Université Paris-Dauphine
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Written exam : Evaluation des actifs financiers et arbitrage
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Exercise 1 : A market with uncertain trade execution

We consider a classical one-period binomial market, where the investor faces some uncertainty on the execution of his trade order. The underlying probability space is

$$\Omega = \{\nu_u, \nu_d\} \times \{0, 1\},$$

we write its elements $\omega = (\nu, \theta)$ and we assume that the historical probability measure $\mathbb{P}$ satisfies $\mathbb{P}(\omega) = \frac{1}{4}$ for each $\omega \in \Omega$.

The market consists in two tradable assets : a risk-free asset with price values $B_0 = 1$ at time 0 and $B_1 = 1 + r$ at time 1, for some fixed $r > 0$, and a risky asset with price $S_0 = 1$,

$$S_1(\nu_u, \theta) = 1 + u, \qquad S_1(\nu_d, \theta) = 1 + d$$

independently of the value of θ, where $-1 < d < u$ are fixed constants.

In this market, a trading strategy (for a self-financing portfolio) is simply a number $\phi \in \mathbb{R}$, representing the number of units of the risky asset the investor would like to hold in the portfolio between time 0 and time 1. We assume that with probability 1/2, the trade is not realized, more precisely :

- If $\omega = (\nu, 1)$, then the trade proceeds as planned,
- If $\omega = (\nu, 0)$, then the trade is cancelled (with no cost for the investor).

We then denote by $X_1^{x,\phi}$ the value at time 1 of a portfolio starting with wealth x at time 0 and trading with strategy ϕ (we assume that initially, all the wealth is invested in the riskless asset).

As usual, if X_i, $i = 0, 1$ is a stochastic process we will call $\tilde{X}$ its discounting, i.e. $\tilde{X}_0 = X_0$ and $\tilde{X}_1 = X_1/(1 + r)$.

(1) For any $x \in \mathbb{R}$ and $\phi \in \mathbb{R}$, show that

$$\tilde{X}_1^{x,\phi}(\nu, \theta) = x + \theta\phi \left(\tilde{S}_1(\nu) - 1\right).$$

Recall that the No Arbitrage condition is

$$\forall \phi \in \mathbb{R}, \quad \left(X_1^{0,\phi}(\omega) \geq 0 \text{ for all } \omega \in \Omega\right) \Rightarrow \left(X_1^{0,\phi}(\omega) = 0 \text{ for all } \omega \in \Omega\right). \qquad (NA)$$

(2) Show that (NA) holds if and only if $d < r < u$.

(3) We let

$$\mathcal{M}(\tilde{S}) = \left\{\mathbb{Q} \sim \mathbb{P} \text{ such that } \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_1\right] = S_0\right\}.$$

Show that (NA) is equivalent to $\mathcal{M}(\tilde{S}) \neq \emptyset$. If this condition holds, compute all the elements of $\mathcal{M}(\tilde{S})$.

We now assume that $d < r < u$ and consider an option with payoff $G = g(S_1)$, for a given function $g : \{1+u, 1+d\} \to \mathbb{R}$.

(4) We denote by $\hat{p}(G)$ the superhedging price of the option. Recall its definition, then show that $\hat{p}(G) = \max_{\omega\in\Omega} \tilde{G}(\omega)$.

(5) Show that, if G is not a constant random variable, then it is not replicable in this market.

(6) The superhedging price being too high of a price, we consider a relaxed requirement on the price and let

$$\tilde{p}(G) = \inf\left\{p \in \mathbb{R},\ \exists\phi \in \mathbb{R},\ \ \mathbb{P}\left(X_1^{p,\phi} \geq G\right) \geq \frac{3}{4}\right\}.$$

Show that for any $\mathbb{Q} \in \mathcal{M}(\tilde{S})$, it holds that

$$\tilde{p}(G) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{G}\right]$$

and compute the associated strategy ϕ.

