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**Paul GASSIAT**

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# Modélisation du risque de liquidité et méthodes de quantification appliquées au contrôle stochastique séquentiel

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Directeur de thèse

**Pr. PHAM Huyên**

Soutenue publiquement le 07/12/2011, devant le jury composé de :

**LAMBERTON Damien**, Professeur, Université de Marne-la-Vallée

**PHAM Huyên**, Professeur, Université Paris Diderot

**RUNGGALDIER Wolfgang**, Professeur, Università degli Studi di Padova

**SULEM Agnès**, Directeur de recherche, INRIA

**TALAY Denis**, Directeur de recherche, INRIA

**TANKOV Peter**, Professeur, Université Paris Diderot

au vu des rapports de :

**BAYRAKTAR Ehran**, Professeur, University of Michigan

**LAMBERTON Damien**, Professeur, Université de Marne-la-Vallée

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# Introduction Générale

Résumé :

Cette thèse est constituée de deux parties pouvant être lues indépendamment.

Dans la première partie on s'intéresse à la modélisation mathématique du risque de liquidité. L'aspect étudié ici est la contrainte sur les dates des transactions, c'est-à-dire que contrairement aux modèles classiques où les investisseurs peuvent échanger les actifs en continu, on suppose que les transactions sont uniquement possibles à des dates aléatoires discrètes. On utilise alors des techniques de contrôle optimal (programmation dynamique, équations d'Hamilton-Jacobi-Bellman) pour identifier les fonctions valeur et les stratégies d'investissement optimales sous ces contraintes. Le premier chapitre étudie un problème de maximisation d'utilité en horizon fini, dans un cadre inspiré des marchés de l'énergie. Dans le deuxième chapitre on considère un marché illiquide à changements de régime, et enfin dans le troisième chapitre on étudie un marché où l'agent a la possibilité d'investir à la fois dans un actif liquide et un actif illiquide, ces derniers étant corrélés.

Dans la deuxième partie on présente des méthodes probabilistes de quantification pour résoudre numériquement un problème de switching optimal. On considère d'abord une approximation en temps discret du problème et on prouve un taux de convergence. Ensuite on propose deux méthodes numériques de quantification : une approche markovienne où on quantifie la loi normale dans le schéma d'Euler, et dans le cas où la diffusion n'est pas contrôlée, une approche de quantification marginale inspirée de méthodes numériques pour le problème d'arrêt optimal.

## 0.1 Première partie : Modélisation du risque de liquidité

Le risque de liquidité est un risque financier majeur, tout particulièrement dans les périodes de crise où les marchés subissent différentes formes d'illiquidité. Il peut être défini comme l'ensemble des contraintes sur la capacité d'un agent à acheter ou vendre un actif et évaluer son portefeuille.

Dans les travaux pionniers de Merton sur l'optimisation de portefeuille et Black Scholes sur la couverture d'option, ainsi que dans la majeure partie de la littérature en mathématiques financières qui a suivi, il est fait l'hypothèse classique que les agents intervenant sur le marché peuvent échanger continûment les actifs financiers sans contraintes et sans impact sur leurs prix. Bien que très pratique d'un point de vue mathématique puisque permettant d'utiliser des outils puissants de calcul stochastique, cette hypothèse n'est pas réaliste en pratique. Dans la dernière décennie, de nombreuses études ont été réalisées dans le but de relaxer cette hypothèse.

Une première approche est de mesurer l'illiquidité en terme de coûts de transaction, voir le livre de Kabanov et Safarian [38] pour un aperçu récent de la théorie. Dans ce contexte, les échanges fréquents d'actifs sont soumis à des coûts potentiellement élevés, mais l'investisseur peut acheter ou vendre des actifs quand il le désire.

D'autre part, il a été observé empiriquement que des transactions à haut volume ont un impact sur le prix de l'actif échangé. On parle alors de modèle de grand investisseur. Ce facteur a été étudié par Cetin, Jarrow et Protter [14], Bank et Baum [7] pour le problème d'arbitrage et de pricing d'options, Schied et Schöneborn [67] pour un problème de liquidation de portefeuille. Ly Vath, Mnif et Pham [50] considèrent un modèle combinant coûts de transaction et effets de grands investisseurs dans un contexte de gestion de portefeuille.

Un autre aspect du risque de liquidité est le retard à l'exécution des ordres de transactions. En pratique, ces ordres ne sont pas exécutés immédiatement et ont besoin d'un certain temps avant d'atteindre le marché (voir par exemple Subramanian et Jarrow [70]). Ce délai à l'exécution a un impact sur la dynamique du portefeuille, et on s'attend donc à ce qu'il modifie les comportements des investisseurs. Ce problème a été étudié dans le contexte de contrôle impulsionnel stochastique par Øksendal et Sulem [56] et Bruder et Pham [11].

Le type d'illiquidité que nous étudions dans cette thèse est la restriction sur les dates de transaction et d'observation. En effet l'hypothèse classique de trading en temps continu est peu réaliste dans le cas de marchés illiquides, où étant donné le faible volume d'ordres traités il peut s'écouler un temps relativement long entre les possibilités successives de transaction. Rogers [65] considère un agent pouvant uniquement rebalancer son portefeuille à des intervalles fixes et montre que la perte causée est relativement faible par rapport à l'incertitude sur les paramètres du prix de l'actif. Rogers et Zane [66], Matsumoto [53] considèrent un modèle où les dates successives de transaction sont données par les temps de saut d'un processus de Poisson d'intensité $\lambda$ constante, et étudient le comportement asymptotique quand $\lambda$ est grand. Dans le même cadre, Pham et Tankov [61, 62] étudient un problème de consommation/investissement en horizon infini, caractérisent la fonction valeur comme unique solution (de viscosité) de l'équation HJB et donnent un schéma numérique pour la calculer. Citons également Bayraktar et Ludkovski [8] qui étudient dans un contexte similaire un problème de liquidation de portefeuille. Nous prolongeons l'approche de ces articles sur trois problèmes différents développés ci-dessous.

### 0.1.1 Investissement optimal dans un marché illiquide avec dates discrètes aléatoires de transaction

Dans le premier chapitre nous étudions un problème de maximisation d'utilité en horizon fini dans un marché illiquide. La contrainte de liquidité s'exprime par le fait que l'agent peut observer le prix de l'actif et effectuer des transactions uniquement à des dates aléatoires discrètes. Une particularité importante de notre modèle est que l'intensité d'arrivée de ces dates est proche de l'infini quand on approche l'horizon en temps $T$. Cette hypothèse est naturelle pour modéliser ce qu'on observe par exemple dans le cas de contrats forward dans les marchés d'énergie : étant donnés la nature physique de sous-jacent, plus on s'approche de la date d'échéance et plus l'activité de trading sur le titre est importante.

Un problème similaire a été étudié par Matsumoto [53] pour une fonction d'utilité logarithmique. Les principale différences avec notre approche, outre le fait que nous prenons en compte des fonctions d'utilité et des processus de prix plus généraux, sont que dans [53] la liquidité est constante, et le prix de l'actif illiquide est observé en continu.

On s'intéresse donc à un marché comportant un actif sans rique (supposé constant sans perte de généralité) et un actif risqué illiquide de processus de prix $(S_t)_{0\leq t\leq T}$. On se donne également une suite de temps d'arrêt $(\tau_n)_{n\geq 0}$ indépendants de $S$, représentant les dates auxquelles l'agent peut observer le prix $S_t$ et effectuer des transactions.

On suppose que $S$ suit une dynamique de type log-Lévy, plus précisément $S_t = \mathcal{E}(L)_t$, où $\mathcal{E}$ dénote l'exponentielle stochastique et

$$L_t = \int_0^t b(u)\,du + \int_0^t c(u)\,dB_u + \int_0^t \int_{-1}^{\infty} y(\mu(dt,dy) - \nu(dt,dy)), \quad 0 \leq t \leq T,$$

est une semimartingale à incréments indépendants de sauts $\Delta L_t > -1$. On suppose de plus des conditions naturelles d'intégrabilité sur les caractéristiques déterministes $(b, c, \nu)$ ainsi qu'une condition de non arbitrage. On notera $Z_{t,s} = \frac{S_s - S_t}{S_t}$ le rendement entre $s$ et $t$ et $p(t, s, dz) = \mathbb{P}[Z_{t,s} \in dz]$ sa distribution.

Les dates $(\tau_n)$ sont données par les temps de saut d'un processus de Poisson inhomogène $(N_t)_{0 \leq t \leq T}$ d'intensité déterministe $\lambda(t)$. On fait l'hypothèse suivante sur $\lambda$ :

$$\int_0^t \lambda(u) du \ < \ \infty, \ \forall \ 0 \leq t < T \text{ et } \int_0^T \lambda(u) du \ = \ \infty.$$

Sous cette condition la suite de temps d'arrêt $(\tau_n)$ satisfait presque sûrement

$$\lim_{n \to \infty} \tau_n = T.$$

On définit la filtration d'observation discrète $\mathcal{F}_n = \sigma\Big\{(\tau_k, Z_{\tau_{k-1}, \tau_k}) \ : \ 1 \leq k \leq n\Big\}$. Une stratégie d'investissement est alors une suite $(\alpha_n)$, où $\alpha_n$, $\mathcal{F}_n$-mesurable, représente le montant détenu en actif risqué sur la période $(\tau_n, \tau_{n+1}]$. Le processus de richesse $(X_{\tau_n})$ associé à une stratégie $\alpha$ vérifie donc

$$X_{\tau_{n+1}} \ = \ X_{\tau_n} + \alpha_n Z_{\tau_n, \tau_{n+1}}.$$

Dans la suite on fixe un capital initial $X_0 > 0$ et on se restreint à l'ensemble $\mathcal{A}$ des stratégies admissibles telles que la richesse de l'investisseur soit positive à toute date : $X_{\tau_n} \geq 0$, $n \geq 0$. Etant donné nos hypothèses sur $S$, $Z_{\tau_n, \tau_{n+1}}$ a pour support $(-1, +\infty)$ conditionellement à $\mathcal{F}_n$, et il est facile de voir que cette contrainte de positivité est équivalente à une interdiction de vente à découvert (à la fois sur l'actif risqué et l'actif sans risque).

Etant donné une fonction d'utilité $U$ satisfaisant des conditions générales, on s'intéresse au problème de contrôle :

$$V_0 \ = \ \sup_{\alpha \in \mathcal{A}} \mathbb{E}[U(X_T)].$$

On s'intéresse donc à résoudre ce problème d'optimisation, c'est-à-dire déterminer $V_0$ et la stratégie optimale $\hat{\alpha}$ correspondante. On va utiliser une approche par Programmation Dynamique directe : on écrit formellement l'équation de Programmation Dynamique (EPD) pour notre problème, puis par des arguments analytiques on montre l'existence d'une solution pour cette EPD, et enfin on conclut par un argument de vérification.

Dans notre contexte l'EPD peut s'écrire comme un problème de point fixe (avec une condition terminale)

$$\begin{cases} \mathcal{L}v = v \\ \lim_{t \nearrow T, x' \to x} v(t, x') = U(x), \end{cases} \tag{0.1.1}$$

où étant donné une fonction $w$ satisfaisant des conditions de croissance appropriées, $\mathcal{L}w$ est défini par :

$$\mathcal{L}w(t, x) = \sup_{\pi \in [0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} w(s, x(1 + \pi z)) p(t, s, dz) ds.$$

Pour montrer l'existence d'une solution à (0.1.1), on adopte une approche par itération de fonctions valeurs, classique dans le cas de problèmes discrets (voir aussi [23]). On considère la suite de fonctions $(v_m)_{m \geq 0}$ définie récursivement par :

$$\begin{aligned} v_0 &= U, \\ v_{m+1} &= \mathcal{L}v_m. \end{aligned}$$

Alors on montre que :

- $v_m$ converge vers une fonction $v^*$, solution de (0.1.1).
- $V_0 = v^*(0, X_0)$, et la stratégie optimale $\hat{\alpha}$ est donnée par :

$$\hat{\alpha}_n = \hat{\pi}(\tau_n, \hat{X}_{\tau_n}) \hat{X}_{\tau_n}, \quad n \geq 0,$$

  où $\hat{\pi}$ est donné par

$$\hat{\pi}(t, x) \in \arg\max_{\pi \in [0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v^*(s, x(1 + \pi z)) p(t, s, dz) ds.$$

De plus $v_m$ correspond au problème de contrôle suivant :

$$v_m(0, X_0) = \sup_{\alpha \in \mathcal{A}_m} \mathbb{E}[U(X_T)],$$

où $\mathcal{A}_m$ est l'ensemble des stratégies admissibles à investissement nul en actif risqué à partir de la $m$-ième date de trading, i.e. $\alpha_n = 0$ pour $n \geq m$.

Dans la dernière partie de ce chapitre on s'intéresse à la convergence de notre problème vers le problème classique de trading en continu. En effet, quand l'intensité d'arrivée de dates de transaction $\lambda$ est très grande à toute date, on s'attend à ce que la valeur correspondant $V_0^\lambda$ soit proche de celle où l'agent peut échanger l'actif en continu, en prenant en compte la contrainte d'interdiction de vente à découvert.

On définit donc

$$V_0^M = \sup_{\pi \in \mathcal{D}(S)} \mathbb{E}\left[U(X_T^\pi)\right],$$

où $\mathcal{D}(S)$ est l'ensemble des stratégies de trading continues sur l'actif $S$ sans vente à découvert.

Le résultat obtenu est alors le suivant : étant donné une suite de fonctions d'intensité $\lambda_k$ telles que

$$\lambda_k(t) \to \infty \quad \text{quand } k \to \infty, \ \forall t \in [0, T],$$

on a la convergence

$$V_0^{\lambda_k} \to V_0^M \quad \text{quand } k \to \infty,$$

Ce chapitre est tiré d'un article rédigé en collaboration avec Huyên Pham et Mihai Sîrbu [29], publié dans *International Journal of Theoretical and Applied Finance.*

### 0.1.2 Investissement/consommation optimaux dans un marché illiquide avec changements de régime

Dans les premiers articles étudiant des modèles de risque de liquidité avec dates de transaction discrètes (par exemple [66], [53], [61]), la fréquence de trading est constante en temps et indépendante du prix des actifs. Cependant en pratique la liquidité du marché subit des fluctuations à la fois déterministes et aléatoires et à différentes échelles de temps. Dans ce chapitre on étudie un modèle simple de marché illiquide avec changements de régime, chaque régime ayant différentes liquidités et dynamique de prix.

Les modèles à changements de régime ont déjà été étudiés à plusieurs reprises dans des applications à la finance, voir les articles [69],[72] ou pour un point de vue statistique la thèse [55]. Plus récemment du point de vue du risque de liquidité, les articles [21] et [49] étudient un marché subissant des chocs de liquidité aux cours desquels l'activité de trading est complètement interrompue.

On considère donc un marché subissant des changements de régime, modélisés par une chaîne de Markov $(I_t)$ à espace d'états fini $\mathbb{I}_d = \{1, \ldots, d\}$ et de générateur infinitésimal $Q = (q_{ij})$. Le marché comporte un actif sans risque supposé constant et un actif risqué de processus de prix $S$. L'investisseur peut effectuer des transactions sur cet actif uniquement à des dates $(\tau_n)_{n\geq 0}$, correspondant aux temps de saut d'un processus de Cox $(N_t)_{t\geq 0}$ d'intensité $\lambda_{I_t}$. Autrement dit à chaque régime $i$ du marché correspond une intensité $\lambda_i$ d'arrivée de dates de transaction. Il est important de noter que contrairement au modèle du premier chapitre ou à l'article [61], la contrainte porte uniquement sur la capacité de transaction, et que le prix $S_t$ est observé en continu par l'agent.

Le prix $S_t$ évolue dans chaque régime suivant un Brownien géométrique : quand $I_t = i$,

$$dS_t = S_t(b_i dt + \sigma_i dW_t),$$

où $W$ est un Brownien standard indépendant de $(I, N)$ et $b_i, \sigma_i$, $i = 1, \ldots, d$ sont des constantes.

On suppose de plus que le prix subit des sauts à chaque changement de régime : quand $I$ passe du régime $i$ au régime $j$ à l'instant $t$,

$$\Delta S_t = -S_{t-}\gamma_{ij},$$

où les $\gamma_{ij} < 1$ sont des constantes.

On considère un agent investissant dans ce marché et consommant en continu ; une stratégie est donc une paire de processus prévisibles $(c, \zeta)$ où $c$ est la consommation et $\zeta$ la stratégie d'investissement. Notant $(X_t, Y_t)$ les variables d'état correspondant à la richesse investie respectivement en actif sans risque et en actif risqué, on a la dynamique :

$$\begin{aligned} dX_t &= -c_t dt - \zeta_t dN_t, \\ dY_t &= Y_{t-}\frac{dS_t}{S_{t-}} + \zeta_t dN_t. \end{aligned}$$

Partant du régime $i$ et des richesses initiales $x$, $y$ on se restreint aux stratégies admissibles (dénotées $\mathcal{A}_i(x, y)$) telles que la richesse totale $R_t := X_t + Y_t$ est positive à toute date de transaction $\tau_n$. Comme dans le chapitre précédent, ceci est équivalent à une contrainte d'interdiction de vente à découvert : $(c,\zeta) \in \mathcal{A}_i(x,y)$ ssi $X_t, Y_t \geq 0$ p.s. pour tout $t$.

On se donne ensuite une fonction d'utilité $U$ satisfaisant les conditions habituelles et une condition de croissance, et pour un facteur d'actualisation $\rho > 0$ on considère le problème d'investissement/consommation en horizon infini :

$$v_i(x,y) \quad = \quad \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right]. \quad (x,y)\in\mathbb{R}_+^2,$$

On introduit également la fonction

$$\hat{v}_i(r) \quad = \quad \sup_{x\in[0,r]} v_i(x,r-x), \quad r \geq 0,$$

correspondant à la valeur obtenue en rebalançant optimalement une richesse initiale $r$ entre actif risqué et sans risque. Autrement dit, $v_i$ est la fonction valeur entre deux dates de transaction, alors que $\tilde{v}_i$ est la fonction valeur à une date de transaction.

Pour résoudre ce problème de contrôle, on caractérise les fonctions valeur $v_i$ comme solutions de l'équation d'Hamilton-Jacobi-Bellman (HJB) associée. L'équation HJB est une équation aux dérivées partielles, équivalent infinitésimal du principe de programmation dynamique de Bellman (voir les livres [24] et [60] pour une introduction au contrôle markovien en temps continu). Pour notre problème cette équation a la forme du système suivant :

$$\begin{aligned}\rho v_i - b_i y\frac{\partial v_i}{\partial y} \; - \; \frac{1}{2}\sigma_i^2 y^2\frac{\partial^2 v_i}{\partial y^2} \; - \; \sup_{c\geq 0}\left[U(c) - c\frac{\partial v_i}{\partial x}\right] & \\ - \sum_{j\neq i} q_{ij}\left[v_j\Big(x, y(1-\gamma_{ij})\Big) - v_i(x,y)\right] & \\ - \lambda_i\big[\sup_{-y\leq\zeta\leq x} v_i(x-\zeta, y+\zeta) - v_i(x,y)\big] & = 0.\end{aligned} \tag{0.1.2}$$

sur $\mathbb{I}_d \times (0,\infty)\times\mathbb{R}_+$, avec les conditions au bord :

$$v_i(0,0) \quad = \quad 0 \tag{0.1.3}$$

$$v_i(0,y) \quad = \quad \mathbb{E}_i\left[\sup_{0\leq\zeta\leq y\frac{S_{\tau_1}}{S_0}} v_{I_{\tau_1}}\left(\zeta, y\frac{S_{\tau_1}}{S_0} - \zeta\right)\right]. \tag{0.1.4}$$

Il est bien connu que dans le cas général les fonctions $v_i$ ne sont pas suffisament différentiables pour interpréter cette équation au sens classique, et qu'il faut recourir à une notion plus faible de solutions, appelées solutions de viscosité (voir par exemple [16]).

En utilisant un principe de programmation dynamique on montre que $v_i$ est solution de viscosité de (0.1.2), et on obtient également un principe de comparaison pour cette équation. Nos fonctions $(v_i)$ sont donc caractérisées comme l'unique solution du système (0.1.2) avec les conditions au bord (0.1.3)-(0.1.4).

On s'intéresse ensuite à l'existence et à la caractérisation de solutions optimales pour notre problème de contrôle. Dans le cas général de solutions de viscosité, il existe des résultats de vérification (cf. [30]-[31]), mais les hypothèses sont trop restrictives pour être appliquées ici. Nous cherchons donc à trouver des conditions sous lesquelles les fonctions $v_i$ seront suffisament différentiables pour pouvoir appliquer les résultats de vérification classiques. Notre équation étant dégénérée (seule la dérivée en $y$ apparait dans les termes du second ordre), dans le cas de fonction d'utilité $U$ générale on ne peut pas espérer appliquer de résultats standards d'existence pour les EDP elliptiques.

Cependant dans le cas particulier d'utilité puissance $U(c) = \frac{c^p}{p}$, on peut réduire la dimension de l'espace d'état. En effet, en faisant le changement de variable

$$\begin{aligned} r &= x + y, \\ z &= \frac{y}{x+y}, \end{aligned}$$

la fonction valeur peut être réécrite

$$v_i(x,y) = U(r)\varphi_i(z).$$

On est donc ramené à résoudre une équation différentielle en $z$, qui cette fois-ci satisfait une condition d'ellipticité uniforme.

Dans ce cas particulier, on montre donc la régularité de la fonction valeur $v_i$, et on en déduit l'existence de contrôles optimaux caractérisés par une formule feedback.

Enfin, on s'intéresse à la résolution numérique de l'équation (0.1.2). La principale difficulté vient des termes non-locaux, que l'on peut contourner par une procédure itérative. On définit

$v_0 = 0$ et récursivement, $v_{n+1}$ est définie comme l'unique solution (de viscosité) de

$$(\rho - q_{ii} + \lambda_i)v_i^{n+1} - b_i y \frac{\partial v_i^{n+1}}{\partial y} - \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 v_i^{n+1}}{\partial y^2} - \sup_{c\geq 0}\left[U(c) - c\frac{\partial v_i^{n+1}}{\partial x}\right]$$
$$= \sum_{j\neq i} q_{ij} v_j^n\Big(x, y(1-\gamma_{ij})\Big) + \lambda_i \sup_{-y\leq\zeta\leq x} v_i^n(x-\zeta, y+\zeta)$$

avec des conditions au bord appropriées.

Comme dans le premier chapitre on peut alors interpréter $v^n$ comme la fonction valeur d'un problème de contrôle :

$$v_i^n(x,y) = \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho t} U(c_t)dt\right].$$

où $\theta_n$ est le $n$-ième temps auquel on a une date de transaction ou un changement de régime. En utilisant cette représentation, on montre que $v^n$ tend vers $v$ et que la vitesse de convergence est exponentielle.

On illustre nos résultats par des tests numériques pour des marchés à 1 ou 2 régimes. Dans le cas de marché à un seul régime, on compare les résultats avec ceux de [61], où l'investisseur observe uniquement l'actif risqué aux dates $(\tau_n)$. Dans le cas d'un marché à 2 régimes on observe que typiquement l'existence de différents régimes augmente le "coût de liquidité" subi par l'agent.

Ce chapitre est tiré d'un article rédigé en collaboration avec Fausto Gozzi et Huyên Pham [27].

### 0.1.3 Investissement/consommation optimaux dans un marché avec actifs liquide et illiquide

La majorité des travaux étudiant le risque de liquidité considèrent des marchés constitués uniquement d'actifs illiquides. Cependant en pratique les marchés sont constitués d'actifs corrélés ayant différents degrés de liquidité. Par example, un indice boursier est souvent beaucoup plus liquide que les actifs individuels suivis par cet indice, et est corrélé positivement avec leurs cours. Un investisseur sur ce marché aura donc la possibilité de couvrir sa position en actif illiquide en investissant dans cet indice et rebalançant fréquemment son investissement dans ce dernier. Tebaldi et Schwartz [68] et Longstaff [48] considèrent un marché constitué d'un actif liquide et un actif illiquide, ce dernier pouvant uniquement être échangé à la date initiale et liquidé à une date finale $T$. Dans ce chapitre, nous prenons une approche moins restrictive et supposons que l'actif illiquide peut être échangé à des dates aléatoires discrètes.

Très récemment un problème similaire a été étudié par Ang, Papanikolaou et Westerfield [2], avec principalement deux différences par rapport à nos résultats. Tout d'abord, les fonctions d'utilités qu'ils considèrent sont de type CRRA avec paramètre d'aversion au risque $\gamma \geq 1$, alors que nous étudions le problème pour une classe de fonctions différentes, non nécessairement de type CRRA. De plus, ils supposent que l'agent observe le prix de l'actif illiquide en continu, alors que dans notre cas l'observation s'effectue uniquement aux dates de transaction. Notre hypothèse semble plus naturelle, puisqu'en pratique les possibilités de transaction et l'observation du prix des actifs coïncident via l'arrivée d'ordres d'achat ou de vente sur le marché.

On considère donc un marché constitué d'un actif sans risque supposé constant et de deux actifs risqués :

- un actif liquide qui peut être échangé en continu, de processus de prix $L$,
- un actif illiquide de processus de prix $I$, qui peut être échangé et observé uniquement à des dates $(\tau_n)$ correspondant aux temps de saut d'un processus de Poisson $N$ d'intensité $\lambda$.

On suppose que $L$ et $I$ suivent une dynamique de Black-Scholes :

$$\begin{aligned} dL_t &= L_t(b_L dt + \sigma_L dW_t), \\ dI_t &= I_t(b_I dt + \sigma_I(\rho dW_t + \sqrt{1-\rho^2} dB_t), \end{aligned}$$

où $W$ et $B$ sont des Browniens indépendants (et indépendants de $N$), et $\rho \in (-1, 1)$ est le coefficient de corrélation.

On définit la filtration d'observation de notre agent :

$$\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}; \quad \mathcal{G}_t = \sigma(\tau_n, I_{\tau_n}; \tau_n \leq t) \vee \mathcal{F}_t^W \vee \mathcal{N},$$

où $\mathcal{F}^W$ est la filtration engendrée par $W$ (ou par $L$) et $\mathcal{N}$ est la tribu engendrée par les ensembles $\mathbb{P}$-négligeables.

Une stratégie d'investissement sur ce marché est alors un triplet $(c, \pi, \alpha)$, où :

- $c = (c_t)$ est un processus $\mathbb{G}$-prévisible représentant le taux de consommation,
- $\pi = (\pi_t)$ également $\mathbb{G}$-prévisible est le montant investi en actif liquide,

- $\alpha = (\alpha_k)$ est une suite de variables aléatoires $\mathcal{G}_{\tau_k}$-mesurables, représentant le montant investi en actif illiquide à la date $\tau_k$.

Etant donnée une richesse initiale $r$, on se restreint à la classe $\mathcal{A}(r)$ de stratégies vérifiant une contrainte d'admissibilité, qui comme dans les chapitres précédents se réduit à une interdiction de vente à découvert. On se donne ensuite une fonction d'utilité $U$ et un facteur d'actualisation $\beta > 0$, et on considère le problème de contrôle :

$$V(r) = \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right].$$

Ce problème de contrôle est un problème non standard, mixte discret/continu de par la nature de la filtration d'observation $\mathcal{G}$. On suit alors la même approche que dans Pham et Tankov [61] : par programmation dynamique on se ramène à étudier le problème entre deux dates de transaction, et on montre que ce problème est équivalent à un problème standard.

Le principe de programmation dynamique pour notre problème a la forme suivante :

$$V(r) \quad = \quad \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta\tau_1} V\left(R_{\tau_1}\right)\right],$$

où

$$R_{\tau_1} \quad = \quad r + \int_0^{\tau_1} (-c_s ds + \pi_s \frac{dL_s}{L_s}) + \alpha_0 \frac{I_{\tau_1} - I_0}{I_0}$$

est la richesse totale à la date $\tau_1$.

On va réécrire le terme de droite de la précédente égalité comme solution d'un problème de contrôle stochastique standard pour la filtration $\mathcal{F}^W$.

Tout d'abord, en notant que comme seule la stratégie avant la date $\tau_1$ intervient dans ce terme, on peut réécrire cette égalité comme

$$V(r) \quad = \quad \sup_{a\leq r} \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta\tau_1} V\left(R_{\tau_1}\right)\right],$$

où étant donné un investissement initial $x$ en richesse liquide, $\mathcal{A}_0(x)$ est l'ensemble des stratégies $(c, \pi)$ $\mathcal{F}^W$-prévisibles satisfaisant des conditions d'admissibilité.

Ensuite on décompose le prix d'actif illiquide en $I_t = E_t J_t$, où

$$\begin{aligned}\frac{dE_t}{E_t} &= \frac{\rho\sigma_I}{\sigma_L}\frac{dL_t}{L_t} = (\rho b_I \frac{\sigma_I}{\sigma_L}dt + \rho\sigma_I dW_t),\\ \frac{dJ_t}{J_t} &= (b_I - \rho b_I\frac{\sigma_I}{\sigma_L})dt + \sigma_I\sqrt{1-\rho^2}dB_t.\end{aligned}$$

Notons qu'alors $(E_t)$ est $\mathcal{F}^W$-adapté, tandis que $(J_t)$ est indépendant de $\mathcal{F}^W$.

Etant donnée une richesse initiale $r = x + y$ répartie initialement en un montant $x$ d'actifs liquides et un montant $y$ d'actifs illiquides, et une stratégie $(c,\pi) \in \mathcal{A}_0(x)$, on considère les variables d'état $X$, $Y$ définies par :

$$\begin{aligned}X_t^{x,c,\pi} &= x + \int_0^t (-c_s ds + \pi_s \frac{dL_s}{L_s}),\\ Y_t^y &= yE_t.\end{aligned}$$

Autrement dit, $X_s$ correspond à la richesse en actifs liquides, alors que $Y_s$ correspond à la richesse investie initialement en actif illiquide modulée par l'information apportée par les variations du prix de l'actif liquide depuis la date initiale.

En définissant l'opérateur $G$ par

$$G[w](t,x,y) \quad = \quad \mathbb{E}\left[w(x+yJ_t)\right],$$

on obtient enfin

$$V(r) \quad = \quad \sup_{0\le a\le r}\ \sup_{(c,\pi)\in\mathcal{A}_0(x)} \mathbb{E}\int_0^\infty e^{-(\beta+\lambda)s}\left(U(c_s) + \lambda G[V]\left(s, X_s^{x,\pi,c}, Y_s^y\right)\right) ds.$$

Ceci est un problème de contrôle stochastique standard (inhomogène en temps). On définit alors la fonction $\hat{V}$, version dynamique définie par :

$$\hat{V}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + \lambda G[V]\left(s, X_s^{t,x,\pi,c}, Y_s^{t,y}\right)\right) ds.$$

On remarque que $V$ et $\hat{V}$ sont reliés par la relation

$$V(r) \quad = \quad \left[\mathcal{H}\hat{V}\right](r) \quad := \quad \sup_{0\le x\le r} \hat{V}(0,x,r-y).$$

Déterminer $V$ revient donc à déterminer $\hat{V}$. Pour ce faire, on va utiliser l'approche classique par équations HJB. L'équation de HJB pour notre problème a la forme suivante :

$$-\hat{V}_t + (\beta+\lambda)\hat{V} - \lambda G[\mathcal{H}\hat{V}](t,x,y) - \sup_{c\geq 0, \pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{V}, D^2_{(x,y)}\hat{V}; c, \pi) = 0, \qquad (0.1.5)$$

où l'hamiltonien $H_{cv}$ est défini par

$$H_{cv}(y,p,A;c,\pi) = \left[U(c) + (\pi b_L - c)p_1 + \frac{\rho b_L \sigma_I}{\sigma_L} y p_2 + \frac{\sigma_L^2 \pi^2}{2} A_{11} + \pi\rho\sigma_I\sigma_L y A_{12} + \rho^2 \frac{\sigma_I^2}{2} y^2 A_{22}\right].$$

Comme dans le chapitre précédent, on montre alors que $\hat{V}$ est l'unique solution de viscosité de (0.1.5) sur $[0,+\infty) \times (0,+\infty) \times \mathbb{R}_+$, satisfaisant la condition au bord

$$\hat{v}(t,0,y) = \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G[\mathcal{H}\hat{v}](s,0,\tilde{Y}_s^{t,y}) ds$$

et une condition de croissance appropriée.

Cette caractérisation permet alors de calculer numériquement $\hat{V}$.

Comme dans les chapitres précédents on a recours à une méthode itérative : on part de $\hat{V}^0 = 0$, et on définit récursivement $\hat{V}^{n+1}$ comme la solution de (0.1.5) où le terme nonlocal est remplacé par $\lambda G[\mathcal{H}\hat{V}^n](t,x,y)$. On a alors des résultats similaires à ceux obtenus au chapitre 2 : on montre que $\hat{V}^n$ correspond au problème de contrôle dans lequel l'agent ne consomme que jusqu'à la date $\tau_n$, et $\hat{V}^n$ converge vers $\hat{V}$ exponentiellement en $n$.

De plus comme on a une EDP à horizon infini, en pratique pour la résoudre on considère une approximation $V^{n,T}$ pour un horizon fixé $T$. On montre que pour $T$ choisi assez grand, $V^{n,T}$ approxime $V^n$ aussi précisément qu'on le souhaite (uniformément en $n$).

Enfin, on illustre numériquement nos résultats. On fixe les paramètres $b_L, \sigma_L, b_I, \sigma_I$, et on observe les variations de la fonction valeur et des stratégies optimales obtenues en faisant varier $\lambda$ et $\rho$.

Ce chapitre est tiré d'un article écrit en collaboration avec Salvatore Federico.

## 0.2 Deuxième partie : Discrétisation en temps et méthodes de quantification appliqués au problème de switching

Dans cette deuxième partie, on propose des schémas numériques pour un problème de switching optimal. Rappelons tout d'abord en quoi consiste ce problème.

On se donne un espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, et un ensemble fini de régimes

$\mathbb{I}_q = \{1, \ldots, q\}$. Un contrôle de switching est alors une suite $(\tau_n, \iota_n)_{n \geq 0}$, où $(\tau_n)$ est une suite croissante de temps d'arrêt et $(\iota_n)$ est une suite de v.a. $\mathcal{F}_{\tau_n}$-mesurables à valeurs dans $\mathbb{I}_q$. A chaque $\alpha$ on associe la diffusion contrôlée

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

où $W$ est un Brownien standard dans $\mathbb{R}^d$, et $\alpha_t = \iota_n$ sur $[\tau_n, \tau_{n+1})$. Le problème de contrôle considéré est alors :

$$v_i(t,x) \quad = \quad \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\Big[ \int_t^T f(X_t, \alpha_t)dt + g(X_T, \alpha_T) - \sum_{\tau_n \leq T} c(X_{\tau_n}, \iota_{n-1}, \iota_n) \Big].$$

Le switching optimal a de nombreuses applications, notamment en finance, et a fait l'objet de nombreuses études : voir par exemple le chapitre 5 dans le livre de Pham [60]. D'un point de vue numérique, la résolution de ces problèmes se fait généralement par une discrétisation en temps, et une procédure de récursion rétrograde qui nécessite le calcul d'espérances conditionnelles. Concernant l'erreur de discrétisation, dans le cas où la diffusion n'est pas contrôlée, des résultats ont été obtenus par des méthodes d'Equations Différentielles Stochastiques Rétrogrades (EDSRs) à réflexion oblique par Chassagneux, Elie et Kharroubi [13] (voir Hamadène et Zhang [34] et Hu et Tang [35] pour les propriétés de ces EDSRs). Quant aux calculs d'espérance conditionnelle, plusieurs méthodes ont été proposées pour le problème d'arrêt optimal : des techniques de calcul de Malliavin (Lions et Regnier [47], Bouchard, Ekeland et Touzi [10]), de régression à la Longstaff-Schwarz (Clément, Lamberton et Protter [15]) ou des méthodes de quantification (Bally-Pagès [5]).

Dans ce chapitre on présente des schémas numériques basés sur cette dernière approche. Rappelons que la quantification optimale consiste à approximer une variable aléatoire $X$ par un quantifieur $\hat{X}$ à support fini, de façon à minimiser l'erreur de quantification $\left\| X - \hat{X} \right\|_p$. On pourra consulter le livre de Graf et Luschgy [32] pour une introduction à la théorie de la quantification. Cette dernière a connu un fort intérêt ces dernières années en Probabilités Numériques, et notamment dans les applications à la finance, voir par exemple l'article [57] pour une présentation globale.

Dans le cas étudié ici, on suppose que toutes les fonctions intervenant sont Lipschitz en la variable d'espace, et que la fonction de coût satisfait une "condition triangulaire" naturelle.

