm{d}B_t. \tag{1.4}$$

The deterministic case ($g \equiv 0$) has been intensively studied, and the level set approach has proven to be very fruitful there (see for instance the monograph [61] and references therein). In the stochastic case, motions of the form (1.4) have first been considered in the physics litterature [83] and then studied rigorously in various contexts by a number of authors, see for instance [54, 35, 115, 118, 76].

Numerous aditional applications of fully nonlinear SPDEs of the form (1.1) are detailed in [89] including nonlinear filtering, interest rate modeling, and pathwise optimal control.

From the mathematical point of view, the analysis of (1.1) is delicate. Since no structure is assumed except degenerate ellipticity, the equation is not expected to have classical (C^2) solutions (even if u_0 is smooth, the solution will only be regular up to a finite time). In the deterministic case (i.e. when $H \equiv 0$), the Crandall-Lions theory of viscosity solutions [32, 31] has been developped to give a notion of solution to fully nonlinear equations. This theory comes with very general existence/uniqueness/stability results. Of course when ξ is smooth, (1.1) appears as a special case of the equations to which this theory applies (actually this can be pushed to $\dot{\xi} \in L^1_{loc}$, see e.g. [7]). However the theory breaks down for rough ξ. Indeed, the basic procedure in viscosity solutions is to use smooth test functions "touching" the solution from above or below. In the stochastic (or rough case), the solution cannot be regular enough in the time variable for such functions to even exist. Making sense of (1.1) therefore requires new methods.

One possible approach to define solutions to (1.1) is by obtaining robust estimates on the Itô-type map

$$\xi \mapsto u.$$

Namely, if ξ^ε is a sequence of smooth approximations to a given ξ, then for each ε there exists a well-defined (viscosity) solution u^ε to (1.1) with ξ replaced by ξ^ε. The question is then to establish the convergence of u^ε to a unique limit. When this convergence holds we can define the solution to (1.1)as this limit. (We note that this is not the only possible definition, for instance Lions and Souganidis [91] have defined a weaker notion of stochastic viscosity solution based on test functions, in the spirit of the deterministic definition, see also [63]).

Due to the fact that u is not expected to be C^2 (or even C^1), the equation cannot be simply interpreted as an infinite dimensional (rough) ODE. To illustrate the difference, recall that in the case of finite dimensional ODEs with one-dimensional noise ($d = 1$), the solution to

$$\dot{X} = V(X)\dot{\xi}$$

is simply given by

$$X(t) = \phi_V(X_0, \xi(t) - \xi(0))$$

where $\phi_V = \phi_V(x,t)$ is the flow associated to the vector field V (namely $\phi_V(x,0) = x$ and $\partial_t\phi_V = V(\phi_V)$). In particular, it is obvious that the map $\xi \mapsto X$ extends continuously to any continuous ξ. (The Doss-Susmann decomposition [36] shows that a similar result remains true when the equation has an additional drift term, while the case when $d \geq 2$ is more subtle and requires Lyons' rough path theory [94]).

In contrast, in the Hamilton-Jacobi case, the solution to

$$\mathrm{d}u = H(Du,x)\mathrm{d}\xi$$

does not depend only on the increment of ξ. This is due to the fact that since H is nonlinear, shocks (discontinuities in the gradient) are created which lead to irreversibility in the dynamics. As a simple example, consider the case when $H(Du,x) = |Du|$ and $u(0,x) = |x|$. Straightforward computations show that the (sub-)level sets $\Gamma_R(t) = \{u(t,\cdot) \leq R\}$ are given by

$$\Gamma_R(t) = \begin{cases} B_0(R - \xi(t)), & t \leq \tau \\ \emptyset, & t > \tau \end{cases}$$

where $B_0(r)$ denotes the ball centered at 0 of radius r and $\tau = \inf\{t > 0, \;\; R - \xi(t) < 0\}$. This is intuitively clear, since the equation corresponds to a geometric motion where spheres move with radial speed given by $\dot{\xi}$. However, there is some irreversibility due to the fact that once a sphere disappears it is not re-created.

In this specific example, one can see that the solution at time t only depends on the increment $\xi(t)$ and $\max_{[0,t]} \xi(s)$ (but this is specific to the choice of H and of the initial condition). These observations show that even though one still expects the solution map to be uniformly continuous (w.r.t. supremum norm when $d = 1$ or rough path distance when $d \geq 2$), this cannot be a simple consequence of infinite dimensional rough path theory and requires an understanding via viscosity solution theory. It is also interesting to understand the structure of the map $\xi \mapsto u$, which is more complicated than in the ODE case.

In the rest of this chapter, I will detail my contributions to the study of these equations. They all deal with the simplest case when the noise is one-dimensional (i.e. $d = 1$), in particular rough path theory does not play a role. In section 1.2 I detail a well-posedness estimate in the case where $H = H(Du,x)$ is quadratic in Du. In section 1.3 I discuss how in the case where $H(Du) = \frac{1}{2}|Du|^2$ is the squared Euclidean norm, the stochastic term in (1.1) leads to a regularizing effect on the solution. Finally, section 1.4 is devoted to results on the so-called speed of propagation (of initial datum) for stochastic Hamilton-Jacobi equations (i.e. $F \equiv 0$ in (1.1)). These results rely on understanding which oscillations of the path cancel at the PDE level, and depend on convexity properties of H.

Notations

For $T > 0$ and $k \geq 0$ we define $C_0^k([0,T]) := \{\xi \in C^k([0,T],\mathbb{R}) : \xi(0) = 0\}$. We define $BUC(\mathbb{R}^N)$ to be the space of all bounded and uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$.

Given $E \subset \mathbb{R}^N$ we let $\|u\|_{\infty,E}$ the usual (essential) supremum norm of a function $u : E \to \mathbb{R}$. If u is Lipschitz continuous we further let $\|Du\|_\infty$ be the Lipschitz constant of u.

We say that a function $u : \mathbb{R}^N \to \mathbb{R}$ is semiconvex (resp. semiconcave) of order C if $x \mapsto u(x) + \frac{1}{2}C|x|^2$ is convex (resp. $x \mapsto u(x) - \frac{1}{2}C|x|^2$ is concave). We let $\|D^2u\|_\infty$ be the smallest C such that u is both semiconcave and semiconvex of order C.

1.2 Eikonal SPDE [FGLS17]

In this section we discuss the results obtained in [FGLS17] (in collaboration with P. Friz, P.L. Lions and P.E. Souganidis). They concern a well-posedness result (in the sense of continuous dependence on ξ) for equations (1.1) in a special case (quadratic Hamiltonian).

Let us briefly recall here the previous well-posedness results. The Lions–Souganidis theory applies to rather general paths when $H = H(Du)$ and, as established in [92] (see also [116]), there is a very precise trade off between the regularity of the paths and H. In the general case when ξ is only continuous, then H is required to be the difference of two convex functions, cf. [88]. The spatial dependent case $H = H(Du, x)$ is in general much more delicate, Lions and Souganidis [92, 116] have obtained some results for general continuous paths (when $d = 1$) and Brownian paths ($d \geq 2$) under certain global structural conditions on H involving higher order derivatives in x and Du.

Some results are also available for $H = H(Du, u, x)$ in the semilinear case i.e. when H depends linearly on Du. In that case, the analysis is greatly simplified since global transformations exist under which (1.1) becomes a classical (deterministic, fully nonlinear) PDE. The case $H = H(u, x)$ was studied in Lions and Souganidis [90] for general F, see also Buckdahn and Ma [18, 19] and then later Diehl and Friz [34]. The case $H = H(Du, u, x)$ was discussed in [20, 21, 47] who obtained stability in the rough path sense.

This section is devoted to the case when the Hamiltonian is quadratic positive definite, namely

$$H(x,p) = \sum_{i,j=1}^{N} g^{ij}(x) p_i p_j \tag{1.5}$$

where there exists $C > 0$ s.t. for all $x \in \mathbb{R}^N$.

$$\frac{1}{C} I_N \leq (g^{ij}(x))_{i,j} \leq C I_N$$

in the sense of symmetric matrices. We further assume that

$$g \in C_b^2(\mathbb{R}^N, \mathbb{R}^{N\times N})$$

The main result in [FGLS17] is an estimate on the difference of solutions to equations (1.1) driven by two different (smooth) noises.

Theorem 1.1. *Assume that F satisfies a small variant of the standard assumptions in viscosity solution theory. Let $\xi, \zeta \in C_0^1([0,T];\mathbb{R})$ $u_0, v_0 \in \mathrm{BUC}(\mathbb{R}^N)$ and $T > 0$. Then there exists a nondecreasing $\Phi : [0,\infty) \to [0,\infty]$, depending only on T and the moduli and* sup*-norms of $u_0, v_0 \in \mathrm{BUC}$, with $\lim_{r\to 0} \Phi(r) = \Phi(0) = 0$, such that, for all $\xi, \zeta \in C_0([0,T])$, if u, v are respectively viscosity sub- and super-solutions of*

$$\partial_t u - F(D^2u, Du, u, x, t) - H(Du, x)\dot{\xi} \leq 0 \quad \text{in } \mathbb{R}^N \times (0,T] \qquad u(0,\cdot) \leq u_0 \quad \text{on } \mathbb{R}^N, \tag{1.6}$$

and

$$\partial_t v - F(D^2 v, Dv, v, x, t) - H(Dv, x)\dot{\zeta} \geq 0 \quad \textit{in } \mathbb{R}^N \times (0,T] \qquad v(0,\cdot) \geq u_0 \quad \textit{on } \mathbb{R}^N, \tag{1.7}$$

then it holds that

$$\|(u-v)_+\|_{\infty;\mathbb{R}^N\times[0,T]} \leq \|(u_0-v_0)_+\|_{\infty;\mathbb{R}^N} + \Phi\left(\|\xi-\zeta\|_{\infty;[0,T]}\right). \tag{1.8}$$

For precise assumptions on F we refer to [FGLS17], but note that they are satisfied when F is a Hamilton-Jacobi-Isaacs operator

$$F(M,p,r,x,t) = \inf_{\alpha}\sup_{\beta}\left\{\operatorname{tr}\left(\sigma_{\alpha\beta}\sigma_{\alpha\beta}^T(p,x)\,M\right) + b_{\alpha\beta}(p,x) - c_{\alpha\beta}(x)r\right\}, \tag{1.9}$$

with

$$\sigma, b, c \text{ bounded and Lipschitz uniformly in } \alpha, \beta.$$

In addition, in the case of the stochastic Hamilton Jacobi equations ($F \equiv 0$), the estimate is simplified : there exists then $C > 0$ such that under the assumptions of the theorem,

$$\sup_{[0,T]\times\mathbb{R}^N}(u-v) \leq \sup_{x,y\in\mathbb{R}^N}\left(u_0(x) - v_0(y) - \frac{C|x-y|^2}{\sup_{s\in[0,T]}(\xi(s)-\zeta(s))}\right) \tag{1.10}$$

(with convention $0/0 = 0$, $1/0 = +\infty$).

The main interest in the above results is that they allow to extend the solution map to all continuous paths.

Corollary 1.2. *Under the assumptions of Theorem 1.1, the solution operator $\mathcal{S}$ to* (1.1)

$$\mathcal{S} : BUC(\mathbb{R}^N) \times C^1([0,T]]) \to BUC(\mathbb{R}^N \times [0,T]), \quad (u_0,\xi) \mapsto u$$

admits a unique continuous extension to $\bar{\mathcal{S}} : BUC(\mathbb{R}^N) \times C([0,\infty);\mathbb{R}) \to BUC(\mathbb{R}^N \times [0,T])$, together with the estimate

$$\left\|\bar{\mathcal{S}}(u_0,\xi) - \bar{\mathcal{S}}(v_0,\zeta)\right\|_{\infty;\mathbb{R}^N\times[0,T]} \leq \|u_0 - v_0\|_{\infty;\mathbb{R}^N} + \Phi\left(\|\xi-\zeta\|_{\infty;[0,T]}\right). \tag{1.11}$$

Let us describe briefly the idea of proof of Theorem 1.1. We start by recalling the idea of the proof of comparison in viscosity solution theory in the most simple case. Let u, v satisfy

$$\partial_t u - H(Du) \leq 0 \leq \partial_t v - H(Dv)$$

in viscosity sense. We then consider for $\lambda > 0$

$$M_\lambda(t) := \sup_{x,y\in\mathbb{R}^N} u(t,x) - v(t,y) - \frac{\lambda}{2}|x-y|^2,$$

which we assume to be attained at some $(\hat{x},\hat{y})$. Then the first order optimality conditions are written as

$$D_x u(t,\hat{x}) = D_y v(t,\hat{y}) = \lambda(\hat{x}-\hat{y})$$

and using the equations satisfied by u and v this implies that

$$\frac{d}{dt}M_\lambda(t) = \partial_t u(t,\hat{x}) - \partial_t v(t,\hat{y}) \leq H(D_x u(t,\hat{x})) - H(D_y v(t,\hat{y})) = 0.$$

(The above is only formal since u, v are not assumed to be C^1, but the definition of viscosity solution and the doubling of variables ensure that the argument is still valid). One then lets $\lambda \to \infty$ to deduce that $\frac{d}{dt}\sup_x (u(t,x) - v(t,x)) \leq 0$.

When treating the "stochastic" case, the above proof needs to be modified since a naive application will lead to the appearance of a term of the form

$$H(\hat{x},\hat{p})\dot{\xi} - H(\hat{y},\hat{p})\dot{\zeta}$$

which has no reason to be small if $(\xi - \zeta)$ is only small in supremum norm. The idea (due to Lions and Souganidis) is that one should replace the penalization $\lambda|x-y|^2$ by a well-chosen function $\Phi^\lambda = \Phi^\lambda(t,x,y)$ satisfying (in a classical sense) the equation

$$\partial_t \Phi^\lambda - H(x, D_x\Phi^\lambda)\dot{\xi} + H(y, -D_y\Phi^\lambda)\dot{\zeta} = 0. \tag{1.12}$$

One can then still proceed as above to conclude that (in the case $F \equiv 0$) if u, v satisfy (1.6)-(1.7), then

$$\frac{d}{dt}\sup_{x,y\in\mathbb{R}^N} u(t,x) - v(t,y) - \Phi^\lambda(t,x,y) \leq 0.$$

Obtaining information on solutions to (1.12) is in general difficult, but it turns out that in the quadratic case (1.5), there exist explicit solutions given by

$$\Phi^\lambda(t,x,y) := \frac{\lambda d_g(x,y)^2}{1-\lambda(\xi_t - \zeta_t)},$$

where d_g is the Riemannian distance associated to the Hamiltonian g. (This will not define a smooth function in general, but one can check that it will be smooth near the diagonal $x = y$ which is enough for our purposes.) Clearly one cannot take $\lambda \to \infty$ as in the deterministic case but taking $\lambda \to \sup(\xi - \zeta)$ yields the estimate (1.10).

The proof of the general case ($F \neq 0$) is more technical but also relies on modifications of classical arguments in viscosity solution theory (together with suitable estimates on the Hessian of d_g).