Exercise 2 : Pricing with different interest rates

We consider a financial market with one risky asset, whose price process satisfies

$$S_t = S_0 + \int_0^t b(S_s)ds + \int_0^t \sigma(S_s)dW_s.$$

where W is a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The functions b, σ are Lipschitz-continuous and b, σ and σ^{-1} are bounded. In the market, there are two different interest rates: $r > 0$ is the rate for lending money, and $R > r$ is the rate for borrowing money. The goal of this exercise is to characterise the hedging price of an European option paying $g(S_T)$ at maturity $T > 0$. Here, g is a continuous function.

(1) Dynamics of the wealth process. For any $t \geq 0$, we denote by ϕ_t the quantity of risky asset held in the portfolio, α_t the amount of money held on the cash account at time t and V_t the value of the wealth process at time t.

   (a) Explicit the infinitesimal change of value of the cash account between the date t and $t + dt$ using r and R.

   (b) Show that for a self-financing strategy

$$V_t = v + \int_0^t \left(rV_s + (b(S_s) - rS_s)\phi_s - (R - r)\left[V_s - \phi_s S_s\right]^-\right) ds + \int_0^t \phi_s \sigma(S_s) dW_s,$$

   where v is the initial wealth and we denote $a^- = -\max(a, 0) = -a\mathbf{1}_{\{a \leq 0\}}$. We work here with strategies $\phi \in \mathcal{A}$, the set of measurable adapted processes such that $\mathbb{E}\left[\int_0^t \phi_s^2 \sigma(S_s)^2 ds\right] < \infty$.

   (c) Compare with the classical framework.

(2) In the sequel, we denote $f(x, y, z) = -ry - z(b(x) - rx)/\sigma(x) + (R - r)[y - zx/\sigma(x)]^-$. In particular the dynamics of the wealth process starting from v and following the strategy ϕ reads

$$V_t^{v,\phi} = v - \int_0^t f(S_s, V_s^{v,\phi}, \sigma(S_s)\phi_s)ds + \int_0^t \phi_s \sigma(S_s) dW_s. \tag{1}$$

We want to characterise the minimal super-hedging price given by

$$p := \inf\left\{v \in \mathbb{R} \text{ such that there exists } \phi \in \mathcal{A} : V_T^{v,\phi} \geq g(S_T)\right\}.$$

We assume that there exists a smooth solution u to the following semilinear PDE

$$\partial_t u + b(x)\partial_x u + \frac{1}{2}\sigma(x)^2 \partial_{xx} u + f(x, u, \sigma(x)\partial_x u) = 0,$$

with $u(T, \cdot) = g$. (We also suppose that u and all its derivatives are bounded).

   (a) Let $Y_t = u(t, S_t)$ and $Z_t = \sigma(S_t)\partial_x u(t, S_t)$. Compute the dynamics of Y and give an expression for Y_T.

   (b) By determining a hedging strategy, deduce that $u(0, S_0) \geq p$. (Hint : use the fact that for fixed v and ϕ, $V^{v,\phi}$ is the unique solution to (1))

(3) We are now going to prove the converse inequality.

(a) Let $v \in \mathbb{R}$, $\phi \in \mathcal{A}$ such that $V_T^{v,\phi} \geq g(S_T)$. We set $\delta := V^{v,\phi} - Y$ and $\beta := \phi\sigma(S) - Z$.Show that

$$\delta_t = \delta_T + \int_t^T (a_s\delta_s + b_s\beta_s)ds - \int_t^T \beta_s dW_s$$

where a and b are some bounded processes (to determine).

(b) Let Γ be the process given by

$$\Gamma_t = e^{\int_0^t (a_s - \frac{1}{2}b_s^2)ds + \int_0^t b_s dW_s}.$$

Compute the dynamics of $(\Gamma_t \delta_t)_{t\in[0,T]}$.

(c) Deduce from the previous question the sign of δ_0 and conclude.