Dans un premier temps on étudie l'impact de la discrétisation en temps (nécessaire pour tout schéma numérique) sur la fonction valeur de notre problème. Etant donné un pas de temps $h$, on considère donc la fonction $v^h$ définie par :

$$v_i^h(t_k,x) \quad = \quad \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})h + g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1}, \iota_n)\Big],$$

où $\mathcal{A}^h_{t_k,i}$ est l'ensemble des contrôles tels que les $\tau_n$ sont à valeur dans $\{\ell h, \ell = k, \ldots, m\}$. On montre que le taux de convergence de $v^h$ vers $v$ est de $h^{1/2-\varepsilon}$, où $h$ est le pas de discrétisation en temps et $\varepsilon > 0$. Ceci étend les résultats obtenus par Chassagneux, Elie et Kharroubi [13] dans le cas où la diffusion n'est pas contrôlée. Quand le coût de changement de régime $c$ ne dépend pas du processus $X$, on obtient un taux de convergence en $h^{1/2}$ comme pour le problème d'arrêt optimal (cf. Lamberton [45]).

Comme la diffusion $X_s$ n'est pas forcément simulable en pratique, on considère donc à la place le schéma d'Euler $\bar{X}_s$ défini récursivement par :

$$\begin{aligned} \bar{X}_{t_k} &= x, \\ \bar{X}_{t_{\ell+1}} &= \bar{X}_{t_\ell} + b(\bar{X}_{t_\ell}, \alpha_{t_\ell})h + \sigma(\bar{X}_{t_\ell}, \alpha_{t_\ell})\sqrt{h}\vartheta_{\ell+1}, \quad k \leq \ell \leq m-1, \end{aligned}$$

où $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$ a pour distribution $\mathcal{N}(0, I_d)$. On montre alors que la fonction valeur correspondante $\bar{v}^h$ converge vers $v^h$ en $h^{1/2}$.

La principale difficulté de ces preuves vient du terme de coût de changement de régime, le nombre de ces changements étant a priori illimités. En utilisant des outils d'EDSRs, on montre des estimations sur les moments de ce nombre de switchings pour une stratégie optimale.

On étudie ensuite deux schémas numériques par quantification :

• Le premier schéma est une approche par quantification markovienne dans la veine de Pagès, Pham et Printems [58]. On considère une grille de discrétisation en espace $\mathbb{X} = (\delta/d)\mathbb{Z}^d \cap B(0,R)$. On approxime le schéma d'Euler de la façon suivante : la gaussienne $\vartheta_{\ell+1}$ est remplacée par sa quantifiée $\hat{\vartheta}_{\ell+1}$, et le résultat obtenu est ensuite projeté sur la grille $\mathbb{X}$. Autrement dit on considère le processus $\hat{X}^{(1)}$ défini par :

$$\begin{aligned}
\hat{X}^{(1)}_{t_k} &= x, \\
\hat{X}^{(1)}_{t_{\ell+1}} &= \mathrm{Proj}_{\mathbb{X}}\left(\hat{X}^{(1)}_{t_\ell} + b(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})h + \sigma(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})\sqrt{h}\hat{\vartheta}_{\ell+1}\right), \quad k \leq \ell \leq m-1,
\end{aligned}$$

et on définit la fonction valeur associée $\hat{v}^{(1)}$ :

$$\hat{v}^{(1)}_i(t_k, x) = \sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})h + g(\hat{X}^{(1)}_{t_m}, \alpha_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\hat{X}^{(1)}_{\tau_n}, \iota_{n-1}, \iota_n)\Big].$$

En pratique, on peut calculer explicitement cette fonction par un algorithme récursif de programmation dynamique :

$$\begin{aligned}
\hat{v}_i(t_m, x) &= g_i(x), \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q \\
\hat{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big[\sum_{l=1}^{N} \pi_l \, \hat{v}_j\Big(t_{k+1}, \mathrm{Proj}_{\mathbb{X}}\left(x + b(x,j)h + \sigma(x,j)\sqrt{h}w_l\right)\Big) + f_j(x)h - c_{ij}(x)\Big], \\
&\qquad (x,i) \in \mathbb{X} \times \mathbb{I}_q, \quad 0 \leq k \leq m-1,
\end{aligned}$$

où $(w_l)_{1 \leq l \leq N}$ est la grille de quantification de la loi normale utilisée, de poids associés $(\pi_l)_{1 \leq l \leq N}$.

En suivant une méthode similaire à [58] (la principale différence étant que dans notre cas la volatilité n'est pas supposée bornée), on obtient le résultat suivant sur la convergence de la fonction $\hat{v}^{(1)}$ :

$$\begin{aligned}
\left|\bar{v}_i(t_k, x) - \hat{v}^{(1)}_i(t_k, x)\right| \leq\; & K \exp\left(K h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2\right)\left(1 + |x| + \frac{\delta}{h}\right)\left\{\frac{\delta}{h} + h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2\left(1 + |x| + \frac{\delta}{h}\right)\right. \\
& \left. + \frac{1}{Rh}\exp(K h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_4)\left(1 + |x|^2 + (\frac{\delta}{h})^2\right)\right\}.
\end{aligned}$$

Cette erreur dépend essentiellement de trois termes : $\frac{\delta}{h}$, $\frac{1}{Rh}$ et $h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2$, et pour avoir une bonne approximation les paramètres de discrétisation doivent donc être choisis de façon à ce que ces termes soient négligeables.

• On propose également une approche par quantification marginale dans le cas particulier où la diffusion n'est pas contrôlée, inspirée du schéma numérique de Bally et Pagès [5] pour le problème d'arrêt optimal. Pour chaque pas de temps $k = 0, \ldots, m$, on se donne une grille $\Gamma_k = \{x^1_k, \ldots, x^{N_k}_k\}$ et on considère la quantification des marginales du schéma d'Euler : $\hat{X}^{(2)}_k = \mathrm{Proj}_k(\bar{X}_{t_k})$. La fonction valeur est alors approximée par $\hat{v}^{(2)}$ définie récursivement par un algorithme de descente d'arbre :

$$\begin{aligned}
\hat{v}_i^{(2)}(t_m, x) &= g_i(x), \quad x \in \Gamma_m \\
\hat{v}_i^{(2)}(t_k, x_k^l) &= \max_{j \in \mathbb{I}_q} \Big[ \sum_{l'=1}^{N_{k+1}} \pi_k^{ll'} \hat{v}_j^{(2)}(t_{k+1}, x_{k+1}^{l'}) + h f_j(x_k^l) - c_{ij}(x_k^l) \Big], \quad l = 1, \ldots, N_k, \\
& \qquad\qquad k = 0, \ldots, m-1,
\end{aligned}$$

où

$$\pi_k^{ll'} = \mathbb{P}[\hat{X}_{k+1} = x_{k+1}^{l'} | \hat{X}_k = x_k^l].$$

On montre alors l'estimation suivante sur la fonction valeur en fonction de l'erreur de quantification :

$$\max_{i \in \mathbb{I}_q} \big| \bar{v}_i(0, x_0) - \hat{v}_0^i(x_0) \big| \leq K(1 + |x_0|) \sum_{k=1}^{m} \big\| \bar{X}_{t_k} - \hat{X}_k \big\|_2.$$

Enfin dans la dernière partie de ce chapitre, on étudie des exemples numériques comparant les résultats de nos schémas numériques aux formules explicites obtenues par Ly Vath et Pham [51].

Cette partie est tirée d'un article réalisé en collaboration avec Idris Kharroubi et Huyên Pham [28].

# General Introduction

Abstract : This thesis is divided into two parts that may be read independently.

The first part is about the mathematical modelling of liquidity risk. The aspect of illiquidity studied here is the constraint on the trading dates, meaning that in opposition to the classical models where investors may trade continuously, we assume that trading is only possible at discrete random times. We then use optimal control techniques (dynamic programming and Hamilton-Jacobi-Bellman equations) to identify the value functions and optimal investment strategies under these constraints. The first chapter focuses on a utility maximisation problem in finite horizon, in a framework inspired by energy markets. In the second chapter we study an illiquid market with regime-switching, and in the third chapter we consider a market in which the agent has the possibility to invest in a liquid asset and an illiquid asset which are correlated.

In the second part we present probabilistic quantization methods to solve numerically an optimal switching problem. We first consider a discrete time approximation of our problem and prove a convergence rate. Then we propose two numerical quantization methods : a markovian approach where we quantize the gaussian in the Euler scheme, and, in the case where the underlying diffusion is not controlled, a marginal quantization approach inspired by numerical methods for the optimal stopping problem.

## 0.1 First part : Liquidity risk modelling

Liquidity risk is one of the most important risks faced by the finance industry, especially during periods of financial crisis when the markets feel various kinds of illiquidity. Roughly speaking, liquidity risk may be defined as the risk associated to the impossibility of the agent to buy or sell assets immediately and/or at each time, as well as to evaluate at each time the value of his portfolio.

In the seminal works of Merton on portfolio management and Black and Scholes on option pricing, as well as in the majority of the following litterature in mathematical finance, it is assumed that investors can buy and sell continuously, with immediate rebalancing, without paying costs for trading and without affecting the assets' price. It is clear that such point of view is quite unrealistic in practice, as investors face various types of assets' illiquidity. In the last decade there have been various approaches to include these types of market's illiquidity, formalize and quantify the different aspects of this financial risk.

A first approach was to study illiquidity in terms of transaction costs, see for instance Kabanov and Safarian's book [38] for a recent overview of the theory. In this context frequent trading of assets may induce potentially high costs, but the investor may buy or sell continuously.

In another direction, the market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying assets. This factor has been studied by Cetin, Jarrow et Protter [14], Bank et Baum [7] for arbitrage and option pricing, Schied and Schöneborn [67] for a portfolio liquidation problem. Ly Vath, Mnif et Pham [50] consider a model combining large investor effects and transaction costs in a portfolio management context.

Another aspect of illiquidity is the one due to the delay in the execution of the trading orders. Trading orders are actually not executed immediately, requiring time to reach the market (see e.g. Subramanian and Jarrow [70]). This time lag has an impact on the dynamics of the portfolio, and consequently they are expected to lead to different investor's choices. The problem of execution delay has been investigated in the context of stochastic impulse control in Øksendal and Sulem [56] for special kind of dynamics and in Bruder and Pham [11] in a quite general setting.

The type of illiquidity that we study in this thesis is the restriction on trading/observation times. The classical assumption of continuous trading is irrealistic in the case of illiquid markets where, because of the low volume of buy/sell orders, a relatively long period may take place between successive trading possibilities. Rogers [65] considers an agent that can only rebalance his portfolio at fixed intervals and shows that the resulting loss is relatively small compared to the uncertainty on the parameters of the asset. Rogers and Zane [66], Matsumoto [53] consider a model where the successive trading dates are given by the jump time of a Poisson process with constant intensity $\lambda$, and study the asymptotic behavior when $\lambda$ is large. In the same framework, Pham and Tankov [61, 62] study an investment/consumption problem over infinite horizon, characterize the value function as the unique (viscosity) solution of the HJB equation and propose a numerical scheme to compute it. Let us also mention Bayraktar and Ludkovski [8] that study in a similar context a portfolio liquidation problem. We extend the approach of these papers over three different problems developped below.

### 0.1.1 Optimal investment on finite horizon with random discrete order flow in illiquid markets

In the first chapter we study a utility maximisation problem over finite horizon in an illiquid market where the agent can only observe and trade the asset at discrete random times. An important feature of our model is that the arrival rate of these dates is close to infinity when the time horizon $T$ is close. This is a natural assumption to modelize what is for instance observed in the case of forward contracts in energy markets : because of the physical nature of the underlying asset, trading activity is really low far from the delivery, and is higher near the delivery.

A similar problem has been studied by Matsumoto [53] in the cas of logarithmic utility. The main differences with our approach, in addition to the fact that we consider a less restrictive class of utility functions and price processes, are that in [53] liquidity is constant in time and the asset price is observed continuously.

We study a market consisting of a riskless asset (assumed constant) and an illiquid risky asset with price process $(S_t)_{0\leq t\leq T}$. The dates at which the agent can observe the price $S_t$ and trade the illiquid asset are given by a sequence of stopping times $(\tau_n)_{n\geq 0}$ independent of $S$.

We assume that $S$ follows log-Levy type dynamics, more precisely $S_t = \mathcal{E}(L)_t$, where $\mathcal{E}$ denotes the stochastic exponential and

$$L_t = \int_0^t b(u)\,du + \int_0^t c(u)\,dB_u + \int_0^t \int_{-1}^{\infty} y(\mu(dt,dy) - \nu(dt,dy)), \ \ 0 \leq t \leq T,$$

is a semimartingale with independant increments and jumps $\Delta L_t > -1$. We further assume natural integrability conditions on the deterministic characteristics $(b, c, \nu)$ and a no-arbitrage condition. We denote by $Z_{t,s} = \frac{S_s - S_t}{S_t}$ the return between $s$ and $t$ and $p(t,s,dz) = \mathbb{P}[Z_{t,s} \in dz]$ its distribution.

The dates $(\tau_n)$ are given by the jump times of an inhomogeneous Poisson process $(N_t)_{0\leq t\leq T}$

of deterministic intensity $\lambda(t)$. We make the following assumption for $\lambda$ :

$$\int_0^t \lambda(u)du \;<\; \infty,\ \forall\ 0 \leq t < T \text{ and } \int_0^T \lambda(u)du \;=\; \infty.$$

Under this condition the sequence of stopping times $(\tau_n)$ satisfies almost surely

$$\lim_{n\to\infty} \tau_n = T.$$

We define the discrete observation filtration $\mathcal{F}_n = \sigma\Big\{(\tau_k, Z_{\tau_{k-1},\tau_k}) \,:\, 1 \leq k \leq n\Big\}$. An investment strategy is then a sequence $(\alpha_n)$, where $\alpha_n$, $\mathcal{F}_n$-mesurable, represents the amount held in the risky asset over the period $(\tau_n, \tau_{n+1}]$. The wealth process $(X_{\tau_n})$ associated to a strategy $\alpha$ verifies

$$X_{\tau_{n+1}} \;=\; X_{\tau_n} + \alpha_n Z_{\tau_n,\tau_{n+1}}.$$

In the sequel we fix an initial capital $X_0 > 0$ and we restrict our attention to the set $\mathcal{A}$ of admissible strategies such that the investor's wealth is nonnegative at all times : $X_{\tau_n} \geq 0$, $n \geq 0$. Given our assumptions on $S$, $Z_{\tau_n,\tau_{n+1}}$ has for support $(-1, +\infty)$ conditionally on $\mathcal{F}_n$, and it is easy to see that this admissibility constraint is equivalent to a no-shortselling constraint (both on the riskless and risky assets).

Given a utility function $U$ satisfying some general conditions, we study the following control problem :

$$V_0 \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T)].$$

We then solve this optimization problem, i.e. we characterize $V_0$ and the corresponding optimal strategy $\hat{\alpha}$. We use a direct Dynamic Programming approach : we formally write down the Dynamic Programming Equation (DPE) for our problem, then by analytical arguments we prove the existence of a solution to this DPE, and we conclude by a verification argument.

In our context the DPE is written as a fixed-point problem (with a terminal condition)

$$\begin{cases} \mathcal{L}v = v \\ \lim_{t\nearrow T, x'\to x} v(t, x') = U(x), \end{cases} \tag{0.1.1}$$

where given a function $w$ verifying appropriate growth conditions, $\mathcal{L}w$ is defined by :

$$\mathcal{L}w(t,x) = \sup_{\pi\in[0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} w(s, x(1+\pi z)) p(t,s,dz) ds.$$

To prove the existence of a solution to (0.1.1), we follow a value iteration approach, standard in the case of discrete problems (see also [23]). We consider the sequence of functions $(v_m)_{m\geq 0}$ defined inductively by :

$$v_0 = U,$$
$$v_{m+1} = \mathcal{L}v_m.$$

We then show that:

- $v_m$ converges to a function $v^*$, solution to (0.1.1).
- $V_0 = v^*(0, X_0)$, and the optimal strategy $\hat{\alpha}$ is given by :

$$\hat{\alpha}_n = \hat{\pi}(\tau_n, \hat{X}_{\tau_n})\hat{X}_{\tau_n}, \quad n \geq 0,$$

  where $\hat{\pi}$ is defined by

$$\hat{\pi}(t,x) \in \arg\max_{\pi\in[0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v^*(s, x(1+\pi z)) p(t,s,dz) ds.$$

Moreover, $v_m$ corresponds to the following control problem :

$$v_m(0, X_0) = \sup_{\alpha\in\mathcal{A}_m} \mathbb{E}[U(X_T)],$$

where $\mathcal{A}_m$ is the set of admissible strategies with no investment in the risky asset after the $m$-th trading date, i.e. $\alpha_n = 0$ for $n \geq m$.

In the last part of this chapter we focus on the convergence of our problem to the standard continuous time trading problem. Indeed, when the intensity $\lambda$ of arrival of trading dates is very large at all times, we expect the corresponding value $V_0^\lambda$ to be very close to the one where the agent may trade continuously, taking into account the no-shortselling constraint.

We thus define

$$V_0^M = \sup_{\pi\in\mathcal{D}(S)} \mathbb{E}\left[U(X_T^\pi)\right],$$

where $\mathcal{D}(S)$ is the set of continous-time trading strategies in the asset $S$ with no shortselling.

We obtain the following result : given a sequence of intensity functions $\lambda_k$ such that

$$\lambda_k(t) \to \infty \quad \text{as } k \to \infty, \ \forall t \in [0, T],$$

we have the convergence

$$V_0^{\lambda_k} \to V_0^M \quad \text{as } k \to \infty,$$

This chapter is based on a paper written in collaboration with Huyên Pham et Mihai Sîrbu [29], published in *International Journal of Theoretical and Applied Finance.*

### 0.1.2 Optimal investment/consumption in an illiquid market with regime switching

In the first papers studying liquidity risk models with discrete trading times (for instance [66], [53], [61]), the trading frequency is constant in time and independent from the assets' price. In practice the liquidity of the market exhibits a cyclical pattern, following both random and deterministic fluctuations at various time scales, and the liquidity of the market is correlated to price dynamics. In this chapter we study a simple model of an illiquid market with regime switching, each regime having different liquidity and price dynamics.

Regime-switching models and their applications to finance have been studied in several works, see e.g. the papers [69],[72] or from a statistical viewpoint the thesis [55]. More recently in a liquidity risk concept, the papers [21] and [49] study a market undergoing liquidity shocks during which trading activity is completely stopped.

We consider a market going through regime switches, modelled by a Markov chain $(I_t)$ with finite state space $\mathbb{I}_d = \{1, \ldots, d\}$ and infinitesimal generator $Q = (q_{ij})$. The market is composed of a riskless asset assumed constant and a risky asset with price process $S$. The agent can only trade this asset at times $(\tau_n)_{n \geq 0}$, corresponding to the jump times of a Cox process $(N_t)_{t \geq 0}$ with intensity $\lambda_{I_t}$. In other words, at each market regime $i$ corresponds an intensity $\lambda_i$ of trading times arrival. It is important to note that unlike in the model of the first chapter or the paper [61], the constraint is only on the trading times, and that the price $S_t$ is observed continuously by the agent.

In each regime, the price $S_t$ follows geometric Brownian motion dynamics :when $I_t = i$,

$$dS_t = S_t(b_i dt + \sigma_i dW_t),$$

where $W$ is a standard Brownian motion independent of $(I, N)$, and $b_i, \sigma_i,\ i = 1, \ldots, d$ are constant.

Furthermore, we assume that the price jumps at each regime change : when $I$ goes from regime $i$ to regime $j$ at time $t$,

$$\Delta S_t = -S_{t-}\gamma_{ij},$$

where the $\gamma_{ij} < 1$ are given constants.

We consider an agent investing in this market and consuming continuously; a strategy is then a pair of predictable processes $(c, \zeta)$ where $c$ is the consumption and $\zeta$ the investment strategy. Denoting by $(X_t, Y_t)$ the state variables corresponding to the wealth invested respectively in the riskless and the risky asset, we have the following dynamics :

$$\begin{aligned} dX_t &= -c_t dt - \zeta_t dN_t, \\ dY_t &= Y_{t-}\frac{dS_t}{S_{t-}} + \zeta_t dN_t. \end{aligned}$$

Starting from regime $i$ and initial wealths $x$, $y$ we consider the admissible strategies (denoted by $\mathcal{A}_i(x, y)$) such that the total wealth $R_t := X_t + Y_t$ is nonnegative at any trading time $\tau_n$. As in the previous chapter, this is equivalent to a no-shortselling constraint : $(c, \zeta) \in \mathcal{A}_i(x, y)$ iff $X_t, Y_t \geq 0$ a.s. for all $t$.

Given a utility function $U$ satisfying standard conditions and a growth condition, and for a discount factor $\rho > 0$ we consider the investment/consumption problem over an infinite horizon :

$$v_i(x, y) = \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right]. \quad (x, y) \in \mathbb{R}_+^2,$$

We also introduce the function

$$\hat{v}_i(r) = \sup_{x\in[0,r]} v_i(x, r - x), \quad r \geq 0,$$

corresponding to the value obtained by rebalancing optimally an initial wealth $r$ between the riskless and the risky asset. In other words, $v_i$ is the value function between two trading times, while $\tilde{v}_i$ is the value function at a trading time.

In order to solve this control problem, we characterize the value functions $v_i$ as solutions to the corresponding Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation is a partial differential equation, infinitesimal equivalent to Bellman's dynamic programming principle (see the books [24] an [60] for an introduction to markovian control in continuous time). In our case this equation is the following system :

$$\begin{aligned} \rho v_i - b_i y \frac{\partial v_i}{\partial y} \; - \; \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2} \; - \; \sup_{c \geq 0} \left[ U(c) - c\frac{\partial v_i}{\partial x} \right] & \\ - \sum_{j \neq i} q_{ij} \left[ v_j\Big(x, y(1-\gamma_{ij})\Big) - v_i(x,y) \right] & \\ - \lambda_i \big[ \sup_{-y \leq \zeta \leq x} v_i(x-\zeta, y+\zeta) - v_i(x,y) \big] & = 0. \end{aligned} \tag{0.1.2}$$

on $\mathbb{I}_d \times (0,\infty) \times \mathbb{R}_+$, with boundary conditions :

$$v_i(0,0) \;=\; 0 \tag{0.1.3}$$

$$v_i(0,y) \;=\; \mathbb{E}_i \left[ \sup_{0 \leq \zeta \leq y \frac{S_{\tau_1}}{S_0}} v_{I_{\tau_1}} \left( \zeta, y\frac{S_{\tau_1}}{S_0} - \zeta \right) \right]. \tag{0.1.4}$$

It is well known that in the general case the functions $v_i$ are not smooth enough to interpret this equation in the classical sense, and a weaker class of solutions, namely viscosity solutions, is required (see e.g. [16]).

Using a dynamic programming principle we show that $v_i$ is a viscosity solution to (0.1.2), and we further prove a comparison principle for this equation. Our functions $(v_i)$ are thus characterized as the unique solution to the system (0.1.2) with boundary conditions (0.1.3)-(0.1.4).

We then focus on the existence and characterization of optimal solutions to our control problem. In the general case of viscosity solutions, there are some verification results (see [30]-[31]), but their assumptions are too restrictive to be applied here. We thus try to find some conditions under which the functions $v_i$ are sufficiently differentiable to apply classical verification results. Since our equation is degenerate (the second order term only contains the derivative with respect to $y$), in the case of a general utility function $U$ we cannot hope to apply standard existence results for elliptic PDEs.

However, in the special case of power utility $U(c) = \frac{c^p}{p}$, we can reduce the dimension of the state space. Indeed, applying the change of variables

$$\begin{aligned} r &= x + y, \\ z &= \frac{y}{x+y}, \end{aligned}$$

the value function may be rewritten as

$$v_i(x,y) = U(r)\varphi_i(z).$$

We are thus led to solving a differential equation in $z$, which this time satisfies a uniform ellipticity condition.

In this particular case, we are then able to prove the regularity of the function $v_i$, and we deduce the existence of optimal controls characterized in feedback form.

Finally, we look into the numerical resolution of the equation (0.1.2). The main difficulty comes from the nonlocal terms, which may be avoided by an iterative procedure. We define $v_0 = 0$ and inductively, $v_{n+1}$ is defined as the unique (viscosity) solution to

$$\begin{aligned} (\rho - q_{ii} + \lambda_i)v_i^{n+1} - b_i y \frac{\partial v_i^{n+1}}{\partial y} \; - \; \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 v_i^{n+1}}{\partial y^2} \; - \; \sup_{c \geq 0}\left[U(c) - c\frac{\partial v_i^{n+1}}{\partial x}\right] \\ = \sum_{j \neq i} q_{ij} v_j^n \Big(x, y(1-\gamma_{ij})\Big) + \lambda_i \sup_{-y \leq \zeta \leq x} v_i^n(x-\zeta, y+\zeta) \end{aligned}$$

with appropriate boundary conditions.

As in the first chapter $v_n$ may be interpreted as the value function for a control problem :

$$v_i^n(x,y) = \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho t} U(c_t)dt\right],$$

where $\theta_n$ is the $n$-th time where there is either a trading date or a regime change. Using this representations, whe show that $v_n$ converges to $v$ and that the convergence speed is exponential.

We illustrate our results with numerical tests for markets with 1 or 2 regimes. In the single-regime case, we compare the results to those of [61], where the agent observes the risky asset only at the dates $(\tau_n)$. In the case of a market with 2 regimes, we observe that typically the existence of several regimes increases the "cost of illiquidity" for the agent.

This chapter is based on a paper written in collaboration with Fausto Gozzi and Huyên Pham [27].

### 0.1.3 Optimal investment/consumption in a market with liquid and illiquid assets

Tha majority of works on liuidity risk focus on an agent investing exclusively in an illiquid asset. However, in practice it is common to have several correlated tradable assets with different liquidity. For instance an index fund over some given financial market will be usually much more liquid than the individual tracked assets, while sharing a positive correlation with those assets. An investor in this market will then have the possibility of hedging his exposure in the less liquid assets by investing in the index and rebalancing his position frequently. Tebaldi and Schwartz [68], Longstaff [48] consider a market constituted of a liquid asset that can be traded continuously, and an illiquid asset that may only be traded at the initial time and is liquidated at a terminal date. Following the line of the latter papers, here we also consider a market composed by a liquid asset and an illiquid one, but we take a less restrictive approach assuming that the illiquid asset may be traded at discrete random times.

To this regard, we have to mention the recent paper by Ang, Papanikolaou and Westerfield [2] that studies a very similar problem to the one studied here. However, we stress that our results are different for two reasons. First, they consider utility functions of CRRA type with risk aversion parameter $\gamma \geq 1$, while we study the problem for a different class of functions, not assumed of CRRA type. Second, they assume that the agent is able to observe the illiquid asset's price continuously, while in our case observation is restricted to the trading dates. We believe this is a more natural assumption, as in practice trading possibilities and observation of the price coincide via the arrival of buy/sell orders on the market.

We consider a market consisting of a riskless asset assumed constant and two risky assets :

- a liquid asset that may be traded continuously, with price process $L$,
- an illiquid asset with price process $I$, that can only be traded and observed at random times $(\tau_n)$ corresponding to the jump times of a Poisson process $N$ with intensity $\lambda$.

We assume that $L$ and $I$ follow Black-Scholes dynamics :

$$\begin{aligned} dL_t &= L_t(b_L dt + \sigma_L dW_t), \\ dI_t &= I_t(b_I dt + \sigma_I(\rho dW_t + \sqrt{1-\rho^2} dB_t), \end{aligned}$$

where $W$ and $B$ are independent Brownian motions (independent of $N$), and $\rho \in (-1, 1)$ is the correlation coefficient.

We define the observation filtration for the agent :

$$\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}; \quad \mathcal{G}_t = \sigma(\tau_n, I_{\tau_n}; \tau_n \leq t) \vee \mathcal{F}_t^W \vee \mathcal{N},$$

where $\mathcal{F}^W$ is the filtration generated by $W$ (or by $L$) and $\mathcal{N}$ is the $\sigma$-algebra generated by the $\mathbb{P}$-null sets.

A consumption/investment strategy on this market is then a triple $(c, \pi, \alpha)$, where:

- $c = (c_t)$ is a $\mathbb{G}$-predictable process corresponding to the consumption rate,
- $\pi = (\pi_t)$ also $\mathbb{G}$-predictable is the amount invested in the liquid asset,
- $\alpha = (\alpha_k)$ is a sequence of $\mathcal{G}_{\tau_k}$-measurable random variables, representing the amount invested in the illiquid asset at time $\tau_k$.

Given a initial wealth $r$, we restrict our attention to the set $\mathcal{A}(r)$ of strategies satisfying an admissibility condition, which like in the previous chapters reduces to a no-shortselling constraint. We are then given a utility function $U$ and a discount factor $\beta > 0$, and consider the control problem :

$$V(r) = \sup_{(c,\pi,\alpha) \in \mathcal{A}(r)} \mathbb{E} \left[ \int_0^\infty e^{-\beta s} U(c_s) ds \right].$$

This control problem is a nonstandard, mixed discrete/continuous (due to the nature of our observation filtration $\mathcal{G}$) problem . We then follow the same approach as Pham and Tankov in [61] : by dynamic programming we are reduced to study the problem between two trading times, and we show that it is equivalent to a standard control problem.

In our context, the dynamic programming principle is written as :

$$V(r) \quad = \quad \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right],$$

where

$$R_{\tau_1} \quad = \quad r + \int_0^{\tau_1}(-c_s ds + \pi_s \frac{dL_s}{L_s}) + \alpha_0 \frac{I_{\tau_1} - I_0}{I_0}$$

is the wealth at time $\tau_1$.

We will rewrite the right-hand side of the previous equality as solution to a standard stochastic control problem for the filtration $\mathcal{F}^W$.

First, noting that only this term only takes into account the strategy before $\tau_1$, we may rewrite this equality as

$$V(r) \quad = \quad \sup_{a\le r} \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right],$$

where given an initial investment $x$ in liquid wealth, $\mathcal{A}_0(x)$ is the set of $\mathcal{F}^W$-predictable strategies satisfying some admissibility conditions.

We then decompose the illiquid asset price as $I_t = E_t J_t$, where

$$\frac{dE_t}{E_t} = \frac{\rho\sigma_I}{\sigma_L}\frac{dL_t}{L_t} = (\rho b_I \frac{\sigma_I}{\sigma_L}dt + \rho\sigma_I dW_t),$$
$$\frac{dJ_t}{J_t} = (b_I - \rho b_I \frac{\sigma_I}{\sigma_L})dt + \sigma_I\sqrt{1-\rho^2}dB_t.$$

Note that $(E_t)$ is $\mathcal{F}^W$-adapted, while $(J_t)$ is independent of $\mathcal{F}^W$.

Given an initial wealth $r = x + y$ split initially between an amount $x$ of liquid wealth and an amount $y$ in the illiquid asset, and a strategy $(c,\pi) \in \mathcal{A}_0(x)$, we consider the state variables $X$, $Y$ defined by :

$$X_t^{x,c,\pi} \quad = \quad x + \int_0^t(-c_s ds + \pi_s \frac{dL_s}{L_s}),$$
$$Y_t^y \quad = \quad yE_t.$$

In other words, $X_s$ corresponds to the liquid wealth, while $Y_s$ corresponds to the wealth initially invested in $I$, modulated by the information brought by the variations of the liquid asset's price since the initial date.

Defining the operator $G$ by

$$G[w](t,x,y) \quad = \quad \mathbb{E}\left[w(x+yJ_t)\right],$$

we finally obtain

$$V(r) \quad = \sup_{0\le a\le r}\ \sup_{(c,\pi)\in\mathcal{A}_0(x)} \mathbb{E}\int_0^\infty e^{-(\beta+\lambda)s}\left(U(c_s)+\lambda G[V]\left(s,X_s^{x,\pi,c},Y_s^y\right)\right)ds.$$

This is a standard (time-inhomogeneous) stochastic control problem. We then define the dynamic value function $\hat{V}$ by :

$$\hat{V}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s)+\lambda G[V]\left(s,X_s^{t,x,\pi,c},Y_s^{t,y}\right)\right)ds.$$

Notice that $\hat{V}$ and $V$ are connected by

$$V(r) \quad = \quad \left[\mathcal{H}\hat{V}\right](r) \quad := \quad \sup_{0\le x\le r} \hat{V}(0,x,r-y).$$

Computing $V$ is thus equivalent to computing $\hat{V}$. To do so, we use the standard approach by HJB equations. The HJB equation for our problem is written as

$$-\hat{V}_t+(\beta+\lambda)\hat{V}-\lambda G[\mathcal{H}\hat{V}](t,x,y)-\sup_{c\ge 0,\pi\in\mathbb{R}} H_{cv}(y,D_{(x,y)}\hat{V},D^2_{(x,y)}\hat{V};c,\pi)=0, \qquad (0.1.5)$$

where the hamiltonian $H_{cv}$ is defined by

$$H_{cv}(y,p,A;c,\pi)=\left[U(c)+(\pi b_L-c)p_1+\frac{\rho b_L\sigma_I}{\sigma_L}yp_2+\frac{\sigma_L^2\pi^2}{2}A_{11}+\pi\rho\sigma_I\sigma_L yA_{12}+\rho^2\frac{\sigma_I^2}{2}y^2A_{22}\right].$$

As in the previous chapter, we then show that $\hat{V}$ is the unique viscosity solution to (0.1.5) on $[0,+\infty)\times(0,+\infty)\times\mathbb{R}_+$, satisfying the boundary condition

$$\hat{v}(t,0,y)=\mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G[\mathcal{H}\hat{v}](s,0,\tilde{Y}_s^{t,y})ds$$

and an appropriate growth condition.

This characterization allows us to compute $\hat{V}$ numerically.

Like in the previous chapters, we follow an iterative method : starting from $\hat{V}^0 = 0$, we define recursively $\hat{V}^{n+1}$ as the solution to (0.1.5) where the nonlocal term is replaced by

$\lambda G[\mathcal{H}\hat{V}^n](t,x,y)$. We then obtain similar results as in chapter 2 : we show that $\hat{V}^n$ corresponds to the control problem in which the agent only consumes up to time $\tau_n$, and $\hat{V}^n$ converges to $\hat{V}$ exponentially in $n$.

Moreover, since the PDE we need to solve is over an infinite horizon, in practice we consider an approximate solution $V^{n,T}$ for a fixed horizon $T$. We prove that for $T$ chosen large enough, $V^{n,T}$ approximates $V^n$ with arbitrary small precision (uniformly in $n$).

Finally we present some numerical illustrations to our results. We fix the parameters $b_L, \sigma_L, b_I, \sigma_I$, and we observe how the value function and the optimal strategies change when $\lambda$ and $\rho$ vary.

This chapter is based on a paper written in collaboration with Salvatore Federico.

## 0.2 Second part : Time discretization and quantization methods for optimal multiple switching problem

In this second part, we present numerical schemes for the optimal switching problem. Let us first recall the definition of this problem.

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and a finite set of regimes $\mathbb{I}_q = \{1, \ldots, q\}$. A switching control is then a sequence $(\tau_n, \iota_n)_{n\geq 0}$, where $(\tau_n)$ is a nondecreasing sequence of stopping times and $(\iota_n)$ is a sequence of $\mathcal{F}_{\tau_n}$-measurables r.v.s valued in $\mathbb{I}_q$. To each $\alpha$ is associated the controlled diffusion

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

where $W$ is a standard Brownian motion in $\mathbb{R}^d$, and $\alpha_t = \iota_n$ sur $[\tau_n, \tau_{n+1})$. The control problem we consider is then :

$$v_i(t,x) \quad = \quad \sup_{\alpha \in \mathcal{A}_{t,i}} \mathbb{E}\Big[\int_t^T f(X_t, \alpha_t)dt + g(X_T, \alpha_T) - \sum_{\tau_n \leq T} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\Big].$$

Optimal switching has numerous applications, in particular in finance, see e.g. chapter 5 in the book [60]. From a numerical point of view, these problems are usually solved by a discretization in time, and a backward inductive procedure that requires the computation of conditional expectations. Regarding the discretization error, in the case where the diffusion is not controlled, some results have been obtained by Chassagneux, Elie and Kharroubi [13], using methods of Backward Stochastic Differential Equations (BSDEs) with oblique reflection (see Hamadène and Zhang [34], Hu and Tang [35] for the properties of these BSDEs). For the computation of conditional expectations, several methods have been developed for the optimal stopping problem : Malliavin calculus techniques (Lions and Regnier [47], Bouchard, Ekeland and Touzi [10]), Longstaff-Schwarz type regressions (Clément, Lamberton and Protter [15]), or quantization methods (Bally-Pagés [5]).

In this chapter we present numerical schemes based on this latter approach. Let us recall that optimal quantization consists in approximating a random variable $X$ by a quantizer $\hat{X}$ with finite support, in such a way that the quantization error $\left\|X - \hat{X}\right\|_p$ is minimized. See for instance the book by Graf and Luschgy [32] for an introduction to quantization theory. In the last decade, optimal quantization has been intensively studied in numerical probability, and in particular in finance, see the paper [57] for an overview.

In our case, we assume that all the functions we consider are Lipschitz in the space variable, and that the cost function satisfies a natural "triangular condition".

We first study the impact of time discretization on the value function of our problem. Given a time step $h$, we consider the function $v^h$ defined by :

$$v_i^h(t_k, x) \quad = \quad \sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})h + g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1}, \iota_n)\Big],$$

where $\mathcal{A}^h_{t_k,i}$ is the set of controls such that the sequence $(\tau_n)$ takes its values in $\{\ell h, \ell = k, \ldots, m\}$. We show that the convergence rate of $v^h$ to $v$ is in $h^{1/2-\varepsilon}$, for any $\varepsilon > 0$. This extends the results obtained by Chassagneux, Elie and Kharroubi [13] in the case where the diffusion is uncontrolled. When the switching cost $c$ does not depend on the process $X$, we recover the same convergence rate in $h^{1/2}$ as in the case of optimal stopping (see Lamberton [45]).