1.3 Regularization properties [GG19]

The results of this section concern the special case when the Hamiltonian is the square of the Euclidean norm

$$H(p) = \frac{1}{2}|p|^2. \tag{1.13}$$

We observed in a joint work with B. Gess [GG19] that, in that case, the stochastic term in (1.1) leads to a regularizing effect in terms of bounds on the second derivatives of the solution.

This work can be seen as a particular example of the *regularization by noise* phenomenon, the general principle of which is that the inclusion of stochastic perturbations in an equation may lead to more regular solutions and in some cases even to the uniqueness of solutions. Historically, possible regularizing effects of additive noise in PDE have been investigated, e.g. for (stochastic) reaction diffusion equations [64] and for Navier-Stokes equations in [43, 44]. In [42, 40], well-posedness and regularization by linear multiplicative noise for transport equations have been obtained. We refer to [41, 57] and references therein for (many) more examples where this phenomenon appears.

In the case of nonlinear noise, fewer results are known. Regularizing effects of *non-linear* stochastic perturbations in the setting of (stochastic) scalar conservation laws have been discovered in [58]. In particular, in [58] it has been shown that the inclusion of a Burgers-type multiplicative noise leads to higher order Sobolev regularity of the solution, compared to the deterministic theory. Subsequently, the results and techniques developed in [58] have been (partially) extended in [59] to a class of parabolic-hyperbolic SPDE.

In our case of fully nonlinear PDE perturbed by a noise driven by a quadratic Hamiltonian, the idea is to use the (well-known) regularization property of the semigroup S_H associated to the equation $\partial_t u = H(Du)$. In fact, this semigroup is explicitely given as sup-convolution

$$S_H(\delta)(\phi)(x) = \sup_y \left\{ \phi(y) - \frac{|x-y|^2}{\delta} \right\}$$

and one immediately reads from this formula that, independently on the regularity of the initial condition, $S_H(\delta)(\phi) + \frac{|\cdot|^2}{\delta}$ is convex, so that $S_H(\delta)(\phi)$ is always semiconvex (i.e. its second derivative is bounded from below). Similarly, the semigroup S_{-H} maps any initial condition to a semiconcave function.

It was further observed by Lasry-Lions [84] that combining S_H and S_{-H} in a suitable way led to two-sided ($C^{1,1}$) bounds. In [GG19] we extend this result by noting that when the signal is stochastic, the term $H(Du)\dot{\xi}(t)$ similarly leads to a two-sided regularizing effect, which could possibly compensate for a loss of regularity created by the deterministic F term.

The main result in [GG19] can be written as follows.

Theorem 1.3. *Suppose that F satisfies the usual assumptions in the theory of viscosity solutions. We further assume that there exist V^+, V^- locally Lipschitz on $(0,\infty)$ s.t. if v is a solution to $\partial_t v - F(D^2 v, Dv, x, t) = 0$, then letting $\ell^+(t) \in [0,\infty)$ be the largest ℓ s.t. $D^2 v(t,\cdot) \leq \frac{1}{\ell} I$ (resp. ℓ^- the largest ℓ s.t. $D^2 v(t,\cdot) \geq -\frac{1}{\ell} I$), it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \ell^{\pm}(t) \geq V_F^{\pm}(\ell^{\pm}(t)). \tag{1.14}$$

Let $u_0 \in BUC(\mathbb{R}^N)$, $\xi \in C(\mathbb{R}_+)$, and let u be the unique viscosity solution to

$$\begin{cases} \mathrm{d}u + \frac{1}{2}|Du|^2 \mathrm{d}\xi(t) = F(D^2 u, Du, x, t)\mathrm{d}t, \\ u(0,\cdot) = u_0. \end{cases} \tag{1.15}$$

Suppose that $-\frac{Id}{\ell_0^-} \leq D^2 u_0 \leq \frac{Id}{\ell_0^+}$ for some $\ell_0^{\pm} \in [0,\infty)$, in the sense of distributions. Then, for each $t \geq 0$,

$$-\frac{Id}{L^-(t)} \leq D^2 u(t,\cdot) \leq \frac{Id}{L^+(t)}, \tag{1.16}$$

in the sense of distributions, where $L^{\pm} : \mathbb{R}_+ \to \mathbb{R}_+$ *is the maximal continuous solution to*

$$\begin{aligned} &\mathrm{d}L^{\pm}(t) = V_F^{\pm}(L^{\pm}(t))\mathrm{d}t \quad \pm \quad \mathrm{d}\xi(t) \ \textit{on} \ \{t \geq 0 : L^{\pm}(t) > 0\}, \quad L \geq 0, \\ &L^{\pm}(0) = \ell_0^{\pm}. \end{aligned} \tag{1.17}$$

The above result allows to obtain two-sided bounds on D^2u at generic times t, for a large class of equations. In contrast, in the deterministic case one usually only has one-sided bounds for solutions to degenerate (or first-order) parabolic equations. Indeed, since usually at least V^+ or V^- is negative, then one of $\ell+$ and ℓ^- willl reach zero in finite time and then stay there. In contrast, the inclusion of the random perturbation in (1.15) and consequently in (1.17) can cause both solutions $L^{\pm}$ to become strictly positive even after previously attaining zero value, thus implying a two sided bound on the second derivative of u via (1.16). In this sense, we observe a regularization by noise effect.

In order to illustrate the above theorem, let us give a few examples. The first one is the very simple case when $F \equiv 0$.

Theorem 1.4. *Consider the solution to*

$$\mathrm{d}u + \frac{1}{2}|Du|^2 \circ \mathrm{d}\xi_t = 0 \quad \textit{on } \mathbb{R}^N, \tag{1.18}$$

with $\xi \in C(\mathbb{R}_+)$ *and* $u(0,\cdot) = u_0 \in BUC(\mathbb{R}^N)$*. Then*

$$\|D^2u(t,\cdot)\|_{\infty;\mathbb{R}^N} \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^+(t) = \xi_t - \min_{s\in[0,t]} \xi_s$, $L^-(t) = \max_{s\in[0,t]} \xi_s - \xi_t$.

Note that the bound above does not depend on the regularity of the initial condition. While this result is already interesting, the roughness of the signal does not appear in an important way (since there are obviously smooth ξ's such that the associated $L^{\pm}$ are zero only at countably many points). Here is another example where roughness of ξ plays a role.

Theorem 1.5. *Consider the solution to*

$$\mathrm{d}u + \frac{1}{2}|Du|^2 \circ \mathrm{d}\beta_t^H = F(Du)\,\mathrm{d}t \quad \textit{on } \mathbb{R}^N, \tag{1.19}$$

where β^H *is a fractional Brownian motion with Hurst parameter* $H \in (0,1)$*,* $F \in C^2(\mathbb{R}^N)$*, and* $u(0,\cdot) = u_0$ *is Lipschitz and bounded. Then, for all* $t > 0$*,*

$$\mathbb{P}(\|D^2u(t,\cdot)\|_{L^\infty} < \infty) = 1,$$

for u *being a solution to* (1.19).

In that case, the equations for $L^{\pm}$ are given by

$$\mathrm{d}L^{\pm} = -C\mathrm{d}t \pm \mathrm{d}\xi(t)$$

for some constant $C > 0$ depending on F and u_0, and here the roughness of ξ (more precisely, the fact that it is not Lipschitz continuous) is crucial to have $L^{\pm}$ be nonzero at almost all times.

Even more interesting is the following example, where we show that whether regularization (i.e. finite bounds on D^2u) occur depend on the strength of the noise.

Theorem 1.6. *Consider the stochastic p-Laplace equation*

$$du + \frac{\sigma}{2}|\partial_x u|^2 \circ d\beta(t) = \frac{1}{6}\partial_x((\partial_x u)^3)\, dt \quad \text{on } \mathbb{R},$$

with $\sigma > 0$, β a Brownian motion and initial condition $u_0 \in BUC(\mathbb{R})$. Then :

- *If $\sigma > 2$,*

$$\forall t > 0, \quad \mathbb{P}\text{-a.s.}, \ \|\partial_{xx} u(t)\|_{\infty;\mathbb{R}} < \infty.$$

- *There exist a (nonempty) class of initial conditions $\mathcal{U}$ s.t. if $u_0 \in \mathcal{U}$ and $\sigma \leq 2$, then*

$$\mathbb{P}\text{-a.s., for all } t \text{ large enough}, \|\partial_{xx} u(t)\|_{\infty;\mathbb{R}} = \infty \quad .$$

Indeed, in that case $L^{\pm}$ are the solutions to the reflected (at 0^+) SDE with dynamics on $(0, \infty)$ given by

$$dL^{\pm} = -\frac{2}{L^{\pm}(t)}dt \pm \sigma d\beta_t, \quad L^{\pm}(0) = \frac{1}{\|(\partial_{xx} u_0)_{\pm}\|_{\infty;\mathbb{R}}},$$

namely they are (up to a time-change) Bessel processes of dimension in $(-\infty, 1)$. The cut-off for σ then corresponds to the dimension 0 for the Bessel processes , which is where the boundary behaviour at 0 changes from a regular boundary to an exit boundary (i.e. in the latter case the processes $L^{\pm}$ cannot leave the point 0 after reaching it).

1.4 Speed of propagation of initial datum [Gas17, GGLS19]

An important feature of (deterministic) Hamilton-Jacobi equations

$$\partial_t u = H(Du, x) \quad \text{on } (0, T) \times \mathbb{R}^N \tag{1.20}$$

is the so-called *finite speed of propagation* : assuming for instance that H is Lipschitz in Du with Lipschitz constant L, then if u^1 and u^2 are two (viscosity) solutions of (1.20), one has

$$u^1(0, \cdot) = u^2(0, \cdot) \text{ on } B_0(R) \ \Rightarrow \ \forall t \geq 0, \ u^1(t, \cdot) = u^2(t, \cdot) \text{ on } B_0(R - Lt) \tag{1.21}$$

where by $B_0(R)$ we mean the ball of radius R centered at 0.

A formal proof is the following : let w solve the equation $\partial_t = L|Dw|$, then the Lax-Oleinik formula yields

$$w(t, x) = \sup \{w(0, y), \quad |y - x| \leq Lt\}.$$

On the other hand, with u^1, u^2 as above, then

$$\partial_t(u^1 - u^2) = H(x, Du^1) - H(x, Du^2) \leq L|D(u^1 - u^2)|.$$

By combining the representation of w with the comparison principle, we can deduce (1.21).

A natural question (raised by Souganidis in [116]) in our context is whether a similar property holds in the stochastic case, for the equation

$$\partial_t u = H(Du, x)\dot{\xi} \text{ in } \mathbb{R}^d \times (0, T] \qquad u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^d \tag{1.22}$$

Let us introduce some notation. Given $T > 0$ and $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ let

$$\begin{aligned}\rho_H(\xi, T) := \sup \Big\{ R \geq 0 : \text{ there exist solutions } u^1, u^2 \text{ of (1.22) and } x \in \mathbb{R}^d, \\ \text{such that } u^1(\cdot, 0) = u^2(\cdot, 0) \text{ in } B_x(R) \text{ and } u^1(x, T) \neq u^2(x, T) \Big\},\end{aligned} \tag{1.23}$$

where $B_x(R)$ is the ball in $\mathbb{R}^d$ centered at x with radius R.

The classical argument presented above yields that, if ξ is a C^1- or, more generally, a BV-path, then

$$\rho_H(\xi, T) \leq L\|\xi\|_{TV([0,T])}, \tag{1.24}$$

where

$$\|\xi\|_{TV([0,T])} := \sup_{0=t_0 \leq \ldots \leq t_n = T} \sum_{i=0}^{n-1} |\xi(t_{i+1}) - \xi(t_i)|$$

is the total variation semi-norm of ξ and L is the Lipschitz constant of H w.r.t. Du. Simple examples show that (1.24) is sharp in the deterministic case when $H = H(Du)$ and $\dot{\xi} \equiv 1$.

Lions and Souganidis [116] had obtained a result of finite speed of propagation for constants.

Theorem 1.7 (Lions-Souganidis). *If $H = H_1 - H_2$ where $H_i = H_i(p)$ convex, $\|D_p H_i\|_\infty \leq 1$, with $H_i(0) = 0$, then for any $A \in \mathbb{R}$,*

$$u(0, \cdot) \equiv A \text{ on } B_0(R) \;\; \Rightarrow u(t, \cdot) \equiv A \text{ on } B_0(R(t))$$

where $R(t) = R - (\max_{s\in[0,t]} \xi(s) - \min_{s\in[0,t]} \xi(s))$.

By comparison, this implies results for level sets, e.g.

$$\{u_0 \geq A\} \supset B_0(R) \Rightarrow \{u(t, \cdot) \geq A\} \supset B_0(R(t)).$$

This means that the evolution of a smooth (in the sense of bounded curvature) level set satisfies a finite speed of propagation, at least until the time when it develops singularities. This is similar in spirit (although a stronger statement) with the observation that if the initial condition u_0 is smooth, then so will be the solution for some time, during which the oscillations of ξ cancel out at the PDE level (which implies finite speed of propagation by the classical theory).

Note that Theorem 1.7 also implies that local bounds propagate with finite speed, i.e.

$$\|u(t, \cdot)\|_{\infty; B_0(R(t))} \leq \|u_0\|_{\infty; B_0(R)}.$$

However this does not imply a finite speed of propagation for (1.22). In the note [Gas17], I obtained a simple counter-example showing that in general, the bound (1.24) cannot be improved.

Theorem 1.8 ([Gas17])**.** *Let*

$$H : \mathbb{R}^2 \times \mathbb{R}^2 \to R, H(p,x) = |p_1| - |p_2|.$$

Then for each $T > 0$ and $\xi \in C([0,T])$,

$$\rho_H(\xi, T) = \|\xi\|_{TV([0,T])}.$$

In particular with that choice of H the equation (1.22) *does not have the finite speed of propagation property if ξ is of unbounded variation.*

An important by-product of the above equality is that for such a Hamiltonian H, all of the oscillations of ξ are relevant for the dynamics of (1.22). Actually, the proof shows that there exists a choice of u_0 such that for any ξ of bounded variation,

$$t < s \Rightarrow u(t,\cdot) \neq u(s,\cdot) \text{ unless } \dot{\xi} \equiv 0 \text{ on } (t,s).$$

In a further work [GGLS19], in collaboration with B.Gess, P.L. Lions and P.E. Souganidis, we specialize to the important case when H is convex in Du. In that case we establish an estimate, which is better than (1.24), and, in particular, implies that the rate of dependence $\rho_H(\xi, T)$ is almost surely finite when ξ is a Brownian path. This new bound relies on a better understanding of which oscillations of the signal ξ are effectively relevant for the dynamics of (1.22). Namely we prove that, if H is convex, then ξ can be replaced by a reduced path $R_{0,T}(\xi)$ which keeps track solely of the oscillations of ξ that are relevant for the dynamics of (1.22).

