Stochastic Hamilton-Jacobi equations and their long time behaviour

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Outline

Fully nonlinear (2nd order parabolic) Stochastic PDEs

Evolution equations of the type

$$\begin{cases} u(0,\cdot) & = u_0, \\ du & = F(x, Du, D^2u)dt + H(x, Du) \circ dB_t, \end{cases}$$

where $u : [0, T] \times \mathbb{R}^N \times \Omega \to \mathbb{R}$.

Introduced by Lions-Souganidis ('98).

Here B is a random path (typically Brownian motion) in $\mathbb{R}^d$, and F is degenerate elliptic i.e. nondecreasing in D^2u.

Motivation : movement of interfaces

Interface Γ moving with normal velocity :

$$V = f(x, \vec{n}, D\vec{n}) + \sum_i g_i(x, \vec{n})\dot{B}^i$$

If we represent Γ as a level set $\{u = 0\}$, we can write a PDE for u :

$$du = F(x, Du, D^2u)dt + \sum_i H_i(x, Du) \circ dB^i_t$$

with

$$F(x, p, A) = -|p| f\left(x, -\frac{p}{|p|}, -Q_p(A)\right)$$

$$H_i = -|p| g_i\left(x, -\frac{p}{|p|}\right).$$

(Then F elliptic ⇔ monotonicity of dynamics)

Stochastic mean curvature flow

Guiding example : mean curvature flow + stochastic perturbation

$$V = -div_\Gamma \vec{n}(x) \quad + \quad \dot{B}(t)$$

Level set PDE

$$\partial_t u = \mathrm{Tr}\left(D^2 u \left(I - \frac{Du \otimes Du}{|Du|^2} \right) \right) \quad + \quad |Du| \circ \dot{B}(t)$$

$\rightarrow$ quasilinear, parabolic, degenerate PDE + non-linear multiplicative noise

Deterministic dynamics have been studied by many authors. Level set PDE : Evans-Spruck ('91), Chen-Giga-Goto ('91),Barles, Soner, Souganidis,...

The stochastic dynamics can be obtained as limit of bistable reaction-diffusion with additive noise

$$\partial_t v^\varepsilon = \Delta v^\varepsilon \; - \; \frac{1}{\varepsilon} F'(v^\varepsilon) + \xi^\varepsilon(t)$$

Funaki ('98), Weber ('10), Lions-Souganidis ('00,'20),...

Solution theory

Pathwise approach : $\xi = B(\omega)$ is fixed, i.e. consider

$$\partial_t u = F(x, u, Du, D^2 u) + H_i(x, u, Du)\dot{\xi}^i(t) \text{ on } (0, T] \times \mathbb{R}^N, \quad u(0, \cdot) = u_0$$

for fixed $u_0 \in BUC(\mathbb{R}^N)$, $\xi \in C^0([0, T]; \mathbb{R}^d)$ $(\xi(0) = 0)$.

- Classical case : $\xi \in C^1$. No classical solution in general. But Crandall-Lions theory of **viscosity solutions** give existence/uniqueness.
- Stochastic case : need to extend (continuously) the map $\xi \mapsto u$ from C^1 to C^0

Irreversibility

Nonlinearity of the equation → even for smooth u_0, "shocks" (discontinuities in Du) typically appear in finite time.

Need some PDE theory for weak solutions, the solution is NOT just an SDE solution in infinite dimensions.

For example :

- Finite-dimensional ODE with 1-dimensional noise

$$\dot{X}(t) = V(X)\dot{\xi}(t).$$

  The solution is simply given by $X(t) = \phi_V(X_0, \xi(t))$.

- For stochastic Hamilton-Jacobi

$$\partial_t u = H(x, Du)\dot{\xi},$$

  this is not so simple !
  (Indeed : the (deterministic) equation

$$\partial_s U = H(x, DU), \quad U(0, \cdot) = u_0$$

  typically does not have a C^1 solution on $(-\infty, \infty) \times \mathbb{R}^N$.)