Since in practice the diffusion $X_s$ may not always be simulated, we consider instead the Euler scheme $\bar{X}_s$ defined inductively by :

$$\begin{aligned}
\bar{X}_{t_k} \quad &= \quad x, \\
\bar{X}_{t_{\ell+1}} \quad &= \quad \bar{X}_{t_\ell} + b(\bar{X}_{t_\ell}, \alpha_{t_\ell})h + \sigma(\bar{X}_{t_\ell}, \alpha_{t_\ell})\sqrt{h}\vartheta_{\ell+1}, \quad k \le \ell \le m-1,
\end{aligned}$$

where $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$ has $\mathcal{N}(0, I_d)$ law. We then prove that the corresponding value function $\bar{v}^h$ converges to $v^h$ in $h^{1/2}$.

The main difficulty in these proofs comes from the regime switching term, the number of regime switches being a priori unbounded. Using some tools from BSDE theory, we prove some estimates for the moments of this number in the case of an optimal strategy.

We then study two quantization schemes :

• The first scheme follows an approach by markovian quantization in the vein of Pagès, Pham et Printems [58]. We consider a space discretization grid $\mathbb{X} = (\delta/d)\mathbb{Z}^d \cap B(0,R)$. We approximate the Euler scheme in the following way : the gaussian $\vartheta_{\ell+1}$ is replaced by its quantizer $\hat{\vartheta}_{\ell+1}$, and the obtained result is then projected on the grid $\mathbb{X}$. Hence we consider the process $\hat{X}^{(1)}$ defined by:

$$\begin{aligned}
\hat{X}^{(1)}_{t_k} &= x, \\
\hat{X}^{(1)}_{t_{\ell+1}} &= \mathrm{Proj}_{\mathbb{X}}\left(\hat{X}^{(1)}_{t_\ell} + b(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})h + \sigma(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})\sqrt{h}\hat{\vartheta}_{\ell+1}\right), \quad k \leq \ell \leq m-1,
\end{aligned}$$

and we defined the associated value function $\hat{v}^{(1)}$ :

$$\hat{v}^{(1)}_i(t_k, x) = \sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\hat{X}^{(1)}_{t_\ell}, \alpha_{t_\ell})h + g(\hat{X}^{(1)}_{t_m}, \alpha_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\hat{X}^{(1)}_{\tau_n}, \iota_{n-1}, \iota_n)\Big].$$

This function can be computed explicitly by a dynamic programming induction :

$$\begin{aligned}
\hat{v}_i(t_m, x) &= g_i(x), \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q \\
\hat{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big[\sum_{l=1}^{N} \pi_l \, \hat{v}_j\Big(t_{k+1}, \mathrm{Proj}_{\mathbb{X}}\left(x + b(x,j)h + \sigma(x,j)\sqrt{h}w_l\right)\Big) + f_j(x)h - c_{ij}(x)\Big], \\
&\qquad (x,i) \in \mathbb{X} \times \mathbb{I}_q, \quad 0 \leq k \leq m-1,
\end{aligned}$$

where $(w_l)_{1 \leq l \leq N}$ is a quantization grid for the gaussian law, with weights $(\pi_l)_{1 \leq l \leq N}$.

Following a similar method of proof as [58] (the main difference being that in our case the volatility is not assumed bounded), we get the following result for the convergence of the function $\hat{v}^{(1)}$ :

$$\begin{aligned}
\left|\bar{v}_i(t_k, x) - \hat{v}^{(1)}_i(t_k, x)\right| \leq\; & K \exp\left(Kh^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2\right)\left(1 + |x| + \frac{\delta}{h}\right)\left\{\frac{\delta}{h} + h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2\left(1 + |x| + \frac{\delta}{h}\right)\right. \\
& \left. + \frac{1}{Rh}\exp(Kh^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_4)\left(1 + |x|^2 + (\frac{\delta}{h})^2\right)\right\}.
\end{aligned}$$

This error is mainly a function of three terms : $\frac{\delta}{h}$, $\frac{1}{Rh}$ et $h^{-1/2}\left\|\vartheta - \hat{\vartheta}\right\|_2$, and to obtain a good approximation the discretization parameters must be chosen so that these terms are small.

• In the special case where the diffusion is not controlled, we also present a marginal quantization approach, inspired by the numerical scheme in Bally-Pagès [5] for the optimal stopping problem. At each time step $k = 0, \ldots, m$ is given a grid $\Gamma_k = \{x_k^1, \ldots, x_k^{N_k}\}$, and we consider the quantization of the marginals of the Euler scheme : $\hat{X}_k^{(2)} = \mathrm{Proj}_k(\bar{X}_{t_k})$. The value function is then approximated by $\hat{v}^{(2)}$ defined inductively by a tree descent algorithm :

$$\begin{aligned}
\hat{v}_i^{(2)}(t_m, x) &= g_i(x), \quad x \in \Gamma_m \\
\hat{v}_i^{(2)}(t_k, x_k^l) &= \max_{j \in \mathbb{I}_q} \big[ \sum_{l'=1}^{N_{k+1}} \pi_k^{ll'} \hat{v}_j^{(2)}(t_{k+1}, x_{k+1}^{l'}) + h f_j(x_k^l) - c_{ij}(x_k^l) \big], \quad l = 1, \ldots, N_k, \\
& \qquad k = 0, \ldots, m-1,
\end{aligned}$$

where

$$\pi_k^{ll'} = \mathbb{P}[\hat{X}_{k+1} = x_{k+1}^{l'} | \hat{X}_k = x_k^l].$$

We then show the following estimate on this value function depending on the quantization error :

$$\max_{i \in \mathbb{I}_q} \left| \bar{v}_i(0, x_0) - \hat{v}_0^i(x_0) \right| \leq K(1 + |x_0|) \sum_{k=1}^{m} \left\| \bar{X}_{t_k} - \hat{X}_k \right\|_2.$$

Finally, in the last part of this chapter we present some numerical tests comparing the results obtained by our numerical schemes to the explicit formulae obtained by Ly Vath and Pham [51].

This part is based on a paper written in collaboration with Idris Kharroubi et Huyên Pham [28].

# Part I

# Liquidity risk modelling

# Chapter 1

# Optimal investment on finite horizon with random discrete order flow in illiquid markets

Abstract : We study the problem of optimal portfolio selection in an illiquid market with discrete order flow. In this market, bids and offers are not available at any time but trading occurs more frequently near a terminal horizon. The investor can observe and trade the risky asset only at exogenous random times corresponding to the order flow given by an inhomogenous Poisson process. By using a direct dynamic programming approach, we first derive and solve the fixed point dynamic programming equation and then perform a verification argument which provides the existence and characterization of optimal trading strategies. We prove the convergence of the optimal performance, when the deterministic intensity of the order flow approaches infinity at any time, to the optimal expected utility for an investor trading continuously in a perfectly liquid market model with no-short sale constraints.

**Key words:** liquidity modelling, discrete order flow, optimal investment, inhomogenous Poisson process, dynamic programming.

## 1.1 Introduction

Financial modeling very often relies on the assumption of continuous-time trading. This is essentially due to the availability of the powerful tool of stochastic integration, which allows for an elegant representation of continuous-time trading strategies, and the analytic tractability of the stochastic calculus, typically illustrated by Itô's formula. Sometimes this assumption may not be very realistic in practice: illiquid markets provide a prime example. Indeed, an important aspect of market liquidity is the time restriction both on trading and observation of the assets. For example, in power markets, trading occurs through a brokered OTC market, and the liquidity is really thin. There could be a possible lack of counterparty for a given order: bids and offers are not available at any time, and may arrive randomly, while the investor can observe the asset only at these arrival times. Moreover, in these markets, because of the physical nature of the underlying asset, trading activity is really low far from the delivery, and is higher near the delivery.

In this paper, we propose a framework that takes into account such liquidity features by considering a discrete order flow. In our model, the investor can observe and trade over a finite horizon only at random times given by an inhomogenous Poisson process encoding the quotes in this illiquid market. To capture the high frequency of trading in the neighborhood of the finite horizon, we assume that the deterministic intensity of this inhomogenous Poisson process approaches infinity as time gets closer to the finite horizon. This is *the crucial feature* of our model, which allows us to compare trading strategies using *expected utility from terminal wealth*.

Optimal investment problems with random discrete trading dates were studied by several authors. Rogers and Zane [66] and Matsumoto [53] considered trading times associated to the jump times of a Poisson process, but assumed that the price process is observed continuously, so that trading strategies are actually in continuous-time. Recently, Pham and Tankov [61] (see also [17]) investigated an optimal portfolio/consumption choice problem over an infinite horizon, where the asset price, essentially extracted from a Black-Scholes model, can be observed and traded only at the random times corresponding to a Poisson process with constant intensity.

Compared to the model of Matsumoto [53], which is closest to our work, the present paper contributes to different levels. First, we allow for a general utility function, as opposed to power/log utility, and the underlying continuous-time asset price process is no longer a Brownian motion but an inhomogenous Lévy process. Second, in our framework, the traded asset is observed only at the sequence of Poisson arrivals, and not continuously. As we prove later in Lemma 1.4.1, in this Markov model actually continuous and discrete observations lead to the same solutions. However, this is not obvious from [53], so Lemma 1.4.1 is a part of our contribution. However, the main contribution of our model is the possibility to account for more and more frequent trading near maturity. The total wealth of an agent can be only defined at the trading times, so we believe that the model we present here is the only case of time-illiquidity where it actually makes sense to consider *expected utility from terminal wealth* (see Remark 1.3.1). If the trading intensity is constant (or does not satisfy the hyper-intensity condition (1.2.1) below), then total wealth cannot be defined at the terminal time horizon since the agent will usually not be able to liquidate his position at maturity.

In order to analyze our model of portfolio selection, we use a direct dynamic programming approach. We first derive the fixed point dynamic programming equation (DPE) and provide a constructive proof for the existence of a solution to this DPE in a suitable functional space by means of an iterative procedure. Then, by proving a verification theorem, we obtain the existence and characterization of optimal policies. We also provide an approximation of the optimal strategies that involves only a finite number of iterations. Finally, we address the natural question of convergence of our optimal investment strategy/expected utility when the arrival intensity rate becomes large at all times. We prove that the value function converges to the value function of an agent who can trade continuously in a perfectly liquid market with no-short sale constraints. A related convergence result was recently obtained by Kardaras and Platen [39] by considering continuous-time trading strategies approximated by simple trading strategies with constraints, but with asset prices observed continuously. Here we face some additional subtleties induced by the discrete observation filtrations: the illiquid market investor has less information coming from observing the asset, compared to the continuous-time investor, but he/she has the additional information coming from the arrival times, which is lacking in the perfectly liquid case.

The rest of the paper is organized as follows. Section 2 describes the illiquid market model with the restriction on the trading times, and sets up all the assumptions of the model. We formulate in Section 3 the optimal investment problem, and solve it by a dynamic programming approach and verification argument. Finally, Section 4 is devoted to the convergence issue when the deterministic intensity of arrivals is very large at all times.

## 1.2 The illiquid market model and trading strategies

We consider an illiquid market in which an investor can trade a risky asset over a finite horizon. In this market, bids and offers are not available at any time, but trading occurs more frequently near the horizon. This is typically the case in power markets with forward contracts. This market illiquidity feature is modelled by assuming that the arrivals of buy/sell orders occur at the jumps of an inhomogeneous Poisson process with an increasing deterministic intensity converging to infinity at the final horizon. In order to obtain an analytically tractable model, we further assume that the discrete-time observed asset prices come from an unobserved continuous-time stochastic process, which is independent of the sequence of arrival times. We may think about the continuous time process as an asset price process based on fundamentals independent of time-illiquidity, which would be actually observed if trading occurred at all times.

More precisely, fixing a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a finite horizon $T < \infty$, we consider the fundamental unobserved positive asset price $(S_t)_{0\leq t\leq T}$. An investor can observe and trade the asset *only* at some exogenous random times $(\tau_n)_{n\geq 0}$, $\tau_0 = 0$, such that $(\tau_n)_{n\geq 0}$ and $(S_t)_{0\leq t\leq T}$ are independent under the physical probability measure $\mathbb{P}$.

In order to obtain a stochastic control problem of Markov type for the utility maximization problem below, we assume an exponential-Lévy structure and some regularity/integrability conditions on the continuous-time positive price process $S$. More precisely, we assume that

$$S_t = \mathcal{E}(L)_t, \qquad 0 \leq t \leq T,$$

where the process $(L_t)_{0\leq t\leq T}$ is a semimartingale on $(\Omega, \mathcal{G}, \mathbb{P})$ with independent increments and jumps strictly greater than minus one. We use $\mathcal{E}(L)$ to denote the Doléans-Dade stochastic exponential of $L$. The assumption $\Delta L > -1$ ensures that the asset $S$, as well as its left-limit $S_-$, are strictly positive at all times. It is well known that a semimartingale with independent increments has *deterministic predictable characteristics*, see e.g. [36]. We then assume that the Lévy-Khintchin-Itô decomposition of $L$ has the form

**(HL)**

$$L_t = \int_0^t b(u)\,du + \int_0^t c(u)\,dW_u + \int_0^t \int_{-1}^{\infty} y(\mu(dt,dy) - \nu(dt,dy)), \quad 0 \leq t \leq T,$$

where $(W_t)_{0\leq T}$ is a Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$ independent on the jump measure $\mu$ with deterministic compensator $\nu$, and integrable large jumps, i.e

$$\int_0^T \int_{-1}^{\infty} y\,\nu(dt,dy) \;<\; \infty.$$

The deterministic functions $b : [0,T] \to \mathbb{R}$, $c : [0,T] \to (0,\infty)$, satisfy

$$\int_0^T |b(u)|du \;<\; \infty, \quad \text{and} \quad \int_0^T c^2(u)du \;<\; \infty.$$

Since in our model the asset $S$ is not observed at the terminal time $T$, there is no loss of generality if we assume that $S_T = S_{T-}$, which can be translated in terms of predictable characteristics as $\nu(\{T\},(-1,\infty)) = 0$. We will make this assumption for the rest of the paper.

We denote by

$$Z_{t,s} \;=\; \frac{S_s - S_t}{S_t} = \left\{ e^{(L_s - L_t - \frac{1}{2}\int_t^s c^2(u)\,du)} \prod_{t<u\leq s} e^{-\Delta L_u}(1+\Delta L_u) \right\} - 1, \qquad 0 \leq t \leq s \leq T,$$

the return between times $t$ and $s$ (if trading is allowed at both times) and denote by

$$p(t,s,dz) \;=\; \mathbb{P}[Z_{t,s} \in dz]$$

the distribution of the return.

The sequence of observation/trading times is represented by the jumps of an inhomogeneous (and independent of $S$) Poisson process $(N_t)_{t\in[0,T]}$ with deterministic intensity function $t \in [0,T)$ $\to \lambda(t) \in (0,\infty)$, such that:

$$\int_0^t \lambda(u)du \;<\; \infty, \; (\forall)\; 0 \leq t < T \quad \text{and} \quad \int_0^T \lambda(u)du \;=\; \infty. \tag{1.2.1}$$

The simplest way to actually define such an inhomogeneous Poisson process is to consider a homogeneous Poisson process $M$ with intensity equal to one, independent of $S$, and define

$$N_t = M_{\int_0^t \lambda(u)du} \quad \text{for} \quad 0 \leq t < T. \tag{1.2.2}$$

Condition (1.2.1) is crucial in our illiquidity modelling, and ensures that the probability of having no jumps between any interval $[t,T]$, $t < T$, is null, and so the sequence $(\tau_n)$ converges increasingly to $T$ almost surely when $n$ goes to infinity. We also know that the process of jump times $(\tau_n)_{n\geq 0}$ is a homogeneous Markov chain on $[0,T)$, and its transition probability admits a density given by:

$$\mathbb{P}[\tau_{n+1} \in ds | \tau_n = t] \quad = \quad \lambda(s) e^{-\int_t^s \lambda(u) du} 1_{\{t \leq s < T\}} \, ds, \tag{1.2.3}$$

(which does not depend on $n$).

An investor trading in this market can only observe/trade the asset $S$ at the discrete arrival times $\tau_n$. Therefore, the only information he/she has is coming from observing the two-dimensional process $(\tau_n, S_{\tau_n})_{n\geq 0}$. Taking this into account, we introduce the discrete observation filtration $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$, with $\mathcal{F}_0$ trivial and

$$\mathcal{F}_n \quad = \quad \sigma\Big\{(\tau_k, Z_k) : 1 \leq k \leq n\Big\}, \quad n \geq 1, \tag{1.2.4}$$

where we denote by

$$Z_n = Z_{\tau_{n-1},\tau_n}, \qquad n \geq 1,$$

the observed return process valued in $(-1, \infty)$.

In this model a (simple) trading strategy is a real-valued $\mathbb{F}$-adapted process $\alpha = (\alpha_n)_{n\geq 0}$, where $\alpha_n$ represents the amount invested in the stock over the period $(\tau_n, \tau_{n+1}]$ after observing the stock price at time $\tau_n$. Assuming that the money market pays zero interest rate, the observed wealth process $(X_{\tau_n})_{n\geq 0}$ associated to a trading strategy $\alpha$ is governed by:

$$X_{\tau_{n+1}} \quad = \quad X_{\tau_n} + \alpha_n Z_{n+1}, \quad n \geq 0, \tag{1.2.5}$$

where $X_0$ is the initial capital of the investor. In order to simplify notation, we fix once and for all an initial capital $X_0 > 0$ and denote by $\mathcal{A}$ the set of trading strategies $\alpha$ such that the wealth process stays nonnegative:

$$X_{\tau_n} \quad \geq \quad 0, \quad n \geq 1. \tag{1.2.6}$$

For the rest of the paper, we will call *simple* these trading strategies where trading occurs only at the discrete times $\tau_n$, $n \geq 0$.

**Remark 1.2.1.** *Constrained strategies.* From assumption **(HL)** we conclude that for each $0 \leq t < s \leq T$ the distribution $p(t,s,dz)$ has full support on $[-1, \infty)$, so $Z_n$ has also full support on $[-1, \infty)$. Taking this into account and using (1.2.5), the admissibility condition (1.2.6) on $\alpha \in \mathcal{A}$ means that we have a no-short sale constraint (on both the risky and savings account asset):

$$0 \ \leq \ \alpha_n \ \leq \ X_{\tau_n}, \quad \text{for all} \ \ n \geq 0. \tag{1.2.7}$$

Moreover, since $Z_n > -1$ a.s. for all $n \geq 1$, the wealth process associated to $\alpha \in \mathcal{A}_0$ is actually strictly positive:

$$X_{\tau_n} \ > \ 0, \quad n \geq 0.$$

For technical reasons, some related to the asymptotic behavior in Section 1.4.1, we need to define some continuous time filtrations along with the discrete filtration $\mathbb{F}$. To avoid confusion, we will denote by $\mathbb{G}$ (with different parameters) all continuous-time filtrations. In this spirit, we define the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ generated by observing continously the process $S$ and the arrival times as

$$\mathcal{G}_t \ = \ \sigma\{(S_u, N_u), 0 \leq u \leq t\} \vee \mathcal{N}, \quad 0 \leq t \leq T, \tag{1.2.8}$$

where $N$ is the inhomogenous Poisson process in (1.2.2) and $\mathcal{N}$ are all the null sets of $\mathcal{G}$ under the historical measure $\mathbb{P}$. We would like to point out that, because of the Lévy structure of the joint process $(S, N)$, the filtration $\mathbb{G}$ is right continous, so it satisfies the *usual conditions.* In addition, we have the strict inclusion:

$$\mathcal{F}_n \subset \mathcal{G}_{\tau_n} \quad \text{for all } n \geq 1.$$

We make an additional assumption on the model, which, among others, precludes arbitrage possibilities:

**(NA)**

$$\int_0^T \frac{b^2(u)}{c^2(u)} du < \infty.$$

Under this assumption we can then define the probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \frac{b(u)}{c(u)} dW_u - \frac{1}{2}\int_0^T \left(\frac{b(u)}{c(u)}\right)^2 du}.$$

Under $\mathbb{Q}$, the process $N$ has the same law as under $\mathbb{P}$, and $(\tau_n)_{n \geq 0}$ and $(S_t)_{0 \leq t \leq T}$ are still independent under $\mathbb{Q}$. Moreover, the process $S$ is a positive $(\mathbb{Q},\mathbb{G})$-local martingale so a supermartingale. This means that the discrete-time process $(S_{\tau_n})_{n\geq 0}$ is a $(\mathbb{Q},\mathbb{F})$ supermartingale as well.

**Remark 1.2.2.** *Embedding in a continuous-time wealth process.* Given $\alpha \in \mathcal{A}$ with corresponding wealth process $(X_{\tau_n})_n$ in (1.2.5), let us define the continuous time process $(X_t)_{0\leq t<T}$ by

$$\begin{aligned} X_t &= X_{\tau_n} + \alpha_n Z_{\tau_n,t}, \quad \tau_n < t \leq \tau_{n+1}, \;\; n \geq 0, \\ &= X_0 + \int_0^t H_u dS_u, \quad 0 \leq t < T, \end{aligned} \tag{1.2.9}$$

where $H$ is the simple and $\mathbb{G}$-predictable process

$$H_t = \sum_{n=0}^{\infty} \frac{\alpha_n}{S_{\tau_n}} \mathbb{I}_{\{\tau_n < t \leq \tau_{n+1}\}}, \qquad 0 \leq t < T,$$

representing the number of shares invested in the risky asset. From (1.2.7) and since $S_t > 0$, so $Z_{\tau_n,t} > -1$, $n \geq 0$, we notice that the continuous time process $X$ is strictly positive: $X_t > 0$ for $0 \leq t < T$. Moreover, since $S$ is a $(\mathbb{Q},\mathbb{G})$-local martingale, we also see that $(X_t)_{0\leq t<T}$ is a $(\mathbb{Q},\mathbb{G})$-local martingale, hence a super-martingale up to $T$. Consequently, we also have $X_{t-} > 0$ for $0 \leq t < T$. The definition of $X_{\tau_n}$ in (1.2.5) is consistent with (1.2.9), so $(X_{\tau_n})_{n\geq 0}$ is a positive $\mathbb{F}$-supermartingale under $\mathbb{Q}$. Therefore, for each $\alpha \in \mathcal{A}$ we may define the terminal wealth value by:

$$X_T = \lim_{n\to\infty} X_{\tau_n} = \lim_{t\nearrow T} X_t = X_0 + \sum_{n=0}^{\infty} \alpha_n Z_{n+1},$$

and, since $S_T = S_{T-}$ we also have

$$X_T = X_0 + \int_0^T H_u dS_u,$$

where the integrand $H$ is related to the simple trading strategy $\alpha$ as described above. The supermartingale property implies the budget constraint

$$\mathbb{E}^{\mathbb{Q}}[X_T] \leq X_0.$$

The continuous time wealth process $X$ has the meaning of a shadow wealth process: it is not observed except for at times $\tau_n$, $n \geq 0$. The no-short sale constraints (1.2.7) is translated in terms of the number of shares held as

$$0 \leq H_t S_{t-} \leq X_{t-}, \quad 0 \leq t < T. \tag{1.2.10}$$

We denote by $\mathcal{X}$ the set of all wealth processes $(X_t)_{0\leq t\leq T}$ given by (1.2.9), by using simple trading strategies under the no-short sale constraint (1.2.7)/(1.2.10). We denote by $\bar{\mathcal{X}}$ the set of all positive wealth processes $(X_t)_{0\leq t\leq T}$ given by (1.2.9), by using *general* $\mathbb{G}$-predictable and $S$-integrable processes $H$ satisfying (1.2.10). We clearly have $\mathcal{X} \subset \bar{\mathcal{X}}$.

For technical reasons, it is sometimes convenient to regard trading strategies equivalently in terms of proportions of wealth. For any continuous time wealth process $X \in \bar{\mathcal{X}}$ associated to a trading strategy $H$ satisfying (1.2.10), let us consider the process $\pi = (\pi_t)_{0\leq t\leq T}$, defined by: $\pi_t = H_t S_{t-}/X_{t-}$, and notice that $\pi$ is valued in $[0,1]$ by (1.2.10). We stress the dependence of the wealth on the proportion $\pi$, and denote by $X^{(\pi)} = X$, which is then written in a multiplicative way as

$$X^{(\pi)}_{\cdot} = X_0 \mathcal{E}\Big(\int_0^{\cdot} \pi \frac{dS}{S_-}\Big) = X_0 \mathcal{E}\Big(\int_0^{\cdot} \pi dL\Big), \tag{1.2.11}$$

where $\mathcal{E}$ is the Doléans-Dade operator. Denote by $\mathcal{D}(\mathbb{G})$ the set of all $\mathbb{G}$-predictable processes $\pi$ valued in $[0,1]$. It is then clear that

$$\bar{\mathcal{X}} = \{X^{(\pi)} | \ \ \pi \in \mathcal{D}(\mathbb{G})\}.$$

## 1.3 Optimal investment problem and dynamic programming

We investigate an optimal investment problem in the illiquid market described in the previous section. Let us consider an utility function $U$ defined on $(0,\infty)$, strictly increasing, strictly concave and $C^1$ on $(0,\infty)$, and satisfying the Inada conditions: $U'(0^+) = \infty$, $U'(\infty) = 0$. We make the following additional assumptions on the utility function $U$:

**(HU)** (i) there exist some constants $C > 0$ and $p \in (0,1)$ such that

$$U^+(x) \leq C(1 + x^p), \quad (\forall)\ x > 0$$

where $U^+ = \max(U, 0)$

(ii) there exist some constants $C' > 0$ and $p' < 0$ such that

$$U^-(x) \quad \leq \quad C'(1 + x^{p'}), \quad (\forall)\ x > 0,$$

where $U^- = \max(-U, 0)$.

The above assumptions include most popular utility functions, in particular those with constant relative risk aversion $1-p>0$, in the form $U(x)=(x^p-1)/p$, $x>0$.

Given the chosen positive initial wealth $X_0>0$, we consider the optimal investment problem:

$$V_0 \quad = \quad \sup_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T)] = \sup_{X\in\mathcal{X}} \mathbb{E}[U(X_T)]. \tag{1.3.1}$$

**Remark 1.3.1.** As noted in the Introduction, the hyper-intensity condition (1.2.1) allows us to define the terminal wealth $X_T$. Without such a condition, while one could always define the continuous-time wealth process $X$ as in Remark 1.2.2 for mathematical convenience, the terminal wealth $X_T$ does not have economic meaning. Therefore, the condition (1.2.1) appears as necessary if one wants to compare portfolios using expected utility from terminal wealth, as we do here.

Our aim is to provide an analytic solution to the control problem (1.3.1) using direct dynamic programming, i.e. *first* solve the Dynamic Programming Equation (DPE) *analytically* and *then* perform a *verification* argument. Therefore, there is no need to either define the value function at later times or to prove the Dynamic Programming Principle (DPP).

The Lemma below provides the intuition behind the (DPE):

**Lemma 1.3.1.** *Assume* **(HL)** *holds true. Let $\alpha\in\mathcal{A}$ and let $(X_{\tau_n})_{n\geq 0}$ be the wealth process associated with the trading strategy $\alpha$. Consider a measurable function $v:[0,T)\times(0,\infty)\to\mathbb{R}$. For a fixed $n\geq 0$, if $v(\tau_{n+1},X_{\tau_{n+1}})\in L^1(\Omega,\mathcal{G},\mathbb{P})$, then*

$$\mathbb{E}[v(\tau_{n+1},X_{\tau_{n+1}})|\mathcal{F}_n] \quad = \quad \int_{\tau_n}^{T}\int_{(-1,\infty)} \lambda(s)e^{-\int_{\tau_n}^{s}\lambda(u)du} v(s,X_{\tau_n}+\alpha_n z)p(\tau_n,s,dz)ds,$$

*where the above equality holds $\mathbb{P}$-a.s.*

**Proof.** Assumption (**HL**) together with the independence of $S$ and $N$ ensures that for all $n\geq 0$, the (regular) distribution of $(\tau_{n+1},Z_{n+1})$ conditioned on $\mathcal{F}_n$ is given as follows:

1. $\mathbb{P}[\tau_{n+1}\in ds|\mathcal{F}_n]=\lambda(s)e^{-\int_{\tau_n}^{s}\lambda(u)du}ds$

2. further conditioning on knowing the next arrival time $\tau_{n+1}$, the return $Z_{n+1}$ has distribution

$$\mathbb{P}[Z_{n+1}\in dz|\mathcal{F}_n\vee\sigma(\tau_{n+1})]=p(\tau_n,\tau_{n+1},dz).$$

We then get

$$
\begin{aligned}
\mathbb{E}\big[v(\tau_{n+1}, X_{\tau_{n+1}})|\mathcal{F}_n\big] &= \mathbb{E}\Big[\mathbb{E}\big[v(\tau_{n+1}, X_{\tau_n} + \alpha_n Z_{n+1})|\mathcal{F}_n \vee \sigma(\tau_{n+1})\big]\Big|\mathcal{F}_n\Big] \\
&= \mathbb{E}\Big[\int_{(-1,\infty)} v(\tau_{n+1}, X_{\tau_n} + \alpha_n z)p(\tau_n, \tau_{n+1}, dz)\Big|\mathcal{F}_n\Big] \\
&= \int_{\tau_n}^{T}\int_{(-1,\infty)} \lambda(s)e^{-\int_{\tau_n}^{s}\lambda(u)du} v(s, X_{\tau_n} + \alpha_n z)p(\tau_n, s, dz)ds.
\end{aligned}
$$

□

Taking into account the above Lemma, we can now *formally* write down the Dynamic Programming Equation as

$$
\begin{aligned}
v(t,x) &= \sup_{a\in[0,x]} \int_t^T \int_{(-1,\infty)} \lambda(s)e^{-\int_t^s \lambda(u)du} v(s, x+az)p(t,s,dz)ds, \\
&= \sup_{\pi\in[0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s)e^{-\int_t^s \lambda(u)du} v(s, x(1+\pi z))p(t,s,dz)ds, \qquad (1.3.2)
\end{aligned}
$$

for all $(t,x) \in [0,T) \times (0,\infty)$ together with the natural terminal condition

$$
\lim_{t\nearrow T, x'\to x} v(t,x') = U(x), \quad x > 0. \qquad (1.3.3)
$$

It appears that the right space of functions to be looking for a solution of (1.3.2)-(1.3.3) is actually the space $\mathcal{C}_U([0,T)\times(0,\infty))$ of measurable functions $w$ on $[0,T)\times(0,\infty)$, such that

1. $w(t,.)$ is concave on $(0,\infty)$ for all $t \in [0,T)$, and

2. for some $C = C(w) > 0$, we have

$$
U(x) \le w(t,x) \le C(1+x), \quad \forall (t,x) \in [0,T)\times(0,\infty). \qquad (1.3.4)
$$

For any $w \in \mathcal{C}_U([0,T)\times(0,\infty))$, we consider the measurable function $\mathcal{L}w$ on $[0,T)\times(0,\infty)$ defined by:

$$
\mathcal{L}w(t,x) = \sup_{\pi\in[0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s)e^{-\int_t^s \lambda(u)du} w(s, x(1+\pi z))p(t,s,dz)ds. \qquad (1.3.5)
$$

Lemma 1.3.2 below shows that the operator

$$
\mathcal{L} : \mathcal{C}_U([0,T)\times(0,\infty)) \to \mathcal{C}_U([0,T)\times(0,\infty)),
$$

is well defined. Therefore, we are looking for a a solution $w \in \mathcal{C}_U([0,T) \times (0,\infty))$ to the DPE:

$$\begin{cases} \mathcal{L}w = w \\ \lim_{t \nearrow T, x' \to x} w(t,x') = U(x). \end{cases} \tag{1.3.6}$$

In order to solve the DPE and perform the verification argument, we need some technical details collected in the subsection below:

### 1.3.1 A supersolution of the DPE and other technical details

**Lemma 1.3.2.** *Assume that* **(HL)** *holds. For any $w \in \mathcal{C}_U([0,T) \times (0,\infty))$, $\mathcal{L}w$ also belongs to $\mathcal{C}_U([0,T) \times (0,\infty))$. For each $(t,x) \in [0,T) \times (0,\infty)$ the supremum in* (1.3.5) *is attained at some $\pi(t,x)$ which can be chosen measurable in $(t,x)$.*

**Proof.** Given $w \in \mathcal{C}_U([0,T) \times (0,\infty))$, let us consider the measurable function $\widehat{w}$ defined on $[0,T) \times (0,\infty) \times [0,1]$ by:

$$\widehat{w}(t,x,\pi) \quad = \quad \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} w(s, x(1+\pi z)) p(t,s,dz) ds,$$

so that $\mathcal{L}w(t,x) = \sup_{\pi \in [0,1]} \widehat{w}(t,x,\pi)$. Under **(HL)**, there exists some positive constant $C$ (which can be chosen actually as $C = 1 + e^{\int_0^T |b(u)|du}$), s.t.

$$\int_{(-1,\infty)} |z| p(t,s,dz) \quad \leq \quad C, \qquad 0 \leq t \leq s < T. \tag{1.3.7}$$

Thus, for $w$ satisfying (1.3.4), we see that $\hat{w}$ is well-defined on $[0,T) \times (0,\infty) \times [0,1]$ and satisfies:

$$-\infty \leq \widehat{w}(t,x,\pi) \quad \leq \quad C(1+x), \quad \forall (t,x,\pi) \in [0,T) \times (0,\infty) \times [0,1], \tag{1.3.8}$$

for some positive constant $C > 0$. Moreover, for each $(t,x) \in [0,T) \times (0,\infty)$, we have

$$\widehat{w}(t,x,0) \quad \geq \quad U(x). \tag{1.3.9}$$

As a matter of fact, one can easily see that $\widehat{w}(t,x,\pi)$ is actually finite for any $\pi \in [0,1)$ and may only equal negative infinity for $\pi = 1$. Consequently, for fixed $(t,x) \in [0,T) \times (0,\infty)$, $\widehat{w}(t,x,.)$ is a proper one-dimensional concave function defined on $[0,1]$ (concavity follows easily from that of $w$). In addition, using the linear growth (1.3.4) together with Fatou lemma, we obtain that $\pi \in [0,1] \to \widehat{w}(t,x,\pi)$ is upper semicontinuous (this refers to the endpoints $\pi = 0,1$ since the function is continuous on $(0,1)$ being finite and concave). Therefore $\mathcal{L}w(t,x) = \max_{\pi \in [0,1]} \widehat{w}(t,x,\pi)$, where the maximum is attained at some $\pi = \pi(t,x)$ which can be chosen measurable in $(t,x)$, see e.g. Ch. 11 in [9]. In addition, since $\pi \to \widehat{w}(t,x,\pi)$ is continuous on $(0,1)$, the function $\mathcal{L}w$ has the additional representation

$$\mathcal{L}w(t,x) \quad = \quad \sup_{\pi\in[0,1]\cap\mathbb{Q}} \widehat{w}(t,x,\pi),$$

which shows that $\mathcal{L}w$ is measurable. The concavity of $w(t,.)$ implies the concavity of $(x,a) \in \{(x,a)\in(0,\infty)\times\mathbb{R} : a\in[0,x]\} \to \widehat{w}(t,x,a/x)$ for all $t\in[0,T)$. This easily implies that $\mathcal{L}w(t,.)$ is also concave on $(0,\infty)$ for all $t\in[0,T)$. Finally, it is clear from (1.3.8) and (1.3.9) that $\mathcal{L}w$ satisfies also the growth condition:

$$U(x) \;\le\; \mathcal{L}w(t,x) \quad\le\quad C(1+x), \quad \forall (t,x)\in[0,T)\times(0,\infty).$$

□

The next lemma constructs a supersolution $f \in C_U([0,T)\times(0,\infty))$ for the DPE:

**Lemma 1.3.3.** *Assume that* (**HL**), (**NA**) *and* (**HU**) *hold. Define*

$$f(t,x) = \inf_{y>0}\left\{\mathbb{E}[\tilde{U}(yY_{t,T})] + yx\right\} \quad (t,x)\in[0,T]\times(0,\infty),$$

*where*

$$Y_{t,T} = e^{-\int_t^T \frac{b(u)}{c(u)}dW_u - \frac{1}{2}\int_t^T\left(\frac{b(u)}{c(u)}\right)^2 du}, \tag{1.3.10}$$

*and $\tilde{U}$ is the Fenchel-Legendre transform of $U$:*

$$\tilde{U}(y) \quad = \quad \sup_{x>0}[U(x) - xy] \;<\; \infty, \quad \forall y>0. \tag{1.3.11}$$

*Then, $f$ lies in the set $\mathcal{C}_U([0,T)\times(0,\infty))$, and satisfies*

$$\begin{cases} \mathcal{L}f \le f \\ \lim_{t\nearrow T, x'\to x} f(t,x') = U(x). \end{cases} \tag{1.3.12}$$

**Proof.** Jensen's inequality gives $\mathbb{E}[\tilde{U}(yY_{t,T})] \ge \tilde{U}(y)$, so

$$f(t,x) \ge \inf_{y>0}\left\{\tilde{U}(y) + yx\right\} = U(x).$$

From the definition of $f$ we know that

$$f(t,x) \le \mathbb{E}[\tilde{U}(yY_{t,T})] + yx, \quad (\forall)\ y > 0. \tag{1.3.13}$$

Fix a $y_0 > 0$. We denote by

$$\mathcal{G}'_t = \sigma\{(S_u - S_t, N_u - N_t),\ t \le u \le T\},$$

the information accumulated from time $t$ to maturity $T$ by observing continuously the asset $S$ and the arrival times (the jumps of $N$). Jensen's inequality together with Assumption **(HU)**(i) shows that

$$\mathbb{E}[\tilde{U}(y_0 Y_{t,T})] = \mathbb{E}[\tilde{U}(y_0 \mathbb{E}[Y_{0,T}|\mathcal{G}'_t])] \le \mathbb{E}[\mathbb{E}[\tilde{U}(y_0 Y_{0,T})|\mathcal{G}'_t]] = \mathbb{E}[\tilde{U}(y_0 Y_{0,T})] < \infty,$$

so

$$f(t,x) \le \mathbb{E}[\tilde{U}(y_0 Y_{t,T})] + y_0 x \le C(1+x) \quad (\forall)\ (t,x) \in [0,T] \times (0,\infty).$$

This shows that $f \in \mathcal{C}_U([0,T) \times (0,\infty))$. Using assumption **(HU)** (both (i) and (ii)) we obtain that there exist constants $\tilde{C} > 0$, $q < 0$ and $0 < q' < 1$ such that

$$\tilde{U}^+(y) \le \tilde{C}(1 + y^q), \quad \tilde{U}^-(y) \le \tilde{C}(1 + y^{q'}), \quad (\forall)\ y > 0.$$

Using the defintion of $Y_{t,T}$ (which is log-normal) and the explicit moments of log-normal random variables we obtain that

$$\sup_{0 \le t < T} \mathbb{E}[|\tilde{U}(yY_{t,T})|^2] < \infty.$$

This means that the collection $(\tilde{U}(yY_{t,T}))_{0 \le t < T}$ of random variables is uniformly integrable, so

$$\lim_{t \nearrow T} \mathbb{E}[\tilde{U}(yY_{t,T})] = \tilde{U}(y),\ (\forall)\ y > 0.$$

We can now use this in (1.3.13) to deduce that

$$U(x) \le \liminf_{t \nearrow T, x' \to x} f(t,x') \le \limsup_{t \nearrow T, x' \to x} f(t,x') \le \tilde{U}(y) + xy,\ (\forall)\ y > 0.$$

Taking the infimum over $y$ we obtain the terminal condition. For each fixed $t$, the function $f(t,\cdot)$

is finite and concave on $(0, \infty)$, so the only thing left to check is the supersolution property. Fix $0 \leq t \leq s \leq T$ and $x > 0$. Denote by $h(z) = \mathbb{E}[\tilde{U}(zY_{s,T})]$ and fix $y > 0$ and $\pi \in [0, 1]$. By the very definition of the function $f$ we have that

$$f(s, x(1 + \pi Z_{t,s})) \leq h(yY_{t,s}) + x(1 + \pi Z_{t,s})yY_{t,s}.$$

Using independence and the definition of $h$, we obtain

$$\begin{aligned}\mathbb{E}[f(s, x(1 + \pi Z_{t,s}))] &\leq \mathbb{E}[h(yY_{t,s}) + x(1 + \pi Z_{t,s})yY_{t,s}] = \\ &= \mathbb{E}[\tilde{U}(yY_{t,s}Y_{s,T})] + \mathbb{E}[x(1 + \pi Z_{t,s})yY_{t,s}] \leq \mathbb{E}[\tilde{U}(yY_{t,T})] + xy.\end{aligned}$$

Taking the inf over all $y$ and recalling the definition of $f(t, x)$ we obtain

$$f(t, x) \geq \mathbb{E}[f(s, x(1 + \pi Z_{t,s}))] = \int_{(-1,\infty)} f(s, x(1 + \pi z))p(t, s, dz)$$

for all $\pi$ and $s$. For a fixed $\pi$, we can integrate over $s$ to obtain

$$f(t, x) \geq \int_t^T \int_{(-1,\infty)} \lambda(s)e^{-\int_t^s \lambda(u)du} f(s, x(1 + \pi z))p(t, s, dz)ds,$$

and then taking the supremum over $\pi$ we obtain $f(t, x) \geq (\mathcal{L}f)(t, x)$, so the proof is complete. Due to the linear growth condition of $f$ and recalling (1.3.7), all expectations/integrals above are well defined, but may be negative infinity. In other words, the positive parts in all expectations/integrals are actually integrable. □

**Remark 1.3.2.** The whole analysis in this paper extends to the case when the Brownian part of the process $L$ is degenerate, as long as the jumps have full support on $(-1, \infty)$ and the jump measure allows for a martingale measure with density process $Y$ that can replace the definition (1.3.10) in the corresponding proofs. In other words, the assumptions **(HL)** and **(NA)** can be relaxed to include the situation when the drift can be removed by changing the jump measure appropriately, if the Gaussian part is missing.