This path is defined in the following way : given $\xi \in C_0([0,T])$, the sequence $(\tau_i)_{i\in\mathbb{Z}}$ of successive extrema of ξ is defined by

$$\tau_0 := \sup\left\{t \in [0,T],\;\; \xi(t) = \max_{0\le s\le T} \xi(s) \text{ or } \xi(t) = \min_{0\le s\le T} \xi(s)\right\}, \tag{1.25}$$

and, for all $i \geq 0$,

$$\tau_{i+1} = \begin{cases} \arg\max_{[\tau_i,T]} \xi & \text{if} \quad \xi(\tau_i) < 0, \\ \arg\min_{[\tau_i,T]} \xi & \text{if} \quad \xi(\tau_i) > 0, \end{cases} \tag{1.26}$$

and, for all $i \leq 0$,

$$\tau_{i-1} = \begin{cases} \arg\max_{[0,\tau_i]} \xi & \text{if} \quad \xi(\tau_i) < 0, \\ \arg\min_{[0,\tau_i]} \xi & \text{if} \quad \xi(\tau_i) > 0. \end{cases} \tag{1.27}$$

The reduced path $R_{0,T}(\xi)$ of $\xi \in C_0([0,T])$ is then defined as the (unique continuous) path that coincides with ξ on each of the τ_i, and is linear on each interval of the form (τ_i, τ_{i+1}), $i \in \mathbb{Z}$ (see Figure 1.1).

Theorem 1.9. *Let u^ξ be the solution to* (1.22) *with H convex in Du. Then it holds that*

$$u^\xi(\cdot, T) = u^{R_{0,T}(\xi)}(\cdot, T). \tag{1.28}$$

In particular, for all $\xi \in C_0([0,T])$, we have

$$\rho_H(\xi, T) \leq L \; \|R_{0,T}(\xi)\|_{TV([0,T])}. \tag{1.29}$$

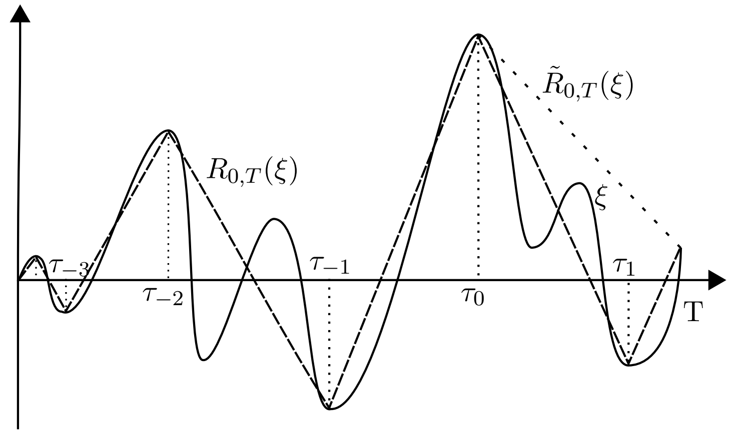


Figure 1.1: The reduced path $R_{0,T}(\xi$ and fully reduced path $\tilde{R}_{0,T}(\xi)$

The proof is rather straightforward, using as a bulding block the inequality

$$S_H(t) \circ S_H(-t)u_0 \leq u_0 \leq S_H(-t) \circ S_H(t)u_0,$$

which follows from the control representation of S_H

Of course, the reduced path $R_{0,T}(\xi)$ may very well not have finite variation, so that the bound (1.29) is not always useful. However we also prove that when $\xi = B(\omega)$ is the path of a Brownian motion, then its reduced path has finite length a.s. (and in fact this length, as a random variable, has "almost Gaussian" tails).

Theorem 1.10. *Let B be a Brownian motion and fix $T > 0$. Then, for each $\gamma \in (0, 2)$, there exists $C = C(\gamma, T) > 0$ such that, for any $x \geq 2$,*

$$\mathbb{P}\left(\|R_{0,T}(B)\|_{TV([0,T])} \geq x\right) \leq C \exp\left(-Cx^{\gamma}\right). \tag{1.30}$$

We also study the sharpness of the upper bound on the range of dependence ρ_H, in the case $H(p) = |p|$. We obtain a lower bound involving now what we call the fully reduced path $\tilde{R}_{0,T}(\xi)$ associated to ξ. It is defined similarly to $R_{0,T}(\xi)$ except that it is now linear on $[\tau_0, T)$, i.e. all the oscillations after the global extremum are forgotten (see again Figure 1.1).

Theorem 1.11. *Let $H(p) = |p|$ on $\mathbb{R}^d$ with $d \geq 1$. Then, for all $T > 0$ and $\xi \in C_0([0, T])$,*

$$\rho_H(\xi, T) \geq \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}. \tag{1.31}$$

When $d = 1$, then the above inequality is in fact an equality.

Note that this result implies in particular that there exist continuous paths ξ such that the equation (1.22) with $H(p) = |p|$ does not satisfy the finite speed of propagation property (even though it will satisfy it for almost every Brownian path).

1.5 Perspectives

Some of the results presented in this section have already been extended by other authors. In particular, the well-posedness result and estimate in Section 1.2 have been extended by Lions and Souganidis [92] to a large class of (strictly) convex Hamiltonians H. The idea is similarly to obtain good estimates on the "fundamental solution" which is represented in terms of a problem of calculus of variations. Some details have been written down by Seeger [113] in a special case.

A natural question is whether the regularization results of Section 1.3 can be extended to more general Hamiltonians $H = H(Du)$. In the case where the dimension $N = 1$, then this is indeed the case, and a similar result holds for all strictly convex H, see [75] and [92]. Suprisingly, this is no longer the case in dimension $N \geq 2$. Indeed, Lions and Souganidis have shown that, even though a similar bound on the Hessian holds in the deterministic case $\partial_t v = H(Dv)$, an anologous result to Theorem 1.5 never holds, unless H is exactly a quadratic function of Du.

A work in progress is concerned with the long-time behaviour of equations of stochastic Hamilton-Jacobi equations ($F \equiv 0$ and $d = 1$ in (1.1)). When $H = H(u_x)$ in one space dimension, we can show that in the periodic setting $u(t, \cdot)$ converges to a (random) constant as $t \to \infty$, under minimal assumptions on H. When $H = H(x, Du)$ is convex in Du, using the representation from Theorem 1.9 we are also able to identify some features of the long-time asymptotic behaviour.

An important limitation of the results exposed in this chapter is that throughout the dimension of the noise d is equal to 1, and it would be very interesting to obtain more robust methods, which would allow to obtain information on solutions to (1.1) without being restricted to considering properties of a single semigroup S_H (note that for instance, considering Section 1.4, we do not even know if finite speed of propagation holds for an equation such as $\mathrm{d}u = H_0(Du)\mathrm{d}t + H_1(Du) \circ \mathrm{d}B_t$ with H_0, H_1 convex).

Chapter 2

Malliavin calculus and singular SPDE

2.1 Introduction

In this chapter we detail the results obtained in [CFG17] (in collaboration with G. Cannizzaro and P. Friz) and in [GL19] (in collaboration with C. Labbé), where we apply Malliavin calculus to singular SPDE in order to obtain information on the law of solutions (more precisely, absolute continuity of certain finite dimensional projections).

The most well-known examples of singular SPDE are the dynamic Φ^4_3 model, given by

$$\partial_t u - \Delta u = -u^3 + \xi, \qquad (\Phi^4_3)$$

the KPZ equation

$$\partial_t u - \partial_{xx} u = \lambda \left(\partial_x u\right)^2 + \xi, \qquad \text{(KPZ)}$$

in both of which ξ is space-time white noise, in spatial dimension $d = 3$ for Φ^4_3 and $d = 1$ for KPZ, as well as the generalized Parabolic Anderson model (gPAM)

$$\partial_t u - \Delta u = g(u)\xi, \qquad \text{(gPAM)}$$

where ξ is spatial white noise and $d = 2$ or 3.

These equations appear in a number of contexts. The (Φ^4_3)equation appears as stochastic quantization of Eculidean quantum field theory [104], as well as scaling limit of statistical physics models for phase coexistence near criticality (e.g. [60, 97, 73]). The KPZ equation, introduced in [82] is a simple model of interface growth, and interpolates between the EW (Gaussian) and the KPZ universality class. It satisfies a so-called weak universality property, and arises as scaling limit of other interface growth models in specific regimes [12, 71]. The linear PAM (i.e. (gPAM) with $g(u) = u$) can be obtained from scaling limits related to random walks in random environment [28].

In all these equations, the regularity of the noise is too low for the equation to be well-posed in a deterministic way, which is why they were called "singular" SPDE. Recall that the product uv of two functions (or Schwartz distributions) u and v taken in the Hölder spaces $\mathcal{C}^\alpha$ and $\mathcal{C}^\beta$

is only well-defined under the condition that $\alpha + \beta > 0$. One can check that in the examples above, the regularity expected for the solution is too low for the nonlinear terms to satisfy this constraint. For instance, consider the case of (Φ^4_3). Since space-time white noise ξ has (parabolic) Hölder regularity smaller than $-\frac{d+2}{2} = -\frac{5}{2}$ in dimension 3, by Schauder regularity theory the solution u should be of regularity (no better than) $-\frac{5}{2} + 2 = -\frac{1}{2} < 0$, and the cubic term u^3 is therefore problematic.

Nevertheless, the above equations satisfy the property of so-called *local subcriticality*. This means that when rescaling the equation in a way that preserves the heat operator and the law of the noise, the non-linear term disappears in the (small scale) limit, or equivalently that when considering regularity on terms of the right-hand side of the equation, the non-linear terms (if somehow defined) will be more regular than the noise. This suggests that the equation may be solved as some perturbation of the linear problem. This is actually almost immediate in some simpler cases (e.g. for the Φ^4_2 equation simply substracting the solution to the linear equation from u suffices to set up a fixed point, cf. Da Prato-Debussche [33]). For the equations above however this is not so simple since one needs to push the expansion to include additional nonlinear terms. This was done by Hairer via his theory of regularity structures in [65] (we will present the basic ideas in the next subsection), alternatively by Gubinelli, Imkeller and Perkowski using so-called paracontrolled distributions [62]. These works allow to obtain local (in time) well-posedness results for renormalized equations. Let us give an example of such statements, for the equations that we will consider below.

Theorem 2.1. *Let $\xi^\varepsilon = \xi * \rho^\varepsilon$ be a regularization of space-time white-noise. Then there exists constants C_ε, such that for a suitable initial condition u_0, there exists a (random) time $T > 0$ such that the solutions to*

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = C_\varepsilon u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon \in (0,T) \times \mathbb{T}^3, \quad u(0,\cdot) = u_0$$

converge in probability to a limit u as $\varepsilon \to 0$. In addition, the limit does not depend on the choice of the regularization ρ^ε.

Similarly, for a regularization ξ_ε of spatial white noise on $\mathbb{T}^2$ there exists constants C'_ε s.t. if g is regular enough, then the solutions to

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = g(u_\varepsilon)\zeta_\varepsilon - C'_\varepsilon g g'(u_\varepsilon) \in (0,T) \times \mathbb{T}^2, \quad u(0,\cdot) = u_0 \tag{2.1}$$

converge to some function u on $[0,T] \times \mathbb{T}^2$ for some random $T > 0$.

In the works [CFG17, GL19], we apply the tools of Malliavin calculus to these equations. As a result, we obtain some information on the law of solutions. The results may be briefly summarized as follows (more general statements will be presented in the sections below).

Theorem 2.2. *(1) Let u be the solution to* (gPAM) *in dimension $d = 2$, with $g > 0$. Then for each $t > 0$, $x \in \mathbb{T}^2$,*

The law of $u(t,x)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}$.

(2) Let u be the solution to (Φ^4_3)*. Then for each linearly independent smooth functions $\phi_1, \ldots, \phi_N$ with compact support in $(0,\infty) \times \mathbb{T}^3$,*

The law of $(\langle u, \phi_1 \rangle, \ldots, \langle u, \phi_N \rangle)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^N$.

Since the seminal works of Malliavin [95], there have been numerous works dedicated to using Malliavin calculus to obtain absolute continuity of the laws of solutions to stochastic equations in various contexts. We do not attempt to review this here and refer to e.g. [100] and references therein, but we note that our works have been strongly influenced by papers combining Malliavin calculus with rough path theory (such as [23, 22, 70, 24, 78]) or with the study of stochastic PDE (for instance [102, 6, 96]).

2.1.1 Regularity structures : general idea

We summarize here the basic of the theory of regularity structures that we will need in the exposition below. We only describe the objects in the theory without giving actual definitions, there are by now a number of lecture notes that the interested reader may consult for more precise descriptions, for instance [66, 27, 67] or [50, ch. 13-15].

We will describe the general procedure, following the example of the Φ^4_3 equation. The basic idea is, after rewriting the equation in its mild formulation

$$u = K * \left(-u^3 + \xi\right) + \kappa u_0, \tag{2.2}$$

(where K is the heat kernel and κu_0 is the solution to the heat equation starting from u_0 at time 0), to look for a solution having local expansions in terms of some objects which are polynomial functions of the noise ξ.

The precise construction requires :

1. A *set of symbols* $\mathcal{T}$, each of them having a certain homogeneity. $\mathcal{T}$ contains the usual (multi-variable) polynomials (with homogeneity given by the degree), as well as additional symbols depending on the equation. One always has a symbol Ξ corresponding to the noise ξ, and then an inductive procedure using an abstract integration operator $\mathcal{I} : \mathcal{T} \to \mathcal{T}$ (corresponding to K) and multilinear operations gives additional symbols. For instance, for (2.2) we need among others

$$\Xi,\ \mathcal{I}(\Xi),\ \mathcal{I}(\Xi)^3,\ ,\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2,\ \ldots$$

   It is often convenient to use a tree notation to describe elements of $\mathcal{T}$: Ξ is represented by a dot, the integration map $\mathcal{I}$ are represented by respectively straight lines and dotted lines, and the product of two symbols is represented by joining the corresponding trees at the root. For example :

$$\mathcal{I}(\Xi)^2 = \vee\,, \quad \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2 =$$ .

2. *Models* encodes concrete objects (functions, or Schwartz distributions) associated to the abstract symbols $\mathcal{T}$. More precisely, for each $z = (t, x) \in D$ in the state space, we are given a map Π_z from $\mathcal{T}$ to the space of Schwartz distributions on D. The family $\Pi = (\Pi_z)_{z \in D}$ is then a model if it satisfies certain algebraic and analytic conditions (the algebraic conditions encode consistency under change of base points, while the analytic conditions ensure that the homogeneity of symbols is respected). For instance, one has :

$$\Pi_z \Xi = \xi, \ \ \Pi_z X^k = (\cdot - z)^k ,$$

as well as

$$\Pi_z \Psi \leftrightarrow (K * \xi)^3, \ \ \Pi_z \Psi\!\!\Psi \leftrightarrow (K * (K * \xi)^3)(K * \xi)^2,$$

the arrows above meaning that the model actually postulates a meaning to these products (which are classically ill-defined), in a way which must be consistent under change of base point.