Irreversibility

- Simple example :

$$\partial_t u = |Du|\,\dot{\xi}(t), \quad u(0,x) = |x|.$$

Then the (sub-)level sets $\Gamma_R(t) = \{u(t,\cdot) \leqslant R\}$ are given by

$$\Gamma_R(t) = \begin{cases} B_0(R - \xi(t)), & t \leqslant \tau \\ \emptyset, & t > \tau \end{cases}$$

where $\tau = \inf\{t > 0, \ \ R - \xi(t) < 0\}$.

Some well-posedness results

$$\partial_t u = F(x, u, Du, D^2 u) + H_i(x, Du)\dot{\xi}^i(t) \text{ on } (0, T] \times \mathbb{R}^N, \quad u(0, \cdot) = u_0$$

- Lions-Souganidis (98-00) : x-independent case. $H = H(Du)$ difference of convex functions (d-dimensional noise), $F = F(Du, D^2 u)$.
- Semilinear case ($H = H(x, u, Du)$ linear in Du) : global transformations, cf. rough path approaches by Friz and coauthors ('10 '13)
- $d = 1$, $H = H(x, Du)$ strictly convex in Du. Friz-G.-Lions-Souganidis ('17), Seeger ('18), Lions-Souganidis.
- Neumann boundary conditions : G.-Seeger ('21).

Proofs based on maximum principle arguments

$$\frac{d}{dt} \max_{x,y} \{u(t, x) - v(t, y) - \Phi(t, x, y)\} \leqslant 0,$$

for suitably chosen test-function Φ.

Rest of the talk

Given a solution u to an equation of the form

$$du = F(x, Du, D^2u)dt + H(x, Du) \circ dB(t), \quad t > 0, \; x \in \mathbb{T}^N$$

what can we say about $u(t, \cdot)$ as $t \to \infty$?

No general theory, but we present results in three regimes :

- Convergence to equilibrium in x-independent equations
- Quantitative rate of convergence in dimension 1 ($x \in \mathbb{T}$)
- Convergence to equilibrium in x dependent equations (when $F \equiv 0$)

Outline

Basic idea

Consider

$$du = F(D^2u, Du)dt + H(Du) \circ dB_t.$$

Up to translating the solution, assume $F(0,0) = H(0) = 0 \rightarrow$ constants are solutions.

- Say we know that solutions to the deterministic equation $\partial_t v = F(D^2 v, Dv)$ converge to constants.
- B Brownian motion : almost surely, B will be very small on large intervals.
- By continuity, this means that on these intervals, u will behave like v, and be close to a constant.

Conclusion : $\lim_{t\rightarrow\infty} u(t,\cdot) = (const.)$.

(Very classical idea, cf e.g. Dirr-Souganidis '03)

Long time behaviour for a stochastic mean curvature flow

Let $D = D' \times \mathbb{R}$, where D is a bounded, C^2, convex domain in $\mathbb{R}^{N-1}$.
Fix Γ_0 closed subset of D, with for some $a < b$

$$D' \times (-\infty, a] \subset \Gamma_0 \subset D' \times (-\infty, b]$$

and consider $\Gamma(t)$ its evolution under stochastic mean curvature flow

$$V = -div_{\partial\Gamma(t)}\vec{n}(x) + \dot{B}(t)$$

with $\Gamma(t) = \Gamma_0$, and right angle boundary condition

$$n_{\Gamma(t)} \cdot n_D = 0 \quad \text{on } \partial\Gamma(t) \cap \partial D.$$

Theorem (G. Seeger '21+)

Almost surely, for some $c \in [a, b]$,

$$\lim_{t\to\infty} dist_H\left(\Gamma(t), \bar{\Gamma}^c(t)\right) = 0,$$

where

$$\bar{\Gamma}^c(t) = D \times (-\infty, c + B(t)].$$

Ideas of proof

We consider the level set formulation

$$\begin{cases} du = \mathrm{Tr}\left(D^2 u \left(I - \frac{Du \otimes Du}{|Du|^2}\right)\right) dt + |Du| d\xi(t) \\ u(0,x) = u_0(x), \qquad \partial_n u = 0 \text{ on } (0,+\infty) \times \partial D \end{cases}$$

Then we use the scheme of proof described above (comparison + related compactness estimates) in the following steps :

- convergence of deterministic dynamics : Giga Ohnuma Sato ('99), $u \to v = v(x_n)$.
- stochastic term $\to v \nearrow$ in x_n.
- convergence in Hausdorff distance requires a Borel-Cantelli argument to rule out "macroscopic holes" for large time.