It turns out that, for the verification arguments below, we also need an assumption on the integrability of jumps.

**(HI)**: (i) there exists $q > 1$ such that

$$\int_0^T \int_0^\infty \Big((1 + y)^q - 1 - qy\Big)\nu(dt, dy) < \infty.$$

(ii) If the utility function $U$ satisfies $U(0) = -\infty$ , then there exists $r < p' < 0$ (where $p'$ is given in **(HU)**(ii)) such that

$$\int_0^T \int_{-1}^0 \Big((1+y)^r - 1 - ry\Big)\nu(dt,dy) < \infty.$$

(iii) there are no predictable jumps, i.e. $\nu(\{t\},(-1,\infty)) = 0$ for each $t$

**Remark 1.3.3.** Using convexity, it is an easy exercise to see that assumption **(HI)** (i) can actually be rephrased as $\nu_q([0,T]) < \infty$, and assumption **(HI)**(ii) as $\nu_r([0,T]) < \infty$ where

$$\nu_l(dt) = \int_{-1}^{\infty} \sup_{\pi\in[0,1]} \Big((1+\pi y)^l - 1 - l\pi y\Big)\,\nu(dt,dy).$$

We now prove a crucial uniform integrability condition, but before that we denote by $\mathcal{T}$ the set of random times $0 \le \tau < T$ which are stopping times with respect to the filtration $\mathbb{G}$.

**Lemma 1.3.4.** *Assume that* **(HL)**, **(HU)**, **(NA)** *and* **(HI)** *hold.*
**(1)** *For any $X \in \bar{\mathcal{X}}$, the family $(f^+(\tau, X_\tau))_{\tau\in\mathcal{T}}$ is uniformly $\mathbb{P}$-integrable.*
**(2)** *For any $X \in \bar{\mathcal{X}}$, the family $(U^-(X_\tau))_{\tau\in\mathcal{T}}$, is uniformly $\mathbb{P}$-integrable.*

**Proof.** Assume $\nu_l([0,T]) < \infty$ for some $l$. Consider $X = X^{(\pi)} \in \bar{\mathcal{X}}$ for some $\pi \in \mathcal{D}(\mathbb{G})$, and recall that

$$dX_t^{(\pi)} \;=\; \pi_t X_{t-}^{(\pi)} dL_t, \quad 0 \le t \le T. \tag{1.3.14}$$

In order to simplify notation, we supress the upper indices of $X$. We apply Itô formula to $(X_t)^l$ to conclude that

$$X_t^l = x^l + \int_0^t (X_{u-})^l \left(l\pi_u b(u) + \frac{l(l-1)}{2}c^2(u)\pi^2(u)\right) du +$$
$$\int_0^t \int_{-1}^{\infty} (X_{u-})^l \Big((1+\pi_u y)^l - 1 - l\pi_u y\Big)\nu(du,dy) +'' local\ martingale''$$

Fix a stopping time $\tau \in \mathcal{T}$. If $T_n'$ is a sequence of localizing stopping times for the local martingale part, denote by $T_n = T_n' \wedge \{\inf t : (X_t)^l \ge n\}$. Observe that $T_n \nearrow T$ a.s. since $X_-$ is locally bounded and locally bounded away from zero. We then have, for each $0 \le t < T$,

$$\begin{aligned}
\mathbb{E}[(X_{t\wedge\tau\wedge T_n})^l] \;&=\; x^l + \mathbb{E}\left[\int_0^{t\wedge\tau\wedge T_n} (X_{u-})^l \left\{\left(l\pi_u b(u) + \frac{l(l-1)}{2}c^2(u)\pi^2(u)\right) du \right.\right.\\
&\quad + \left.\left. \int_{-1}^{\infty} \Big((1+\pi_u y)^l - 1 - l\pi_u y\Big)\nu(du,dy)\right\}\right] \qquad (1.3.15)\\
\le x^l \;&+\; \mathbb{E}\left[\int_0^{t\wedge\tau\wedge T_n} (X_{u-})^l \left\{\left(|lb(u)| + \frac{|l(l-1)|}{2}c^2(u)\right) du + \nu_l(du)\right\}\right]
\end{aligned}$$

Since $(X_{u-})^l \le n$ for $0 \le u \le \tau \wedge T_n$ and $\nu_l([0,T]) < \infty$, we conclude that

$$\mathbb{E}[(X_{t\wedge\tau\wedge T_n})^l] < \infty, \quad 0 \le t < T. \tag{1.3.16}$$

In addition, since the paths of the process $X^l$ are RCLL and $\nu_l(\{u\}) = 0$ for each $0 \le u \le T$ (because of **(HI)** part (iii)), we have that, for each $0 \le t < T$, with $\mathbb{P}$-probability one

$$\begin{aligned}\int_0^{t\wedge\tau\wedge T_n} (X_{u-})^l \left\{ \left( |lb(u)| + \frac{|l(l-1)|}{2} c^2(u) \right) du + \nu_l(du) \right\} &= \\ \int_0^{t\wedge\tau\wedge T_n} (X_u)^l \left\{ \left( |lb(u)| + \frac{|l(l-1)|}{2} c^2(u) \right) du + \nu_l(du) \right\} &\le \\ \int_0^t (X_{u\wedge\tau\wedge T_n})^l \left\{ \left( |lb(u)| + \frac{|l(l-1)|}{2} c^2(u) \right) du + \nu_l(du) \right\}&.\end{aligned}$$

Replacing this in (1.3.16) and using Fubini, we obtain

$$\mathbb{E}[(X_{t\wedge\tau\wedge T_n})^l] \le x^l + \int_0^t \mathbb{E}[(X_{u\wedge\tau\wedge T_n})^l] \left\{ \left( |lb(u)| + \frac{|l(l-1)|}{2} c^2(u) \right) du + \nu_l(du) \right\}. \tag{1.3.17}$$

Now, using (1.3.16) and $\nu_l([0,T]) < \infty$, we can apply Gronwall in (1.3.17) to conclude that $\mathbb{E}[(X_{t\wedge\tau\wedge T_n})^l] \le M(l) < \infty$, for each $0 \le t < T$, where $M(l)$ does not depend on $\tau$ or $n$. Letting $n \to \infty$ and $t \to T$, by Fatou, we obtain $\mathbb{E}[(X_\tau)^l] \le M(l)$ for each stopping time $\tau \in \mathcal{T}$.

We can finish the proof considering $l = q$ for item (i) and $l = r$ for item (ii), and also using the upper bound $f(t,x) \le C(1+x)$ as well as Assumption **(HU)** part (ii). □

**Remark 1.3.4.** In case $U(0) = -\infty$, we can follow the arguments in the Proof of Lemma 1.3.4 for the case $\pi = 1$ and $l = r$ (taking into account that $X_t = X_0 S_t$ for $0 \le t < T$) to conclude that

$$\mathbb{E}[(S_t)^r] = \mathbb{E}[(1 + Z_{0,t})^r] \ < \ \infty \text{ for } 0 \le t < T.$$

(we assumed that $S_0 = 1$ above, and we also used that the times $0 \le t < T$, because are deterministic, belong to $\mathcal{T}$). The same argument actually works if we start at any time $0 \le t < T$, so we have

$$\mathbb{E}[(1 + Z_{t,s})^r] = \int_{(-1,\infty)} (1+z)^r p(t,s,dz) \ < \ \infty, \text{ for } \ t \le s < T.$$

### 1.3.2 Construction of a solution for the DPE

We provide a constructive proof for the existence of a solution of (1.3.6) using an iteration scheme. Let us define inductively the sequence of functions $(v_m)_m$ in $\mathcal{C}_U([0,T)\times(0,\infty))$ by:

$$v_0 \;=\; U, \quad v_{m+1} \;=\; \mathcal{L}v_m, \quad m \geq 0.$$

**Lemma 1.3.5.** *Assume* **(HL)**, **(NA)**, *and* **(HU)**. *Then the sequence of functions $v_m$ satisfies*

$$v_m \leq v_{m+1} \leq f, \quad m \geq 0.$$

**Proof.** We do the proof by induction. We obviously have $U = v_0 \leq v_1$. In addition, since the operator $\mathcal{L}$ is monotone and $U \leq f$ we have

$$v_1 = \mathcal{L}U \leq \mathcal{L}f \leq f,$$

so the statement is true for $m = 0$. Assume now the statement is true for $m$. We use again the monotonicity of $\mathcal{L}$ to get

$$v_{m+2} = \mathcal{L}v_{m+1} \geq \mathcal{L}v_m = v_{m+1}, \qquad v_{m+2} = \mathcal{L}v_{m+1} \leq \mathcal{L}f \leq f,$$

so the proof is finished. □

Under the conditions of the above Lemma, the nondecreasing sequence $(v_m)_m$ converges pointwise, and we may define

$$v^* \;=\; \lim_{m\to\infty} v_m \;\leq\; f. \tag{1.3.18}$$

We show next that $v^*$ satisfies the fixed point DP equation.

**Theorem 1.3.1.** *Assume that* **(HL)**, **(NA)**, **(HU)** *and* **(HI)** *hold. Then, $v^*$ is solution to the fixed point DP (1.3.6).*

**Proof.** Fix $\pi \in [0,1]$. We know by construction that

$$v_{m+1}(t,x) \;\;\geq\;\; \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v_m(s, x(1+\pi z))p(t,s,dz)ds.$$

If $0 \leq \pi < 1$, then $v_m(s, x(1+\pi z) \geq U(x(1-\pi))$ so the integral on the right hand side is clearly finite. If $\pi = 1$, according to Remark 1.3.4, the integral on the right hand side is still finite for each $m \geq 0$. Therefore, we can let $m \nearrow \infty$ and use the monotone convergence theorem to obtain

$$v^*(t,x) \geq \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v^*(s, x(1+\pi z)) p(t,s,dz) ds.$$

Since this happens for each $\pi$, taking the supremum over $\pi$ we get $v^* \geq \mathcal{L}v^*$. Conversely, for $\varepsilon > 0$ there exists $m$ such that $v^*(t,x) - \varepsilon \leq v_{m+1}(t,x)$ and (because of convexity the maximum is attained) $\pi^m(t,x)$ such that

$$v_{m+1}(t,x) = \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v_m(s, x(1+\pi^m(t,x)z)) p(t,s,dz) ds.$$

Since $v_m \leq v^*$ it follows that

$$\begin{aligned} v^*(t,x) - \varepsilon &\leq \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v^*(s, x(1+\pi^m(t,x)z)) p(t,s,dz) ds \\ &\leq \mathcal{L}v^*(t,x). \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $v^* = \mathcal{L}v^*$. Finally, since $U(t,x) \leq v^*(t,x) \leq f(t,x)$ and the function $f$ satisfies the boundary condition (1.3.3) by Lemma 1.3.3, we conclude that $v^*$ is a solution to the fixed point DP equation (1.3.6). □

**Remark 1.3.5.** The previous theorem shows the existence of a fixed point to the DP equation (1.3.6), and gives also an iterative procedure for constructing a fixed point. In the next subsection, we shall prove that such a fixed point is equal to the value function $v$, which implies in particular the uniqueness for the fixed point equation (1.3.6).

### 1.3.3 Verification and optimal strategies

Consider the solution $v^*$ to the fixed point DP equation (1.3.6), constructed in Theorem 1.3.1. We now state a verification theorem for the fixed point equation (1.3.6), which provides the optimal portfolio strategy in feedback form.

**Theorem 1.3.2.** *Assume that* **(HL)**, **(NA)**, **(HU)** *and* **(HI)** *hold. Then,*

$$V_0 = v^*(0, X_0),$$

*and an optimal control* $\hat{\alpha} \in \mathcal{A}$ *is given by*

$$\hat{\alpha}_n = \hat{\pi}(\tau_n, \hat{X}_{\tau_n}) \hat{X}_{\tau_n}, \quad n \geq 0, \tag{1.3.19}$$

*where* $\hat{\pi}$ *is a measurable function on* $[0,T) \times (0,\infty)$ *solution to*

$$\hat{\pi}(t,x) \quad \in \quad \arg\max_{\pi \in [0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u)du} v^*(s, x(1+\pi z)) p(t,s,dz) ds,$$

*and* $(\hat{X}_{\tau_n})_{n\geq 0}$ *is the wealth given by*

$$\hat{X}_{\tau_{n+1}} \quad = \quad \hat{X}_{\tau_n} + \hat{\alpha}_n Z_{n+1}, \quad n \geq 0,$$

*and starting from* $\hat{X}_0 = X_0$*.*

**Proof.** Consider $\alpha \in \mathcal{A}$ and the corresponding positive wealth process $(X_{\tau_n})_{n\geq 0}$. From Lemma 1.3.4, we know that

$$\mathbb{E}[|v^*(\tau_n, X_{\tau_n})|] < \infty, \qquad (\forall)\ n \geq 0.$$

We apply Lemma 1.3.1 to get for any $n \geq 0$:

$$\begin{aligned} \mathbb{E}[v^*(\tau_{n+1}, X_{\tau_{n+1}})|\mathcal{F}_n] &= \int_{\tau_n}^T \int_{(-1,\infty)} \lambda(s) e^{-\int_{\tau_n}^s \lambda(u)du} v^*(s, X_{\tau_n} + \alpha_n z) p(\tau_n, s, dz) ds \\ &\leq \mathcal{L}v^*(\tau_n, X_{\tau_n}) \ = \ v^*(\tau_n, X_{\tau_n}), \end{aligned} \tag{1.3.20}$$

so the process $\{v^*(\tau_n, X_{\tau_n}), n \geq 0\}$ is a $(\mathbb{P}, \mathbb{F})$-supermartingale. Recalling that $v^*(t,.) \geq U$ we obtain

$$\mathbb{E}[U(X_{\tau_n})] \ \leq \ \mathbb{E}[v^*(\tau_n, X_{\tau_n})] \ \leq \ v^*(0, X_0), \qquad (\forall)\ n \geq 0.$$

Now, by Lemma 1.3.4, the sequence $(U(X_{\tau_n}))_n$ is uniformly integrable. By sending $n$ to infinity into the last inequality, we then get

$$\mathbb{E}[U(X_T)] \quad \leq \quad v^*(0, X_0).$$

Since $\alpha$ is arbitrary, we obtain $V_0 \leq v^*(0, X_0)$.

Conversely, let $\hat{\alpha} \in \mathcal{A}$ be the portfolio strategy given by (1.3.19), and $(\hat{X}_{\tau_n})_{n\geq 0}$ the associated wealth process. Then, by the same calculations as in (1.3.20), we have now the equalities:

$$\begin{aligned} \mathbb{E}[v^*(\tau_{n+1}, \hat{X}_{\tau_{n+1}})|\mathcal{F}_n] &= \int_{\tau_n}^T \int_{(-1,\infty)} \lambda(s) e^{-\int_{\tau_n}^s \lambda(u)du} v^*(s, X_{\tau_n} + \hat{\alpha}_n z) p(\tau_n, s, dz) ds \\ &= \mathcal{L}v^*(\tau_n, \hat{X}_{\tau_n}) \ = \ v^*(\tau_n, \hat{X}_{\tau_n}), \quad n \geq 0, \end{aligned}$$

by definition of $\mathcal{L}$ and $\hat{\alpha}$. This means that the process $\{v^*(\tau_n, \hat{X}_{\tau_n}), n \geq 0\}$ is a $(\mathbb{P}, \mathbb{F})$-martingale, and so:

$$\mathbb{E}[v^*(\tau_n, \hat{X}_{\tau_n})] = v^*(0, X_0), \qquad (\forall)\ n \geq 0.$$

From the the bounds $U \leq v^* \leq f$ and Lemma 1.3.4, we know that the sequence $(v^*(\tau_n, \hat{X}_{\tau_n}))_n$ is uniformly integrable. By sending $n$ to infinity into the last equality, and recalling the terminal condition for $v^*$, we then get

$$\mathbb{E}[U(\hat{X}_T)] = v^*(0, X_0).$$

Together with the inequality, $V_0 \leq v^*(0, X_0)$, this proves that $V_0 = v^*(0, X_0)$ and $\hat{\alpha}$ is an optimal control. □

An identical verification argument to the proof of Theorem 1.3.2 can be performed for an investor starting at time $t$ with initial capital $x$: this way we prove that $v^*$ is actually the value function of the control problem. In addition, this shows that the Dynamic Programming Equation (1.3.6) has a unique solution. For the sake of avoiding the heavy notation associated with strategies starting at time $t$, we decided to only do the verification for time $t = 0$.

The Proposition below shows that actually we can approximate the optimal control, and not only the maximal expected utility, using a finite number of iterations. The approximate optimal control is actually very simple, since after the $m$-th arrival time all the wealth is invested in the money market. In addition, a stochastic control representation for the iteration $v_m$ is provided.

**Proposition 1.3.1.** *Assume that* **(HL)**, **(NA)**, **(HU)** *and* **(HI)** *hold. Then*

$$v_m(0, X_0) = \sup_{\alpha \in \mathcal{A}_m} \mathbb{E}[U(X_T)], \tag{1.3.21}$$

*where $\mathcal{A}_m$ is the set of admissible controls $\alpha = (\alpha_n)_{\geq 0} \in \mathcal{A}$ such that all money is invested in the money market after $m$ arrivals, i.e. $\alpha_n = 0$ for $n \geq m$.*

*For any $0 \leq n \leq m-1$, consider the measurable function $\hat{\pi}^n(\cdot, \cdot)$ defined by*

$$\hat{\pi}^n(t, x) = \arg\max_{\pi \in [0,1]} \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u) du} v_{m-n-1}(s, x(1 + \pi z)) p(t, s, dz) ds,$$

*so that*

$$v_{m-n}(t, x) = \int_t^T \int_{(-1,\infty)} \lambda(s) e^{-\int_t^s \lambda(u) du} v_{m-n-1}(s, x(1 + \pi^n(t, x) z)) p(t, s, dz) ds.$$

*Define in feedback form the admissible strategy* $\hat{\alpha}^m \in \mathcal{A}_m$ *by* $\alpha_n^m = \hat{\pi}^n(\tau_n, \hat{X}_{\tau_n}^m)\hat{X}_{\tau_n}^m$ *for* $0 \leq n \leq m-1$ *and* $\alpha_n^m = 0$ *for* $n \geq m$*, where the wealth processes* $(\hat{X}_{\tau_n}^m)_{n \geq 0}$ *is given by*

$$\hat{X}_{\tau_{n+1}}^m = \hat{X}_{\tau_n}^m + \hat{\alpha}_n^m Z_{n+1}, \quad 0 \leq n \leq m-1, \qquad \hat{X}_{\tau_n}^m = \hat{X}_{\tau_m}^m, \quad n \geq m,$$

*starting from the initial wealth* $X_0$*. Then* $\alpha^m \in \mathcal{A}_m$ *is an optimal control for* (1.3.21).

**Proof.** The proof is based on similar arguments to the proof of Theorem 1.3.2. Namely, for each $\alpha \in \mathcal{A}_m$, one can use Lemma 1.3.1 to conclude that $(v_{m-n}(\tau_n, X_n))_{n=0,1,\ldots,m}$ is a supermartingale and, for the particular choice of the control $\alpha^m$ described above we actually have that $(v_{m-n}(\tau_n, \hat{X}_n^m))_{n=0,1,\ldots,m}$ is a true martingale. Since $v_0(t,x) = U(x)$ and for each $\alpha \in \mathcal{A}_m$ the wealth process $X$ is constant after the arrival time $\tau_m$, it is easy to finish the proof. □

Theorems 1.3.1 and 1.3.2 together show how we can compute by iterations the maximal expected utility and the optimal control. Since the control problem is finite-horizon in time and infinite horizon in $n$, taking into account Proposition 1.3.1, the iteration procedure represents exactly the approximation of the infinite horizon problem by a sequence of finite horizon problems.

## 1.4 Convergence in the illiquid market model

So far, we have considered the optimal investment problem (1.3.1) for a *fixed* arrival rate function $\lambda : [0,T) \to [0,\infty)$ satisfying condition (1.2.1). To emphasize the dependence on the arrival rate, let us denote by $V_0^\lambda$ the value in (1.3.1). When the arrival rate is very large *at all times* (in some sense to be precised), one would expect that $V_0^\lambda$ is very close to the optimal expected utility of an agent who can trade at all times (therefore continuously) in the asset $S$. It is also expected that the constraint (1.2.7), which is *implicitly* contained in the admissibility condition (1.2.6) in the time-illiquid case, becomes and *explicit* no-short sale constraint (1.2.10) in the continuous time limit. This section is devoted to proving that this is actually true.

First, we need to define the optimization problem for the agent who can trade continuously. We remind the reader that continuous time trading strategies can be defined by (1.2.9). We denote by $\mathcal{X}^S$ the set of positive wealth processes $(X_t)_{0 \leq t \leq T}$ given by (1.2.9), by using $\mathbb{G}^S$-predictable and $S$-integrable processes $H$ satisfying the no-short sale constraint (1.2.10). The filtration $\mathbb{G}^S = (\mathcal{G}_t^S)_{0\le t\le T}$ is defined by

$$\mathcal{G}_t^S = \sigma\{S_s, 0 \le s \le t\} \vee \mathcal{N},$$

and represents the information one can get from following the asset $S$. Because of the Lévy structure of $S$, $\mathbb{G}^S$ satisfies the usual conditions. We also denote by $\mathcal{D}(S)$ the set of all $\mathbb{G}^S$-predictable processes $\pi$ valued in $[0,1]$. It is then clear that

$$\mathcal{X}^S = \{X^{(\pi)} |\ \ \pi \in \mathcal{D}(S)\}.$$

The optimization problem for an agent trading continuously, under no-short selling constraints can be formulated as

$$V_0^M \quad = \quad \sup_{X\in\mathcal{X}^S} \mathbb{E}[U(X_T)] = \sup_{\pi\in\mathcal{D}(S)} \mathbb{E}[U(X_T^{(\pi)})]. \tag{1.4.1}$$

The main result of this section is:

**Theorem 1.4.1.** *Under Assumptions* **(HL)***,* **(NA)***,* **(HU)** *and* **(HI)***, consider a sequence* $(\lambda_k)_k$ *of intensity functions such that each* $\lambda_k$ *satisfies* (1.2.1)*, and*

$$\lambda_k(t) \to \infty \quad \text{as } k \to \infty, \ \ \forall t \in [0,T]. \tag{1.4.2}$$

*Then* $V_0^{\lambda_k} \to V_0^M$ *as* $k \to \infty$*, where* $V_0^M$ *is defined by* (1.4.1)*.*

In order to prove Theorem 1.4.1, we first have to put all the optimization problems (1.3.1) on *the same physical probability space*, independent of the intensity function $\lambda$. This is an easy task actually. We consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ supporting two independent processes: the continuous time stock price process $(S_t)_{0\le t\le T}$ (which has all the desired properties) and a Poisson process $(M_t)_{0\le t<\infty}$ with intensity equal to one. After that, for each intensity function $\lambda$ we define the nonhomogenous Poisson process $N^\lambda$ (actually its sequence of jumps) by (1.2.2). Therefore, for different intensities, we still have the same physical space. We now denote by $\mathbb{F}^\lambda$ and $\mathbb{G}^\lambda$ the discrete and continuous time filtrations on $(\Omega, \mathcal{G}, \mathbb{P})$ defined by (1.2.4) and (1.2.8) corresponding to the intensity $\lambda$, and by $\tau_n^\lambda$ the associated jump times.

The main obstacle in proving Theorem 1.4.1 is the fact that the filtration $\mathbb{F}^\lambda$ only observes the process $S$ at the arrival times, while the filtration $\mathbb{G}^S$ used by the investor in (1.4.1) observes the stock continuously. This problem is overcome in three steps

**Step 1**: first, we show that in (1.3.1), the discrete-time filtration $\mathbb{F}^\lambda = (\mathcal{F}_n^\lambda)_{n\geq 0}$ can be replaced by the larger filtration $(\mathcal{G}_{\tau_n^\lambda}^\lambda)_{n\geq 0}$. In other words, due to the Markov structure of the model, an investor who can only trade at the discrete arrival times, cannot improve his/her expected utility by continuously observing the evolution of the stock between the arrival times. This is done in Lemma 1.4.1 below.

**Lemma 1.4.1.** *Fix an intensity function* $\lambda$ *and define*

$$V_0^{\lambda,c} \quad := \quad \sup_{\alpha\in\mathcal{A}_c^\lambda} \mathbb{E}[U(X_T)], \tag{1.4.3}$$

*where* $\mathcal{A}_c^\lambda$ *is the set of simple admissible strategies* $\alpha = (\alpha_n)_{n\geq 0}$ *with continuous observation, which means that for each* $n \geq 0$ *we have* $\alpha_n \in \mathcal{G}_{\tau_n^\lambda}^\lambda$ *and* $\alpha$ *satisfies the constraint* (1.2.7) *for the wealth process* $(X_{\tau_n})_{n\geq 0}$ *defined by* (1.2.5).

*Then, under Assumptions* **(HL)**, **(NA)**, **(HU)** *and* **(HI)**, *we have* $V_0^{\lambda,c} = V_0^\lambda$.

**Proof.** Assumption (**HL**) together with the independence of $S$ and $N$ ensures that for all $n \geq 0$, the (regular) distribution of $(\tau_{n+1}, Z_{n+1})$ conditioned on $\mathcal{G}_{\tau_n^\lambda}^\lambda$ is given by:

1. $\mathbb{P}[\tau_{n+1}^\lambda \in ds|\mathcal{G}_{\tau_n^\lambda}^\lambda] = \lambda(s)e^{-\int_{\tau_n}^s \lambda(u)du}ds$
2. further conditioning on knowing the next arrival time $\tau_{n+1}^\lambda$, the return $Z_{n+1}$ has distribution

$$\mathbb{P}[Z_{n+1} \in dz|\mathcal{G}_{\tau_n^\lambda}^\lambda \vee \sigma(\tau_{n+1}^\lambda)] = p(\tau_n^\lambda, \tau_{n+1}^\lambda, dz).$$

Therefore, in Lemma 1.3.1, one can replace the filtration $\mathbb{F}^\lambda$ by the larger filtration $(\mathcal{G}_{\tau_n^\lambda}^\lambda)_{n\geq 0}$ and obtain that, for each $\alpha \in \mathcal{A}_c^\lambda$ we have

$$\mathbb{E}[v(\tau_{n+1}^\lambda, X_{\tau_{n+1}^\lambda})|\mathcal{G}_{\tau_n^\lambda}^\lambda] \quad = \quad \int_{\tau_n^\lambda}^T \int_{(-1,\infty)} \lambda(s)e^{-\int_{\tau_n^\lambda}^s \lambda(u)du} v(s, X_{\tau_n^\lambda} + \alpha_n z)p(\tau_n^\lambda, s, dz)ds.$$

After that, one can just follow the verification arguments in the proof of Theorem 1.3.2 to show that

$$V_0^{\lambda,c} = v^{*,\lambda}(0, X_0),$$

which ends the proof. □

**Step 2**: we define the continuous time filtration $\mathbb{G}^\infty = (\mathcal{G}_t^\infty)_{0\leq t\leq T}$ which contains all the information from the arrival times right at time zero. This corresponds to an investor who knows in advance all the jumps of the homogeneous Poisson process $M$ and also observes continuously the stock $S$ up to time $t$:

$$\mathcal{G}_t^\infty = \mathcal{G}_t^S \vee \sigma(M_u :\ 0 \le u < \infty), \quad 0 \le t < T.$$

Because the information added is independent, the process $S$ is still a semimartingale with respect to the larger filtration $\mathbb{G}^\infty$, which satisfies the usual conditions as well. Using again the independence property, Lemma 1.4.2 below shows that if the investor in (1.4.1) has the additional information in $\mathbb{G}^\infty$, he/she cannot improve the maximal expected utility.

**Lemma 1.4.2.** *Consider the set $\bar{\mathcal{X}}_c$ of wealth processes defined by* (1.2.9) *where the general integrand $H$ is $\mathbb{G}^\infty$-predictable, $S$-integrable and satisfies* (1.2.10)*. Define*

$$V_0^\infty \ := \ \sup_{X \in \bar{\mathcal{X}}_c} \mathbb{E}[U(X_T)].$$

*Then $V_0^\infty = V_0^M$.*

**Proof.** Since $\mathcal{X}^S \subset \bar{\mathcal{X}}_c$, we obviously have $V_0^M \le V_0^\infty$. Now take some arbitrary $X \in \bar{\mathcal{X}}_c$ associated to a no-short sale trading strategy $H$, which is $\mathbb{G}^\infty$-predictable. Consider the $\mathbb{G}^S$-predictable projection of $H$: $\hat{H}_s = \mathbb{E}[H_s | \mathcal{G}_{s-}^S]$, $t \le s \le T$. We then have (see e.g. Lemma 2.2.4 in [59])

$$\hat{X}_t \ := \ \mathbb{E}[X_t | \mathcal{G}_t^S] \ = \ X_0 + \int_0^t \hat{H}_u dS_u, \quad 0 \le t \le T.$$

This means that the process $\hat{X}$ lies in $\mathcal{X}^S$. Since $U$ is concave, we get by the law of iterated conditional expectations and Jensen's inequality

$$\mathbb{E}[U(X_T)] \ = \ \mathbb{E}\Big[\mathbb{E}[U(X_T)|\mathcal{G}_T^S]\Big] \le \mathbb{E}\Big[U\Big(E[X_T|\mathcal{G}_T^S]\Big)\Big] \ = \ \mathbb{E}[U(\hat{X}_T)] \ \le \ V_0^M.$$

We conclude from the arbitrariness of $X$ in $\bar{\mathcal{X}}_c$. □

**Step 3**: Once we prove the step above and transform the Merton problem in a utility maximization problem with no short-sale constraints under the filtration $\mathbb{G}^\infty$, we can basically follow the arguments in Theorems 3.1, 3.3 and 4.1 in [39], to finish the proof. In order to use the ideas in [39] we still need the following technical lemma.

**Lemma 1.4.3.** *Consider a sequence $(\lambda_k)_k$ of intensity functions such that each $\lambda_k$ satisfies*

(1.2.1), *and*

$$\lambda_k(t) \to \infty \quad \text{as } k \to \infty, \ \ \forall t \in [0,T].$$

*Then, the jump times* $\{(\tau_n^k)_n, k \in \mathbb{N}\}$ *of* $\{N^{\lambda_k}, k \in \mathbb{N}\}$*, satisfies up to a subsequence (in* $k$*),*

$$\sup_n \left|\tau_{n+1}^k - \tau_n^k\right| \to 0 \ \text{a.s.} \quad \text{when } k \to \infty. \tag{1.4.4}$$

**Proof.** Consider subdivisions $0 = t_0^k < \ldots < t_{M_k}^k = T$ such that $\sup_{i=0,\ldots,M_k-1}(t_{i+1} - t_i) \leq 1/2k$. By Fatou's lemma, $\lim_{l\to\infty} \int_s^t \lambda_l(u)du = \infty$ for all $0 \leq s < t \leq T$, so up to a subsequence indexed by $l_k$ we can assume that

$$\sum_{i=0}^{M_k-1} \exp\Big(-\int_{t_i^k}^{t_{i+1}^k} \lambda_{l_k}(u)du\Big) \ \leq \ 2^{-k}, \ \ \forall k \geq 0.$$

Now, from the relations

$$\begin{aligned}
\mathbb{P}\Big[\sup_n \left|\tau_{n+1}^{l_k} - \tau_n^{l_k}\right| > 1/k\Big] \ &\leq \ \sum_{i=0}^{M_k-1} \mathbb{P}\Big[\exists n, \tau_n^{l_k} \leq t_i^k < t_{i+1}^k < \tau_{n+1}^{l_k}\Big] \\
&= \ \sum_{i=0}^{M_k-1} \mathbb{P}\Big[N_{t_i^k}^{\lambda_{l_k}} = N_{t_{i+1}^k}^{\lambda_{l_k}}\Big] \\
&= \ \sum_{i=0}^{M_k-1} \exp\Big(-\int_{t_i^k}^{t_{i+1}^k} \lambda_{l_k}(u)\,du\Big),
\end{aligned}$$

we see that for all $\epsilon > 0$,

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}\Big[\sup_n \left|\tau_{n+1}^{l_k} - \tau_n^{l_k}\right| > \epsilon\Big] \ &\leq \ \sum_{k<\epsilon^{-1}} \mathbb{P}\Big[\sup_n \left|\tau_{n+1}^{l_k} - \tau_n^{l_k}\right| > \epsilon\Big] + \sum_{k \geq \epsilon^{-1}} 2^{-k} \\
&< \ \infty,
\end{aligned}$$

and thus by Borel-Cantelli we have

$$\sup_n \left|\tau_{n+1}^{l_k} - \tau_n^{l_k}\right| \to 0 \text{ a.s.} \quad \text{when } k \to \infty,$$

which ends the proof of the Lemma. □

**Proof of Theorem 1.4.1**. Using Lemma 1.4.3 and taking further subsequences of any sequence, it is enough to prove Theorem 1.4.1 under the additional assumption (1.4.4).