3. The space of *modelled distributions* $\mathcal{D}^\gamma$ (which depends on the model Π) is a set of functions $F : D \to \mathcal{T}_{<\gamma}$ (the set of symbols of homogeneity less than γ) satisfying certain conditions of Hölder type. The idea is that $F(z)$ should encode a local expansion near z up to order γ, and the conditions on F ensure that these expansions are in a sense compatible. This is made precise via the fundamental *reconstruction theorem*, which ensures the existence of a map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{S}'(D)$ such that $\mathcal{R}F$ is indeed described locally by $\Pi_z F$ up to an error of order γ.

The solution procedure then consists in two separate steps : in the first (probabilistic) step, one needs to define the model corresponding to the irregular noise ξ. This is non-trivial and requires to consider renormalizations $\hat{\Pi}^\varepsilon = \mathfrak{R}^\varepsilon \Pi^\varepsilon$ of the canonical models obtained from regularizations. For instance, in the case of (Φ^4_3),

$$\hat{\Pi}_z \vee = \lim_\varepsilon \hat{\Pi}^\varepsilon_z \vee = \lim_\varepsilon (K * \xi_\varepsilon)^2 - C^1_\varepsilon$$

for some constants C^1_ε (and another renormalization constant arises from considering $\Psi\!\!\vee$).

In the second (analytic) step, we write an "abstract" fixed point equation in $\mathcal{D}^\gamma$ spaces corresponding to the PDE in consideration, for instance, for (Φ^4_3), given the model Π we look for $U \in \mathcal{D}^\gamma$ for some suitable γ, solution to

$$U = \mathcal{K}\left(-U^3 + \Xi\right) + \kappa u_0, \tag{2.3}$$

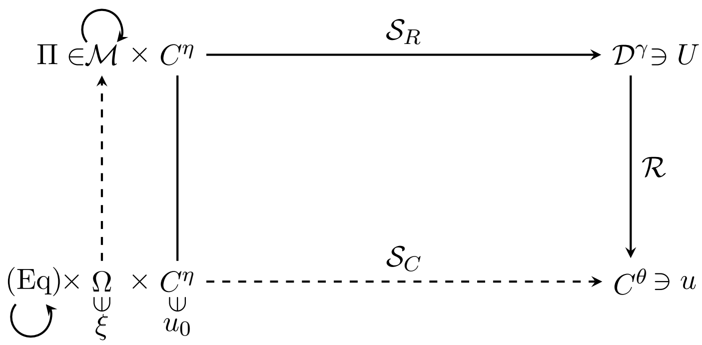
and U will in this case be of the form

$$U(z) = \mathbf{I} + U_1(z)\mathbf{1} - \Upsilon + U_{\Upsilon}(z)\Upsilon + \sum_{i=1}^{3} U_{X_i}(z)\mathbf{X}_i .$$

The solution to (Φ^4_3) is then defined as $u = \mathcal{R}U$.

Finally, one can see the impact of the renormalization of the model at the level of the equation satisfied by the sequence $u^\varepsilon = \mathcal{R}(U(\hat{\Pi}^\varepsilon))$, which allows to obtain statements such as Theorem 2.1.

The solution procedure can be summarized by the following diagramme :



One important point is that the theory of regularity structures (as its precursor rough path theory [94]) comes with built-in robustness properties, exemplified in the fact that the arrows with solid lines above are actually continuous maps.

We conclude this presentation by noting that the inital results in [65] have by now been vastly generalized to a framework capable of treating a very general class of singular (locally subcritical) SPDE in the three papers [17, 26, 16].

2.1.2 Malliavin calculus

We describe briefly the basics of Malliavin calculus that we use. For a more complete reference see for instance [100]. We let $(\Omega, \mathbb{P}, \mathcal{H})$ be an abstract Wiener space, i.e. $\mathbb{P}$ is a Gaussian measure on the separable Banach space Ω, and $\mathcal{H} \subset \Omega$ is the associated Cameron-Martin space. The example to have in mind is the case of white noise on a domain D, where $\Omega = \mathcal{C}^{0,\alpha}(D)$ (the closure of smooth functions in $\mathcal{C}^\alpha$) for $\alpha < 0$ small enough, and $\mathcal{H} = L^2(D)$.

Given random variables F, i.e. measurable maps defined on Ω, Malliavin calculus studies certain regularity properties of these maps (which are probabilistic in nature and different from the usual Banach space calculus, recall that for instance the solution map to an SDE is never continuous on Ω but is smooth in the Malliavin sense). One then has integration by parts formulae (coming from the Gaussian structure) which are at the heart of the theory.

The pathwise nature of regularity structures allows to take a rather strong definition of Malliavin differentiability (compared to the more classical Sobolev type definition). We say that a map $F : \Omega \to \mathbb{R}^N$ is in $\mathcal{C}^1_{\mathcal{H}-loc}$ if for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$h \in \mathcal{H} \mapsto F(\omega + h) \text{ is Frechet-differentiable in a neighbourhood of } 0.$$

We then call $DF(\omega)$ $(\in \mathcal{H})$ the derivative at 0. The classical Bouleau-Hirsch criterion then gives a sufficient condition for F to have an absolutely continuous law w.r.t. Lebesgue measure.

Theorem 2.3 (Bouleau-Hirsch [15]). *Assume that $F : \Omega \to \mathbb{R}^N$ is in $\mathcal{C}^1_{\mathcal{H}-loc}$, and that DF is non-degenerate in the sense that*

$$\mathbb{P} - a.e.\ \omega, \text{ the map } h \in \mathcal{H} \mapsto \langle DF(\omega), h\rangle \in \mathbb{R}^N \text{ is surjective.}$$

Then F admits a density w.r.t. the Lebesgue measure (on $\mathbb{R}^N$).

2.2 Results and ideas of proofs

We will now present in more details the results and methods of [CFG17, GL19] which allow to obtain Theorem 2.2. In both cases, the general approach is the same : one wants to apply the Bouleau-Hirsch criterion, and therefore this requires to prove, in a first step, Malliavin differentiability of (a projection of) the solution to the PDE, and in a second step, non-degeneracy of the Malliavin derivative. These two steps are essentially independent. Since the precise methods we use (for each of these two steps) in [CFG17]and [GL19] are quite different from one another, we present the results in two separate subsections. (In the statements below we will implicitely assume that the existence time for the equations is always $T \equiv +\infty$, this is not essential but simplifies the presentation).

2.2.1 The case of (gPAM) [CFG17]

Theorem 2.4 ([CFG17])**.** *(1) Let u be the solution to* (gPAM)*. Fix $(t,x) \in (0,\infty) \times \mathbb{T}^2$. Then $F = u(t,x)$ is $\mathcal{C}^1_{\mathcal{H}-loc}$, with derivative given by*

$$\langle DF, h\rangle = v^h(t,x), \text{ where } v^h = \lim_{\varepsilon} v^h_\varepsilon,$$

$$(\partial_t - \Delta)v^h_\varepsilon = g(u_\varepsilon)h_\varepsilon + v^h_\varepsilon\left(g'(u_\varepsilon)\xi_\varepsilon - C_\varepsilon(gg'' + (g')^2)(u_\varepsilon)\right), \quad v^h_\varepsilon(0) = 0.$$

(In the above equation, ξ_ε, u_ε, C_ε are defined as in Theorem 2.1.)

(2) Assume $g \geq 0$, and $g(u_0)$ is not identically 0*.*

Then a.s., for each $(t,x) \in (0,\infty)\times\mathbb{T}^2$, if $h > 0$, then $v^h(t,x) > 0$. In particular, the random variable $F = u(t,x)$ is non-degenerate.

Theorem 2.2 (1) then follows from combining this theorem with the Bouleau-Hirsch criterion.

Let us give an idea of the proof. Given $h \in \mathcal{H} = L^2(\mathbb{T}^2)$, we consider the equation with shifted noise

$$\partial_t u - \Delta u = g(u)\left(\xi + h\right), \tag{2.4}$$

and we need to differentiate u with respect to h. Differentiating formally, one sees that the derivative in the h direction should be the solution to

$$(\partial - \Delta)v = g(u)h + vg'(u)\xi. \tag{2.5}$$

The difficulty is that the products appearing are not well-posed if one only thinks in terms of Hölder regularity (since measured on that scale h has essentially no better regularity than ξ). However h is much more regular when measured in the L^2 scale.

The approach followed in [CFG17] to solve the above equations is to introduce an extended model (on an extended symbol space), giving sense now not only to polynomials in ξ, but to polynomials in the pair (ξ, h). The symbol set $\mathcal{T}^H$ ($\supset \mathcal{T}$) now contains all symbols where instances of Ξ may be replaced by H, i.e.

$$\Xi, H, \Xi \cdot \mathcal{I}H, H \cdot \mathcal{I}\Xi, H \cdot \mathcal{I}H, \ldots$$

We then show (deterministically) that given a model Π on $\mathcal{T}$ and $h \in \mathcal{H}$, there exists a unique model Π^h on $\mathcal{T}^H$ such that :

$$\Pi^h = \Pi \text{ on } \mathcal{T}, \;\; \Pi^h(H) = h, \;\; \Pi^h(H\mathcal{I}\Xi) = h \cdot \Pi(\mathcal{I}\Xi), ...$$

and the map $(\Pi, h) \mapsto \Pi^h$ is locally Lipschitz.

This essentially boils down to the fact that multiplication is well-defined on $C^\beta \times H^\gamma$ (resp. Hölder and Sobolev spaces), provided $\beta + \gamma > 0$, with suitable Hölder-type estimates, such as :

$$\begin{aligned} &\xi \in C^\alpha, K * h \in H^2 \\ &\Rightarrow \xi \cdot (K * h - (K * h)(x)) \text{ of order } \alpha + 2 - \frac{d}{2} - \varepsilon (\geq 2\alpha + 2) \text{ at } x. \end{aligned}$$

At this stage we rely crucially on the fact that we consider specifically equation (gPAM), which is the simplest equation for which regularity structures is needed. Indeed, there is essentially only one non-linear term to make sense of in the model, i.e. $\xi K * \xi$ (this would be similar to the "level 2" case in rough path theory), so that the computations mentioned above can be done "by hand" by considering only a few symbols.

Once this is done, we can solve (2.4)-(2.5) at the abstract level by considering U^h, V^h solutions to

$$\begin{aligned} U^h &= \mathcal{K}(F(U^h) \cdot (\Xi + H)) + \kappa u_0, \\ V^h &= \mathcal{K}\left(F(U) \cdot H + V^h \cdot F'(U) \cdot \Xi\right), \end{aligned}$$

after which it is reasonably straightforward to prove that (letting $v^h = \mathcal{R}V^h$, $u^h = \mathcal{R}U^h$)

$$u^h = u + v^h + o(\|h\|_{\mathcal{H}}),$$

which is the required Fréchet differentiability.

As for the proof of the nondegeneracy, the key step is to use a strong maximum principle, which may be written as follows.

Proposition 2.5. *Let w be the solution to a linear heat equation*

$$(\partial_t - \Delta) w = w\tilde{\xi}, \quad w(0, \cdot) = w_0$$

where $\tilde{\xi}$ is such that the theory of regularity structures applies. Then

$$w_0 \geq 0, \; w_0 \textit{ not identically } 0 \;\Rightarrow w(t, \cdot) > 0 \textit{ for all } t > 0.$$

The proof follows an idea due to Mueller (originally in the context of the stochastic heat equation [98]) : writing the equation in integral form

$$w = K * (w\tilde{\xi}) + \kappa w_0,$$

and using the estimates from the theory, one can see that the first term is negligible for small t. This means that if $w_0 > 0$ on an open set B_0, then at times t_1 small enough, $w(t_1, \cdot)$ will be strictly positive on a bigger open set $B_1 \supset B_0$. This argument can be quantified and then iterated to show that actually w is strictly positive everywhere for all times $t > 0$.

We also note that Proposition 2.5 is interesting by itself, for instance it proves that solutions to linear PAM starting from nonnegative initial conditions are then strictly positive at times $t > 0$.

2.2.2 The case of (Φ^4_3) [GL19]

We consider the solution u to

$$\begin{cases} u(0) = u_0 \\ \partial_t u = \Delta u - u^3 + Cu + \xi \quad \text{on } (0,\infty)\times\mathbb{T}^3, \end{cases} \tag{Φ^4_3}$$

The noise ξ is assumed to be of the form $\xi = R * \zeta$ where ζ is space-time white noise, and the kernel R is such that for each multi-index k

$$|\partial_k R(z)| \lesssim \|z\|_{\mathfrak{s}}^{-|\mathfrak{s}|-k+\beta} \tag{2.6}$$

for z near 0, with $\beta \geq 0$. Note that this means that R is β-regularizing, so that ξ is at least as regular as space-time white noise. (The case $R = \delta$ of space-time white noise is actually included as a limiting case).

We then have

Theorem 2.6. *Under the above assumption, letting u be the (renormalized) solution to (Φ^4_3), for any test function φ smooth enough on $(0,T)\times\mathbb{T}^3$, the random variable $\langle u, \varphi\rangle$ is in $\mathcal{C}^1_{\mathcal{H}-loc}$, with derivative in the h-direction given by $\langle v^h, \varphi\rangle$, where*

$$v^h = \lim_{\varepsilon} v^h_\varepsilon,$$

$$(\partial_t - \Delta) v^h_\varepsilon = -3u^2 v^h_\varepsilon + C_\varepsilon \varepsilon + h_\varepsilon, \;\; v^h_\varepsilon(0,\cdot) = 0.$$

For the non-degeneracy of the Malliavin derivative , we need further assumptions on the kernel R. In fact we propose two different assumptions. The first one is that the noise is non-degenerate :

Assumption (D)**.** The Cameron-Martin space

$$\mathcal{H} = \left\{R * h, h \in L^2\right\}$$

is dense in $L^2([0,T]\times\mathbb{T}^3)$.

A second assumption is that the noise is "rough enough" on small scales. To present the assumption we introduce some notation. For $C > 1$ and $n \geq 0$, let

$$A^C_n = \left\{\xi \in \mathbb{R}^4 : C^{-1}2^n \leq |\xi| \leq C2^n\right\}$$

and

$$B^C_n = \left\{(\xi,\xi') \in (\mathbb{R}^4)^2 : \; \xi, \xi', \xi+\xi' \in A^C_n\right\}.$$

The assumption is then written as

Assumption (R)**.** One has $\beta < \frac{1}{2}$ and for some $C \geq 1$,

$$\limsup_{n\to\infty} 2^{3n\beta} \sup_{(\xi,\xi')\in B^C_n} \left|\hat{R}(\xi)\hat{R}(\xi')\hat{R}(-\xi-\xi')\right| > 0 \tag{2.7}$$

where $\hat{R}$ is the Fourier transform of R.

A slightly more explicit sufficient condition can be obtained by writing a (parabolic) Littlewood-Paley decomposition of space-time white noise $\zeta = \sum_{n\geq 0} \Delta_n \zeta$ where

$$\Delta_n \zeta = \zeta * \left(2^{n(|\mathfrak{s}|)d+2)} \rho \left(2^n \cdot \right) \right)$$

for some suitable function ρ. Then Assumption (R) is satisfied if

$$\xi = \sum_{n\geq 0} \alpha_n \Delta_n \zeta$$

with $\alpha_n \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} 2^{n\beta} |\alpha_n| \in (0, +\infty) \text{ with } \beta < \frac{1}{2}.$$

The non-degeneracy result is then as follows.