(Similar results were obtained by different methods in the case of a periodic graph by Es-Sahrir & von Renesse ('12) and Dabrock-Hofmanova-Röger ('21))

Outline

Stochastic mean curvature in $1D$

For illustration purposes, consider the stochastic mean curvature flow evolution of a periodic $1D$ graph :

$$\partial_t h = \frac{\partial_{xx} h}{1 + (\partial_x h)^2} + \sqrt{1 + (\partial_x h)^2} \circ \dot{B}(t), \text{ on } [0, \infty) \times \mathbb{T},$$
$$h(0, \cdot) = h_0 \in C(\mathbb{T}),$$

where B is a scalar Brownian motion.

From Section 2 we expect (know) that

$$\operatorname{osc}(u, t\cdot) := \max_{\mathbb{T}} u(t, \cdot) - \min_{\mathbb{T}} u(t, \cdot) \to_{t \to \infty} 0.$$

What about quantitative statements ?

Theorem (G. Gess Lions Souganidis ('22+))

There exists $c > 0$ and a $\mathbb{P}$-a.s. finite random variable $C(\omega)$, such that, for any h solution to the previous equation, then,

$$\forall t \geqslant 0, \quad \mathrm{osc}(h(t,\cdot)) \leqslant C(\omega)\mathrm{osc}(h_0)e^{-ct}.$$

Remark : this is an improvement over both the deterministic ($B \equiv 0$) and the purely stochastic case :

- due to the degeneracy as $\partial_x h \to \infty$, there exist solutions h^R to $\partial_t h = \frac{\partial_{xx} h}{1+(\partial_x h)^2}$ s.t.

$$\mathrm{osc}(h^R(0,\cdot)) = R, \quad \mathrm{osc}(h^R(t,\cdot)) \geqslant \frac{R}{2} \text{ for all } t \leqslant cR.$$

- for $\partial_t h = \sqrt{1+(\partial_x h)^2} \circ \dot{B}(t)$,

$$\mathrm{osc}(h(t,\cdot)) \approx_{t\to\infty} \frac{C}{\max_{[0,t]} B - \min_{[0,t]} B} \approx \frac{C}{t^{1/2}}.$$

- Stochastic structure of B not so important, would also work for $B(t) = t$, $C\sin(t)$ or fractional BM.

Sketch of proof

Rewrite the equation as

$$\partial_t u = \partial_x F(u_x) + H(u_x)\dot{B}(t)$$

(with $F(r) = \arctan(r)$ and $H(r) = \sqrt{1+r^2} - 1$). The proof proceeds in 3 steps :

1. The stochastic term guarantees that

$$\mathrm{osc}(u(\tau_1,\cdot)) \leqslant C, \quad \text{for some } \tau_1 \leqslant \ln(\mathrm{osc}(u_0)) + C(\omega).$$

   Proof : This follows from

$$\frac{d}{dt}\int_{\mathbb{T}} u = \int_{\mathbb{T}} \partial_x(F(u_x)) \; + \; \left(\int_{\mathbb{T}} H(u_x)\right) \dot{B}(t)$$

   (where $H(p) = \sqrt{1+p^2} - 1$), which leads to

$$\mathrm{osc}(u(t+h)) \leqslant \int_{\mathbb{T}} H(u_x(t+h,\cdot)) \leqslant C\frac{\mathrm{osc}(u(t))}{\mathrm{osc}(B)_{t,t+h}}$$

   (where $H(p) = \sqrt{1+p^2} - 1$),and then using that $[0,T]$ can be subdivided in T/C intervals where $\mathrm{osc}(B)$ is of order C', this gives

$$\mathrm{osc}(u(T)) \leqslant C(\omega)e^{-cT}\mathrm{osc}(u_0).$$

2. At a time $\tau_2 = \tau_1 + C'(\omega)$,

$$\|u_x(\tau_2)\|_{L^\infty} \leqslant C.$$

Proof : A similar argument to the previous step gives

$$\|G(u_x(t+h,\cdot))\|_\infty \leqslant \frac{\mathrm{osc}(u(t,\cdot))}{\sup_{[u,v]\subset[t,t+h],\ B(u)=\min_{[u,v]} B} \int_u^v (B(r)-B(u))dr},$$