We first represent continuous time trading strategies with no short sale constraints in terms of proportion of wealth, for the larger filtration $\mathbb{G}^{\infty}$. For any continuous time wealth process $X$

$\in \bar{\mathcal{X}}_c$ associated to a trading strategy $H$ satisfying (1.2.10), we still denote by $\pi_t = H_t S_{t-}/X_{t-}$, and notice that the process $(\pi_t)_{0\le t\le T}$ is valued in $[0,1]$ by (1.2.10). We also denote by $X^{(\pi)}$ the process defined by (1.2.11) and define $\mathcal{D}^\infty$ to be the set of all $\mathbb{G}^\infty$-predictable processes $\pi$ valued in $[0,1]$. It is then clear that

$$V_0^\infty \quad = \quad \sup_{\pi\in\mathcal{D}^\infty} \mathbb{E}[U(X_T^{(\pi)})].$$

Using Lemma 1.4.1 and Lemma 1.4.2 we have, for each intensity function $\lambda$ that

$$V_0^\lambda = V_0^{\lambda,c} \le V_0^\infty = V_0^M. \tag{1.4.5}$$

Let now $\pi \in \mathcal{D}(S) \subset \mathcal{D}^\infty$ be a (proportional) trading strategy in (1.4.1) and let $(\lambda_k)_k$ a sequence of intensity functions as in Theorem 1.4.1. We will follow the arguments in [39] to approximate this continuous time trading strategy by a sequence of simple strategies $\alpha^k \in \mathcal{A}_c^{\lambda_k}$, which are discrete, but *use information from continuous observations.* First, according to Lemma 3.4 and 3.5 in [39], there exists a sequence $\pi^m \in \mathcal{D}(S)$ such that each $\pi^m$ is LCRL (left continuous with right limits) and such that

$$uc\mathbb{P} - \lim_{m\to\infty} X^{(\pi^m)} = X^{(\pi)},$$

so, in order to approximate $\pi$ we can actually assume it is LCRL. Here, by $uc\mathbb{P}$-convergence, we mean the usual convergence of processes in probability, uniformly on $[0,T]$.

Next, let us define

$$\pi_n^k = \pi_{\tau_n^{\lambda^k}+}, \quad n \ge 0,$$

where $\pi_{t+} = \lim_{u\searrow t}\pi_t$. Because the filtration $\mathbb{G}^{\lambda^k}$ satisfies the usual conditions we have that the process $(\pi_{t+})_{0\le t\le T}$ is optional with respect to $\mathbb{G}^{\lambda_k}$ for each $k$. Since, in addition, $\tau_n^{\lambda_k}$ are stopping times with respect to $\mathbb{G}^{\lambda_k}$ we obtain that, for each $k$,

$$\pi_n^k \in \mathcal{G}_{\tau_n^{\lambda_k}}^{\lambda^k}, \qquad k \ge 0. \tag{1.4.6}$$

Therefore, if we define, for each fixed $k$ the discrete-time wealth process by

$$X_{\tau_{n+1}^{\lambda_k}} \quad = \quad X_{\tau_n^{\lambda_k}}(1+\pi_n^k Z_{n+1}), \quad n \ge 0, \tag{1.4.7}$$

and denote by $\alpha_n^k = \pi_n^k X_{\tau_n^{\lambda_k}}$ we have $\alpha^k = (\alpha_n^k)_{n\geq 0} \in \mathcal{A}_c^{\lambda_k}$. To each of the above defined $\alpha^k$, we can associate by Remark 1.2.2 a continuous time simple integrand $H^k$ which is $\mathbb{G}^{\lambda_k}$-predictable and the continuous time wealth process $(X_t^k)_{0\leq t\leq T}$. The fundamental observation is now that all $H^k$ are predictable with respect to the same “large" filtration $\mathbb{G}^\infty$. Using this universal filtration, we can now follow the proof of Theorem 3.1 in [39], which actually works for stochastic partitions under condition (1.4.4), to conclude that

$$X^{(\pi)} = uc\mathbb{P} - \lim_{k\to\infty} X^k.$$

Therefore, we can approximate any continuous time strategy in the Merton problem (1.4.1) by simple trading strategies $\alpha^k \in \mathcal{A}_c^{\lambda_k}$. The rather obvious details on how approximation of strategies leads to approximation of optimal expected utility are identical to the arguments in [39] Section 4, and are omitted: this means that for all $\pi \in \mathcal{D}(S)$, $\mathbb{E}[U(X_T^{(\pi)})] = \lim_k \mathbb{E}[U(X_T^k)]$, and so $V_0^M \leq \liminf_k V_0^{\lambda_k,c}$. Together with (1.4.5), this concludes the proof of Theorem 1.4.1.

□

# Chapter 2

# Investment/consumption problem in illiquid markets with regime switching

Abstract: We consider an illiquid financial market with different regimes modeled by a continuous-time finite-state Markov chain. The investor can trade a stock only at the discrete arrival times of a Cox process with intensity depending on the market regime. Moreover, the risky asset price is subject to liquidity shocks, which change its rate of return and volatility, and induce jumps on its dynamics. In this setting, we study the problem of an economic agent optimizing her expected utility from consumption under a non-bankruptcy constraint. By using the dynamic programming method, we provide the characterization of the value function of this stochastic control problem in terms of the unique viscosity solution to a system of integro-partial differential equations. We next focus on the popular case of CRRA utility functions, for which we can prove smoothness $C^2$ results for the value function. As an important byproduct, this allows us to get the existence of optimal investment/consumption strategies characterized in feedback forms. We analyze a convergent numerical scheme for the resolution to our stochastic control problem, and we illustrate finally with some numerical experiments the effects of liquidity regimes in the investor's optimal decision.

**Key words :** Optimal consumption, liquidity effects, regime-switching models, viscosity solutions, integro-differential system.

## 2.1 Introduction

A classical assumption in the theory of optimal portfolio/consumption choice as in Merton [54] is that assets are continuously tradable by agents. This is not always realistic in practice, and illiquid markets provide a prime example. Indeed, an important aspect of market liquidity is the time restriction on assets trading: investors cannot buy and sell them immediately, and have to wait some time before being able to unwind a position in some financial assets. In the past years, there was a significant strand of literature addressing these liquidity constraints. In [66], [53], the price process is observed continuously but the trades succeed only at the jump times of a Poisson process. Recently, the papers [61], [17], [29] relax the continuous-time price observation by considering that asset is observed only at the random trading times. In all these cited papers, the intensity of trading times is constant or deterministic. However, the market liquidity is also affected by long-term macroeconomic conditions, for example by financial crisis or political turmoil, and so the level of trading activity measured by its intensity should vary randomly over time. Moreover, liquidity breakdowns would typically induce drops on the stock price in addition to changes in its rate of return and volatility.

In this paper, we investigate the effects of such liquidity features on the optimal portfolio choice. We model the index of market liquidity as an observable continuous-time Markov chain with finite-state regimes, which is consistent with some cyclicality observed in financial markets. The economic agent can trade only at the discrete arrival times of a Cox process with intensity depending on the market regimes. Moreover, the risky asset price is subject to liquidity shocks, which switch its rate of return and volatility, while inducing jumps on its dynamics. In this hybrid jump-diffusion setting with regime switching, we study the optimal investment/consumption problem over an infinite horizon under a nonbankruptcy state constraint. We first prove that dynamic programming principle (DPP) holds in our framework. Due to the state constraints in two dimensions, we have to slightly weaken the standard continuity assumption, see Remark 2.3.1. Then, using DPP, we characterize the value function of this stochastic control problem as the unique constrained viscosity solution to a system of integro-partial differential equations. In the particular case of CRRA utility function, we can go beyond the viscosity properties, and prove $C^2$ regularity results for the value function in the interior of the domain. As a consequence, we show the existence of optimal strategies expressed in feedback form in terms of the derivatives of the value function. Due to the presence of state constraints, the value function is not smooth at the boundary, and so the verification theorem cannot be proved with the classical arguments of Dynkin's formula. To overcome this technical problem, we use an ad hoc approximation procedure (see Proposition 2.5.2). We also provide a convergent numerical scheme for solving the system of equations characterizing our control problem, and we illustrate with some numerical results the effect of liquidity regimes in the agent's optimal investment/consumption. We also measure the impact of continuous time observation with respect to a discrete time observation of the stock prices. Our paper contributes and extends the existing literature in several ways. First, we extend the papers [66] and [53] by considering stochastic intensity trading times and regime switching in the asset prices. For a two-state Markov chain modulating the market liquidity, and in the limiting case where the intensity in one regime goes to infinity, while the other one goes to zero, we recover the setup of [21] and [49] where an investor can trade continuously in the perfectly liquid regime but faces a threat of trading interruptions during a period of market freeze. On the other hand, regime switching models in optimal investment problems was already used in [72], [69] or [63] for continuous-time trading.

The rest of the paper is structured as follows. Section 2 describes our continuous-time market model with regime-switching liquidity, and formulates the optimization problem for the investor. In Section 3 we state some useful properties of the value function of our stochastic control problem. Section 4 is devoted to the analytic characterization of the value function as the unique viscosity solution to the dynamic programming equation. The special case of CRRA utility functions is studied in Section 5: we show smoothness results for the value functions, and obtain the existence of optimal strategies via a verification theorem. Some numerical illustrations complete this last section. Finally two appendices are devoted to the proof of two technical results: the dynamic programming principle, and the existence and uniqueness of viscosity solutions.

## 2.2 A market model with regime-switching liquidity

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Let $I$ be a continuous-time Markov chain valued in the finite state space $\mathbb{I}_d = \{1, \ldots, d\}$, with intensity matrix $Q = (q_{ij})$. For $i \neq j$ in $\mathbb{I}_d$, we can associate to the jump process $I$, a Poisson process $N^{ij}$ with intensity rate $q_{ij} \geq 0$, representing the number of switching from state $i$ to $j$. We interpret the process $I$ as a proxy for market liquidity with states (or regimes) representing the level of liquidity activity, in the sense that the intensity of trading times varies with the regime value. This is modeled through a Cox process $(N_t)_{t\geq 0}$ with intensity $(\lambda_{I_t})_{t\geq 0}$, where $\lambda_i > 0$ for each $i \in \mathbb{I}_d$ . For example, if $\lambda_i < \lambda_j$, this means that trading times occur more often in regime $j$ than in regime $i$. The increasing sequence of jump times $(\tau_n)_{n\geq 0}$, $\tau_0 = 0$, associated to the counting process $N$ represents the random times when an investor can trade a risky asset of price process $S$.

**Remark 2.2.1.** Notice that the jumps of $I$ and $N$ are a.s. disjoint. Indeed, for any $n$,

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\{\Delta I_{\tau_n}\neq 0\}}|I, \tau_{n-1}\right] &= \int_{\tau_{n-1}}^{\infty} \mathbf{1}_{\{\Delta I_t \neq 0\}} \lambda_{I_t} e^{-\int_{\tau_{n-1}}^{t} \lambda_{I_u} du} dt \\ &= 0 \text{ a.s.}, \end{aligned}$$

since almost surely $I$ has countably many jumps.

In the liquidity regime $I_t = i$, the stock price follows the dynamics

$$dS_t = S_t(b_i dt + \sigma_i dW_t),$$

where $W$ is a standard Brownian motion independent of $(I, N)$, and $b_i \in \mathbb{R}$, $\sigma_i \geq 0$, for $i \in \mathbb{I}_d$. Moreover, at the times of transition from $I_{t^-} = i$ to $I_t = j$, the stock changes as follows:

$$S_t = S_{t-}(1 - \gamma_{ij})$$

for a given $\gamma_{ij} \in (-\infty, 1)$, so the stock price remains strictly positive, and we may have a relative loss (if $\gamma_{ij} > 0$), or gain (if $\gamma_{ij} \leq 0$). Typically, there is a drop of the stock price after a liquidity breakdown, i.e. $\gamma_{ij} > 0$ for $\lambda_j < \lambda_i$. Overall, the risky asset is governed by a regime-switching jump-diffusion model:

$$dS_t = S_{t^-}\Big(b_{I_{t^-}} dt + \sigma_{I_{t^-}} dW_t - \gamma_{I_{t^-}, I_t} dN_t^{I_{t^-}, I_t}\Big). \tag{2.2.1}$$

*Portfolio dynamics under liquidity constraint.* We consider an agent investing and consuming in this regime-switching market. We denote by $(Y_t)$ the total amount invested in the stock, and by $(c_t)$ the consumption rate per unit of time, which is a nonnegative adapted process. Since the number of shares $Y_t/S_t$ in the stock held by the investor has to be kept constant between two trading dates $\tau_n$ and $\tau_{n+1}$, then between such trading times, the process $Y$ follows the dynamics:

$$dY_t \quad = \quad Y_{t^-}\frac{dS_t}{S_{t^-}}, \quad \tau_n \leq t < \tau_{n+1}, \ n \geq 0,$$

The trading strategy is represented by a predictable process $(\zeta_t)$ such that at a trading time $t = \tau_{n+1}$, the rebalancing on the number of shares induces a jump $\zeta_t$ in the amount invested in the stock :

$$\Delta Y_t \quad = \quad \zeta_t.$$

Overall, the càdlàg process $Y$ is governed by the hybrid controlled jump-diffusion process

$$dY_t \quad = \quad Y_{t^-}\Big(b_{I_{t^-}}dt + \sigma_{I_{t^-}}dW_t - \gamma_{I_{t^-},I_t}dN_t^{I_{t^-},I_t}\Big) + \zeta_t dN_t. \tag{2.2.2}$$

Assuming a constant savings account, i.e. zero interest rate, the amount $(X_t)$ invested in cash then follows

$$dX_t \quad = \quad -c_t dt - \zeta_t dN_t. \tag{2.2.3}$$

The total wealth is defined at any time $t \geq 0$, by $R_t = X_t + Y_t$, and we shall require the non-bankruptcy constraint at any trading time:

$$R_{\tau_n} \quad \geq \quad 0, \quad a.s. \quad \forall n \geq 0. \tag{2.2.4}$$

Actually, this non-bankruptcy constraint means a no-short sale constraint on both the stock and savings account, as showed by the following Lemma.

**Lemma 2.2.1.** *The nonbankruptcy constraint (2.2.4) is formulated equivalently in the no-short sale constraint:*

$$X_t \geq 0, \quad and \quad Y_t \geq 0, \quad \forall t \geq 0. \tag{2.2.5}$$

*This is also written equivalently in terms of the controls as:*

$$-Y_{t^-} \ \leq \ \zeta_t \ \leq \ X_{t^-}, \quad t \geq 0, \tag{2.2.6}$$

$$\int_t^{\tau_{n+1}} c_s ds \quad \leq \quad X_t, \quad \tau_n \leq t < \tau_{n+1}, \ n \geq 0. \tag{2.2.7}$$

**Proof.** By writing by induction the wealth at any trading time as

$$R_{\tau_{n+1}} \quad = \quad R_{\tau_n} + Y_{\tau_n}\Big(\frac{S_{\tau_{n+1}}}{S_{\tau_n}} - 1\Big) - \int_{\tau_n}^{\tau_{n+1}} c_t dt, \quad n \geq 0,$$

and since (conditionally on $\mathcal{F}_{\tau_n}$) the stock price $S_{\tau_{n+1}}$ has full support in $(0, \infty)$, we see that the nonbankruptcy condition $R_{\tau_{n+1}} \geq 0$ is equivalent to a no-short sale constraint:

$$0 \ \leq \ Y_{\tau_n} \ \leq \ R_{\tau_n}, \quad n \geq 0, \tag{2.2.8}$$

together with the condition on the nonnegative consumption rate

$$\int_{\tau_n}^{\tau_{n+1}} c_t dt \quad \leq \quad R_{\tau_n} - Y_{\tau_n} = X_{\tau_n}, \quad n \geq 0. \tag{2.2.9}$$

Since $Y_{\tau_n} = Y_{\tau_n -} + \zeta_{\tau_n}$, and since $R_{\tau_n} = R_{\tau_n -}$ by Remark 2.2.1, the no-short sale constraint (2.2.8) means equivalently that (2.2.6) is satisfied for $t = \tau_n$. Since $\zeta$ is predictable, this is equivalent to (2.2.6) being satisfied $d\mathbb{P} \otimes dt$ almost everywhere. Indeed, letting $H_t = \mathbf{1}_{\{\zeta_t < -Y_{t-} \text{ or } \zeta_t > X_{t-}\}}$, $H$ is predictable, so that $\forall t \geq 0$, $0 = \mathbb{E}\Big[\sum_{\tau_n \leq t} H_{\tau_n}\Big] = \mathbb{E}\Big[\int_0^t H_s \lambda_{I_s} ds\Big]$, and we deduce that $H_t = 0$ $d\mathbb{P} \otimes dt$ a.e. since $\lambda_{I_t} > 0$.

Moreover, since $X_t = X_{\tau_n} - \int_{\tau_n}^t c_s ds$ for $\tau_n \leq t < \tau_{n+1}$, the condition (2.2.9) is equivalent to (2.2.7). By rewriting the conditions (2.2.8)-(2.2.9) as

$$Y_{\tau_n} \geq 0, \quad X_{\tau_n} \geq 0, \quad X_{(\tau_{n+1})-} \geq 0, \quad \forall n \geq 0,$$

and observing that for $\tau_n \leq t < \tau_{n+1}$,

$$Y_t \ = \ \frac{S_t}{S_{\tau_n}} Y_{\tau_n}, \qquad X_{\tau_n} \ \geq \ X_t \ \geq X_{(\tau_{n+1})-},$$

we see that they are equivalent to (2.2.5). □

**Remark 2.2.2.** Under the nonbankruptcy (or no-short sale constraint), the wealth $(R_t)_{t \geq 0}$ is nonnegative, and follows the dynamics:

$$dR_t \quad = \quad R_{t^-} Z_{t^-}\Big(b_{I_{t^-}} dt + \sigma_{I_{t^-}} dW_t - \gamma_{I_{t^-}, I_t} dN_t^{I_{t^-}, I_t}\Big) - c_t dt, \tag{2.2.10}$$

where $Z$ valued in $[0, 1]$ is the proportion of wealth invested in the risky asset:

$$Z_t \quad = \quad \begin{cases} \frac{Y_t}{R_t}, & R_t > 0 \\ 0, & R_t = 0, \end{cases}$$

evolving according to the dynamics:

$$
\begin{aligned}
dZ_t &= Z_{t^-}(1-Z_{t^-})\Big[(b_{I_{t^-}} - Z_{t^-}\sigma^2_{I_{t^-}})dt + \sigma_{I_{t^-}} dW_t - \frac{\gamma_{I_{t^-},I_t}}{1-Z_{t^-}\gamma_{I_{t^-},I_t}} dN_t^{I_{t^-},I_t}\Big] \\
& \quad + \frac{\zeta_t}{R_{t^-}} dN_t \; + \; Z_{t^-}\frac{c_t}{R_{t^-}} dt,
\end{aligned}
\tag{2.2.11}
$$

for $t < \tau = \inf\{t \geq 0 : R_t = 0\}$.

Given an initial state $(i,x,y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+$, we shall denote by $\mathcal{A}_i(x,y)$ the set of investment/consumption control process $(\zeta, c)$ such that the corresponding process $(X,Y)$ solution to (2.2.2)-(2.2.3) with a liquidity regime $I$, and starting from $(I_{0^-}, X_{0^-}, Y_{0^-}) = (i,x,y)$, satisfy the non-bankruptcy constraint (2.2.5) (or equivalently (2.2.6)-(2.2.7)).

*Optimal investment/consumption problem.* The preferences of the agent are described by a utility function $U$ which is increasing, concave, $C^1$ on $(0,\infty)$ with $U(0) = 0$, and satisfies the usual Inada conditions: $U'(0) = \infty$, $U'(\infty) = 0$. We assume the following growth condition on $U$ : there exist some positive constant $K$, and $p \in (0,1)$ s.t.

$$
U(x) \;\leq\; Kx^p, \quad x \geq 0. \tag{2.2.12}
$$

We denote by $\tilde{U}$ the convex conjugate of $U$, defined from $\mathbb{R}$ into $[0,\infty]$ by:

$$
\tilde{U}(\ell) \;=\; \sup_{x \geq 0}[U(x) - x\ell],
$$

which satisfies under (2.2.12) the dual growth condition on $\mathbb{R}_+$:

$$
\tilde{U}(\ell) \;\leq\; \tilde{K}\ell^{-\tilde{p}}, \quad \forall \ell \geq 0, \quad \text{with } \tilde{p} \;=\; \frac{p}{1-p} \;>\; 0, \tag{2.2.13}
$$

for some positive constant $\tilde{K}$.

The agent's objective is to maximize over portfolio/consumption strategies in the above illiquid market model the expected utility from consumption rate over an infinite horizon. We then consider, for each $i \in \mathbb{I}_d$, the value function

$$
v_i(x,y) \;=\; \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right], \quad (x,y) \in \mathbb{R}^2_+, \tag{2.2.14}
$$

where $\rho > 0$ is a positive discount factor. We also introduce, for $i \in \mathbb{I}_d$, the function

$$\hat{v}_i(r) \;=\; \sup_{x\in[0,r]} v_i(x, r-x), \quad r \geq 0, \tag{2.2.15}$$

which represents the maximal utility performance that the agent can achieve starting from an initial nonnegative wealth $r$ and from the regime $i$. More generally, for any locally bounded function $w_i$ on $\mathbb{R}^2_+$, we associate the function $\hat{w}_i$ defined on $\mathbb{R}_+$ by: $\hat{w}_i(r) = \sup_{x\in[0,r]} w_i(x, r-x)$, so that:

$$\hat{w}_i(x+y) \;=\; \sup_{e\in[-y,x]} w_i(x-e, y+e), \quad (x,y) \in \mathbb{R}^2_+.$$

In the sequel, we shall often identify a $d$-tuple function $(w_i)_{i\in\mathbb{I}_d}$ defined on $\mathbb{R}^2_+$ with the function $w$ defined on $\mathbb{R}^2_+ \times \mathbb{I}_d$ by $w(x,y,i) = w_i(x,y)$.

In this paper, we focus on the analytic characterization of the value functions $v_i$ (and so $\hat{v}_i$), $i \in \mathbb{I}_d$, and on their numerical approximation.

## 2.3 Some properties of the value function

We state some preliminary properties of the value functions that will be used in the next section for the PDE characterization. We first need to check that the value functions are well-defined and finite. Let us consider for any $p > 0$, the positive constant:

$$k(p) \;:=\; \max_{i\in\mathbb{I}_d, z\in[0,1]} \Big[ pb_i z - \frac{\sigma_i^2}{2} p(1-p)z^2 + \sum_{j\neq i} q_{ij}((1-z\gamma_{ij})^p - 1) \Big] \;<\; \infty.$$

We then have the following lemma.

**Lemma 2.3.1.** *Fix some initial conditions $(i,x,y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+$, and some $p > 0$. Then:*

**(1)** *For any admissible control $(\zeta, c) \in \mathcal{A}_i(x,y)$ associated with wealth process $R$, the process $(e^{-k(p)t} R_t^p)_{t\geq 0}$ is a supermartingale. So, in particular, for $\rho > k(p)$,*

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}[R_t^p] \;=\; 0. \tag{2.3.1}$$

**(2)** *For fixed $T \in (0,\infty)$, the family $(R^p_{T\wedge\tau})_{\tau,\zeta,c}$ is uniformly integrable, when $\tau$ ranges over all stopping times, and $(\zeta, c)$ runs over $\mathcal{A}_i(x,y)$.*

**Proof.** **(1)** By Itô's formula and (2.2.10), we have

$$\begin{aligned} d(e^{-k(p)t}R_t^p) &= -k(p)e^{-k(p)t}R_t^p dt + e^{-k(p)t}d(R_t^p) \\ &= e^{-k(p)t}\Big[ -k(p)R_t^p + pR_{t-}^{p-1}\left(-c_t + b_{I_{t-}}R_{t-}Z_{t-}\right) + \frac{p(p-1)}{2}R_{t-}^{p-2}(\sigma_{I_{t-}}R_{t-}Z_{t-})^2 \\ &\qquad + \sum_{j\neq I_{t-}} q_{I_{t-},j}(R_{t-}^p(1-\gamma_{I_{t-}j}Z_{t-})^p - R_{t-}^p)\Big]dt + dM_t, \end{aligned}$$

where $M$ is a local martingale. Now, by definition of $k(p)$, we have

$$\begin{aligned} pR_{t-}^{p-1}\left(-c_t + b_{I_{t-}}R_{t-}Z_{t-}\right) + \frac{p(p-1)}{2}R_{t-}^{p-2}(\sigma_{I_{t-}}R_{t-}Z_{t-})^2 & \\ + \sum_{j\neq I_{t-}} q_{I_{t-},j}(R_{t-}^p(1-\gamma_{I_{t-}j}Z_{t-})^p - R_{t-}^p) &\leq -pc_tR_{t-}^{p-1} + k(p)R_{t-}^p \\ &\leq k(p)R_{t-}^p. \end{aligned}$$

Since $R$ has countable jumps, $R_t = R_{t-}$, $d\mathbb{P}\otimes dt$ a.e., and so the drift term in $d(e^{-k(p)t}R_t^p)$ is nonpositive. Hence $(e^{-k(p)t}R_t^p)_{t\geq 0}$ is a local supermartingale, and since it is nonnegative, it is a true supermartingale by Fatou's lemma. In particular, we have

$$0 \leq e^{-\rho t}\mathbb{E}[R_t^p] \leq e^{-(\rho-k(p))t}(x+y)^p \tag{2.3.2}$$

which shows (2.3.1).

**(2)** For any $q > 1$, we get by the supermartingale property of the process $(e^{-k(pq)t}R_t^{pq})_{t\geq 0}$ and the optional sampling theorem:

$$\mathbb{E}\big[(R_{T\wedge\tau}^p)^q\big] \;\leq\; e^{k(pq)T}(x+y)^{pq} \;<\; \infty, \quad \forall(\zeta,c)\in\mathcal{A}_i(x,y),\ \tau \text{ stopping time },$$

which proves the required uniform integrability. □

The next proposition states a comparison result, and, as a byproduct, a growth condition for the value function.

**Proposition 2.3.1.**

**(1)** *Let* $w = (w_i)_{i\in\mathbb{I}_d}$ *be a* $d$*-tuple of nonnegative functions on* $\mathbb{R}_+^2$*, twice differentiable on*

$\mathbb{R}^2_+ \setminus \{(0,0)\}$ *such that*

$$\rho w_i - b_i y \frac{\partial w_i}{\partial y} - \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 w_i}{\partial y^2} - \sum_{j \neq i} q_{ij}[w_j(x, y(1-\gamma_{ij})) - w_i(x,y)]$$

$$- \lambda_i[\hat{w}_i(x+y) - w_i(x,y)] - \tilde{U}\left(\frac{\partial w_i}{\partial x}\right) \quad \geq \quad 0, \quad (2.3.3)$$

*for all* $i \in \mathbb{I}_d$, $(x,y) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$. *Then, for all* $i \in \mathbb{I}_d$, $v_i \leq w_i$, *on* $\mathbb{R}^2_+$.

**(2)** *Under* (2.2.12), *suppose that* $\rho > k(p)$. *Then, there exists some positive constant* $C$ *s.t.*

$$v_i(x,y) \quad \leq \quad C(x+y)^p, \quad \forall (i,x,y) \in \mathbb{I}_d \times \mathbb{R}^2_+. \quad (2.3.4)$$

**Proof. (1)** First notice that for $(x,y) = (0,0)$, the only admissible control in $\mathcal{A}_i(x,y)$ is the zero control $\zeta = 0$, $c = 0$, so that $v_i(0,0) = 0$. Now, fix $(x,y) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, $i \in \mathbb{I}_d$, and consider an arbitrary admissible control $(\zeta, c) \in \mathcal{A}_i(x,y)$. By Itô's formula to $e^{-\rho t} w(X_t, Y_t, I_t)$, we get:

$$\begin{aligned} d[e^{-\rho t} w(X_t, Y_t, I_t)] \quad = \quad & e^{-\rho t}\Big[ -\rho w - c_t \frac{\partial w}{\partial x} + b_{I_{t^-}} Y_{t^-} \frac{\partial w}{\partial y} + \frac{1}{2}\sigma^2_{I_{t^-}} Y^2_{t^-} \frac{\partial^2 w}{\partial y^2} \\ & + \sum_{j \neq I_{t^-}} q_{I_{t^-}j}[w(X_{t^-}, Y_{t^-}(1-\gamma_{I_{t^-}j}), j) - w(X_{t^-}, Y_{t^-}, I_{t^-})] \\ & + \lambda_{I_{t^-}}\left[w(X_{t^-} - \zeta_t, Y_{t^-} + \zeta_t, I_{t^-}) - w(X_{t^-}, Y_{t^-}, I_{t^-})\right]\Big] dt \\ & + e^{-\rho t}\sigma^2_{I_{t^-}} Y_{t^-} \frac{\partial w}{\partial y}(X_{t^-}, Y_{t^-}, I_{t^-}) dW_t \\ & + e^{-\rho t} \sum_{j \neq I_{t^-}} [w(X_{t^-}, Y_{t^-}(1-\gamma_{I_{t^-}j}), j) - w(X_{t^-}, Y_{t^-}, I_{t^-})](dN^{I_{t^-}j} - q_{I_{t^-}j} dt) \\ & + e^{-\rho t}\big[w(X_{t^-} - \zeta_t, Y_{t^-} + \zeta_t, I_{t^-}) - w(X_{t^-}, Y_{t^-}, I_{t^-})\big]\Big](dN_t - \lambda_{I_{t^-}} dt). (2.3.5) \end{aligned}$$

Denote by $\tau = \inf\{t \geq 0 : (X_t, Y_t) = (0,0)\}$, and consider the sequence of bounded stopping times $\tau_n = \inf\{t \geq 0 : X_t + Y_t \geq n \text{ or } X_t + Y_t \leq 1/n\} \wedge n$, $n \geq 1$. Then, $\tau_n \nearrow \tau$ a.s. when $n$ goes to infinity, and $c_t = 0$, $X_t = Y_t = 0$ for $t \geq \tau$, and so

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t} U(c_t) dt\Big] \quad = \quad \mathbb{E}\Big[\int_0^\tau e^{-\rho t} U(c_t) dt\Big]. \quad (2.3.6)$$

From Itô's formula (2.3.5) between time $t = 0$ and $t = \tau_n$, and observing that the integrands of the local martingale parts are bounded for $t \leq \tau_n$, we obtain after taking expectation:

$$\begin{aligned} w(x,y,i) &= \mathbb{E}\Big[e^{-\rho\tau_n} w(X_{\tau_n}, Y_{\tau_n}, I_{\tau_n}) \\ &\quad + \int_0^{\tau_n} e^{-\rho t}\Big(\rho w + c_t \frac{\partial w}{\partial x} - b_{I_{t-}} Y_{t-} \frac{\partial w}{\partial y} - \frac{1}{2}\sigma^2_{I_{t-}} Y^2_{t-} \frac{\partial^2 w}{\partial y^2} \\ &\quad - \sum_{j \neq I_{t-}} q_{I_{t-}j}[w(X_{t-}, Y_{t-}(1-\gamma_{I_{t-}j}), j) - w(X_{t-}, Y_{t-}, I_{t-})] \\ &\quad - \lambda_{I_{t-}}[w(X_{t-} - \zeta_t, Y_{t-} + \zeta_t, I_{t-}) - w(X_{t-}, Y_{t-}, I_{t-})]\Big) dt\Big] \\ &\geq \mathbb{E}\Big[e^{-\rho\tau_n} w(X_{\tau_n}, Y_{\tau_n}, I_{\tau_n}) + \int_0^{\tau_n} e^{-\rho t} U(c_t) dt\Big] \;\geq\; \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} U(c_t) dt\Big], \end{aligned}$$

where we used (2.3.3), and the nonnegativity of $w$. By sending $n$ to infinity with Fatou's lemma, and (2.3.6), we obtain the required inequality: $w_i \geq v_i$ since $(c, \zeta)$ are arbitrary.

**(2)** Consider the function $w_i(x,y) = C(x+y)^p$. Then, for $(x,y) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, and denoting by $z = y/(x+y) \in [0,1]$, a straightforward calculation shows that

$$\begin{aligned} &\rho w_i - b_i y \frac{\partial w_i}{\partial y} - \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 w_i}{\partial y^2} - \sum_{j \neq i} q_{ij}[w_j(x, y(1-\gamma_{ij})) - w_i(x,y)] \\ &- \lambda_i[\hat{w}_i(x+y) - w_i(x,y)] - \tilde{U}(\frac{\partial w_i}{\partial x}) \\ = \;& C(x+y)^p\Big[\rho - p b_i z + \frac{\sigma_i^2}{2} p(1-p) z^2 - \sum_{j \neq i} q_{ij}((1 - z\gamma_{ij})^p - 1)\Big] - \tilde{U}((x+y)^{p-1} pC) \\ \geq \;& (x+y)^p\Big(C(\rho - k(p)) - \tilde{K}(pC)^{-\frac{p}{1-p}}\Big) \end{aligned} \tag{2.3.7}$$

by (2.2.13). Hence, for $\rho > k(p)$, and for $C$ sufficiently large, the r.h.s. of (2.3.7) is nonnegative, and we conclude by using the comparison result in assertion 1). □

In the sequel, we shall assume the standing condition that $\rho > k(p)$ so that the value functions are well-defined and satisfy the growth condition (2.3.4). We now prove continuity properties of the value functions.

**Proposition 2.3.2.** *The value functions $v_i$, $i \in \mathbb{I}_d$, are concave, nondecreasing in both variables, and continuous on $\mathbb{R}^2_+$. This implies also that $\hat{v}_i$, $i \in \mathbb{I}_d$, are nondecreasing, concave and continuous on $\mathbb{R}_+$. Moreover, we have the boundary conditions for $v_i$, $i \in \mathbb{I}_d$, on $\{0\} \times \mathbb{R}_+$:*

$$v_i(0,y) = \begin{cases} 0, & \text{if } y = 0 \\ \mathbb{E}\Big[e^{-\rho\tau_1} \hat{v}_{I^i_{\tau_1}}(y \frac{S_{\tau_1}}{S_0})\Big], & \text{if } y > 0. \end{cases} \tag{2.3.8}$$

*Here $I^i$ denotes the continuous-time Markov chain $I$ starting from $i$ at time* $0$.

**Proof.** Fix some $(x, y, i) \in \mathbb{R}_+^2 \times \mathbb{I}_d$, $\delta_1 \geq 0$, $\delta_2 \geq 0$, and take an admissible control $(\zeta, c) \in \mathcal{A}_i(x, y)$. Denote by $R$ and $R'$ the wealth processes associated to $(\zeta, c)$, starting from initial state $(x, y, i)$ and $(x+\delta_1, y+\delta_2, i)$. We thus have $R' = R + \delta_1 + \delta_2 S/S_0$. This implies that $(\zeta, c)$ is also an admissible control for $(x + \delta_1, y + \delta_2, i)$, which shows clearly the nondecreasing monotonicity of $v_i$ in $x$ and $y$, and thus also the nondecreasing monotonicity of $\hat{v}_i$ by its very definition.

The concavity of $v_i$ in $(x, y)$ follows from the linearity of the admissibility constraints in $X, Y, \zeta, c$, and the concavity of $U$. This also implies the concavity of $\hat{v}_i(r)$ by its definition.

Since $v_i$ is concave, it is continuous on the interior of its domain $\mathbb{R}_+^2$. From (2.3.4), and since $v_i$ is nonnegative, we see that $v_i$ is continuous on $(x_0, y_0) = (0, 0)$ with $v_i(0, 0) = 0$. Then, $\hat{v}_i$ is continuous on $\mathbb{R}_+$ with $\hat{v}_i(0) = 0$. It remains to prove the continuity of $v_i$ at $(x_0, y_0)$ when $x_0 = 0$ or $y_0 = 0$. We shall rely on the following implication of the dynamic programming principle

$$
\begin{aligned}
v_i(x, y) &= \sup_{c \in \mathcal{C}(x)} \mathbb{E}\Big[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_{I^i_{\tau_1}}(R_{\tau_1})\Big] && (2.3.9)\\
&= \sup_{c \in \mathcal{C}(x)} \mathbb{E}\Big[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} \hat{v}_{I^i_{\tau_1}}(x - \int_0^{\tau_1} c_t dt + y \frac{S_{\tau_1}}{S_0})\Big], \ \forall (x, y) \in \mathbb{R}_+^2,
\end{aligned}
$$

where $\mathcal{C}(x)$ denotes the set of nonnegative adapted processes $(c_t)$ s.t. $\int_0^{\tau_1} c_t dt \leq x$ a.s.