Theorem 2.7. *In addition to* (2.6)*, let assumption (D) or (R) hold. Let* $\varphi_1, \dots, \varphi_N$ *be smooth enough with compact support in* $(0, \infty) \times \mathbb{T}^3$*, and let*

$$F = \left(\langle u, \varphi_i \rangle \right)_{i=1,\dots,N} .$$

Then if $\varphi_1, \dots, \varphi_N$ *are linearly independent,* DF *is non-degenerate.*

Again, Theorem 2.2 (2) is a direct corollary. Let us now present the ideas of proofs. For the Malliavin differentiability, as in the case of (gPAM), the difficulty is to make sense "at the same time" for all $h \in \mathcal{H}$ of

$$(\partial_t - \Delta) u = -u^3 + \xi, \quad u(0, \cdot) = u_0,$$

$$(\partial_t - \Delta) u_h = -u_h^3 + \xi + h, \quad u_h(0, \cdot) = u_0,$$

and

$$(\partial_t - \Delta) v_h = -3u^2 v_h + h \quad v_h(0, \cdot) = 0.$$

Recall that in [CFG17], we made sense of these shifted and tangent equations as modelled distributions based on expansions in polynomial functions of the pair (ξ, h). However in the case of (Φ^4_3), since the equation is more complicated, obtaining the right analytic bounds on this extended model is difficult. Instead, we chose in [GL19] to look for solutions as expansions only in ξ, but now with coefficients of Sobolev regularity (instead of Hölder).

Concretely, this means that we consider Besov-type modelled distribution spaces, denoted $\mathcal{D}^\gamma_p$, where $p \in [1, \infty]$ denotes the integrability index (the usual Hölder-like spaces corresponding to $p = \infty$). These spaces have been previously introduced and studied by Hairer and Labbé [68] (we actually need to slightly modify the spaces from [68] by adding a weight near $t = 0$). This then allows to solve the following equations :

$$U_h = \mathcal{K}(-U_h^3 + \Xi) + \mathcal{K}h + \kappa u_0,$$

$$V_h = \mathcal{K}(-U^2 V_h) + \mathcal{K}h.$$

A crucial ingredient from [68] that we use in the fixed point theorem is the $\mathcal{D}_p^\gamma$ version of multiplication and Sobolev-type embeding theorems. For instance, solving for fixed point equation in $\mathcal{D}_2^\gamma$ for $Y_h = U_h - U_0$ given by

$$Y_h = \mathcal{P}(-Y_h^3 - 3Y_h^2 U_0 - 3Y_h U_0^2) + \mathcal{P}h$$

and considering the first term in the r.h.s., we have the following chain of implications :

$$Y \in D_{2,\geq 0}^\gamma \Rightarrow Y \in D_{6,\geq 0}^{\gamma-5/3} \Rightarrow Y^3 \in D_{2,\geq 0}^{\gamma-5/3} \Rightarrow \mathcal{P}(Y^3) \in D_{2,\geq 0}^{\gamma+1/3}$$

(where the subscript "≥ 0" means that only symbols of nonnegative regularity appear in the expansion), and with similar arguments for the other terms this allows to solve the equation for Y_h. As in the case of (gPAM), once these fixed point equations are set up, proving the required Fréchet differentiability is straightforward.

As for the non-degeneracy, it is sufficient to prove that almost surely,

$$\forall \varphi \text{ smooth enough }, \Big(\langle v_h, \varphi \rangle_{L^2([0,T]\times\mathbb{T}^3)} = 0 \quad \forall h \in \mathcal{H} \Big) \Rightarrow \varphi = 0.$$

Following [102] (and also e.g. [96, 5]), we work with a backward representation of the Malliavin derivative, namely for a given φ supported in $(0,T)\times\mathbb{T}^3$, we consider w_φ which is (formally) solution to

$$(-\partial_t - \Delta) w = -3u^2 w + \varphi, \quad w(T,\cdot) = 0, \tag{2.8}$$

since by integration by parts one then has for each pair (h,φ) the identity

$$\langle v_h, \varphi \rangle_{L^2([0,T]\times\mathbb{T}^3)} = \langle h, w_\varphi \rangle_{L^2([0,T]\times\mathbb{T}^3)} .$$

Note that the product $u^2 w$ is actually ill-defined, so to make rigorous sense of (2.8) we work again in a suitable set of modelled distributions defined on an extended regularity structure (to incorporate the backward heat kernel), namely we let $w = \mathcal{R}W$ where

$$W = \overleftarrow{\mathcal{K}}(-3U^2 W) + \overleftarrow{\mathcal{K}}\varphi.$$

We are then reduced to proving

$$\Big(\langle w_\varphi, h \rangle_{L^2([0,T]\times\mathbb{T}^3)} = 0 \quad \forall h \in \mathcal{H} \Big) \;\Rightarrow\; \varphi = 0,$$

where $\mathcal{H}$ is the Cameron-Martin space associated to the noise. Using the equation satisfied by W, a simple induction argument gives the implication

$$w_\varphi = 0 \Rightarrow \varphi = 0,$$

so that when $\mathcal{H}$ is dense in L^2 (Assumption (D)) the result follows immediately. When the noise is degenerate, one has near each point z the local expansion for the r.h.s. of (2.8)

$$-3u^2 w + \varphi = -3w(z)\mathbf{V} + R_z$$

where $\vee$ is the (renormalized) square $\vee = (\,\mathsf{I}\,)^2$, with $(\partial_t - \Delta)\,\mathsf{I} = \xi$, and our roughness assumption (Assumption (R)) implies that R_z is of homogeneity near z strictly greater than that of $-3\vee w$. By testing against suitable localized elements of $\mathcal{H}$, we can then separate the contributions of the two terms to obtain that under the orthogonality condition, $w = 0$ a.e.. Note that this type of argument based on the separation of scales appears frequently in this context (of proving the non-degeneracy of Malliavin derivatives), and is already present in the classical Malliavin proof of Hörmander's theorem (via the uniqueness in the decomposition of a continuous semimartingale as the sum of a martingale and a bounded variation process). The precise argument then takes a different form based on the structure of the problem under consideration, for instance in the context of rough differential expansions this led to the notion of "true roughness", cf. [70, 48, 50]. The theory of regularity structures is particularly well-suited for this kind of argument, since as soon as the theory is used to solve an equation, it automatically gives a Taylor-like expansion (with terms of successively higher homogeneity) for the solutions.

2.3 Perspectives

Malliavin differentiability of solution to singular SPDEs has been recently obtained in a very general context by Schoenbauer [112]. Following the procedure outlined in section 2.2.1 in the case of (gPAM), he proves that one can define models on the extended structure on a set of full measure. The proof is based on an elegant observation that estimates involving the Cameron-Martin norm can be obtained from probabilistic estimates (essentially, replacing occurences of h by independent noises $\hat{\xi}$ with same distribution as ξ), which then allows to use directly the results from Chandra-Hairer [26]. Schoenbauer also obtains results for existence of densities in non-degenerate settings.

A natural question is to go further than mere existence of densities and obtain for instance higher order regularity, tail estimates, etc. The first ingredient to prove such results would be to obtain tail estimates for solutions to linear singular SPDE. For instance, let W be a solution to an equation of the form

$$W = \mathcal{K}(W\tilde{\Xi}) + \kappa(w_0)$$

where the associated model comes from a process with Gaussian tails $\tilde{\xi}$, is it true that $\|W\|_{\mathcal{D}^\gamma}$ has moments of all orders ? We recall that in the case of ordinary (rough) differential equations, tail estimates were obtained by Cass-Litterer-Lyons [25], by a suitable (random) decomposition of the domain (interval). Whether such results can be extended to a PDE setting is a challenging question.

Chapter 3

Rough volatility modelling

3.1 Introduction

It is well known that the classical Black-Scholes model where stock prices follow log-normal distributions fails to reproduce many features empirically observed in financial markets. This can for instance be seen by considering the implied volatility surface. Recall that implied volatility $\sigma_I(K,T)$ for a given strike K and maturity T is defined as the unique σ such that

$$C_m(K,T) = C_{BS}(K,T,\sigma)$$

where $C_m(K,T)$ is the market price for the call option of strike K and maturity T, and $C_{BS}(K,T,\sigma)$ is the price for such a call given by the Black-Scholes formula).

Then a typical volatility surface (the collection of $\sigma_I(K,T)$ for varying T and K) is far from flat, which means that the Black-Scholes model is not consistent with market observed prices. For this reason, a vast amount of literature has been devoted to the study of more sophisticated *stochastic* volatility models, where the stock price follows dynamics of the form (under a risk neutral measure and assuming no interest rate to simplify)

$$\frac{dS_t}{S_t} = \sigma_t dZ_t$$

where $(\sigma_t)_{t\geq 0}$ is itself a stochastic process, possibly correlated with the Brownian motion Z driving the stock price.

In classical volatility models such as the Heston model, σ is a continuous semimartingale, typically given as solution to a SDE. However these models, while possessing analytical tractability rendering them practically attractive, are unable to produce volatility surfaces which are consistent with market observations for the entire range of traded maturities. For instance, the so-called ATM volatility skew defined as

$$\psi(\tau) = \left.\frac{\partial \sigma_I(k,\tau)}{\partial k}\right|_{k=0}$$

(here and below we parametrize the volatility surface by log-moneyness $k = \ln(K/S_t)$ and maturity $\tau = T - t$ where t is current time) can be empirically observed to follow power-law $\psi(\tau) \sim \tau^\alpha$ with $\alpha \approx -0.4$. In contrast, in classical volatility models, $\psi \sim 1$ for short maturities.

One way to recover this power law behaviour of the skew is to take σ_t modelled on *fractional* Brownian motion. Indeed, it was observed in the pioneering work [2] (see also [52]) that in that case the skew is of order $\psi(\tau) \sim \tau^{H-1/2}$, compatible with empirical observations when the Hurst index H is of order 0.1. (Note that since, in that case, $H < 1/2$, the sample paths of volatility in this class of models will be much less regular than those of Brownian motion, hence the name "rough volatility".) In a recent important paper, Bayer, Friz and Gatheral [8] proposed a model where the volatility is log-normal and function of a fractional Brownian motion, which they called the rough Bergomi model, and observed that this model allows for very good fits to the observed volatility surface (at both short and long expiries), even though the model only requires 3 parameters.

In addition to rough volatility being consistent with short maturities in the volatility surface, analysis of historical price time-series [56] also indicates that realized volatility sample paths are consistent with roughness paramaters H as low as 0.05, and log-normal distributions (see also the more recent [11, 53] for more empirical evidence). In addition, even though the rough volatility models themselves are continuous, they have been shown [39, 81] to appear as scaling limits of market microstructure models based on (discrete) Hawkes processes, as soon as natural assumptions are made on the microstructure dynamics.

Despite these advantages, the main drawback of these models from a practical (mathematical) point of view is that fractional Brownian motion lacks both the semimartingale property and the Markov property, which complicates considerably the task of both proving specific properties of the models as well as numerical approximations, since in particular all the usual PDE techniques are not available. Nevertheless there has been a considerable amount of academic activity devoted to their analysis in the last 4 years.

In this section I will summarize my contributions to the study of these models. In the article [BFG+19] we explained how the pathwise approach (more precisely : regularity structures) could be useful to prove mathematical properties of these models. In particular, we crucially used this approach in the following paper [FGP18] where we obtained precise large deviation estimates for option prices. Finally I will explain the results of the note [Gas19] which deal with martingale property and moments in the rough Bergomi model.

3.1.1 Notations

In the rest of this chapter, we will focus on stock price dynamics given by

$$\mathrm{d}S_t = S_t \sigma(t, \hat{W}_t) \mathrm{d}\tilde{W}_t \tag{3.1}$$

where

$$\hat{W}_t = \int_0^t K_H(t,s) \mathrm{d}B_s, \text{ with } K_H(t,s) = C_H (t-s)_+^{H-1/2}, \tag{3.2}$$

i.e. $\hat{W}$ is a (Riemann-Liouville) fractional Brownian motion of Hurst index $H \in (0, 1/2)$, $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a fixed function, and

$$\mathrm{W} = (W, \bar{W}) \text{ is a 2d B.M.}, \quad \tilde{W} = \rho W + \bar{\rho} \bar{W}.$$

Equivalently this may be rewritten in term of the log-price X as

$$S_t = S_0 \exp(X_t), \quad X_t = \int_0^t \sigma(\hat{W}_s)\mathrm{d}\tilde{W}_s - \frac{1}{2}\int_0^t \sigma(\hat{W}_s)^2 \mathrm{d}s. \tag{3.3}$$

In this model the (normalized) call and put prices for time to maturity $t \geq 0$ and log-moneyness $k \in \mathbb{R}$ are respectively given by

$$c(t,k) = \mathbb{E}\left[(e^{X_t} - e^k)_+\right], \quad p(t,k) = \mathbb{E}\left[(e^k - e^{X_t})_+\right].$$

3.2 Pathwise analysis of rough volatility models and applications

3.2.1 The regularity structure framework [BFG$^+$19]

In the paper [BFG$^+$19], which is a joint work with C. Bayer, P. Friz, J. Martin and B. Stemper, we show how a rough paths-type framework may be useful in the context of the above model. To be more precise, we use the framework of Hairer's regularity structures [65], see section 2.1.1 above for introductions and notations.

Unlike the singular SPDE setting where the theory of regularity structures is needed even for making sense of the objects under consideration, here we are essentially only considering the object

$$\int_0^T \sigma\left(t, \hat{W}_t\right) d\tilde{W}_t \tag{3.4}$$

which is a well-defined Itô integral. However the pathwise formalism gives an *analytically robust* way of studying this object via a factorisation of the Itô integration map

$$\mathrm{W} \longmapsto \mathbf{W} \longmapsto \int_0^T \sigma\left(\hat{W}_t\right) d\tilde{W}_t \tag{3.5}$$

such that the second arrow above is continuous. This has a number of (mathematical) advantages, notably when proving approximation or asymptotic results, as we detail below (this observation is classical in the context of rough paths theory, which allowed among other achievements for more transparent proofs of previously known results on SDE, cf. e.g. [51]).