where

$$G(r) = \int_0^r (F'H'')(u)du \, (\sim_{r\to 0} cr)$$

Indeed : $N(t) = \int_{\mathbb{T}} H(u_x(\cdot,t))dx$ satisfies

$$\frac{dN(r)}{dr} \leqslant -\int_{\mathbb{T}} (F'H'')(u_x(x,r))(u_{xx}(x,r))^2 dx \leqslant -\|G(u(\cdot,r)\|_\infty.$$

and we have

$$\int u(\cdot,t) - \int u(\cdot,s) \geqslant -\int_s^t (B(u)-B(s))\frac{dN(u)}{du}du$$

3. Once the gradient is bounded, we obtain

$$\|u_x(\tau_2 + h)\|_{L^2} \leqslant e^{-ch} \|u_x(\tau_2)\|_{L^2}.$$

Proof : This follows from the uniform ellipticity of the deterministic part :

$$\frac{d}{dt}\int_{\mathbb{T}} u_x^2 = -\int_{\mathbb{T}} F'(u_x)(u_{xx})^2 - \int_{\mathbb{T}} \underbrace{\left(H''(u_x)u_x u_{xx}\right)}_{\partial_x(\dots)} \dot{B}(t)$$

$$\leqslant -c(\|u_x\|_\infty) \int_{\mathbb{T}} u_x^2.$$

(Remark : more estimates / examples can be found in the paper).

Outline

In this section, we consider solutions to

$$du = H(x, Du) \circ dB(t), \text{ on } (0, +\infty) \times \mathbb{T}^d$$

where H is **convex** in Du.

Corresponds for instance to motion of a (periodic) graph where velocity is given by

$$V(t, z) = a(x)\dot{B}(t)$$

(where $z = (x, z_n)$ and $a(\cdot) \geqslant 0$), in which case

$$H(x, p) = a(x)\sqrt{1 + |p|^2}.$$

Main result

Theorem (G. Gess Lions Souganidis ('22+))

Let $H = H(x, Du)$ be convex in Du and (for instance) strictly convex in Du. Then, there exists c_H s.t., for any solution u to

$$du + H(x, Du) \circ dB(t) = 0 \text{ on } (0, \infty) \times \mathbb{T}^d,$$

it holds that

$$u(t, x) = c_H B(t) + \psi(t, x) + o_{t \to \infty}(1).$$

Here ψ is a statistically stationary solution to

$$\partial_t \psi + (H(x, D\psi) - c_H) \circ dB(t) = 0,$$

satisfying

$$\phi^-(x) \leqslant \psi(t, x) \leqslant \phi^+(x) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{T}^d.$$

with

$$\pm \left(c + H(x, D\phi^\pm)\right) = 0 \text{ in } \mathbb{T}^d \text{ (in viscosity sense).}$$

- The proof is based on repeated application of the inequality

$$S_H(\delta)S_{-H}(\delta) \leqslant Id \leqslant S_H(\delta)S_{-H}(\delta)$$

which crucially relies on convexity of H. (cf. Barron et al. '98, GGLS '20).

- The long-time behaviour of $u(t,\cdot)$ is therefore given by :
(i) the macroscopic position ($cB(t)$)
(ii) the selected limiting upper/profile (ϕ^+ / ϕ^-)
(iii) the actual position of the associated stationary solution ψ.
In fact, these are roughly independent in the $t \to \infty$ limit :

$$\left(\frac{B(t)}{\sqrt{t}}, u(t,\cdot) - cB(t)\right) \underset{t\to\infty}{\to} \left(Z, \psi^{\Phi^+(B),B'}(0,\cdot)\right).$$

where $Z \sim \mathcal{N}(0,1)$, B, B' are independent.

Conclusion : some open questions

- Good understanding of x-dependent equations
- Well-posedness / selection by noise for ill-defined mean curvature flow ?
- Regularity theory ?

$$\partial_t u = \lambda \Delta u + H(Du) \circ \dot{B}(t) \ ^? \Rightarrow^? \ u(t,\cdot) \in C^1$$

(scaling critical !)