(i) We first consider the case $x_0 = 0$ (and $y_0 > 0$).

In this case, the constraint on consumption $c$ in $\mathcal{C}(x_0)$ means that $c_t = 0$, $t \leq \tau_1$, so that (2.3.9) implies (2.3.8). Now, since $v_i$ is nondecreasing in $x$, we have: $v_i(x, y) \geq v_i(0, y)$. Moreover, by concavity and thus continuity of $v_i(0, .)$, we have: $\lim_{y \to y_0} v_i(0, y) = v_i(0, y_0)$. This implies that $\liminf_{(x,y) \to (0, y_0)} v_i(x, y) \geq v_i(0, y_0)$. The proof of the converse inequality requires more technical arguments. For any $x, y \geq 0$, we have:

$$
\begin{aligned}
v_i(x, y) &= \sup_{c \in \mathcal{C}(x)} \mathbb{E}\Big[\int_0^{\tau_1} e^{-\rho s} U(c_s) ds + e^{-\rho \tau_1} \hat{v}_{I^i_{\tau_1}}(x - \int_0^{\tau_1} c_s ds + y \frac{S_{\tau_1}}{S_0})\Big]\\
&\leq \sup_{c \in \mathcal{C}(x)} \mathbb{E}\Big[\int_0^{\tau_1} e^{-\rho s} U(c_s) ds\Big] \ + \ \mathbb{E}\Big[e^{-\rho \tau_1} \hat{v}_{I_{\tau_1}}(x + y \frac{S_{\tau_1}}{S_0})\Big]\\
&=: \ E_1(x) + E_2(x, y). && (2.3.10)
\end{aligned}
$$

Now, by Jensen's inequality, and since $U$ is concave, we have:

$$
\int_0^\infty U\left(c_s \mathbf{1}_{\{s \leq \tau_1\}}\right) \rho e^{-\rho s} ds \ \leq \ U\left(\int_0^\infty c_s \mathbf{1}_{\{s \leq \tau_1\}} \rho e^{-\rho s} ds\right),
$$

and thus:

$$\int_0^{\tau_1} e^{-\rho s} U(c_s) ds \quad \leq \quad \frac{U(\rho x)}{\rho}, \qquad a.s. \quad \forall c \in \mathcal{C}(x), \tag{2.3.11}$$

by using the fact that $\int_0^{\tau_1} c_t dt \leq x$ a.s. By continuity of $U$ in 0 with $U(0) = 0$, this shows that $E_1(x)$ converges to zero when $x$ goes to $x_0 = 0$. Next, by continuity of $\hat{v}_i$, we have: $\hat{v}_{I^i_{\tau_1}}(x + y\frac{S_{\tau_1}}{S_0}) \to \hat{v}_{I^i_{\tau_1}}(y_0\frac{S_{\tau_1}}{S_0})$ a.s. when $(x, y) \to (0, y_0)$. Let us check that this convergence is dominated. Indeed from (2.3.4), there is some positive constant $C$ s.t.

$$\hat{v}_{I^i_{\tau_1}}(x + y\frac{S_{\tau_1}}{S_0}) \quad \leq \quad C(x + y\frac{S_{\tau_1}}{S_0})^p \; \leq \; C(x+y)^p\Big(1 \vee \Big(\frac{S_{\tau_1}}{S_0}\Big)^p\Big).$$

Moreover,

$$\mathbb{E}\Big[e^{-\rho\tau_1}\Big(\frac{S_{\tau_1}}{S_0}\Big)^p \Big| I, W\Big] \quad = \quad \int_0^\infty \lambda_{I_t} e^{-\int_0^t \lambda_{I_s}} e^{-\rho t}\Big(\frac{S_t}{S_0}\Big)^p dt \;\leq\; \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty e^{-\rho t}\left(\frac{S_t}{S_0}\right)^p dt,$$

and so

$$\begin{aligned}
\mathbb{E}\Big[e^{-\rho\tau_1}\Big(\frac{S_{\tau_1}}{S_0}\Big)^p\Big] \quad &\leq \quad \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty \mathbb{E}\Big[e^{-\rho t}\Big(\frac{S_t}{S_0}\Big)^p\Big] dt \\
&\leq \quad \max_{i \in \mathbb{I}_d} \lambda_i \int_0^\infty e^{-(\rho - k(p))t} dt \;<\; \infty,
\end{aligned}$$

where we used in the second inequality the supermartingale property in Lemma 2.3.1 (and, more precisely, equation (2.3.2)) for $x = 0, y = 1, c \equiv \zeta \equiv 0$. One can then apply the dominated convergence theorem to $E_2(x, y)$, to deduce that $E_2(x, y)$ converges to $\mathbb{E}\Big[e^{-\rho\tau_1}\hat{v}_{I^i_{\tau_1}}(y_0\frac{S_{\tau_1}}{S_0})\Big]$ when $(x, y) \to (0, y_0)$. This, together with (2.3.8), (2.3.10), proves that $\limsup_{(x,y)\to(0,y_0)} v_i(x, y) \leq v_i(0, y_0)$, and thus the continuity of $v_i$ at $(0, y_0)$.

(ii) We consider the case $y_0 = 0$ (and $x_0 > 0$).

Similarly, as in the first case, from the nondecreasing and continuity properties of $v_i(., 0)$, we have: $\liminf_{(x,y)\to(x_0,0)} v_i(x, y) \geq v_i(x_0, 0)$. Conversely, for any $x \geq 0$, and $c \in \mathcal{C}(x)$, let us consider the stopping time $\tau_c = \inf\big\{t \in\geq 0 : \int_0^t c_s ds = x_0\big\}$. Then, the nonnegative adapted process $c'$ defined by: $c'_t = c_t \mathbf{1}_{\{t \leq \tau_c \wedge \tau_1\}}$, lies obviously in $\mathcal{C}(x_0)$. Furthermore,

$$\begin{aligned}
\int_0^{\tau_1} e^{-\rho s} U(c_s) ds \quad &= \quad \int_0^{\tau_c \wedge \tau_1} e^{-\rho s} U(c'_s) ds + \int_{\tau_c \wedge \tau_1}^{\tau_1} e^{-\rho s} U(c_s) ds \\
&\leq \quad \int_0^{\tau_1} e^{-\rho s} U(c'_s) ds \;+\; \frac{U(\rho(x - x_0)_+)}{\rho},
\end{aligned} \tag{2.3.12}$$

by the same Jensen's arguments as in (2.3.11), and for all $y \geq 0$,

$$\begin{aligned}\hat{v}_{I^i_{\tau_1}}\Big(x - \int_0^{\tau_1} c_t dt + y\frac{S_{\tau_1}}{S_0}\Big) &\leq \hat{v}_{I^i_{\tau_1}}\Big(x_0 - \int_0^{\tau_1} c'_t dt + (x-x_0)_+ + y\frac{S_{\tau_1}}{S_0}\Big) \\ &\leq \hat{v}_{I^i_{\tau_1}}\Big(x_0 - \int_0^{\tau_1} c'_t dt\Big) + \hat{v}_{I^i_{\tau_1}}\Big((x-x_0)_+ + y\frac{S_{\tau_1}}{S_0}\Big), \quad (2.3.13)\end{aligned}$$

where we have used the fact that $\hat{v}_i$ is nondecreasing, and subadditive (as a concave function with $\hat{v}_i(0) \geq 0$). By adding the two inequalities (2.3.12)-(2.3.13), and taking expectation, we obtain from (2.3.9):

$$v_i(x,y) \leq v_i(x_0,0) + \frac{U(\rho(x-x_0)_+)}{\rho} + \mathbb{E}\Big[e^{-\rho\tau_1}\hat{v}_{I^i_{\tau_1}}\Big((x-x_0)_+ + y\frac{S_{\tau_1}}{S_0}\Big)\Big],$$

and by the same domination arguments as in the first case, this shows that

$$\limsup_{(x,y)\to(x_0,0)} v_i(x,y) \leq v_i(x_0,0),$$

which ends the proof. □

**Remark 2.3.1.** The above proof of continuity of the value functions at the boundary by means of the dynamic programming principle is somehow different from other similar proofs that one can find e.g. in [20, 61, 72]. Indeed in such problems the proof of dynamic programming principle is done (or referred to) in two parts: the "easy" one ($\leq$) which does not require continuity of the value function, and the 'difficult" one ($\geq$) which requires the continuity of the value function up to the boundary. The proof of continuity at the boundary in such cases uses only the "easy" inequality. In our case, due to the specific boundary condition of our problem, the "easy" inequality is not enough to prove the continuity at the boundary. We need also the "hard" inequality. For this reason we give, in Appendix A, a proof of the dynamic programming principle in our case that, in the "hard" inequality part, uses the continuity of $v_i$ in the interior and the continuity of its restriction to the boundary (which are both implied by the concavity and by the growth condition (2.3.4)).

We shall also need the following lemma.

**Lemma 2.3.2.** *There exists some positive constant $C > 0$ s.t.*

$$\frac{\partial v_i}{\partial x}(x^+,y) := \lim_{\delta\downarrow 0}\frac{v_i(x+\delta,y)-v_i(x,y)}{\delta} \geq C\,U'(2x), \quad \forall\, x,y \in \mathbb{R}_+, i \in \mathbb{I}_d. \quad (2.3.14)$$

**Proof.** Fix some $x, y \geq 0$, and set $x_1 = x + \delta$ for $\delta > 0$. For any $(\zeta, c) \in \mathcal{A}_i(x,y)$ with associated cash/amount in shares $(X, Y)$, notice that $(\tilde{\zeta}, \tilde{c}) := (\zeta, c + \delta\mathbf{1}_{[0,1\wedge\tau_1]})$ is admissible for $(x_1, y)$.

Indeed, the associated cash amount satisfies

$$\tilde{X}_t \;=\; X_t + (x_1 - x) - \int_0^t \delta \mathbf{1}_{[0,1\wedge\tau_1]}(s)ds \;\geq\; X_t \;\geq\; 0,$$

while the amount in cash $\tilde{Y}_t = Y_t \geq 0$ since $\zeta$ is unchanged. Thus, $(\tilde{\zeta}, \tilde{c}) \in \mathcal{A}_i(x_1, y)$, and we have

$$\begin{aligned} v_i(x_1, y) &\geq \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t)dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)dt\right] + \mathbb{E}\left[\int_0^{1\wedge\tau_1} e^{-\rho t}\left(U(c_t+\delta) - U(c_t)\right)dt\right]. \end{aligned} \tag{2.3.15}$$

Now, by concavity of $U$: $U(c_t + \delta) - U(c_t) \geq \delta U'(c_t + \delta)$, and

$$\begin{aligned} \int_0^{1\wedge\tau_1} e^{-\rho t}(U(c_t+\delta) - U(c_t))dt &\geq \int_0^{1\wedge\tau_1} e^{-\rho t}\delta U'(c_t+\delta)dt \\ &\geq \delta e^{-\rho(1\wedge\tau_1)} \int_0^{1\wedge\tau_1} U'(c_t+\delta)dt \\ &\geq \delta e^{-\rho(1\wedge\tau_1)} U'(2x+\delta) \int_0^{1\wedge\tau_1} \mathbf{1}_{\{c_t<2x\}}dt. \end{aligned} \tag{2.3.16}$$

Moreover,

$$2x \int_0^{1\wedge\tau_1} \mathbf{1}_{\{c_t\geq 2x\}}dt \leq \int_0^{1\wedge\tau_1} c_t dt \leq x,$$

since $(\zeta, c)$ is admissible for $(x, y)$, so that

$$\int_0^{1\wedge\tau_1} \mathbf{1}_{\{c_t<2x\}}dt \geq (1\wedge\tau_1) - \left(\frac{1}{2}\wedge\tau_1\right) \geq \frac{1}{2}\mathbf{1}_{\{\tau_1\geq 1\}}. \tag{2.3.17}$$

By combining (2.3.16) and (2.3.17), and taking the expectation, we get

$$\mathbb{E}\left[\int_0^{1\wedge\tau_1} e^{-\rho t}(U(c_t+\delta) - U(c_t))dt\right] \;\geq\; \delta U'(2x+\delta)\mathbb{E}\Big[e^{-\rho(1\wedge\tau_1)}\frac{1}{2}\mathbf{1}_{\{\tau_1\geq 1\}}\Big].$$

By taking the supremum over $(\zeta, c)$ in (2.3.15), we thus obtain with the above inequality

$$v_i(x+\delta, y) \;\geq\; v_i(x, y) + \delta U'(2x+\delta)\mathbb{E}\left[e^{-\rho(1\wedge\tau_1)}\frac{1}{2}\mathbf{1}_{\{\tau_1\geq 1\}}\right].$$

Finally, by choosing $C = \mathbb{E}\big[e^{-\rho(1\wedge\tau_1)}\frac{1}{2}\mathbf{1}_{\{\tau_1\geq 1\}}\big] > 0$, and letting $\delta$ go to 0, we obtain the required inequality (2.3.14). □

## 2.4 Dynamic programming and viscosity characterization

In this section, we provide an analytic characterization of the value functions $v_i$, $i \in \mathbb{I}_d$, to our control problem (2.2.14), by relying on the dynamic programming principle, which is shown to hold and formulated as:

**Proposition 2.4.1.** *(Dynamic programming principle) For all $(x, y, i) \in \mathbb{R}_+^2 \times \mathbb{I}_d$, and any stopping time $\tau$, we have*

$$v_i(x,y) \;=\; \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\Big[\int_0^\tau e^{-\rho t}U(c_t)dt + e^{-\rho\tau} v_{I_\tau}(X_\tau, Y_\tau)\Big]. \tag{2.4.1}$$

**Proof.** See Appendix A. □

The associated dynamic programming system (also called Hamilton-Jacobi-Bellman or HJB system) for $v_i$, $i \in \mathbb{I}_d$, is written as

$$\begin{aligned}\rho v_i - b_i y\frac{\partial v_i}{\partial y} \;-\; \frac{1}{2}\sigma_i^2 y^2\frac{\partial^2 v_i}{\partial y^2} \;-\; \tilde{U}\left(\frac{\partial v_i}{\partial x}\right) & \\ -\sum_{j\neq i} q_{ij}\Big[v_j(x, y(1-\gamma_{ij})) - v_i(x,y)\Big] & \\ -\,\lambda_i\,\left[\hat{v}_i(x+y) - v_i(x,y)\right] \;&=\; 0, \quad (x,y)\in(0,\infty)\times\mathbb{R}_+,\ i\in\mathbb{I}_d,\end{aligned} \tag{2.4.2}$$

together with the boundary condition (2.3.8) on $\{0\} \times \mathbb{R}_+$ for $v_i$, $i \in \mathbb{I}_d$. Notice that, arguing as one does for the deduction of the HJB system above, the boundary condition (2.3.8) may also be written as:

$$\begin{aligned}\rho v_i(0,.) - b_i y\frac{\partial v_i}{\partial y}(0,.) \;-\; \frac{1}{2}\sigma_i^2 y^2\frac{\partial^2 v_i}{\partial y^2}(0,.) & \\ -\sum_{j\neq i} q_{ij}\Big[v_j(0, y(1-\gamma_{ij})) - v_i(0,y)\Big] & \\ -\,\lambda_i\big[\hat{v}_i(y) - v_i(0,y)\big] \;&=\; 0, \quad y>0,\ i\in\mathbb{I}_d.\end{aligned} \tag{2.4.3}$$

Notice that in this boundary condition the term $\tilde{U}\left(\frac{\partial v_i}{\partial x}\right)$ has disappeared. This implicitly comes from the fact that, on the boundary $x = 0$ the only admissible consumption rate is $c = 0$. We will say more on this in studying the case of CRRA utility function in Section 5.1.

In our context, the notion of viscosity solution to the non local second-order system $(E)$ is defined as follows.

**Definition 2.4.1.** *(i) A $d$-tuple $w = (w_i)_{i\in\mathbb{I}_d}$ of continuous functions on $\mathbb{R}_+^2$ is a viscosity* *supersolution (resp. subsolution) to* (2.4.2) *if*

$$\rho\varphi_i(\bar{x},\bar{y}) - b_i\bar{y}\frac{\partial\varphi_i}{\partial y}(\bar{x},\bar{y}) \;-\; \frac{1}{2}\sigma_i^2\bar{y}^2\frac{\partial^2\varphi_i}{\partial y^2}(\bar{x},\bar{y}) \;-\; \tilde{U}\left(\frac{\partial\varphi_i}{\partial x}(\bar{x},\bar{y})\right)$$
$$-\sum_{j\neq i} q_{ij}\Big[\varphi_j(\bar{x},\bar{y}(1-\gamma_{ij})) - \varphi_i(\bar{x},\bar{y})\Big]$$
$$-\;\lambda_i\;\left[\hat{\varphi}_i(\bar{x}+\bar{y}) - \varphi_i(\bar{x},\bar{y})\right] \;\;\geq\;(\;\textit{resp.}\;\;\leq)\;\;\; 0,$$

*for all d-tuple* $\varphi = (\varphi_i)_{i\in\mathbb{I}_d}$ *of* $C^2$ *functions on* $\mathbb{R}_+^2$, *and any* $(\bar{x},\bar{y},i) \in (0,\infty)\times\mathbb{R}_+\times\mathbb{I}_d$, *such that* $w_i(\bar{x},\bar{y}) = \varphi_i(\bar{x},\bar{y})$, *and* $w \geq$ *(resp.* $\leq$*)* $\varphi$ *on* $\mathbb{R}_+^2\times\mathbb{I}_d$.

*(ii) A d-tuple* $w = (w_i)_{i\in\mathbb{I}_d}$ *of continuous functions on* $\mathbb{R}_+^2$ *is a viscosity solution to* (2.4.2) *if it is both a viscosity supersolution and subsolution to* (2.4.2).

The main result of this section is to provide an analytic characterization of the value functions in terms of viscosity solutions to the dynamic programming system.

**Theorem 2.4.1.** *The value function* $v = (v_i)_{i\in\mathbb{I}_d}$ *is the unique viscosity solution to* (2.4.2) *satisfying the boundary condition* (2.3.8)*, and the growth condition* (2.3.4).

**Proof.** The proof of viscosity property follows as usual from the dynamic programming principle. The uniqueness and comparison result for viscosity solutions is proved by rather standard arguments, up to some specificities related to the non local terms and state constraints induced by our hybrid jump-diffusion control problem. We postponed the details in Appendix B. □

## 2.5 The case of CRRA utility

In this section, we consider the case where the utility function is of CRRA type in the form:

$$U(x) \;\;=\;\; \frac{x^p}{p}, \quad x > 0, \quad \text{for some } \; p \in (0,1). \tag{2.5.1}$$

We shall exploit the homogeneity property of the CRRA utility function, and go beyond the viscosity characterization of the value function in order to prove some regularity results, and provide an explicit characterization of the optimal control through a verification theorem. We next give a numerical analysis for computing the value functions and optimal strategies, and illustrate with some tests for measuring the impact of our illiquidity features.

### 2.5.1 Regularity results and verification theorem

For any $(i,x,y) \in \mathbb{I}_d \times \mathbb{R}_+^2$, $(\zeta, c) \in \mathcal{A}(x,y)$ with associated state process $(X,Y)$, we notice from the dynamics (2.2.3)-(2.2.2) that for any $k \geq 0$, the state $(kX, kY)$ is associated to the control $(k\zeta, kc)$. Thus, for $k > 0$,we have $(\zeta, c) \in \mathcal{A}_i(x,y)$ iff $(k\zeta, kc) \in \mathcal{A}(kx, kc)$, and so from the homogeneity property of the power utility function $U$ in (2.5.1), we have:

$$v_i(kx, ky) \quad = \quad k^p v_i(x,y), \quad \forall (i,x,y) \in \mathbb{I}_d \times \mathbb{R}_+^2, \ k \in \mathbb{R}_+. \tag{2.5.2}$$

Let us now consider the change of variables:

$$(x,y) \in \mathbb{R}_+^2 \setminus \{(0,0)\} \quad \longrightarrow \quad (r = x+y, z = \frac{y}{x+y}) \ \in \ (0,\infty) \times [0,1].$$

Then, from (2.5.2), we have $v_i(x,y) = v_i(r(1-z), rz) = r^p v_i(1-z, z)$, and we can separate the value function $v_i$ into:

$$v_i(x,y) \quad = \quad U(x+y)\varphi_i\Big(\frac{y}{x+y}\Big), \quad \forall (i,x,y) \in \mathbb{I}_d \times (\mathbb{R}_+^2 \setminus \{(0,0)\}) \tag{2.5.3}$$

where $\varphi_i(z) = p\ v_i(1-z,z)$ is a continuous function on $[0,1]$. By substituting this transformation for $v_i$ into the dynamic programming equation (2.4.2) and the boundary condition (2.4.3), and after some straightforward calculations, we see that $\varphi = (\varphi_i)_{i \in \mathbb{I}_d}$ should solve the system of (nonlocal) ordinary differential equations (ODEs):

$$\begin{aligned}
&(\rho - pb_i z + \frac{1}{2}p(1-p)\sigma_i^2 z^2)\varphi_i - (1-p)\Big(\varphi_i - \frac{z}{p}\varphi_i'\Big)^{-\frac{p}{1-p}} \\
&\quad - z(1-z)(b_i - z(1-p)\sigma_i^2)\varphi_i' - \frac{1}{2}z^2(1-z)^2\sigma_i^2\varphi_i'' \\
&\quad - \sum_{j \neq i} q_{ij}\Big[(1-z\gamma_{ij})^p\varphi_j\Big(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\Big) - \varphi_i(z)\Big] \\
&\quad - \lambda_i \sup_{\pi \in [0,1]}[\varphi_i(\pi) - \varphi_i(z)] \quad = \quad 0, \quad z \in [0,1), \ \ i \in \mathbb{I}_d,
\end{aligned} \tag{2.5.4}$$

together with the boundary condition for $z = 1$:

$$\begin{aligned}
&(\rho - pb_i + \frac{1}{2}p(1-p)\sigma_i^2)\varphi_i(1) \\
&- \sum_{j \neq i} q_{ij}[(1-\gamma_{ij})^p\varphi_j(1) - \varphi_i(1)] - \lambda_i \sup_{\pi \in [0,1]}[\varphi_i(\pi) - \varphi_i(1)] \quad = \quad 0, \quad i \in \mathbb{I}_d.
\end{aligned} \tag{2.5.5}$$

The following boundary condition for $z = 0$, obtained formally by taking $z = 0$ in (2.5.4),

$$\begin{aligned} & \rho\varphi_i(0) - (1-p)\big(\varphi_i(0)\big)^{-\frac{p}{1-p}} \\ - \sum_{j\neq i} q_{ij}\left[\varphi_j(0) - \varphi_i(0)\right] - \lambda_i \sup_{\pi\in[0,1]} & \left[\varphi_i(\pi) - \varphi_i(0)\right] \;=\; 0, \quad i \in \mathbb{I}_d, \end{aligned} \tag{2.5.6}$$

is proved rigorously in the below Proposition.

**Proposition 2.5.1.** *The d-tuple $\varphi = (\varphi_i)_{i\in\mathbb{I}_d}$ is concave on $[0,1]$, $C^2$ on $(0,1)$. We further have*

$$\lim_{z\to 0} z\varphi_i'(z) = 0, \tag{2.5.7}$$

$$\lim_{z\to 0} z^2\varphi_i''(z) = 0, \tag{2.5.8}$$

$$\lim_{z\to 1} (1-z)\varphi_i'(z) = 0, \tag{2.5.9}$$

$$\lim_{z\to 1} (1-z)^2\varphi_i''(z) = 0, \tag{2.5.10}$$

$$\lim_{z\to 1} \varphi_i'(z) = -\infty, \tag{2.5.11}$$

*and $\varphi$ is the unique bounded classical solution of* (2.5.4) *on* $(0,1)$*, with boundary conditions* (2.5.5)-(2.5.6)*.*

**Proof.** Since $\varphi_i(z) = p\, v_i(1-z, z)$, and by concavity of $v_i(.,.)$ in both variables, it is clear that $\varphi_i$ is concave on $[0,1]$. From the viscosity property of $v_i$ in Theorem 2.4.1, and the change of variables (2.5.3), this implies that $\varphi$ is the unique bounded viscosity solution to (2.5.4) on $[0,1)$, satisfying the boundary condition (2.5.5). Now, recalling that $q_{ii} = -\sum_{j\neq i} q_{ij}$, we observe that the system (2.5.4) can be written as:

$$\begin{aligned} & (\rho - q_{ii} + \lambda_i - pb_iz + \frac{1}{2}p(1-p)\sigma_i^2z^2)\varphi_i - z(1-z)(b_i - z(1-p)\sigma_i^2)\varphi_i' \\ & - \frac{1}{2}z^2(1-z)^2\sigma_i^2\varphi_i'' - (1-p)(\varphi_i - \frac{z}{p}\varphi_i')^{-\frac{p}{1-p}} \\ = \; & \sum_{j\neq i} q_{ij}\left[(1-z\gamma_{ij})^p\varphi_j\Big(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\Big)\right] + \lambda_i \sup_{\pi\in[0,1]} \varphi_i(\pi), \quad z \in (0,1),\ i \in \mathbb{I}_d. \end{aligned} \tag{2.5.12}$$

Let us fix some $i \in \mathbb{I}_d$, and an arbitrary compact $[a,b] \subset (0,1)$. By standard results, see e.g. [16], we know that the second-order ODE:

$$
\begin{aligned}
& (\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2}p(1-p)\sigma_i^2 z^2)w_i - z(1-z)(b_i - z(1-p)\sigma_i^2)w_i' \\
& -\frac{1}{2}z^2(1-z)^2\sigma_i^2 w_i'' - (1-p)(w_i - \frac{z}{p}w_i')^{-\frac{p}{1-p}} \\
= & \sum_{j\neq i} q_{ij}\left[(1-z\gamma_{ij})^p \varphi_j\Big(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\Big)\right] + \lambda_i \sup_{\pi\in[0,1]} \varphi_i(\pi)
\end{aligned}
\tag{2.5.13}
$$

has a unique viscosity solution $w_i$ satisfying $w_i(a) = \varphi_i(a)$, $w_i(b) = \varphi_i(b)$, and that this solution $w_i$ is twice differentiable on $[a,b]$ since the second term $z(1-z)\sigma_i^2$ is uniformly elliptic on $[a,b]$, see [44]. Since $\varphi_i$ is a viscosity solution to (2.5.13) by (2.5.12), we deduce by uniqueness that $\varphi_i = w_i$ on $[a,b]$. Since $a,b$ are arbitrary, this means that $\varphi$ is $C^2$ on $(0,1)$. By concavity of $\varphi_i$, we have for all $z \in (0,1)$,

$$
\frac{\varphi_i(1) - \varphi_i(z)}{1-z} \leq \varphi_i'(z) \leq \frac{\varphi_i(z) - \varphi_i(0)}{z}.
$$

Letting $z \to 0$ and $z \to 1$, and by continuity of $\varphi_i$, we obtain (2.5.7) and (2.5.9).

Now letting $z$ go to 0 in (2.5.4), we obtain $\lim_{z\to 0} z^2\varphi_i''(z) = l$ for some finite $l \leq 0$. If $l < 0$, $z^2\varphi_i''(z) \leq \frac{l}{2}$ whenever $z \leq \eta$, for some $\eta > 0$. By writing that

$$
z(\varphi_i'(z) - \varphi_i'(\eta)) = z\int_\eta^z \varphi_i''(u)du \geq -\frac{l}{2}z\int_z^\eta \frac{du}{u^2} = \frac{l}{2}z\left(\frac{1}{\eta} - \frac{1}{z}\right),
$$

and sending $z \to 0$, we get $\liminf_{z\to 0} z\varphi_i'(z) \geq -l/2$, which contradicts (2.5.7). Thus $l = 0$, and the boundary condition (2.5.6) follows by letting $z \to 0$ in (2.5.4). In the same way, letting $z \to 1$ in (2.5.4) and comparing with (2.5.5), we have

$$
\lim_{z\to 1} \frac{1}{2}(1-z)^2\varphi_i''(z) \quad = \quad (\varphi_i(1) - \varphi_i'(1-))^{-\frac{p}{1-p}} \quad \in [0,\infty].
$$

(2.5.9) implies that this limit is 0, and we obtain (2.5.10) and (2.5.11). □

**Remark 2.5.1.** From (2.5.3) and the above Proposition, we deduce that the value functions $v_i$, $i \in \mathbb{I}_d$, are $C^2$ on $(0,\infty) \times (0,\infty)$, and so are solutions to the dynamic programming system (2.4.2) on $(0,\infty)\times(0,\infty)$ in classical sense.

We now provide an explicit construction of the optimal investment/consumption strategies in feedback form in terms of the smooth solution $\varphi$ to (2.5.4)-(2.5.6)-(2.5.5). We start with the following Lemma.

**Lemma 2.5.1.** *For any $i \in \mathbb{I}_d$, let us define:*

$$
\begin{aligned}
c^*(i,z) &= \begin{cases} \Big(\varphi_i(z) - \frac{z}{p}\varphi_i'(z)\Big)^{\frac{-1}{1-p}} & \text{when } 0 < z < 1 \\ (\varphi_i(0))^{\frac{-1}{1-p}} & \text{when } z = 0 \\ 0 & \text{when } z = 1 \end{cases}, \\
\pi^*(i) &\in \arg\max_{\pi \in [0,1]} \varphi_i(\pi).
\end{aligned}
$$

*Then for each $i \in \mathbb{I}_d$, $c^*(i,.)$ is continuous on $[0,1]$, $C^1$ on $(0,1)$, and given any initial conditions $(r,z) \in \mathbb{I}_d \times \mathbb{R}_+ \times [0,1]$, there exists a solution $(\hat{R}_t, \hat{Z}_t)_{t \geq 0}$ valued in $\mathbb{R}_+ \times [0,1]$ to the SDE:*

$$
d\hat{R}_t = \hat{R}_{t^-}\hat{Z}_{t^-}\Big(b_{I_{t^-}}dt + \sigma_{I_{t^-}}dW_t - \gamma_{I_{t^-},I_t}dN_t^{I_{t^-},I_t}\Big) - \hat{R}_{t^-}c^*(I_{t^-},\hat{Z}_{t^-})dt, \tag{2.5.14}
$$

$$
\begin{aligned}
d\hat{Z}_t &= \hat{Z}_{t^-}(1-\hat{Z}_{t^-})\Big[(b_{I_{t^-}} - \hat{Z}_{t^-}\sigma^2_{I_{t^-}})dt + \sigma_{I_{t^-}}dW_t - \frac{\gamma_{I_{t^-},I_t}}{1-\hat{Z}_{t^-}\gamma_{I_{t^-},I_t}}dN_t^{I_{t^-},I_t}\Big] \\
&\quad + (\pi^*(I_{t^-}) - \hat{Z}_{t^-})dN_t + \hat{Z}_t c^*(I_{t^-},\hat{Z}_{t^-})dt.
\end{aligned} \tag{2.5.15}
$$

*Moreover, if $r > 0$, then $\hat{R}_t > 0$, a.s. for all $t \geq 0$.*

**Proof.** First notice that Lemma 2.3.2, written in terms of the variables $(r,z)$, is formulated equivalently as

$$
\varphi_i(z) - \frac{z}{p}\varphi_i'(z) \geq C2^{p-1}(1-z)^{p-1}, \quad z \in (0,1).
$$

This implies that $c^*(i,.)$ is well-defined on $(0,1)$, and $C^1$ since $\varphi$ is $C^2$. The continuity of $c^*(i,.)$ at 0 and 1 comes from (2.5.7) and (2.5.11).

Let us show the existence of a solution $Z$ to the SDE (2.5.15). We start by the existence of a solution for $t < \tau_1$ (recall that $(\tau_n)$ is the sequence of jump times of $N$). In the case where $z = 1$ (resp. $z = 0$), then $Z_t \equiv 1$ (resp. $Z_t \equiv 0$) is clearly a solution on $[0,\tau_1)$. Consider now the case where $z \in (0,1)$. From the local Lipschitz property of $z \mapsto zc^*(i,z)$, and recalling that $\gamma_{ij} < 1$, we know, adapting e.g. the result of Theorem 38, page 303 of [64], that there exists a solution to

$$
\begin{aligned}
d\hat{Z}_t &= \hat{Z}_{t^-}(1-\hat{Z}_{t^-})\Big[(b_{I_{t^-}} - \hat{Z}_{t^-}\sigma^2_{I_{t^-}})dt + \sigma_{I_{t^-}}dW_t - \frac{\gamma_{I_{t^-},I_t}}{1-\hat{Z}_{t^-}\gamma_{I_{t^-},I_t}}dN_t^{I_{t^-},I_t}\Big] \\
&\quad + \hat{Z}_t c^*(I_{t^-},\hat{Z}_{t^-})dt,
\end{aligned} \tag{2.5.16}
$$

which is valued in $[0,1]$ up to time $t < \tau_1' := \tau_1 \wedge \left(\lim_{\varepsilon\to 0}\inf\left\{t \geq 0 | \hat{Z}_t(1-\hat{Z}_t) \leq \varepsilon\right\}\right)$. By noting that $\hat{Z}_t \geq Z_t^0$, where

$$Z_t^0 = \frac{z\frac{S_t}{S_0}}{z\frac{S_t}{S_0} + (1-z)}, \quad t \geq 0,$$

is the solution to (2.5.16) without the consumption term, and since $S$ is locally bounded away from 0, we have $\lim_{t\to\tau_1'} Z_t = 1$ on $\{\tau_1' < \tau_1\}$. By extending $\hat{Z}_t \equiv 1$ on $[\tau_1', \tau_1)$, we obtain actually a solution on $[0, \tau_1)$. Then at $\tau_1$, by taking $\hat{Z}_{\tau_1} = \pi^*(I_{\tau_1 -})$, we obtain a solution to (2.5.15) valued in $[0,1]$ on $[0,\tau_1]$. Next, we obtain similarly a solution to (2.5.15) on $[\tau_1, \tau_2]$ starting from $\hat{Z}_{\tau_1}$. Finally, since $\tau_n \nearrow \infty$, a.s., by pasting we obtain a solution to (2.5.15) for $t \in \mathbb{R}_+$.

Given a solution $\hat{Z}$ to (2.5.15), the solution $\hat{R}$ to (2.5.14) starting from $r$ at time 0 is determined by the stochastic exponential:

$$\hat{R}_t = r \cdot \mathcal{E}\left(\int_0^\cdot \hat{Z}_{s-}\left(b_{I_{s-}} ds + \sigma_{I_{s-}} dW_s - \gamma_{I_{s-}, I_s} dN_s^{I_{s-}, I_s}\right) - c^*(I_{s-}, \hat{Z}_{s-})dt\right)_t.$$

Since $-\hat{Z}_{t-}\gamma_{I_{t-}, I_t} > -1$, we see that $R_t > 0$, $t \geq 0$, whenever $r > 0$, while $R \equiv 0$ if $r = 0$. □

**Proposition 2.5.2.** *Given some initial conditions $(i, x, y) \in \mathbb{I}_d \times (\mathbb{R}_+^2 \setminus \{(0,0)\})$, let us consider the pair of processes $(\hat{\zeta}, \hat{c})$ defined by:*

$$\hat{\zeta}_t = \hat{R}_{t-}(\pi^*(I_{t-}) - \hat{Z}_{t-}) \tag{2.5.17}$$
$$\hat{c}_t = \hat{R}_{t-} c^*(I_{t-}, \hat{Z}_{t-}), \tag{2.5.18}$$

*where the functions $(c^*, \pi^*)$ are defined in Lemma 2.5.1, and $(\hat{R}, \hat{Z})$ are solutions to* (2.5.14)-(2.5.15)*, starting from $r = x + y$, $z = y/(x+y)$, with $I$ starting from $i$. Then, $(\hat{\zeta}, \hat{c})$ is an optimal investment/consumption strategy in $\mathcal{A}_i(x,y)$, with associated state process $(\hat{X}, \hat{Y}) = (\hat{R}(1-\hat{Z}), \hat{R}\hat{Z})$, for $v_i(x,y) = U(r)\varphi_i(z)$.*

**Proof.** For such choice of $(\hat{\zeta}, \hat{c})$, the dynamics of $(\hat{R}, \hat{Z})$ evolve according to (2.2.10)-(2.2.11) with a feedback control $(\hat{\zeta}, \hat{c})$, and thus correspond (via Itô's formula) to a state process $(\hat{X}, \hat{Y}) = (\hat{R}(1-\hat{Z}), \hat{R}\hat{Z})$ governed by (2.2.2)-(2.2.3), starting from $(x, y)$, and satisfying the nonbankruptcy constraint (2.2.5). Thus, $(\hat{\zeta}, \hat{c}) \in \mathcal{A}_i(x,y)$. Moreover, since $r = x + y > 0$, this implies that $\hat{R} > 0$, and so $(\hat{X}, \hat{Y})$ lies in $\mathbb{R}_+^2 \setminus \{(0,0)\}$.