Let us first describe quickly the idea in our context (to simplify the exposition we adopt a rough-path type formalism as in [FGP18] although the details are spelled out in [BFG$^+$19] in the language of regularity structures). Let M be an integer such that $(MH - \frac{1}{2}) > 0$. We define a model $\mathbf{W}$ as a collection of objects

$$\mathbf{W} = \left(W, \bar{W}, \int \hat{W}\mathrm{d}W, \int \hat{W}\mathrm{d}\bar{W}, \int \hat{W}^2\mathrm{d}W, ..., \int \hat{W}^M\mathrm{d}\bar{W}\right) ,$$

satisfying certain algebraic conditions. The above objects are indexed by $(t,s) \in [0,T]^2$, e.g.

$$W_{s,t} = W_t - W_s, \qquad \left(\int \hat{W}^M \mathrm{d}W\right)_{s,t} = \int_s^t \hat{W}^M_{s,r}\mathrm{d}W_r$$

where equalities actually mean that the left hand side postulates the value of the right hand side. We also define the model norm

$$\|\mathbf{W}\| := \|W\|_{1/2-\kappa} + \|\bar{W}\|_{1/2-\kappa} + ... + \Big\| \int \hat{W}^M d\bar{W} \Big\|_{M(H-\kappa)+1/2-\kappa}, \tag{3.6}$$

generalizing the α-Hölder norm (with $\alpha = 1/2 - \kappa$) of the Brownian noise $\mathrm{W} = (W, \bar{W})$ ($\kappa > 0$ is fixed but small enough). Here, for instance, and working on $[0, T]$, the final summand is spelled out as

$$\sup_{0 \le s < t \le T} \frac{\left| \int_s^t \hat{W}^M_{s,r} \mathrm{d}\bar{W}_r \right|}{|t-s|^{M(H-\kappa)+1/2-\kappa}} .$$

where $\hat{W}_{s,r} = \hat{W}_r - \hat{W}_s$. The *model distance* between two models $\mathbf{W}$ and $\mathbf{V}$ is given by $\|\mathbf{W}; \mathbf{V}\| := \|\mathbf{W} - \mathbf{V}\|$.

It is straigthforward to check that interpreting the integrals as Itô integrals gives rise to a random model $\mathbf{W}^{Ito}(\omega)$ for almost every ω. Furthermore, the reconstruction theorem [65] shows that for σ smooth enough, one can define a map

$$\Phi : \mathbf{W} \mapsto \int \sigma(t, \hat{W}) \mathrm{d}\mathbf{W}$$

which is locally Lipschitz for model metric, and is consistent with Itô integration in the sense that

$$\Big(\int \sigma(t, \hat{W}) \mathrm{d}\tilde{W} \Big)(\omega) = \int \sigma(t, \hat{W}(\omega)) \mathrm{d}\mathbf{W}^{Ito}(\omega) \quad \text{a.s.}$$

where the left-hand side is a classical Itô integral.

Let us now describe three applications of the above construction as detailed in [BFG+19].

A Wong-Zakai approximation

Recall that the Wong-Zakai theorem states that when replacing a Brownian motion W by (for instance) piecewise linear approximation W^ε, the integrals $\int f(W^\varepsilon) \mathrm{d}W^\varepsilon$ converge to the Stratonovich integral $\int f(B) \circ \mathrm{d}B$. In the case of rough volatility, the Stratonovich integral $\int f(\hat{W}) \circ \mathrm{d}\tilde{W}$ does not make sense since the quadratic covariation $[\hat{W}, \tilde{W}] = \pm\infty$ for $H < \frac{1}{2}$ (and $\rho \neq 0$). However one can still recover a non-trivial limit via a suitable renormalization, as the following result shows (this is similar in spirit to a previous result by Hairer and Pardoux [69] in the (more complicated) case of the stochastic heat equation).

Theorem 3.1. *Let $(W^\varepsilon, \bar{W}^\varepsilon)$ be piecewise linear approximation of step ε of $(W, \bar{W})$ and let $\hat{W}^\varepsilon_t = \int_0^t K_H(t,s) \dot{W}^\varepsilon(s) \mathrm{d}s$. Then there exist functions $\mathcal{C}^\varepsilon : [0, T] \to \mathbb{R}$ such that for all smooth function $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$, one has*

$$\int_0^T \sigma(t, \hat{W}_t) \mathrm{d}\tilde{W}_t = \lim_{\varepsilon \to 0} \int_0^T \sigma(t, \hat{W}^\varepsilon_s) \mathrm{d}\tilde{W}^\varepsilon_t - \int_0^T \mathcal{C}^\varepsilon(t) \partial_2 \sigma(t, \hat{W}^\varepsilon_t) \mathrm{d}t$$

in probability.

In fact, by continuity of the reconstruction map, this theorem is obtained by proving convergence of suitable renormalized models of the form

$$\widehat{\mathbf{W}}^{\varepsilon}(\omega) = \Big(W^{\varepsilon}, \bar{W}^{\varepsilon}, \int \hat{W}^{\varepsilon} \mathrm{d}W - \int \mathcal{C}^{\varepsilon} \mathrm{d}t, \int \hat{W}^{\epsilon} \mathrm{d}\bar{W}^{\epsilon},$$
$$\ldots, \int (\hat{W}^{\varepsilon})^{M} \mathrm{d}\bar{W} - M \int (\hat{W}^{\varepsilon})^{M-1} \mathrm{d}t, \int (\hat{W}^{\varepsilon})^{M} \mathrm{d}\bar{W}\Big)$$

to the Itô model, where the functions $\mathcal{C}^{\varepsilon}$ are explicit and of order $\varepsilon^{H-1/2}$, in particular they diverge as $\varepsilon \to 0$. The fact that we require renormalization functions and not just constants is due to the fact that piecewise linear approximations are not stationary in time (see also [119] for a similar result in the case of a singular SPDE). We actually expect that the convergence above still holds with $\mathcal{C}^{\varepsilon}$ replaced by its average value C^{ε} (we actually prove it in [BFG$^+$19] for $H > \frac{1}{4}$ and check it numerically for smaller values of H).

Of course this theorem allows to obtain approximations S^{ε} for the stock price S^{ε}, and in [BFG$^+$19] several numerical experiments are performed based on such approximations.

Large deviations

Recall the *contraction principle* from Large Deviation theory : if a family $(X_{\delta})_{\delta}$ satisfies a Large Deviation Principle (LDP) with rate function I and ϕ is a continuous map, then $X_{\delta} = \phi(Y_{\delta})$ also satisfies a LDP. One advantage of (rough) pathwise approaches to stochastic analysis is that, due to the fact that the integration (or solution) map is continuous, one can immediately obtain (a series of) large deviation corollaries, assuming that one has proven a LDP at the level of the rough path (or model).

In the context of regularity structures, this was done by Hairer and Weber in [72] (recall that models encode polynomial functions of the noise, so that their work build on large deviations results from Borell for Gaussian polynomials in Banach spaces, see e.g. the exposition in Ledoux [85]). We can essentially re-use their result to get a LDP on the space of models in our context.

In order to state the theorem let us introduce some notation. The Cameron-Martin space is

$$\mathrm{H} = \left\{ \mathrm{h} = (h, \bar{h}) \in (H^1([0,1]))^2 \right\}$$

with norm

$$\|\mathrm{h}\|^2_{\mathrm{H}} := \int_0^1 \left(\dot{h}(s)^2 + \dot{\bar{h}}(s)^2 \right) ds.$$

An element $\mathrm{h} \in \mathrm{H}$ admits a canonical lift as a model

$$\mathbf{h} = \left(h, \bar{h}, \int \hat{h} \mathrm{d}h, \int \hat{h} \mathrm{d}\bar{h}, \int \hat{h}^2 \mathrm{d}h, ..., \int \hat{h}^M \mathrm{d}\bar{h} \right)$$

with $\hat{h}_t = \int_0^t K(t,s) \dot{h}(s) \mathrm{d}s$. The integrals above are well-defined as Young integrals, as can be seen by e.g. the Besov-variation embedding from [49]. One then has the following small noise LDP :

Theorem 3.2. *Let*

$$\mathbf{W}^\delta(\omega) = \left(\delta W, \delta \bar{W}, \delta^2 \int \hat{W} \mathrm{d}W, \delta^2 \int \hat{W} \mathrm{d}\bar{W}, \delta^3 \int \hat{W}^2 \mathrm{d}W, ..., \delta^{M+1} \int \hat{W}^M \mathrm{d}\bar{W}\right)$$

be the Itô model obtained by multiplying each instance of W or $\bar{W}$ by $\delta > 0$. Then, as $\delta \to 0$, the family $\mathbf{W}^\delta$ satisfies a LDP with rate δ^2 and rate function defined on the space of models by

$$\mathcal{I}(\mathbf{W}) = \begin{cases} \frac{1}{2}\|\mathrm{h}\|^2_{\mathrm{H}}, & \textit{if } \mathbf{W} = \mathbf{h}, \\ +\infty & \textit{otherwise.} \end{cases}$$

By the contraction principle, this immediately implies a LDP for the random variables $\Phi(\mathbf{W}^\delta)$ (recall that $\Phi : \mathcal{M} \to \mathbb{R}$ is the map $\mathbf{W} \mapsto \int_0^T \sigma\left(t, \hat{W}_t\right) \mathrm{d}\mathbf{W}_t$), with rate function

$$\Lambda(x) = \inf\left\{\frac{1}{2}\|\mathrm{h}\|^2_{\mathrm{H}}, \ \int_0^1 \sigma(\hat{h}) d\tilde{h} = x\right\}, \tag{3.7}$$

where $\tilde{h} = \rho h + \bar{\rho}\bar{h}$. The main reason why this is useful is that via a scaling argument, the small-noise regime above may be transformed into small-time asymptotics, with immediate pricing applications. However the scaling is different than that of usual SDE, and when considering e.g. call prices, the log-moneyness needs to be rescaled by a time-dependent factor.

Corollary 3.3. *Let X_t be given by* (3.3)*. Then for fixed $x > 0$, letting $k_t = xt^{1/2-H}$, one has*

$$\lim_{t\to 0} t^{2H} \log \mathbb{P}\left(X_t \geq k_t\right) = -\Lambda(x).$$

If σ has linear growth (or if S has finite moments of order $p > 1$) one also has

$$\lim_{t\to 0} t^{2H} \log c(t, k_t) = -\Lambda(x).$$

A similar statement holds for $x < 0$ and in-the-money put option prices.

We note that the above result had already been obtained by Forde and Zhang [46] (with slightly different assumptions). For practical purposes it is useful to refine the LDP above into precise asymptotics, this is detailed in Section 3.2.2 below.

Non-simple rough volatility models

Another advantage of the regularity structure approach is its flexibility. Namely, once one has proven results at the level of the model $\mathbf{W}$, one immediately gets as corollary results for any process which can be written as a continuous map of $\mathbf{W}$ (i.e. one can change the second map in (3.5)).

As an example, we give in [BFG+19] large deviation and approximation results similar to those above for stock price models where the volatility is of the form $\sigma(Z_t)$ with Z solution to a fractional equation

$$Z_t = Z_0 + \int_0^t K(t,s)\left(a(Z_s)ds + b(Z_s)dW_s\right),$$

where a and b are smooth functions. We actually prove these results for $H > 1/4$ only, since in that case the same regularity structure as in the case of a simple model suffices. (The results are certainly also true for arbitrary H but would require considering more and more symbols as $H \downarrow 0$, in order to set up the abstract fixed point corresponding to Z).

3.2.2 Precise asymptotics [FGP18]

In a further joint work with P. Friz and P. Pigato, building on the previous work, we prove precise large deviation estimates for short-time call prices. The main result is as follows.

Theorem 3.4 ([FGP18]). *Assume linear growth of σ or a moment assumption on S. Then for $x > 0$ small enough, letting $k_t = xt^{1/2-H}$, it holds that*

$$c(t, k_t) \sim_{t\to 0} \exp\left(-\frac{\Lambda(x)}{t^{2H}}\right) t^{1/2+2H} \frac{A(x)}{2\Lambda'(x)\sqrt{\Lambda(x)}\sqrt{\pi}}$$

for some function $A(x)$ with $A(0^+) = 1$.

A similar result holds for in-the-money put options.

The result above deals with the large deviation scale where k_t is of order $t^{1/2-H}$. In contrast with classical volatility models ($H = 1/2$), this regime requires "zooming-in" near the money as $t \to 0$. In practice this is natural since quoted strikes tend to be closer to spot price as time to maturity decreases. The case where $t^{1/2} \ll k_t \ll t^{1/2-H}$ corresponds to the *moderate deviation* regime, and by checking uniformity of some estimates as $x \to 0$ in the above theorem we can also obtain some asymptotics in this case. The regime where $k_t \lesssim t^{1/2}$ however is ruled by Central Limit type estimates which require completely different methods (precise asymptotics have been obtained in that case by El Euch and coauthors [38]).

Note that call price asymptotics can be translated into implied volatility asymptotics by results from Gao-Lee [55], which in our case yields :

Corollary 3.5. *Writing $k_t = xt^{1/2-H}$, for x fixed, $t \downarrow 0$:*

$$\sigma^2_{BS}(t, k_t) = v^2(x) + t^{2H}a(x) + o(t^{2H})$$

where

$$v(x) = \frac{x}{\sqrt{2\Lambda(x)}}$$

and

$$a(x) = \frac{x^2}{2\Lambda(x)^2} \log\left(\frac{2A(x)\Lambda(x)}{\Lambda'(x)x}\right).$$

This type of asymptotics is very useful in practice since it allows quick calibration of model parameters (which typically will still need to be complemented by Monte Carlo but at least gives a starting point when starting the calibration procedure, see for instance the discussion in [45] in the context of the classical Heston model).

Sketch of proof

We want to write the short-time asymptotics as small-noise asymptotics.

Let $\varepsilon = t^{1/2}$, $\bar{\varepsilon} = t^H$, then

$$X_t \approx^{(d)} \frac{\varepsilon}{\bar{\varepsilon}} X_1^{\bar{\varepsilon}}$$

(up to term of order $\varepsilon^2 = o(\varepsilon\bar{\varepsilon})$), with

$$X_1^{\bar{\varepsilon}} = \int_0^1 \sigma(\varepsilon \hat{W}_s) d(\bar{\varepsilon}\tilde{W}_t) =: \Phi(\bar{\varepsilon}\mathbf{W}).$$

Furthermore, for x small enough, there exists a unique minimizer h^x in the definition of $\Lambda(x)$

$$\Lambda(x) = \frac{1}{2}\|\mathrm{h}^x\|^2.$$

From the Large Deviation principle, most of the mass in the expectation $\mathbb{E}[(e^{X_t} - e^{k_t})_+]$ will be concentrated for W close to h^x. The idea of Laplace method on Wiener space (following classical works by Azencott [4], Ben Arous [10] among others) is to make this more precise by performing a (stochastic) Taylor expansion around h^x. Let us describe the idea by a formal computation. The first step is to do a change a base-point, writing

$$\mathbb{E}\left[\left(\Phi\left(\bar{\varepsilon}W\right) - \Phi(\mathrm{h}^x)\right)_+\right] = \mathbb{E}\left[\left(\Phi\left(\mathrm{h}^x + \bar{\varepsilon}\mathrm{W}\right) - \Phi(\mathrm{h}^x)\right)_+ \exp\left(-\frac{1}{\bar{\varepsilon}}\int_0^1 \dot{\mathrm{h}}^x d\mathrm{W}\right)\right] \exp(-\frac{1}{2\bar{\varepsilon}^2}\|\mathrm{h}^x\|^2)$$

using a Girsanov (Cameron-Martin) transformation. Note that the last term on the right is exactly the LDP factor $\exp\left(-\frac{\Lambda(x)}{\bar{\varepsilon}^2}\right)$. Then we perform a (stochastic) Taylor expansion in $\bar{\varepsilon}$

$$\Phi\left(\mathrm{h}^x + \bar{\varepsilon}\mathrm{W}\right) = x + \bar{\varepsilon} g^1(\mathrm{W}) + \bar{\varepsilon}^2 g^2(\mathrm{W}) + O(\bar{\varepsilon}^3),$$

where g^1 is linear in W, hence Gaussian with variance σ_x^2, and g^2 is quadratic in W.