As in the proof of the standard verification theorem, we would like to apply Itô's formula to the function $e^{-\rho t} v(\hat{X}_t, \hat{Y}_t, I_t)$ (denoting by $v(x,y,i) = v_i(x,y) = U(x+y)\varphi_i(y/(x+y))$). However this is not immediately possible since the process $(\hat{X}_t, \hat{Y}_t)$ may reach the boundary of $\mathbb{R}^2$ where the derivatives of $v$ do not have classical sense. To overcome this problem, we approximate the function $\varphi_i$ (and so $v(x,y,i)$) as follows. We define, for every $\varepsilon > 0$ a function $\varphi^\varepsilon = (\varphi^\varepsilon)_{i\in\mathbb{I}_d} \in C^2([0,1],\mathbb{R}^d)$ as in the proof of Theorem 4.24 in [20], such that $\varphi_i^\varepsilon = \varphi_i$ on $[\varepsilon, 1-\varepsilon]$, and $\varphi_i^\varepsilon$ is affine on $[0,\varepsilon]$ and $[1-\varepsilon, 1]$. It is then easy to see that this implies

- $\varphi_i^\varepsilon \to \varphi_i$ uniformly on $[0,1]$ as $\varepsilon \to 0$,
- $z(1-z)(\varphi_i^\varepsilon)' \to z(1-z)\varphi_i'$ uniformly on $[0,1]$ as $\varepsilon \to 0$,
- $z^2(1-z)^2(\varphi_i^\varepsilon)'' \to z^2(1-z)^2\varphi_i''$ uniformly on $[0,1]$ as $\varepsilon \to 0$,
- $((\varphi_i^\varepsilon)')^{-1} \to (\varphi_i')^{-1}$ uniformly on $[0,\eta]$ for some $\eta > 0$ as $\varepsilon \to 0$.

Now we can apply Dynkin's formula to the function $v^\varepsilon(x,y,i) = U(x+y)\varphi_i^\varepsilon(y/(x+y))$ calculated on the process $(\hat{X},\hat{Y},I)$ between time 0 and $\tau_n \wedge T$, where $\tau_n = \inf\{t \geq 0 : \hat{X}_t + \hat{Y}_t \geq n\}$ :

$$
\begin{aligned}
v^\varepsilon(x,y,i) \quad = \quad & \mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)} v^\varepsilon(\hat{X}_{\tau_n\wedge T}, \hat{Y}_{\tau_n\wedge T}, I_{\tau_n\wedge T}) \\
& + \int_0^{\tau_n\wedge T} e^{-\rho t}\Big(\rho v^\varepsilon + \hat{c}_t \frac{\partial v^\varepsilon}{\partial x} - b_{I_{t-}} \hat{Y}_{t-} \frac{\partial v^\varepsilon}{\partial y} - \frac{1}{2}\sigma_{I_{t-}}^2 \hat{Y}_{t-}^2 \frac{\partial^2 v^\varepsilon}{\partial y^2} \\
& - \sum_{j\neq I_{t-}} q_{I_{t-}j}[v^\varepsilon(\hat{X}_{t-}, \hat{Y}_{t-}(1-\gamma_{I_{t-}j}), j) - v^\varepsilon(\hat{X}_{t-}, \hat{Y}_{t-}, I_{t-})] \\
& - \lambda_{I_{t-}}\left[v^\varepsilon(\hat{X}_{t-} - \hat{\zeta}_t, \hat{Y}_{t-} + \hat{\zeta}_t, I_{t-}) - v^\varepsilon(\hat{X}_{t-}, \hat{Y}_{t-}, I_{t-})\right]\Big) dt\Big] \qquad (2.5.19)
\end{aligned}
$$

We denote by $\hat{\zeta}(i,r,z) = r(\pi^*(i) - z)$, $\hat{c}(i,r,z) = rc^*(i,z)$, and define $g^\varepsilon$ on $(\mathbb{R}_+^2 \setminus \{(0,0)\}) \times \mathbb{I}_d$ by

$$
\begin{aligned}
& \rho v_i^\varepsilon - b_i y \frac{\partial v_i^\varepsilon}{\partial y} - \frac{1}{2}\sigma_i^2 y^2 \frac{\partial^2 v_i^\varepsilon}{\partial y^2} + \hat{c}(i, x+y, \frac{y}{x+y})\frac{\partial v_i^\varepsilon}{\partial x} - U(\hat{c}(i, x+y, \frac{y}{x+y})) \\
& \qquad - \sum_{j\neq i} q_{ij}\Big[v_j^\varepsilon(x, y(1-\gamma_{ij})) - v_i^\varepsilon(x,y)\Big] \\
& - \lambda_i\Big[v_i^\varepsilon\Big(x - \hat{\zeta}(i, x+y, \frac{y}{x+y}), y + \hat{\zeta}(i, x+y, \frac{y}{x+y})\Big) - v_i^\varepsilon(x,y)\Big] \quad =: \quad g_i^\varepsilon(x,y),
\end{aligned}
$$

so that from (2.5.19):

$$
\begin{aligned}
v^\varepsilon(i,x,y) \quad = \quad & \mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)} v^\varepsilon(\hat{X}_{\tau_n\wedge T}, \hat{Y}_{\tau_n\wedge T}, I_{\tau_n\wedge T}) \\
& + \int_0^{\tau_n\wedge T} e^{-\rho t}(U(\hat{c}_t) + g^\varepsilon(\hat{X}_t, \hat{Y}_t, I_t))dt\Big]. \qquad (2.5.20)
\end{aligned}
$$

Notice that denoting $r = x + y$, $z = y/(x+y)$, we have

$$\begin{aligned}
\hat{c}(i, x+y, \frac{y}{x+y})\frac{\partial v_i}{\partial x} &= r^p\left(\varphi_i - \frac{z}{p}\varphi_i'\right)^{-\frac{p}{1-p}}, \\
y\frac{\partial v_i}{\partial y} &= r^p z\left(\varphi_i + \frac{1-z}{p}\varphi_i'\right), \\
y^2\frac{\partial^2 v_i}{\partial y^2} &= r^p z^2\left((p-1)\varphi_i - 2(1-z)\frac{(1-p)}{p}\varphi_i' + \frac{(1-z)^2}{p}\varphi_i''\right),
\end{aligned}$$

so that the properties of $\varphi^\varepsilon$ imply :

- $v_i^\varepsilon = v_i$ on $\left\{\varepsilon \leq \frac{y}{x+y} \leq 1-\varepsilon\right\}$,
- $v_i^\varepsilon \to v_i$ uniformly on bounded subsets of $\mathbb{R}_+^2$,
- $\hat{c}(i, x+y, \frac{y}{x+y})\frac{\partial v_i^\varepsilon}{\partial x} \to \begin{cases} c(i, x+y, \frac{y}{x+y})\frac{\partial v_i}{\partial x}, & x > 0 \\ 0, & x = 0 \end{cases}$ uniformly on bounded subsets of $\mathbb{R}_+^2$,
- $y\frac{\partial v_i^\varepsilon}{\partial y} \to \begin{cases} y\frac{\partial v_i}{\partial y}, & y > 0 \\ 0, & y = 0 \end{cases}$ uniformly on bounded subsets of $\mathbb{R}_+^2$,
- $y^2\frac{\partial^2 v_i^\varepsilon}{\partial y^2} \to \begin{cases} y^2\frac{\partial^2 v_i}{\partial y^2}, & y > 0 \\ 0, & y = 0 \end{cases}$ uniformly on bounded subsets of $\mathbb{R}_+^2$.

Since $v$ is a classical solution of (2.4.2) on $(0,\infty)\times(0,\infty)$, this implies that $g^\varepsilon$ converges to 0 uniformly on bounded subsets of $\mathbb{R}_+^2$ when $\varepsilon$ goes to 0. We then obtain by letting $\varepsilon \to 0$ in (2.5.20):

$$v(x,y,i) = \mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)}v(\hat{X}_{\tau_n\wedge T}, \hat{Y}_{\tau_n\wedge T}, I_{\tau_n\wedge T}) + \int_0^{\tau_n\wedge T} e^{-\rho t}U(\hat{c}_t)dt\Big],$$

From the growth condition (2.3.4) we get

$$\mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)}v(\hat{X}_{\tau_n\wedge T}, \hat{Y}_{\tau_n\wedge T}, I_{\tau_n\wedge T})\Big] \leq C\mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)}R^p_{\tau_n\wedge T}\Big].$$

So, using Lemma 2.3.1, sending $n$ to infinity, and then $T$ to infinity, we get

$$\lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}\Big[e^{-\rho(\tau_n\wedge T)}v(\hat{X}_{\tau_n\wedge T}, \hat{Y}_{\tau_n\wedge T}, I_{\tau_n\wedge T})\Big] = 0.$$

Applying monotone convergence theorem to the second term in the r.h.s. of (2.5.20), we then obtain

$$v_i(x,y) \;=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t} U(\hat{c}_t)dt\Big],$$

which proves the optimality of $(\hat{\zeta}, \hat{c})$. □

### 2.5.2 Numerical analysis

We focus on the numerical resolution of the system of ODEs (2.5.4)-(2.5.6)-(2.5.5) satisfied by $(\varphi_i)_{i\in\mathbb{I}_d}$, and rewritten for all $i \in \mathbb{I}_d$ as:

$$\begin{aligned}
&(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2}p(1-p)\sigma_i^2 z^2)\varphi_i - z(1-z)(b_i - z(1-p)\sigma_i^2)\varphi_i' \\
&- \frac{1}{2}z^2(1-z)^2\sigma_i^2\varphi_i'' - (1-p)(\varphi_i - \frac{z}{p}\varphi_i')^{-\frac{p}{1-p}} \\
= \;& \sum_{j\neq i} q_{ij}\left[(1-z\gamma_{ij})^p \varphi_j\Big(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\Big)\right] + \lambda_i \sup_{\pi\in[0,1]} \varphi_i(\pi), \quad z \in (0,1),
\end{aligned}$$

$$\begin{aligned}
(\rho - q_{ii} + \lambda_i)\varphi_i(0) - (1-p)\varphi_i(0)^{-\frac{p}{1-p}} &= \sum_{j\neq i} q_{ij}\varphi_j(0) \;+\; \lambda_i \sup_{\pi\in[0,1]} \varphi_i(\pi), \\
(\rho - q_{ii} + \lambda_i - pb_i + \frac{1}{2}p(1-p)\sigma_i^2)\varphi_i(1) &= \sum_{j\neq i} q_{ij}(1-\gamma_{ij})^p\varphi_j(1) + \lambda_i \sup_{\pi\in[0,1]} \varphi_i(\pi).
\end{aligned}$$

The main numerical difficulty comes from the nonlocal terms in the r.h.s. of these equations. We shall adopt an iterative method as follows: starting with $\varphi^0 = (\varphi_i^0)_{i\in\mathbb{I}_d} = 0$, we solve $\varphi^{n+1} = (\varphi_i^{n+1})_{i\in\mathbb{I}_d}$ as the (classical) solution to the local ODEs where the non local terms are calculated from $(\varphi_i^n)$ :

$$\begin{aligned}
(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2}p(1-p)\sigma_i^2 z^2)\varphi_i^{n+1} &- z(1-z)(b_i - z(1-p)\sigma_i^2)(\varphi_i^{n+1})' \\
&-\frac{1}{2}z^2(1-z)^2\sigma_i^2(\varphi_i^{n+1})'' - (1-p)(\varphi_i^{n+1} - \frac{z}{p}(\varphi_i^{n+1})')^{-\frac{p}{1-p}} \\
&= \sum_{j\neq i} q_{ij}\left[(1-z\gamma_{ij})^p\varphi_j^n\Big(\frac{z(1-\gamma_{ij})}{1-z\gamma_{ij}}\Big)\right] \;+\; \lambda_i \sup_{\pi\in[0,1]} \varphi_i^n(\pi),
\end{aligned}$$

with boundary conditions

$$
\begin{aligned}
(\rho - q_{ii} + \lambda_i)\varphi_i^{n+1}(0) - (1-p)\varphi_i^{n+1}(0)^{-\frac{p}{1-p}} &= \sum_{j\neq i} q_{ij}\varphi_j^n(0) + \lambda_i \sup_{\pi\in[0,1]} \varphi_i^n(\pi), \\
(\rho - q_{ii} + \lambda_i - pb_i + \frac{1}{2}p(1-p)\sigma_i^2)\varphi_i^{n+1}(1) &= \sum_{j\neq i} q_{ij}(1-\gamma_{ij})^p\varphi_j^n(1) + \lambda_i \sup_{\pi\in[0,1]} \varphi_i^n(\pi).
\end{aligned}
$$

Let us denote by:

$$
v_i^n(x,y) = \begin{cases} U(x+y)\varphi_i^n\Big(\frac{y}{x+y}\Big), & \text{for } (i,x,y)\in\mathbb{I}_d\times(\mathbb{R}_+^2\setminus\{(0,0)\}) \\ 0, & \text{for } i\in\mathbb{I}_d,\ (x,y)=(0,0). \end{cases}
$$

A straightforward calculation shows that $v^n = (v_i^n)_{i\in\mathbb{I}_d}$ are solutions to the iterative local PDEs:

$$
\begin{aligned}
& (\rho - q_{ii} + \lambda_i)v_i^{n+1} - b_i y\frac{\partial v_i^{n+1}}{\partial y} - \frac{1}{2}\sigma_i^2 y^2\frac{\partial^2 v_i^{n+1}}{\partial y^2} - \tilde{U}(\frac{\partial v_i^{n+1}}{\partial x}) \\
= & \sum_{j\neq i} q_{ij}v_j^n(x,y(1-\gamma_{ij})) + \lambda_i\hat{v}_i^n(x+y), \quad (x,y)\in(0,\infty)\times\mathbb{R}_+,\ i\in\mathbb{I}_d, \qquad (2.5.21)
\end{aligned}
$$

together with the boundary condition (2.3.8) on $\{0\}\times(0,\infty)$ for $v_i$, $i\in\mathbb{I}_d$:

$$
\begin{aligned}
& (\rho - q_{ii} + \lambda_i)v_i^{n+1}(0,.) - b_i y\frac{\partial v_i^{n+1}}{\partial y}(0,.) - \frac{1}{2}\sigma_i^2 y^2\frac{\partial^2 v_i^{n+1}}{\partial y^2}(0,.) \\
= & \sum_{j\neq i} q_{ij}v_j^n(0,y(1-\gamma_{ij})) + \lambda_i\hat{v}_i^n(y), \quad y>0,\ i\in\mathbb{I}_d. \qquad (2.5.22)
\end{aligned}
$$

We then have the stochastic control representation for $v^n$ (and so for $\varphi^n$).

**Proposition 2.5.3.** *For all $n\geq 0$, we have*

$$
v_i^n(x,y) = \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\Big[\int_0^{\theta_n} e^{-\rho t}U(c_t)dt\Big], \quad (i,x,y)\in\mathbb{I}_d\times\mathbb{R}_+^2, \qquad (2.5.23)
$$

*where the sequence of random times $(\theta_n)_{n\geq 0}$ are defined by induction from $\theta_0 = 0$, and:*

$$
\theta_{n+1} = \inf\Big\{t>\theta_n : \Delta N_t \neq 0 \text{ or } \Delta N_t^{I_{t-},I_t}\neq 0\Big\},
$$

*i.e. $\theta_n$ is the $n$-th time where we have either a change of regime or a trading time.*

**Proof.** Denoting by $w_i^n(x,y)$ the r.h.s. of (2.5.23), we need to show that $w_i^n = v_i^n$. First (with a similar proof to Proposition 2.4.1) we have the following Dynamic Programming Principle for the $w^n$ : for each finite stopping time $\tau$,

$$\begin{aligned} w_i^{n+1}(x,y) &= \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\Bigg[\int_0^{\tau\wedge\theta_1} e^{-\rho t}U(c_t)dt + \mathbf{1}_{\{\tau\geq\theta_1\}}e^{-\rho\theta_1}w^n_{I_{\theta_1}}(X_{\theta_1},Y_{\theta_1}) \\ &\quad +\mathbf{1}_{\{\tau<\theta_1\}}e^{-\rho\tau}w^{n+1}_{I_\tau}(X_\tau,Y_\tau)\Bigg] \end{aligned} \tag{2.5.24}$$

The only difference with the statement of Proposition 2.4.1 is the fact that when $\tau \geq \theta_1$, we substitute $w^{n+1}$ with $w^n$ since there are only $n$ stopping times remaining before consumption is stopped due to the finiteness of the horizon in the definition of $w^n$.

By using (2.5.24), we can show as in Theorem 2.4.1 that $w^n$ is the unique viscosity solution to (2.5.21), satisfying boundary condition (2.5.22) and growth condition (2.3.4) (it is actually easier since there are only local terms in this case). Since we already know that $v^n$ is such a solution, it follows that $w^n = v^n$. □

As a consequence, we obtain the following convergence result for the sequence $(v^n)_n$.

**Proposition 2.5.4.** *The sequence $(v^n)_n$ converges increasingly to $v$, and there exists some positive constants $C$ and $\delta < 1$ s.t.*

$$0 \leq v_i - v_i^n \leq C\delta^n(x+y)^p, \quad \forall(i,x,y) \in \mathbb{I}_d \times \mathbb{R}_+^2. \tag{2.5.25}$$

**Proof.** First let us show that

$$\delta := \sup_{\substack{(c,\zeta)\in\mathcal{A}_i(x,y) \\ \{(x,y)\in\mathbb{R}_+^2\,:\,x+y=1\}}} \mathbb{E}\Big[e^{-\rho\theta_1}R^p_{\theta_1}\Big] < 1. \tag{2.5.26}$$

By writing that $e^{-\rho t}R_t^p = D_tL_t$, where $(L_t)_t = (e^{-k(p)t}R_t^p)_t$ is a nonnegative supermartingale by Lemma 2.3.1, and $(D_t)_t = (e^{-(\rho-k(p))t})_t$ is a decreasing process, we see that $(e^{-\rho t}R_t^p)_t$ is also a nonnegative supermartingale for all $(\zeta,c) \in \mathcal{A}_i(x,y)$, and so:

$$\begin{aligned} \mathbb{E}\left[e^{-\rho\theta_1}R^p_{\theta_1}\right] &\leq \mathbb{E}\left[e^{-\rho(\theta_1\wedge 1)}R^p_{\theta_1\wedge 1}\right] \\ &= \mathbb{E}\left[e^{-(\rho-k(p))(\theta_1\wedge 1)}e^{-k(p)(\theta_1\wedge 1)}R^p_{\theta_1\wedge 1}\right]. \end{aligned}$$

Now, since $e^{-(\rho-k(p))(\theta_1\wedge 1)} < 1$ a.s., $\mathbb{E}\left[e^{-k(p)(\theta_1\wedge 1)}R^p_{\theta_1\wedge 1}\right] \leq 1$, for all $(\zeta,c) \in \mathcal{A}_i(x,y)$ with $x+y = 1$ (recall the supermartingale property of $(e^{-k(p)t}R_t^p)_t$), and by using also the uniform integrability of the family $\left(e^{-k(p)(\theta_1\wedge 1)}R^p_{\theta_1\wedge 1}\right)_{c,\zeta}$ from Lemma 2.3.1, we obtain the relation (2.5.26).

The nondecreasing property of the sequence $(v_i^n)_n$ follows immediately from the representation (2.5.23), and we have: $v_i^n \leq v_i^{n+1} \leq v$ for all $n \geq 0$. Moreover, the dynamic programming principle (2.5.24) applied to $\tau = \theta_1$ gives

$$v_i^{n+1}(x,y) = \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^{\theta_1} e^{-\rho t}U(c_t)dt + e^{-\rho\theta_1} v^n_{I_{\theta_1}}(X_{\theta_1}, Y_{\theta_1})\right] \tag{2.5.27}$$

Let us show (2.5.25) by induction on $n$. The case $n = 0$ is simply the growth condition (2.3.4) since $v^0 = 0$. Assume now that (2.5.25) holds true at step $n$. From the dynamic programming principle (2.4.1) and (2.5.27) for $v$ and $v^{n+1}$, we then have:

$$\begin{aligned} v_i^{n+1}(x,y) &\geq v_i(x,y) - \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\Big[e^{-\rho\theta_1}(v_{I_{\theta_1}} - v^n_{I_{\theta_1}})(X_{\theta_1}, Y_{\theta_1})\Big] \\ &\geq v_i(x,y) - \sup_{(\zeta,c)\in\mathcal{A}_i(x,y)} \mathbb{E}\Big[e^{-\rho\theta_1} C\delta^n R^p_{\theta_1}\Big] \\ &= v_i(x,y) - C\delta^{n+1}(x+y)^p, \end{aligned}$$

by definition of $\delta$. This proves the required inequality at step $n+1$, and ends the proof. □

In the next section, we solve the local ODEs for $\varphi^n$ with Newton's method by a finite-difference scheme (see section 3.2 in [42]).

### 2.5.3 Numerical illustrations

#### Single-regime case

In this paragraph, we consider the case where there is only one regime ($d = 1$). In this case, our model is similar to the one studied in [61], with the key difference that in their model, the investor only observes the stock price at the trading times, so that the consumption process is piecewise-deterministic. We want to compare our results with [61], and take the same values for our parameters $\rho = 0.2, b = 0.4, \sigma = 1$.

Defining the *cost of liquidity* $P(x)$ as the extra amount needed to have the same utility as in the Merton case : $v(x + P(x)) = v_M(x)$, we compare the results in our model and in the discrete observation model in [61]. The results in Table 1 indicate that the impact of the lack of continuous observation is quite large, and more important than the constraint of only being able to trade at discrete times.

In Figure 1 we have plotted the graph of $\varphi(z)$ and of the optimal consumption rate $c^*(z)$ for

| $\lambda$ | Discrete observation | Continuous observation |
|---|---|---|
| 1 | 0.275 | 0.153 |
| 5 | 0.121 | 0.016 |
| 40 | 0.054 | 0.001 |

Table 2.1: Cost of liquidity $P(1)$ as a function of $\lambda$.

different values of $\lambda$. Notice how the value function, the optimal proportion and the optimal consumption rate converge to the Merton values when $\lambda$ increases.

We observe that the optimal investment proportion is increasing with $\lambda$. When $z$ is close to 1 i.e. the cash proportion in the portfolio is small, the investor faces the risk of "having nothing more to consume" and the further away the next trading date is the smaller the consumption rate should be, i.e. $c^*$ is increasing in $\lambda$. When $z$ is far from 1 it is the opposite : when $\lambda$ is smaller the investor will not be able to invest optimally to maximize future income and should consume more quickly.

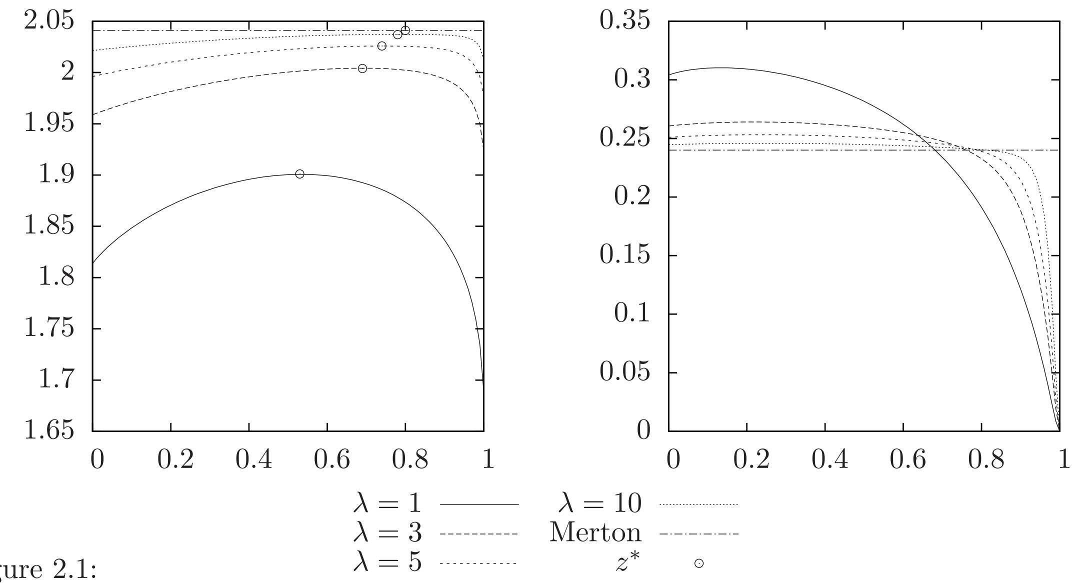


Figure 2.1:
Value function $\varphi(z)$ (left) and optimal consumption rate $c^*(z)$(right) for different values of $\lambda$

**Two regimes**

In this paragraph, we consider the case of $d = 2$ regimes. We assume that the asset price is continuous, i.e. $\gamma_{12} = \gamma_{21} = 0$. In this case, the value functions and optimal strategies for the continuous trading (Merton) problem are explicit, see [69]: $v_{i,M}(r) = \frac{r^p}{p}\varphi_{i,M}$ where $(\varphi_{i,M})_{i=1,2}$

is the only positive solution to the equations:

$$\Big(\rho - q_{ii} - \frac{b_i^2 p}{2\sigma_i^2(1-p)}\Big)\varphi_{i,M} - (1-p)\varphi_{i,M}^{-\frac{p}{1-p}} \;=\; q_{ij}\varphi_{j,M}, \quad i,j \in \{1,2\},\ i \neq j.$$

The optimal proportion invested in the asset $\pi^*_{i,M} = \frac{b_i}{(1-p)\sigma_i^2}$ is the same as in the single-regime case, and the optimal consumption rate is $c^*_{i,M} = (\varphi_{i,M})^{-\frac{1}{p}}$. We take for values of the parameters

$$\begin{aligned} p &= 0.5, \\ q_{12} = q_{21} &= 1, \\ b_1 = b_2 &= 0.4, \\ \sigma_1 = 1, & \qquad \sigma_2 = 2, \end{aligned}$$

i.e. the difference between the two market regimes is the volatility of the asset. In Figure 2.2, we plot the value function and optimal consumption for each of the two regimes in this market, for various values of the liquidity parameters $(\lambda_1, \lambda_2)$. As in the single-regime case, when the liquidity increases, $\varphi$ and $c^*$ converge to the Merton value.

To quantify the impact of regime-switching on the investor, it is also interesting to compare the cost of liquidity with the single-regime case, see Tables 2.2 and 2.3. We observe that, for equivalent trading intensity, the cost of liquidity is higher in the regime-switching case. This is economically intuitive : in each regime the optimal investment proportion is different, so that the investor needs to rebalance his portfolio more often (at every change of regime).

| $(\lambda_1, \lambda_2)$ | $P_1(1)$ | $P_2(1)$ |
|---|---|---|
| (1,1) | 0.257 | 0.224 |
| (5,5) | 0.112 | 0.103 |
| (10,10) | 0.069 | 0.064 |

Table 2.2: Cost of liquidity $P_i(1)$ as a function of $(\lambda_1, \lambda_2)$.

| $\lambda$ | $P_1(1)$ | $P_2(1)$ |
|---|---|---|
| 1 | 0.153 | 0.087 |
| 5 | 0.015 | 0.042 |
| 10 | 0.004 | 0.024 |

Table 2.3: Cost of liquidity $P_i(1)$ for the single-regime case.

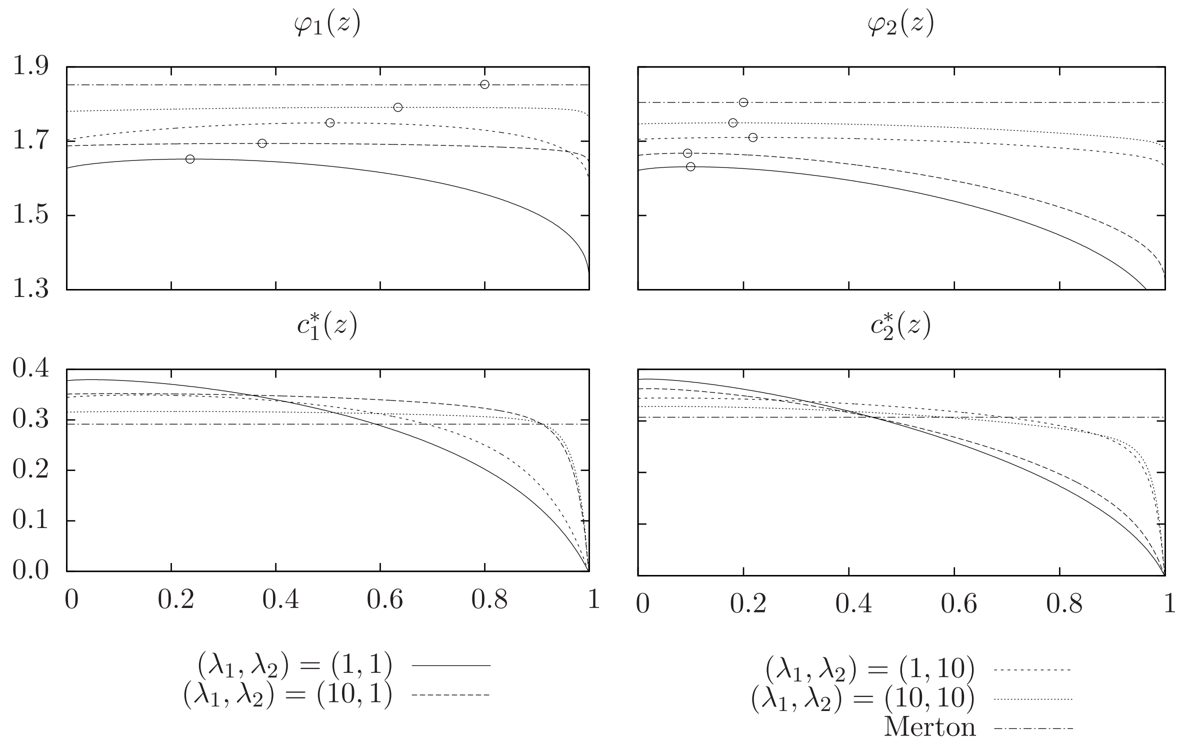


Figure 2.2: $\varphi_i$ and $c_i^*$ for different values of $(\lambda_1, \lambda_2)$

## 2.6 Appendix A : Dynamic Programming Principle

We introduce the weak formulation of the control problem.

**Definition 2.6.1.** *Given $(i, x, y) \in \mathbb{I}_d \times \mathbb{R}_+ \times \mathbb{R}_+$, a control $\mathcal{U}$ is a 9-tuple $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, W, I, N, c, \zeta)$, where :*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a filtered probability space satisfying the usual conditions.*
2. *$I$ is a Markov chain with space state $\mathbb{I}_d$ and generator $Q$, $I_0 = i$ a.s., $N$ is a Cox process with intensity $(\lambda_{I_t})$, and $W$ is an $\mathbb{F}$-Brownian motion independent of $(I, N)$.*
3. *$\mathcal{F}_t = \sigma(W_s, I_s, N_s; s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets of $\mathcal{F}$.*
4. *$(c_t)$ is $\mathbb{F}$-progressively measurable, $(\zeta_t)$ is $\mathbb{F}$-predictable.*

*We say that $\mathcal{U}$ is* admissible, *(writing $\mathcal{U} \in \mathcal{A}_i^w(x, y)$), if the solution $(X, Y)$ to* (2.2.3)-(2.2.2) *with $X_0 = x, Y_0 = y$, satisfies $X_t \geq 0$, $Y_t \geq 0$ a.s.*

Given $\mathcal{U} \in \mathcal{A}_i^w(x,y)$, define $J(\mathcal{U}) = \mathbb{E}\left[\int_0^\infty e^{-\rho s}U(c_s)ds\right]$, and the value function

$$v_i(x,y) = \sup_{\mathcal{U}\in\mathcal{A}_i^w(x,y)} J(\mathcal{U}).$$

**Proposition 2.6.1.** *For every finite stopping time $\tau$ and initial conditions $i, x, y$,*

$$v_i(x,y) \quad = \sup_{(\zeta,c)\in\mathcal{A}_i^w(x,y)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}U(c_t)dt + e^{-\rho\tau}v_{I_\tau}(X_\tau, Y_\tau)\right]. \tag{2.6.1}$$

Before proving this proposition we state some technical lemmas.

**Lemma 2.6.1.** *Given $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t), W, I, N)$ satisfying the conditions of Definition 2.6.1, define $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$, where $\mathcal{F}_t^0 = \sigma(W_s, I_s, N_s; s \leq t)$. Then if $(c_t)$ is $\mathbb{F}$-progressively measurable (resp. predictable), there exists $c_1$ $\mathbb{F}^0$-progressively measurable (resp. predictable) such that $c = c_1$ $d\mathbb{P}\otimes dt$ a.e..*

**Proof.** We only give a sketch as the arguments is standard. We first use Lemma 3.2.4 page 133 in [41] to find, for each $n \in \mathbb{N}$, an approximating $\mathcal{F}_t$-simple process $c^n$ converging to $c$ in the $L^2(dt\otimes d\mathbb{P})$ norm. Then, using Lemma 1.25 page 13 in [40], we can change every $c^n$ on a null-set and find a sequence of $\mathcal{F}_s^{t,0}$-simple process $c_1^n(t)$ that again converges to $c$ in the $L^2(dt \otimes d\mathbb{P})$ norm. We now extract a subsequence (denoted again by $c_1^n$) such that $c_1^n \to c$ a.e. and we define $c_1 := \liminf_{n\to+\infty} c_1^n$. This is $\mathcal{F}_s^{t,0}$-progressively measurable and $c = c_1$, $dt \otimes d\mathbb{P}$ a.e. on $[0, +\infty) \times \Omega$. This concludes the proof. □

**Remark 2.6.1.** With the notations of the previous lemma, it is easy to check that $(X^{c',\zeta'}, Y^{c',\zeta'}) \sim (X^{c,\zeta}, Y^{c,\zeta})$ in law. Hence without loss of generality we can assume that $c$ is $\mathbb{F}^0$-progressively measurable and $\zeta$ is $\mathbb{F}^0$-predictable.

Define $\mathcal{W}$ as the space of continuous functions on $\mathbb{R}_+$, $\mathcal{I}$ the space of cadlag $\mathbb{I}_d$-valued functions, $\mathcal{N}$ the space of nondecreasing cadlag $\mathbb{N}$-valued functions. On $\mathcal{W}\times\mathcal{I}\times\mathcal{N}$, define the filtration $(\mathcal{B}_t^0)_{t\geq 0}$, where $\mathcal{B}_t^0$ is the smallest $\sigma$-algebra making the coordinate mappings for $s \leq t$ measurable, and define $\mathcal{B}_{t+}^0 = \bigcap_{s>t}\mathcal{B}_s^0$.

**Lemma 2.6.2.** *If $c$ is $\mathbb{F}^0$-progressively measurable (resp. $\mathbb{F}^0$-predictable), there exists a $\mathcal{B}_{t+}^0$-progressively measurable (resp. $\mathcal{B}_t^0$-predictable) process $f_c : \mathbb{R}_+ \times \mathcal{W}\times\mathcal{I}\times\mathcal{N} \to \mathbb{R}$, such that*

$$c_t \quad = \quad f_c(t, W_{.\wedge t}, I_{.\wedge t}, N_{.\wedge t}), \quad \textit{for } \mathbb{P}-a.e\ \ \omega, \quad \textit{for all } t\in\mathbb{R}_+$$

**Proof.** For the progressively measurable part one can see e.g. Theorem 2.10 in [71]. For $c$ predictable, notice that this is true if $c = X\mathbf{1}_{(t,s]}$, where $X$ is $\mathcal{F}^0_t$-measurable, and conclude with a monotone class argument. □

**Proof of Proposition A.1.** Let $V_i(x,y)$ be the right hand side of (2.6.1).
*Step 1.* $v_i(x,y) \leq V_i(x,y)$: Take $\mathcal{U} \in \mathcal{A}^w_i(x,y)$. Then

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\,|\mathcal{F}_\tau\right] = \int_0^\tau e^{-\rho t}U(c_t)dt + e^{-\rho\tau}\mathbb{E}\left[\int_0^\infty e^{-\rho s}U(c_{\tau+s})ds\,|\mathcal{F}_\tau\right]. \qquad (2.6.2)$$

By Remark 2.6.1, w.l.o.g. we can assume that $c$ is $\mathbb{F}^0$-progressively measurable (resp. $\zeta$ $\mathbb{F}^0$-predictable). For $\omega_0 \in \Omega$, define the shifted control $\tilde{\mathcal{U}}^{\omega_0} = (\Omega, \tilde{F}^\tau, \mathbb{P}_{\omega_0}, \tilde{\mathcal{F}}^\tau_t, \tilde{W}, \tilde{I}, \tilde{N}, \tilde{c}, \tilde{\zeta})$, where :

- $\mathbb{P}_{\omega_0} = \mathbb{P}(.|\mathcal{F}_\tau)(\omega_0)$
- $\tilde{W}_t = W_{\tau+t} - W_\tau$
- $\tilde{I}_t = I_{\tau+t}$
- $N'_t = N_{\tau+t} - N_\tau$
- $\tilde{F}^\tau$ is the augmentation of $\mathcal{F}$ by the $\mathbb{P}_{\omega_0}$-null sets, and $\tilde{\mathcal{F}}^\tau_t$ is the augmented filtration generated by $(\tilde{W}, \tilde{I}, \tilde{N})$.
- $\tilde{c}_t = c_{t+\tau}$, $\tilde{\zeta}_t = \zeta_{t+\tau}$

Then we can check that for almost all $\omega_0$, $\tilde{\mathcal{U}}^{\omega_0}$ satisfies the conditions of Definition 2.6.1 (with initial conditions $(I_\tau(\omega_0), X_\tau(\omega_0), Y_\tau(\omega_0)))$ : 2. comes from the independence of $W$ and $(I,N)$ and the strong Markov property, and 4. is verified because for almost all $\omega_0$ $\mathcal{F}^0_{t+\tau} \subset \tilde{\mathcal{F}}^\tau_t$.