In addition the optimality of h^x has for consequence that the stochastic integral in the exponential may be identified :

$$g^1(\mathrm{W}) = \Lambda'(x)\int_0^1 \dot{\mathrm{h}}^x d\mathrm{W}.$$

Finally we decompose g^2 as

$$g^2 = \Delta_0 (g^1)^2 + \Delta_1 g^1 + \Delta_2$$

where Δ_i, $i = 0, 1, 2$ are independent from g^1. The exponential term in the expectation forces g_1 to be small so that the correlated terms in g_2 may be ignored and we then have :

$$\begin{aligned} c(\varepsilon^2, k_\varepsilon) &\sim e^{-\frac{\Lambda(x)}{\bar{\varepsilon}^2}} \varepsilon \mathbb{E}\left[\exp\left(-\Lambda'(x)\frac{g^1}{\bar{\varepsilon}}\right)\left(g_x^1 + \bar{\varepsilon}\left(\Delta_0 (g^1)^2 + \Delta_1 g^1\right) + \bar{\varepsilon}\Delta_2\right)_+\right] \\ &\sim e^{-\frac{\Lambda(x)}{\bar{\varepsilon}^2}} \varepsilon \mathbb{E}\left[\exp\left(-\Lambda'(x)\frac{g^1}{\bar{\varepsilon}}\right)\left(g^1 + \bar{\varepsilon}\Delta_2\right)_+\right]. \end{aligned}$$

This leads to

$$c(\varepsilon^2, k_\varepsilon) \sim e^{-\frac{\Lambda(x)}{\bar{\varepsilon}^2}} \varepsilon\bar{\varepsilon}^2 \frac{1}{\sqrt{2\pi}\Lambda'(x)^2 \sigma_x^2} \mathbb{E}\left[\exp(\Delta_2)\right]$$

which is the expression from the theorem, with

$$A(x) = \mathbb{E}\left[\exp(\Delta_2)\right].$$

The fact that $A(x) = \mathbb{E}[\exp(\Delta_2)]$ is finite is not obvious since Δ_2 is in the second Wiener chaos (quadratic function of Gaussians). However the finiteness can be expressed as the non-degeneracy of the minimizer h^x, and be shown to hold for x small enough.

In the sketch above we have ignored the remainder terms, but in the actual proof, we crucially use the rough path / regularity structure approach described in Section 3.2.1. More precisely, when writing the Taylor expansion above, we actually have

$$\Phi(T_{\mathrm{h}}\delta_{\bar{\varepsilon}}\mathbf{W}) = G_0^{\mathrm{h}} + \bar{\varepsilon}G_1^{\mathrm{h}}(\mathbf{W}) + \bar{\varepsilon}^2 G_2^{\mathrm{h}}(\mathbf{W}) + R_3^{\mathrm{h},\varepsilon}(\mathbf{W}) \ .$$

with deterministic control (in terms of the model $\mathbf{W}$) on all the terms. This simplifies Azencott's method which relied on complicated probabilistic estimates. The fact that rough path arguments simplify Laplace method on Wiener space actually goes back to Aida [1], and also Inahama-Kawabi (e.g. [79, 77]) (in the context of rough differential equations or classical SDE).

Finally, we note that expressions for g^1 and g^2 can be obtained by formal differentiation in the expression of $\Phi(\mathrm{h}^x + \bar{\varepsilon}\mathrm{W})$, for instance

$$g^2(\mathrm{W}) = \int_0^1 2f'(\hat{h})\hat{W}d\tilde{W} + f''(\hat{h})\hat{W}^2 d\tilde{h},$$

and it holds that

$$\Delta_2(\mathrm{W}) = g^2(\mathrm{V}), \quad \mathrm{V} = \mathrm{W} - g_1\mathrm{v},$$

where $\mathrm{v} \in \mathrm{H}$ is deterministic such that W is independent from g^1. In principle that means that once h^x is known, the value of $A(x)$ may be computed. In practice this is not so simple, but we can obtain an expansion near 0 of $x \mapsto \mathrm{h}^x$ which gives expansions of $\Lambda(x)$ and $A(x)$.

3.3 Martingale property and moments in the rough Bergomi model [Gas19]

In this section we explain the results from the note [Gas19], which concern martingality and moments in the rough Bergomi model from [8]. In this model stock price dynamics are as in (3.1)-(3.2) above, with a volatility function of the form

$$\sigma(t, y) = \zeta(t)\exp(\eta y) \, .$$

for some strictly positive function ζ. The main result in [Gas19] is as follows.

Theorem 3.6 ([Gas19])**.** *In the rough Bergomi model :*

1. *S is a martingale $\Leftrightarrow \rho \leq 0$.*
2. *For $\rho \leq 0$ and $m > (1-\rho^2)^{-1}$, $\mathbb{E}[S_t^m] = +\infty$ for all $t > 0$.*

The first result gives a necessary and sufficient condition for when the price process S, which is obviously a local martingale (and a supermartingale) is a true martingale. The true martingale property is very important in practice, since using a strict local martingale measure for pricing has some obvious drawbacks. For instance : if S is a strict local martingale then $\mathbb{E}[S_T] < S_0$

for some $T > 0$, so that already the price given by the model for holding one unit of stock until time T does not coincide with market data (this suggests that the asset price is greater than its actual "fundamental" value and for this reason strict local martingale models have been used in the modelisation of bubbles, see [105] and references therein). Note that in the rough Bergomi model, due to the superlinear growth of σ, Novikov's criterion for martingality is never satisfied.

The second result concerns finiteness of moments of the stock price. This is important for instance in Monte Carlo simulation (to know that the Monte Carlo error is ruled by CLT estimates, finite variance is needed) and in asymptotic formulae (to go from stock price large deviations to call price asymptotics, as in Section 3.2.2, some information is needed on tail asymptotics of S such as existence of a nontrivial moment of order $p > 1$). However our result is only negative, and a converse result would be very useful.

We remark that in the Brownian case ($K \equiv 1$), both of the above results are well known, cf. [114, 80, 86], and in that case the condition $\rho^2 > \frac{m-1}{m}$ is also a sufficient condition for the moments to be finite. We note that the exponential form of σ does not play a role in our result, which applies as soon as the volatility function satisfies a suitable superlinear growth condition as $y \to +\infty$.

The proof of Theorem 3.6 1. follows the classical argument (found already in the aforementioned [114, 80, 86], see also [13, 110] for additional references) relating the martingale property of stochastic exponentials with explosions of a SDE (in our case, this will be a Volterra SDE). The case $\rho \leq 0$ is then essentially immediate, while the proof of the reverse implication follows from the fact that the Volterra SDE may blow up in arbitrarily short time with positive probability if $\rho > 0$.

The proof of Theorem 3.6 2. relies on the Boué-Dupuis formula, which expresses the expectation of exponentials of Brownian functionals as values of (here : Volterra) stochastic control problems. We then show that for the considered values of the parameters, we may choose a feedback control such that, as in the previous proof, the process (and the value) blow up in arbitrarily small time.

3.4 Perspectives

Let us mention a few open questions on rough volatility models which might be interesting to study. As mentioned in the previous section, a result on the tails of lognormal rough volatility models such as rough Bergomi (e.g. existence of moments for the stock price) would be useful for practical applications, but is still unknown at the moment. The results in the classical Markovian case (e.g. [86]) heavily rely on PDE techniques (or equivalently, clever applications of Itô's formula), so that it is likely that one would need a new approach for the rough case.

A good understanding of the *weak* rate of convergence for numerical schemes in the context of rough volatility is also missing. The strong rate of convergence (w.r.t. the grid size) is known to be equal to the Hurst index H (which means that convergence is very slow since H will usually be close to 0), but it is expected that the weak rate of convergence, which is the one of practical interest in option pricing, is higher (recall that in the classical case $H = 1/2$, the weak rate is equal to 1). It would be very interesting to identify the correct rate for various numerical schemes (or even to obtain asymptotic expansions for the weak error, which then would allow for instance to use Richardson-Romberg methods).

Finally, some recent works (e.g. [81]) go even further and propose "hyper-rough" volatility models where the volatility has negative regularity (so that is does not even admit continuous sample paths). It would be interesting to develop methods allowing to obtain asymptotics and/or numerical schemes for this class of models.

Chapter 4

Around Root's solution to the Skorokhod Embedding Problem

4.1 Introduction

The classical Skorokhod Embedding Problem (SEP) consists in the following task : given two probability measures μ and ν on the real line, find a stopping time τ such that

$$\text{if } X_0 \sim \mu \text{ then } X_\tau \sim \nu \text{ and } (X_{\tau\wedge t})_{t\geq 0} \text{ is uniformly integrable,} \qquad (\mathrm{SEP}_{\mu,\nu})$$

where $X = B$ is a standard Brownian motion. When μ is a Dirac mass at 0 such a stopping time exists if (and only if) ν has a finite first moment and is centered, whereas in general the existence of a solution to ($\mathrm{SEP}_{\mu,\nu}$) requires $\mu \leq \nu$ in the convex order, namely $\mu(f) \leq \nu(f)$ for all positive convex function f.

Skorokhod was the first to give a solution to this problem (via a randomized stopping time), and since then numerous different ways of solving the SEP have been discovered, each one typically optimizing a specific criterion, let us mention for instance the solutions of Azéma-Yor [3] and Vallois [117] with optimality properties related to respectively the running maximum and the local time at 0. We refer to the survey by Obloj [101] for a detailed description of more than 20 different solutions to the SEP. While the SEP was initially developped in order to transfer results from Brownian motion to random walks and vice versa (cf e.g. [111]), it has been more recently shown in the seminal works of Hobson that it was closely related to the problem of obtaining *robust* (i.e. model-free) bounds on option prices (see for instance the lecture notes [74]), which has led to an important regain of interest in the last decade from the mathematical finance community.

In this chapter we present results related to a specific solution to SEP, namely that discovered by Root [107]. In his paper, Root showed that there exists a solution given as a hitting time for the space-time process $(t, B_t)_{t\geq 0}$. Namely, there exists a closed subset $R \subset (0,\infty) \times \mathbb{R}$ satisfying the so-called barrier property

$$(t, x) \in R \Rightarrow (t, \infty) \times \{x\} \subset R \qquad (4.1)$$

such that

$$\tau_R := \inf\{t \geq 0, \quad (t, B_t) \in R\}$$

is a solution to SEP (to be precise, Root showed this in the case where $\mu = \delta_0$ and ν has finite second moment). It was then shown by Loynes [93] that R is essentially unique. In a very nice paper [109], Rost showed that Root's solution to the SEP had the following optimality property :

$$\mathbb{E}[F(\tau_R)] \leq \mathbb{E}[F(\tau)] \text{ for every convex function } F \text{and solution } \tau \text{ of } (\mathrm{SEP}_{\mu,\nu}). \tag{4.2}$$

Rost's approach was to first prove existence of a solution to SEP that optimizes the above criterion, then to observe that by a path-swapping argument this implies that such a stopping time must be given as a hitting time to a barrier. Such a procedure has been recently pursued in a very general context for general optimality criteria by Beiglboeck, Cox, Huessmann [9] with an influence from optimal transport theory.

Rost further discovered that there exists another related solution of the SEP as a hitting time for (t, X_t), where now the set R satisfies a reversed barrier property, and this solution satisfies a reversed optimality criterion (namely (4.2) with "$\geq$" instead of "$\leq$").

However, these classical works remained mostly theoretical, and did not adress the question of computing the barrier R. This question was considered (with a motivation coming from robust finance, since Root's solution's optimality property means that it is linked to variance options) by Dupire [37] and Cox and Wang [30], who further extended the results to the case when B is replaced by a one-dimensional diffusion $X = \int_0^\cdot \sigma(X_s)\mathrm{d}B_s$.

My contributions to the subject, which I will describe in this chapter, are concerned with obtaining representations of the Root barrier which allow to compute it in practice : in [GOdR15] we revisited Dupire and Cox-Wang's free boundary PDE representation, in [GMO15] we described how to compute the barrier function as solution to a nonlinear integral equation, and finally in [GOZ19]we generalized the free boundary representation to a large class of Markov processes.

To conclude this introduction, I will note that there have been in recent years many papers by other authors studying Root and Rost's solution to the SEP, for instance the extension of this solution to the multi-marginal case has been obtained by Cox, Obloj and Touzi [29] (and even further extended to full marginal specification by Richard, Tan and Touzi [106]).

4.2 PDE approach to Root's solution [GOdR15]

Given a measure μ on $\mathbb{R}$ with finite first moment, define its potential u_μ by

$$u_\mu(x) := -\int_{\mathbb{R}} |x - y| \, \mu(\mathrm{d}y) \tag{4.3}$$

Two probability measures μ and ν on $\mathbb{R}$ are said to be in *convex order* (denoted $\mu \prec_{cx} \nu$) if $u_\nu \leq u_\mu$ on $\mathbb{R}$.

In this chapter we consider the Skorokhod Embedding Problem ($\mathrm{SEP}_{\mu,\nu}$) when X is given as a one-dimensional diffusion

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\mathrm{d}B_s, \quad t \geq 0,$$

where $\sigma \in C([0,\infty)\times\mathbb{R},\mathbb{R})$ is Lipschitz in space and of linear growth, both uniformly in time, and locally bounded away from 0. The main result from [GOdR15] is then as follows.

Theorem 4.1 ([GOdR15])**.** *Let σ be as above and $\mu \prec_{cx} \nu$ be two probability measures on $\mathbb{R}$ with finite first moment. Let u be the (unique linearly growing) viscosity solution to*

$$\begin{cases} \min\left(u-u_\nu, \partial_t u - \frac{\sigma^2}{2}\partial_{xx}u\right) = 0 \text{ on } (0,\infty)\times\mathbb{R}, \\ u(0,.) = u_\mu(.), \end{cases} \tag{4.4}$$

and let

$$R = \{(t,x)\in[0,\infty]\times[-\infty,\infty] : u(t,x) = u_\nu(x)\}. \tag{4.5}$$

Then R is a closed barrier and

$$\tau^R = \inf\{t>0 : (t,X_t)\in R\}$$

is a solution to (SEP$_{\mu,\nu}$)*, which satisfies the optimality property* (4.2)*.*

We note that this theorem is only a minor extension of previous results (compared to [30], the only novelty is that σ may depend on t and ν is only assumed to have a first moment). However the main interest of [GOdR15] was to present a new proof of these results which is short and self-contained (whereas for instance [30] relied on Root's original result).

Sketch of the proof that τ_R solves the SEP. In the first step, we show that for any closed barrier R, letting $u^R(t,x) := -\mathbb{E}^\nu\left[|x-B_{t\wedge\tau^R}|\right]$, then

$$\begin{cases} \min\left(u^R-u_{\mu^R}, \partial_t u^R - \frac{1}{2}\frac{\sigma^2}{2}\partial_{xx}u^R\right) = 0 \text{ on } (0,\infty)\times\mathbb{R}, \\ u^R(0,x) = u_\nu(x) \end{cases}$$

in viscosity sense, where μ^R is the law of B_{τ^R}. In fact one can show that $(\partial_t - \frac{1}{2}\frac{\sigma^2}{2}\partial_{xx})u^R(t,x) = \frac{d}{dx}\mathbb{P}(B_\tau\in dx, \tau<t)$ from which the above equation follows.

Letting then u be the solution to the PDE (4.13) and $R=\{u=u_\mu\}$, we need to show $u=u^R$, which implies $B_{\tau^R}\equiv\mu$. Let $R_\varepsilon \subset R \subset R^\varepsilon$ be obtained from R by shifting the time coordinate of $\pm\varepsilon$. Then using Step 1., one can see that both $u^{R_\varepsilon}-u$, $u-u^{R^\varepsilon}$ solve

$$\partial_t w - \left(\frac{\sigma^2}{2}\partial_{xx}w\right)_+ \le 0, \quad w(0,\cdot)=0$$

in viscosity sense. By the maximum principle, this yields $u^{R_\varepsilon}\le u\le u^{R^\varepsilon}$. It follows from basic sample path properties of one-dimensional diffusions that almost surely

$$\lim_{\varepsilon\to0}\tau_{R_\varepsilon} = \lim_{\varepsilon\to0}\tau_{R^\varepsilon} = \tau_R,$$

and finally this gives $u=\lim_\varepsilon u^{R_\varepsilon} = \lim_\varepsilon u^{R^\varepsilon} = u^R$. □

Essentially the same proof also allows to recover Rost's solution to the SEP, the equation is then

$$\begin{cases} \partial_t v - \min\left(0, \frac{\sigma^2}{2}\partial_{xx} v\right) = 0 \text{ on } (0,\infty)\times\mathbb{R}, \\ v(0,x) = u_\mu(x) - u_\nu(x) \end{cases}$$

and the reversed barrier is

$$\hat{R} = \{(t,x) \in (0,\infty)\times\mathbb{R},\ \ v(t,x) = v(0,x)\}.$$

We further explain in [GOdR15] how these PDE are related to solutions of FBSDE and can be solved numerically to determine the barriers.

4.3 An integral equation for the barrier function [GMO15]

We present in [GMO15] an alternative way to compute the Root barrier R, at least in the case when X is a Brownian motion and the initial measure μ is the Dirac at 0. Note that the barrier property (4.1) implies that there exists a unique *barrier function* $r : \mathbb{R} \to [0,\infty]$ such that R may be written as

$$R = \{(t,x) \in (0,\infty)\times\mathbb{R},\ \ t \geq r(x)\}.$$

Theorem 4.2 ([GMO15]). *Denote for $t \geq 0, x \in \mathbb{R}$*

$$g(t,x) = \sqrt{\frac{2t}{\pi}} e^{-\frac{x^2}{2t}} - |x| \operatorname{Erfc}\left(\frac{|x|}{\sqrt{2t}}\right),$$

where p is the heat kernel, and let ν be an atom-free, zero-mean probability measure on $\mathbb{R}$. Then the barrier function r of the Root solution for $SEP_{\delta_0,\nu}$ solves the nonlinear Volterra integral equation

$$u_{\delta_0}(x) - u_\nu(x) = g(r(x),x) - \int_{\{y: r(y) < r(x)\}} g(r(x) - r(y), x-y)\,\nu(\mathrm{d}y) \quad \forall x \in (-\infty,\infty). \tag{4.6}$$

In addition, if $r : \mathbb{R} \to [0,\infty]$ is continuous, it is the unique continuous solution to (4.6).

The fact that r solves (4.6) is an easy consequence of the Itô-Tanaka formula, whereas the uniqueness relies on the PDE representation (Theorem 4.1). In general, the above representation does not give a more efficient way to compute R than via the PDE representation, due to the dependence of the domain of integration on the unknown r (indeed, that means that for instance if one were to discretize (4.6), at each time-step one needs to consider all space points, as one would when solving a PDE numerically). However, if one knows a priori some information on the shape of r such as monotonicity, then the above equation simplifies.

Corollary 4.3. *In the context of Theorem 4.2, assume that*

$$r \text{ is symmetric around } 0,\ \text{ continuous and non-increasing on } [0,\infty]. \tag{4.7}$$

Then r solves the nonlinear Volterra integral equation of the first kind

$$\begin{aligned} u_{\delta_0}(x) - u_\nu(x) &= g(r(x),x) \\ &\quad - \int_x^\infty \left(g(r(x) - r(y), x+y) + g(r(x)-r(y), x-y)\right)\nu(\mathrm{d}y) \quad \forall x \in (0,\infty). \end{aligned} \tag{4.8}$$

We further discuss in [GMO15] sufficient conditions for (4.7) to hold. In particular, we prove that it does tor the family of measures symmetric and given by

$$\nu_{k,\alpha}\left([-x,x]\right) = c_{k,\alpha}\left(\frac{x}{k}\right)^{\alpha}, \quad 0 \le x \le k,$$

for any $k > 0$, $\alpha \ge 1$. In particular, this includes the family of uniform distributions $\mathcal{U}\,[-k,k]$. The integral equation may then be solved numerically by standard methods which allow very fast computation of the barrier.

We further discuss in [GMO15] how the Root barrier corresponding to the uniform distribution allows to simulate bounded increments of space-time Brownian motion. Namely is τ is the corresponding hitting time then (t, B_t) is bounded for $0 \le t \le \tau$, and (τ, B_τ) has same law as $(r(U), U)$ with $U \sim \mathcal{U}[-1,1]$ which is immediate to simulate once r is known. This leads for instance to Monte Carlo algorithms to obtain the exit distribution of (t, B_t) from a time-space domain D via a variant of Muller's classical random walk on spheres algorithm [99], consisting here as considering iterated Root hitting times, where the Root barrier is scaled according to the distance to the boundary.

4.4 Root's barrier for Markov processes [GOZ19]

In the recent preprint [GOZ19], we extend the results from section 4.2 to the case when X is a rather general Markov process. Let us introduce some notation. Let X be a standard Markov process (in the sense of [14]) with state space E and semigroup $P = (P_t)_{t\ge 0}$. We will assume some transience so that the uniform integrability can be dropped in the SEP, and given measures ν, μ on E we are simply looking for (possibly randomized) stopping times τ such that

$$\text{if } X_0 \sim \mu \text{ then } X_\tau \sim \nu. \qquad (\mathrm{SEP}'_{\mu,\nu})$$

Let $U = \int_0^\infty P_t dt$ be the potential kernel, the potential of a measure μ is the measure μU. We say that μ is smaller than ν in balayage order (denoted $\mu \prec_b \nu$) if $\mu U \ge \nu U$. Then Rost [108] proved that if μU and νU are σ-finite measures,

$$\nu \prec_b \mu \;\Leftrightarrow\; \text{ there exists a solution to}(\mathrm{SEP}'_{\mu,\nu}).$$

(The $\Leftarrow$ is immediate since in that case $\mu U(A) = \nu U(A) + \mathbb{E}\int_0^\tau 1_A(X_s)ds$ for any Borel set A, but the other implication is non-trivial and Rost's construction relied on the so-called filling scheme).

We make the following assumption on X :

Assumption 4.4. X is a standard Markov process with semigroup P, which is in duality with another standard process $\widehat{X}$ with semigroup $\widehat{P}$ on the same probability space, w.r.t. some σ-finite measure ξ on E, namely for all $t \ge 0$ and $f, g \ge 0$

$$\int_E (P_t f) g \mathrm{d}\xi = \int_E f(\widehat{P}_t g)\mathrm{d}\xi. \tag{4.9}$$

Furthermore, the semigroups of X and $\widehat{X}$ are absolutely continuous w.r.t. ξ:

$$P_t(x,\cdot) \ll \xi, \quad \widehat{P}_t(y,\cdot) \ll \xi, \qquad \forall x,y \in E. \tag{4.10}$$

Under this assumption, any potential measure μU is absolutely continuous w.r.t. ξ, and has a unique density $\mu\widehat{U}$ which is excessive w.r.t. $\hat{P}$. (Recall that a nonnegative function f on E is excessive w.r.t. a semigroup P if $P_t f \leq f$ for all $t \geq 0$ and $\lim_{t\to 0} P_t f = f$ pointwise.)

Let $\hat{Q}$ the (co-)space-time semigroup associated to $\hat{X}$ (i.e. the semigroup corresponding to the process $(\hat{t}, \hat{X}_t)$, where $d\hat{t} = -dt$). Our main result may be written as follows.

Theorem 4.5 ([GOZ19])**.** *Let X be a Markov process for which Assumption 4.4 holds. Let μ, ν be two measures such that μU and νU are σ-finite measures and such that ν charges no semipolar set* [1]*. Let $\mu \prec_b \nu$ and define*

$$u^{\mu,\nu}(t,x) := \inf\left\{g \ \widehat{Q}\text{-excessive:} \ \ g \geq \mu\widehat{U}(x)\mathbb{1}_{\{t\leq 0\}} + \nu\widehat{U}(x)\mathbb{1}_{\{t>0\}}\right\}, \tag{4.11}$$

and

$$R = \left\{(t,x) \in \mathbb{R}_+ \times E \ \mid \ u^{\mu,\nu}(t,x) = \nu\widehat{U}(x)\right\}. \tag{4.12}$$

Then R is a barrier and

$$\tau^R = \inf\{t > 0 : (t, X_t) \in R\}$$

is a solution to $(\mathrm{SEP}'_{\mu,\nu})$*, which satisfies the optimality property* (4.2) *among solutions of* $(\mathrm{SEP}'_{\mu,\nu})$*.*

The proof of Theorem 4.5 is done in two steps. In the first step, we show the existence of a Root stopping time solving the SEP. Here we rely on the work of Rost [109], that shows that $(\mathrm{SEP}'_{\mu,\nu})$ has as solution stopping time that lies between the hitting times of two barriers which differ only by a time-space graph. We show that these hitting times are necessarily equal under our absolute continuity assumption (4.10). In the second step, we show that one may take the Root barrier R given by the free boundary (4.12). From a conceptual point of view, this step is similar to the case of one-dimensional diffusions as studied for instance in [30]. However, there the analysis is greatly simplified due to the existence of local times. Since local times do not necessarily exist in our setting, the situation becomes more delicate and requires the analysis of negligible sets via potential theory.

Informally $u^{\mu,\nu}$ in (4.11) is the solution of the obstacle problem

$$u(0,\cdot) = \mu\widehat{U}, \quad \min\left[(\partial_t - \widehat{\mathcal{L}})u, u - \nu\widehat{U}\right] = 0 \qquad \text{on } (0,+\infty)\times E, \tag{4.13}$$

where $\widehat{\mathcal{L}}$ is the generator of the dual process $\widehat{X}$, so that Theorem 4.5 is indeed a generalization of Theorem 4.1.

Besides existence and optimality of a Root stopping time, the main interest of Theorem 4.5 is that it provides a way to compute the Root barrier for a large class of Markov processes. Concretely, it allows to use classical optimal stopping and the dynamic programming algorithm to compute $u^{\mu,\nu}$ and hence R. Several applications are discussed in [GOZ19] ranging from Lévy processes to hypo-elliptic diffusions, and concrete examples of computed barriers are presented there.

[1] Under Assumption 4.4, a semipolar set is a set which is visited a.s. at at most countably many times by the process. This assumption on ν is satisfied if for instance ν is absolutely continuous w.r.t. ξ.

Another important application of Theorem 4.5 is to identify via a time-change new solutions to the SEP optimizing a different criterion. Namely let A be an additive functional of the form $A_t = \int_0^t a(X_s)ds$ where a is bounded and bounded away from 0. Considering the associated time-change of X, one can show that there exists a barrier R^A (given as a free boundary to a certain obstacle problem) such that

$$\tau_A := \inf\left\{t > 0, (A_t, X_t) \in R^A\right\}$$

is a solution to $(\mathrm{SEP}'_{\mu,\nu})$, which satisfies

$$\mathbb{E}[F(A_{\tau_A})] \leq \mathbb{E}[F(A_\tau)] \text{ for every convex function } F \text{ and solution } \tau \text{ of } (\mathrm{SEP}'_{\mu,\nu}).$$

4.5 Perspectives

It would be interesting to extend the construction of Theorem 4.5 for general Markov processes to the multi-marginal case. The case of one-dimensional diffusions was recently treated in [29] who identified the correct obstacle problem. It should be possible to prove a similar result for Markov processes, however the procedure of proof from [GOZ19] should be modified since we can no longer rely on already known existence results. One could try to adapt the proof of [GOdR15] to this context.

Another natural question would be to find a general framework allowing to compute solutions to the SEP with specific optimality criteria, for instance minimizing $E[F(S_t)]$ where S is an auxiliary nondecreasing process and F is a (possibly convex or concave) function. In the case where X is linear Brownian motion and F is convex, Theorem 4.1 allows via time-change to solve the case when $dS_t = a(S_t, B_t)dt$ if a is continuous and strictly positive. However the time-change argument does not work in general (e.g. when S is local time at 0 then the time-changed process is constant). Identifying rather general conditions on S for which this holds is challenging and will be the subject of future work.

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RÉSUMÉ

Ce mémoire d'HDR s'articule en quatre parties :

- la première partie étudie les équations d'Hamilton-Jacobi stochastiques (leur caractère bien-posé ainsi que certaines propriétés qualitatives des solutions telles que régularité et rayon de dépendance par rapport aux données initiales),
- la deuxième partie porte sur l'étude d'EDPS singulières, plus précisément à l'application du calcul de Malliavin dans ce contexte pour obtenir l'existence de densités pour la loi des solutions,
- la troisième partie étudie les modèles à volatilité stochastique rugueuse, en particulier l'application à ces modèles de techniques trajectorielles pour prouver des formules asymptotiques,
- la dernière partie présente des résultats obtenus sur la solution de Root au problème de plongement de Skorokhod : calcul via une formulation EDP, équation intégrale, et extension des résultats aux processus de Markov généraux.

MOTS CLÉS

analyse stochastique, méthodes trajectorielles, trajectoires rugueuses, EDP stochastiques, équations d'Hamilton-Jacobi, modèles à volatilité stochastique, plongement de Skorokhod.

ABSTRACT

This habilitation thesis is made of four chapters:

- the first chapter studies stochastic Hamilton-Jacobi equations (well-posedness aspects as well as qualitative properties such as regularity and range of dependence on initial data)
- the second chapter shows how Malliavin calculus may be applied to the study of singular SPDEs to obtain existence of densities of the law of solutions,
- the third chapter is concerned with rough stochastic volatility models, and in particular the application of pathwise methods in this context (notably to prove asymptotic formulae),
- the fourth chapter presents results around Root's solution to the Skorokhod embedding problem : PDE formulation, integral equation, extension to general Markov processes.

KEYWORDS

stochastic analysis, pathwise methods, rough paths, stochastic PDE, Hamilton-Jacobi equations, stochastic volatility models, Skorokhod embedding.