Moreover, there is a modification $(X', Y')$ of $(X,Y)$ s.t. $(X'_{\tau+t}, Y'_{\tau+t})$ is $\tilde{F}^\tau$-adapted, and a solution of (2.2.3)-(2.2.2) for $(\tilde{W}, \tilde{I}, \tilde{N})$. Hence $\tilde{\mathcal{U}}^{\omega_0} \in \mathcal{A}^w_{I_\tau(\omega_0)}(X_\tau(\omega_0), Y_\tau(\omega_0))$, and

$$\mathbb{E}\left[\int_0^\infty e^{-\rho s}U(c_{\tau+s})ds\,|\mathcal{F}_\tau\right](\omega_0) = J(\tilde{\mathcal{U}}^{\omega_0}) \leq v_{I_\tau}(X_\tau, Y_\tau)(\omega_0).$$

Hence taking the expectation over $\omega_0$ in (2.6.2),

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right] \leq \mathbb{E}\left[\int_0^\tau e^{-\rho t}U(c_t)dt + e^{-\rho\tau}v_{I_\tau}(X_\tau, Y_\tau)\right],$$

and taking the supremum over $\mathcal{U}$, we obtain $v_i(x,y) \le V_i(x,y)$.

*Step 2.* $v_i(x,y) \ge V_i(x,y)$: Recall that in the proof of Proposition 2.3.2 we only needed the DPP to prove the continuity of $v_i$ up to the boundary. Hence we know a priori that $v_i$ is continuous on $Int(\mathbb{R}^2_+)$, and that the restriction of $v_i$ to the boundary is continuous. One can then find a countable sequence $(U_k)_{k\ge 0}$ s.t.

(i) $(U_k)_k$ is a partition of $\mathbb{R}^2_+$,

(ii) $\forall (x,y),(x',y') \in U_k, \forall i, |v_i(x,y) - v_i(x',y')| \le \varepsilon$,

(iii) $U_k$ contains its bottom-left corner $(x_k,y_k) = \Big(\min_{(x,y)\in U_k} x, \min_{(x,y)\in U_k} y\Big)$.

Indeed, we can construct such a partition in the following way: $v_i$ is continuous on the boundary so we can partition each of the boundary lines into a countable number of segments verifying (ii) and (iii). Then in the interior we have first a partition in "squared rings" : $Int(\mathbb{R}^2_+) = \cup_{n\ge 1} K_n$, where $K_n = [1/(n+1), n+1]^2 \setminus [1/n, n]^2$. Since $v_i$ is continuous on the interior, we can partition each $K_n$ into a finite number of squares verifying (ii) and (iii). By taking the union of the line segments and the squares for each $K_n$, we obtain a sequence $(U_k)$ satisfying (i)-(iii).

Notice that (iii) implies the inclusion $\mathcal{A}_i(x_k,y_k) \subset \mathcal{A}_i(x,y)$, for all $(x,y) \in U_k$. For each $k$, take $\mathcal{U}^{i,k} = (\Omega^{i,k}, \mathcal{F}^{i,k}, \mathbb{P}^{i,k}, \mathbb{F}^{i,k}, W^{i,k}, I^{i,k}, N^{i,k}, c^{i,k}, \zeta^{i,k})$ $\varepsilon$-optimal for $(i, x_k, y_k)$, and $f_c^{i,k}, f_\zeta^{i,k}$ associated to $(c^{i,k}, \zeta^{i,k})$ by Lemma 2.6.2. Then for each $(c,\zeta) \in \mathcal{A}_i(x,y)$, let us define $\tilde{c}, \tilde{\zeta}$ by :

$$\tilde{c}_t = \begin{cases} c_t & \text{when } t < \tau \\ f_c^{i,k}(t-\tau, \tilde{W}(.\wedge(t-\tau)), \tilde{I}(.\wedge(t-\tau)), \tilde{N}(.\wedge(t-\tau))) & \text{when } t \ge \tau, I_\tau = i, (X_\tau, Y_\tau) \in U_k. \end{cases}$$

Then $\tilde{c}$ (resp. $\tilde{\zeta}$) is $\mathbb{F}$- progressively measurable (resp. predictable). Furthermore, for almost all $\omega_0$, with $i = I_\tau(\omega_0)$ and $(X_\tau, Y_\tau)(\omega_0) \in U_k$,

$$\mathcal{L}_{\mathbb{P}^{\omega_0}}(\tilde{W}, \tilde{I}, \tilde{N}, (\tilde{c}_{t+\tau}), (\tilde{\zeta}_{t+\tau})) = \mathcal{L}_{\mathbb{P}^{i,k}}(W^{i,k}, I^{i,k}, N^{i,k}, c^{i,k}, \zeta^{i,k}),$$

and since $\mathcal{A}_i(x_k,y_k) \subset \mathcal{A}_{I_\tau(\omega_0)}(X_\tau(\omega_0), Y_\tau(\omega_0))$, this implies $X_t^{\tilde{c},\tilde{\zeta}}, Y_t^{\tilde{c},\tilde{\zeta}} \ge 0$ a.s., and $(\tilde{c},\tilde{\zeta}) \in$

$\mathcal{A}_i(x,y)$. We also have

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-\rho s} U(\tilde{c}_{\tau+s}) ds \,|\mathcal{F}_\tau\right](\omega_0) &= \mathbb{E}^{i,k}\left[\int_0^\infty e^{-\rho s} U(c_s^{i,k}) ds\right] \\ &\geq v_i(x_k, y_k) - \varepsilon \\ &\geq v_{I_\tau}(X_\tau, Y_\tau)(\omega_0) - 2\varepsilon. \end{aligned}$$

By taking expectation in (2.6.2), we have

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t) dt\right] \geq \mathbb{E}\left[\int_0^\tau e^{-\rho t} U(c_t) dt + e^{-\rho \tau} v_{I_\tau}(X_\tau, Y_\tau)\right] - 2\varepsilon.$$

Finally, by taking the supremum over $\mathcal{U}$, and letting $\varepsilon$ go to 0, we obtain $v_i(x,y) \geq V_i(x,y)$. □

**Remark 2.6.2.** Actually the weak value function is equal to the value function defined in (2.2.14) for any $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W, I, N)$ satisfying (1)-(3) in Definition 2.6.1. Indeed, given any $\mathcal{U}' = (\Omega', \mathcal{F}', \mathbb{P}', \mathbb{F}', W', I', N') \in \mathcal{A}_i^w(x,y)$, letting $f_{c'}$ and $f_{\zeta'}$ being associated to $c'$ and $\zeta'$ by Lemmas 2.6.1 and 2.6.2, and defining (almost surely) $c_t = f_{c'}(t, W, I, N)$, $\zeta_t = f_{\zeta'}(t, W, I, N)$, by the same arguments as in the Proof of Proposition 2.6.1, $\mathcal{U} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W, I, N, c, \zeta) \in \mathcal{A}_i^w(x,y)$, and $J(\mathcal{U}) = J(\mathcal{U}')$. Hence

$$\sup_{\mathcal{U}' \in \mathcal{A}_i^w(x,y)} J(\mathcal{U}') = \sup_{(c,\zeta) \in \mathcal{A}_i(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho s} U(c_s) ds\right].$$

## 2.7 Appendix B : Viscosity characterization

We first prove the viscosity property of the value function to its dynamic programming system (2.4.2), written as:

$$F_i(x, y, v_i(x,y), Dv_i(x,y), D^2 v_i(x,y)) + G_i(x,y,v) = 0, \ (x,y) \in (0,\infty) \times \mathbb{R}_+,$$

for any $i \in \mathbb{I}_d$, where $F_i$ is the local operator defined by:

$$F_i(x,y,u,p,A) = \rho u - b_i y p_2 - \frac{1}{2}\sigma_i^2 y^2 a_{22} - \tilde{U}(p_1)$$

for $(x,y) \in (0,\infty)\times\mathbb{R}_+$, $u \in \mathbb{R}$, $p = (p_1\ p_2) \in \mathbb{R}^2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \in \mathcal{S}^2$ (the set of symmetric $2\times 2$ matrices), and $G_i$ is the nonlocal operator defined by:

$$G_i(x,y,w) \;=\; -\sum_{j\neq i} q_{ij}\big[w_j(x,y(1-\gamma_{ij})) - w_i(x,y)\big] - \lambda_i\big[\hat{w}_i(x+y) - w_i(x,y)\big]$$

for $w = (w_i)_{i\in\mathbb{I}_d}$ $d$-tuple of continuous functions on $\mathbb{R}^2_+$.

**Proposition 2.7.1.** *The value function $v = (v_i)_{i\in\mathbb{I}_d}$ is a viscosity solution of (E).*

**Proof.** *Viscosity supersolution*: Let $(i,\bar{x},\bar{y}) \in \mathbb{I}_d \times (0,\infty)\times\mathbb{R}_+$, $\varphi = (\varphi_i)_{i\in\mathbb{I}_d}$, $C^2$ test functions s.t. $v_i(\bar{x},\bar{y}) = \varphi_i(\bar{x},\bar{y})$, and $v \geq \varphi$. Take some arbitrary $e \in (-\bar{y},\bar{x})$, and $c \in \mathbb{R}_+$. Since $\bar{x} > 0$, there exists a strictly positive stopping time $\tau > 0$ a.s. such that the control process $(\bar{\zeta},\bar{c})$ defined by:

$$\bar{\zeta}_t \;=\; e1_{t\leq\tau}, \qquad \bar{c}_t \;=\; c1_{t\leq\tau}, \quad t\geq 0, \tag{2.7.1}$$

with associated state process $(\bar{X},\bar{Y},I)$ starting from $(x,y,i)$ at time 0, satisfies $\bar{X}_t \geq 0$, $\bar{Y}_t \geq 0$, for all $t$. Thus, $(\bar{\zeta},\bar{c}) \in \mathcal{A}_i(x,y)$. Let $\mathcal{V}$ be a compact neighbourhood of $(x,y,i)$ in $(0,\infty)\times\mathbb{R}_+\times\mathbb{I}_d$, and consider the sequence of stopping time: $\theta_n = \theta\wedge h_n$, where $\theta = \inf\Big\{t\geq 0 : (\bar{X}_t,\bar{Y}_t,I_t)\notin\mathcal{V}\Big\}$, and $(h_n)$ is a strictly positive sequence converging to zero. From the dynamic programming principle (2.4.1), and by applying Itô's formula to $e^{-\rho t}\varphi(\bar{X}_t,\bar{Y}_t,I_t)$ between 0 and $\theta_n$, we get:

$$\begin{aligned}
\varphi(\bar{x},\bar{y},i) \;=\; v(x,y,i) \;&\geq\; \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho t}U(\bar{c}_t)dt + e^{-\rho\theta_n}v(\bar{X}_{\theta_n},\bar{Y}_{\theta_n},I_{\theta_n})\right] \\
&\geq\; \mathbb{E}\Big[\int_0^{\theta_n} e^{-\rho t}U(\bar{c}_t)dt + e^{-\rho\theta_n}\varphi(\bar{X}_{\theta_n},\bar{Y}_{\theta_n},I_{\theta_n})\Big] \\
&=\; \varphi(\bar{x},\bar{y},i) + \mathbb{E}\Big[\int_0^{\theta_n} e^{-\rho t}\Big(U(\bar{c}_t) - \rho\varphi - \bar{c}_t\frac{\partial\varphi}{\partial x} \\
&\qquad + b_{I_{t-}}\bar{Y}_{t^-}\frac{\partial\varphi}{\partial y} + \frac{1}{2}\sigma^2_{I_{t-}}\bar{Y}^2_{t^-}\frac{\partial^2\varphi}{\partial y^2} \\
&\qquad + \sum_{j\neq I_{t-}} q_{I_{t-}j}[\varphi(\bar{X}_{t^-},\bar{Y}_{t^-}(1-\gamma_{I_{t-}j}),j) - \varphi(\bar{X}_{t^-},\bar{Y}_{t^-},I_{t^-})] \\
&\qquad + \lambda_{I_{t-}}[\varphi(\bar{X}_{t^-}-\bar{\zeta}_t,\bar{Y}_{t^-}+\bar{\zeta}_t,I_{t^-}) - \varphi(\bar{X}_{t^-},\bar{Y}_{t^-},I_{t^-})]\Big)dt\Big],
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}\Big[\frac{1}{h_n}\int_0^{\theta_n} e^{-\rho t}\Big(\rho\varphi - U(\bar{c}_t) + \bar{c}_t\frac{\partial\varphi}{\partial x} - b_{I_{t-}}\bar{Y}_{t-}\frac{\partial\varphi}{\partial y} - \frac{1}{2}\sigma^2_{I_{t-}}\bar{Y}^2_{t-}\frac{\partial^2\varphi}{\partial y^2} \\
- \sum_{j\neq I_{t-}} q_{I_{t-}j}[\varphi(\bar{X}_{t-},\bar{Y}_{t-}(1-\gamma_{I_{t-}j}),j) - \varphi(\bar{X}_{t-},\bar{Y}_{t-},I_{t-})] \\
-\lambda_{I_{t-}}\left[\varphi(\bar{X}_{t-}-\bar{\zeta}_t,\bar{Y}_{t-}+\bar{\zeta}_t,I_{t-}) - \varphi(\bar{X}_{t-},\bar{Y}_{t-},I_{t-})\right]\Big)dt\Big] \quad \geq \quad 0 \qquad (2.7.2)
\end{aligned}$$

Now, we have almost surely for $n$ large enough, $\theta \geq h_n$, i.e. $\theta_n = h_n$, so that by using also (2.7.1)

$$\begin{aligned}
&\frac{1}{h_n}\int_0^{\theta_n} e^{-\rho t}\Big(\rho\varphi - U(\bar{c}_t) + \bar{c}_t\frac{\partial\varphi}{\partial x} - b_{I_{t-}}\bar{Y}_{t-}\frac{\partial\varphi}{\partial y} - \frac{1}{2}\sigma^2_{I_{t-}}\bar{Y}^2_{t-}\frac{\partial^2\varphi}{\partial y^2} \\
&- \sum_{j\neq I_{t-}} q_{I_{t-}j}[\varphi(\bar{X}_{t-},\bar{Y}_{t-}(1-\gamma_{I_{t-}j}),j) - \varphi(\bar{X}_{t-},\bar{Y}_{t-},I_{t-})] \\
&-\lambda_{I_{t-}}\left[\varphi(\bar{X}_{t-}-\bar{\zeta}_t,\bar{Y}_{t-}+\bar{\zeta}_t,I_{t-}) - \varphi(\bar{X}_{t-},\bar{Y}_{t-},I_{t-})\right]\Big)dt\Big] \\
\longrightarrow\quad &\rho\varphi_i(\bar{x},\bar{y}) - U(c) + c\frac{\partial\varphi_i}{\partial x}(\bar{x},\bar{y}) - b_i\bar{y}\frac{\partial\varphi_i}{\partial y}(\bar{x},\bar{y}) - \frac{1}{2}\sigma_i^2\bar{y}^2\frac{\partial^2\varphi_i}{\partial y^2}(\bar{x},\bar{y}) \\
&- \sum_{j\neq i} q_{ij}[\varphi_j(\bar{x},\bar{y}(1-\gamma_{ij})) - \varphi_i(\bar{x},\bar{y})] - \lambda_i[\varphi_i(\bar{x}-e,\bar{y}+e) - \varphi_i(\bar{x},\bar{y})], \quad a.s.
\end{aligned}$$

when $n$ goes to infinity. Moreover, since the integrand of the Lebesgue integral term in (2.7.2) is bounded for $t \leq \theta$, one can apply the dominated convergence theorem in (2.7.2), which gives:

$$\begin{aligned}
\rho\varphi_i(\bar{x},\bar{y}) - U(c) + c\frac{\partial\varphi_i}{\partial x}(\bar{x},\bar{y}) - b_i\bar{y}\frac{\partial\varphi_i}{\partial y}(\bar{x},\bar{y}) - \frac{1}{2}\sigma_i^2\bar{y}^2\frac{\partial^2\varphi_i}{\partial y^2}(\bar{x},\bar{y}) \\
- \sum_{j\neq i} q_{ij}[\varphi_j(\bar{x},\bar{y}(1-\gamma_{ij})) - \varphi_i(\bar{x},\bar{y})] - \lambda_i[\varphi_i(\bar{x}-e,\bar{y}+e) - \varphi_i(\bar{x},\bar{y})] \quad \geq \quad 0.
\end{aligned}$$

Since $c$ and $e$ are arbitrary, we obtain the required viscosity supersolution inequality by taking the supremum over $c \in \mathbb{R}_+$ and $e \in (-\bar{y},\bar{x})$.

*Viscosity subsolution*: Let $(\bar{i},\bar{x},\bar{y}) \in \mathbb{I}_d \times (0,\infty) \times \mathbb{R}_+$, $\varphi = (\varphi_i)_{i\in\mathbb{I}_d}$, $C^2$ test functions s.t. $v(\bar{x},\bar{y},\bar{i}) = \varphi(\bar{x},\bar{y},\bar{i})$, and $v \leq \varphi$. We can also assume w.l.o.g. that $v < \varphi$ outside $(\bar{x},\bar{y},\bar{i})$. We argue by contradiction by assuming that

$$\begin{aligned}
\rho\varphi_{\bar{i}}(\bar{x},\bar{y}) - b_{\bar{i}}\bar{y}\frac{\partial\varphi_{\bar{i}}}{\partial y}(\bar{x},\bar{y}) - \frac{1}{2}\sigma^2_{\bar{i}}\bar{y}^2\frac{\partial^2\varphi_{\bar{i}}}{\partial y^2}(\bar{x},\bar{y}) - \tilde{U}\Big(\frac{\partial\varphi_{\bar{i}}}{\partial x}(\bar{x},\bar{y})\Big) \\
- \sum_{j\neq\bar{i}} q_{\bar{i}j}[\varphi_j(\bar{x},\bar{y}(1-\gamma_{\bar{i}j})) - \varphi_{\bar{i}}(\bar{x},\bar{y})] - \lambda_{\bar{i}}[\hat{\varphi}_{\bar{i}}(\bar{x}+\bar{y}) - \varphi_{\bar{i}}(\bar{x},\bar{y})] \quad > \quad 0.
\end{aligned}$$

By continuity of $\varphi$, and of its derivatives, there exist some compact neighbourhood $\bar{\mathcal{V}}$ of $(\bar{x},\bar{y},\bar{i})$ in $(0,\infty)\times\mathbb{R}_+\times\mathbb{I}_d$, and $\varepsilon>0$, such that

$$\begin{aligned}&\rho\varphi_i(x,y)-b_iy\frac{\partial\varphi_i}{\partial y}(x,y)-\frac{1}{2}\sigma_i^2y^2\frac{\partial^2\varphi_i}{\partial y^2}(x,y)-\tilde{U}\Big(\frac{\partial\varphi_i}{\partial x}(x,y)\Big) \\ &-\sum_{j\neq i}q_{ij}[\varphi_j(x,y(1-\gamma_{ij}))-\varphi_i(x,y)]-\lambda_i[\hat{\varphi}_i(x+y)-\varphi_i(x,y)] \;\geq\; \varepsilon, \quad \forall(x,y,i)\in\bar{\mathcal{V}}.\end{aligned} \tag{2.7.3}$$

Since $v<\varphi$ outside $(\bar{x},\bar{y},\bar{i})$, there exists some $\delta>0$ s.t. $v<\varphi-\delta$ outside of $\bar{\mathcal{V}}$. We can also assume that $\varepsilon\leq\delta\rho$. By the DPP (2.4.1), there exists $(\zeta,c)\in\mathcal{A}_{\bar{i}}(\bar{x},\bar{y})$ s.t.

$$v(\bar{x},\bar{y},\bar{i})-\varepsilon\frac{1-e^{-\rho}}{2\rho} \;\leq\; \mathbb{E}\left[\int_0^{\theta\wedge 1}e^{-\rho t}U(c_t)dt+e^{-\rho(\theta\wedge 1)}v(X_{\theta\wedge 1},Y_{\theta\wedge 1},I_{\theta\wedge 1})\right],$$

where $(X,Y,I)$ is controlled by $(\zeta,c)$, and we take $\theta=\inf\Big\{t\geq 0:(X_t,Y_t,I_t)\notin\bar{\mathcal{V}}\Big\}$. We then get:

$$\begin{aligned}&\varphi(\bar{x},\bar{y},\bar{i})-\varepsilon\frac{1-e^{-\rho}}{2\rho}\\ =\;& v(\bar{x},\bar{y},\bar{i})-\varepsilon\frac{1-e^{-\rho}}{2\rho}\\ \leq\;& \mathbb{E}\left[\int_0^{\theta\wedge 1}e^{-\rho t}U(c_t)dt+e^{-\rho(\theta\wedge 1)}\varphi(X_{\theta\wedge 1},Y_{\theta\wedge 1},I_{\theta\wedge 1})-e^{-\rho\theta}\delta\mathbf{1}_{\{\theta<1\}}\right]\\ =\;& \varphi(\bar{x},\bar{y},\bar{i})+\mathbb{E}\Big[\int_0^{\theta\wedge 1}e^{-\rho t}\Big(U(c_t)-\rho\varphi-c_t\frac{\partial\varphi}{\partial x}\\ &\qquad +b_{I_{t-}}Y_{t-}\frac{\partial\varphi}{\partial y}+\frac{1}{2}\sigma^2_{I_{t-}}Y^2_{t-}\frac{\partial^2\varphi}{\partial y^2}\\ &\quad+\sum_{j\neq I_{t-}}q_{I_{t-}j}[\varphi(X_{t-},Y_{t-}(1-\gamma_{I_{t-}j}),j)-\varphi(X_{t-},Y_{t-},I_{t-})]\\ &\quad+\lambda_{I_{t-}}[\varphi(X_{t-}-\zeta_t,Y_{t-}+\zeta_t,I_{t-})-\varphi(X_{t-},Y_{t-},I_{t-})]\Big)dt-e^{-\rho\theta}\delta\mathbf{1}_{\{\theta<1\}}\Big]\\ \leq\;& \varphi(\bar{x},\bar{y},\bar{i})+\mathbb{E}\left[\int_0^{\theta\wedge 1}-\varepsilon e^{-\rho t}dt-e^{-\rho\theta}\delta\mathbf{1}_{\{\theta<1\}}\right]\end{aligned}$$

where we applied Itô's formula in the second equality, and used (2.7.3) in the last inequality. This means that

$$\begin{aligned}-\varepsilon\frac{1-e^{-\rho}}{2\rho} \;\leq\;& \mathbb{E}\left[\int_0^{\theta\wedge 1}-\varepsilon e^{-\rho t}dt-e^{-\rho\theta}\delta\mathbf{1}_{\{\theta<1\}}\right]\\ =\;& \mathbb{E}\left[-\frac{\varepsilon}{\rho}+\frac{\varepsilon}{\rho}e^{-\rho(\theta\wedge 1)}-e^{-\rho\theta}\delta\mathbf{1}_{\{\theta<1\}}\right] \;\leq\; -\frac{\varepsilon}{\rho}(1-e^{-\rho}),\end{aligned}$$

since $\varepsilon/\rho \leq \delta$, and we get the required contradiction. □

Let us now prove comparison principle for our dynamic programming system. As usual, it is convenient to formulate an equivalent definition for viscosity solutions to (2.4.2) in terms of semi-jets. We shall use the notation $X = (x, y)$ for $\mathbb{R}_+ \times \mathbb{R}_+$-valued vectors. Given $w = (w_i)_{i\in\mathbb{I}_d}$ a d-tuple of continuous functions on $\mathbb{R}^2_+$, the second-order *superjet* of $w_i$ at $X \in \mathbb{R}^2_+$ is defined by:

$$\begin{aligned}\mathcal{P}^{2,+}w_i(X) &= \Big\{(p,A) \in \mathbb{R}^2 \times \mathcal{S}^2 \text{ s.t. } w_i(X') \leq w_i(X) + \langle p, X' - X\rangle \\ &\quad + \frac{1}{2}\langle A(X'-X), X'-X\rangle + o\left(|X'-X|^2\right) \text{ as } X' \to X\Big\},\end{aligned}$$

and its closure $\overline{\mathcal{P}}^{2,+}w_i(X)$ as the set of elements $(p, A) \in \mathbb{R}^2 \times \mathcal{S}^2$ for which there exists a sequence $(X_m, p_m, A_m)_m$ of $\mathbb{R}^2_+ \times \mathcal{P}^{2,+}w_i(X_m)$ satisfying $(X_m, p_m, A_m) \to (X, p, A)$. We also define the second-order *subjet* $\mathcal{P}^{2,-}w_i(X) = -\mathcal{P}^{2,+}(-w_i)(X)$, and $\overline{\mathcal{P}}^{2,-}w_i(X) = -\overline{\mathcal{P}}^{2,+}(-w_i)(X)$. By standard arguments (see e.g. [3] for equations with nonlocal terms), one has an equivalent definition of viscosity solutions in terms of semijets:

A $d$-tuple $w = (w_i)_{i\in\mathbb{I}_d}$ of continuous functions on $\mathbb{R}^2_+$ is a viscosity supersolution (resp. subsolution) of (2.4.2) if and only if for all $(i, x, y) \in \mathbb{I}_d \times (0, \infty) \times \mathbb{R}_+$, and all $(p, A) \in \overline{\mathcal{P}}^{2,-}w_i(x, y)$ (resp. $\overline{\mathcal{P}}^{2,+}w_i(x, y)$):

$$F_i(x, y, w_i(x, y), p, A) + G_i(x, y, w) \;\; \geq \;\; 0, \quad (\textit{resp.} \;\leq\; 0).$$

We then prove the following comparison theorem.

**Theorem 2.7.1.** *Let $V = (V_i)_{i\in\mathbb{I}_d}$ (resp. $W = (W_i)_{i\in\mathbb{I}_d}$) be a viscosity subsolution (resp. supersolution) of* (2.4.2)*, satisfying the growth condition* (2.3.4)*, and the boundary conditions*

$$V_i(0,0) \;\leq\; 0 \tag{2.7.4}$$

$$V_i(0,y) \;\leq\; \mathbb{E}_i\left[\hat{V}_{I^i_{\tau_1}}\,(y\frac{S_{\tau_1}}{S_0})\right], \quad \forall y > 0, \tag{2.7.5}$$

*(resp. $\geq$ for $W$). Then $V \leq W$.*

**Proof.** *Step 1*: Take $p' > p$ such that $k(p') < \rho$, and define $\psi_i(x, y) = (x + y)^{p'}$, $i \in \mathbb{I}_d$. Let us check that $W^n = W + \frac{1}{n}\psi$ is still a supersolution of (E). Notice that $\mathcal{P}^{2,-}W^n_i =$

$\mathcal{P}^{2,-}W_i + \frac{1}{n}(D\psi_i, D^2\psi_i)$, and we have for all $(p, A) \in \mathcal{P}^{2,-}W_i(x, y)$:

$$\begin{aligned}
& F_i(x, y, W_i^n(x, y), p + \frac{1}{n}D\psi_i, A + \frac{1}{n}D^2\psi_i) + G_i(x, y, W^n) \\
= \ & F_i(x, y, W_i(x, y), p, A) + G_i(x, y, W) \\
& + \frac{1}{n}(x+y)^{p'}\Big(\rho - p'b_i\frac{y}{x+y} + p'(1-p')\frac{\sigma_i^2}{2}\left(\frac{y}{x+y}\right)^2 - \sum_{j\neq i} q_{ij}((1-\frac{y}{x+y}\gamma_{ij})^{p'} - 1)\Big) \\
& + \tilde{U}(p_1) - \tilde{U}(p_1 + \frac{1}{n}p'x^{p'-1}) \qquad (2.7.6) \\
\geq \ & 0.
\end{aligned}$$

Indeed, the three lines in the r.h.s. of (2.7.6) are nonnegative: the first one since $W$ is a supersolution, the second one by $k(p') < \rho$, and the last one since $\tilde{U}$ is nonincreasing.

Moreover, by the growth condition (2.3.4) on $V$ and $W$, we have:

$$\lim_{r\to\infty} \max_{i\in\mathbb{I}_d}(\hat{V}_i - \hat{W}_i^n)(r) \quad = \quad -\infty. \qquad (2.7.7)$$

In the next step, our aim is to show that for all $n \geq 1$, $V \leq W^n$, which would imply that $V \leq W$. We shall argue by contradiction.

*Step 2*: Assume that there exists some $n \geq 1$ s.t.

$$M := \sup_{i\in\mathbb{I}_d, (x,y)\in\mathbb{R}_+^2} (V_i - W_i^n)(x, y) > 0.$$

By (2.7.7), there exists $i \in \mathbb{I}_d$, some compact subset $\mathcal{C}$ of $\mathbb{R}_+^2$, and $\overline{X} = (\overline{x}, \overline{y}) \in \mathcal{C}$ such that

$$M \quad = \quad \max_{\mathcal{C}}(V_i - W_i^n) \;\; = \;\; (V_i - W_i^n)(\overline{x}, \overline{y}). \qquad (2.7.8)$$

Note that by (2.7.4), $(\overline{x}, \overline{y}) \neq (0, 0)$. We then have two possible cases:

• Case 1 : $\overline{x} = 0$. Notice that the boundary condition (2.7.5) implies the viscosity subsolution property for $V_i$ also at $\bar{X} = (0, \bar{y})$:

$$F_i(\bar{X}, V_i(\bar{X}), p, A) + G_i(\bar{X}, V) \quad \leq \quad 0, \quad \forall (p, A) \in \overline{\mathcal{P}}^{2,+}V_i(\bar{X})$$

However the viscosity supersolution property fot $W^n$ does not hold at $(0, \bar{y})$. Let $(X_k)_k = (x_k, y_k)_k$ be a sequence converging to $\overline{X}$, with $x_k > 0$, and $\varepsilon_k := \left|X_k - \overline{X}\right|$. We then consider the function

$$
\begin{aligned}
\Phi_k(X,X') &= V_i(X) - W_i^n(X') - \psi_k(X,X'),\\
\psi_k(x,y,x',y') &= x^4 + (y-\overline{y})^4 + \frac{|X-X'|^2}{2\varepsilon_k} + \left(\frac{x'}{x_k}-1\right)_-^3
\end{aligned}
$$

Since $\Phi_k$ is continuous, there exists $(\widehat{X}_k, \widehat{X}'_k) \in \mathcal{C}^2$ s.t.

$$M_k := \sup_{\mathcal{C}^2} \Phi_k = \Phi_k(\widehat{X}_k, \widehat{X}'_k),$$

and a subsequence, still denoted $(\widehat{X}_k, \widehat{X}'_k)$, converging to some $(\widehat{X}, \widehat{X}')$ as $k$ goes to $\infty$. By writing that $\Phi_k(\overline{X}, X_k) \leq \Phi_k(\widehat{X}_k, \widehat{X}'_k)$, we have :

$$
\begin{aligned}
& V_i(\overline{X}) - W_i^n(X_k) - \frac{\left|\overline{X} - X_k\right|}{2} && (2.7.9)\\
\leq\ & V_i(\widehat{X}_k) - W_i^n(\widehat{X}'_k) - (\hat{x}_k^4 + (\hat{y}_k - \overline{y})^4) - R_k && (2.7.10)\\
\leq\ & V_i(\widehat{X}_k) - W_i^n(\widehat{X}'_k) - (\hat{x}_k^4 + (\hat{y}_k - \overline{y})^4), && (2.7.11)
\end{aligned}
$$

where we set

$$R_k = \frac{\left|\widehat{X}_k - \widehat{X}'_k\right|^2}{2\varepsilon_k} + \left(\frac{\hat{x}'_k}{x_k} - 1\right)_-^3$$

Since $V_i$ and $W_i^n$ are bounded on $\mathcal{C}$, we deduce by inequality (2.7.10) the boundedness of the sequence $(R_k)_{k\geq 0}$, which implies $\widehat{X} = \widehat{X'}$. Then by sending $k$ to infinity in (2.7.9) and (2.7.11), with the continuity of $V_i$ and $W_i^n$, we obtain $M = V_i(\overline{X}) - W_i^n(\overline{X}) \leq V_i(\widehat{X}) - W_i^n(\widehat{X}) - (\hat{x}_k^4 + (\hat{y}_k - \overline{y})^4)$, and by definition of $M$ this shows

$$\widehat{X} = \widehat{X}' = \overline{X} \qquad (2.7.12)$$

Sending again $k$ to infinity in (2.7.9)-(2.7.10)-(2.7.11), we obtain $M \leq M - \limsup_k R_k \leq M$, and so

$$\frac{\left|\widehat{X}_k - \widehat{X}'_k\right|^2}{2\varepsilon_k} + \left(\frac{\widehat{x}'_k}{x_k} - 1\right)_-^3 \ \to\ 0, \qquad (2.7.13)$$

as $k$ goes to infinity. In particular for $k$ large enough $\hat{x}'_k \geq \frac{x_k}{2} > 0$. We can then apply Ishii's lemma (see Theorem 3.2 in [16]) to obtain $A, A' \in \mathcal{S}^2$ s.t.

$$(p, A) \in \overline{\mathcal{P}}^{2,+} V_i(\widehat{X}_k), \quad (p', A') \in \overline{\mathcal{P}}^{2,-} W_i^n(\widehat{X}'_k) \tag{2.7.14}$$

$$\begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \leq D + \varepsilon_k D^2, \tag{2.7.15}$$

where

$$p = D_X \psi_k(\widehat{X}_k, \widehat{X'}_k), \quad p' = D_{X'} \psi_k(\widehat{X}_k, \widehat{X}'_k),\ D = D^2_{X,X'} \psi_k(\widehat{X}_k, \widehat{X'}_k).$$

Now, we write

$$\begin{aligned} \rho M \ \leq \ \rho M_k \ &\leq \ \rho(V_i(\hat{X}_k) - W_i^n(\widehat{X'}_k)) \\ &= \ F_i(\widehat{X}_k, V_i(\widehat{X}_k), p, A) - F_i(\widehat{X}_k, W_i^n(\widehat{X'}_k), p, A) \\ &= \ F_i(\widehat{X}_k, V_i(\widehat{X}_k), p, A) + G_i(\widehat{X}_k, V) \\ &\quad - F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - G_i(\widehat{X}'_k, W^n) \\ &\quad + G_i(\widehat{X}'_k, W^n) - G_i(\widehat{X}_k, V) \\ &\quad + F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - F_i(\widehat{X}_k, W_i^n(\widehat{X}'_k), p, A) \end{aligned} \tag{2.7.16}$$

From the viscosity subsolution property for $V$ at $\hat{X}_k$, and the viscosity supersolution property for $W^n$ at $\hat{X}'_k$, the first two lines in the r.h.s. of (2.7.16) are nonpositive. For the third line, by sending $k$ to infinity, we have:

$$\begin{aligned} & G_i(\widehat{X}'_k, W^n) - G_i(\widehat{X}_k, V) \\ \to \ & G_i(\overline{X}, W^n) - G_i(\overline{X}, V) \\ = \ & \sum_{j \neq i} q_{ij} \Big[ (V_j - W_j^n)\Big(\overline{x}, \overline{y}(1-\gamma_{ij})\Big) - (V_i - W_i^n)(\overline{x}, \overline{y}) \Big] \\ & + \lambda_i \Big[ (\hat{V}_i - \hat{W}_i^n)(\overline{x} + \overline{y}) - (V_i - W_i^n)(\overline{x}, \overline{y}) \Big] \\ \leq \ & 0 \end{aligned}$$

by (2.7.8). For the fourth line of (2.7.16), we have

$$\begin{aligned} & F_i(\widehat{X}'_k, W_i^n(\widehat{X}'_k), p', A') - F_i(\widehat{X}_k, W_i^n(\widehat{X}'_k), p, A) \\ = \;& b_i(\hat{y}_k p_2 - \hat{y}'_k p'_2) + \tilde{U}(p_1) - \tilde{U}(p'_1) + \frac{\sigma_i^2}{2}\left(\hat{y}_k^2 a_{22} - (\hat{y}'_k)^2 a'_{22}\right) \end{aligned}$$

Now

$$\begin{aligned} \hat{y}_k p_2 - \hat{y}'_k p'_2 &= \hat{y}_k\left(4(\hat{y}_k - \overline{y})^3 + \frac{\hat{y}_k - \hat{y}'_k}{\varepsilon_k}\right) - \hat{y}'_k\left(\frac{\hat{y}_k - \hat{y}'_k}{\varepsilon_k}\right) \\ &\leq 4\hat{y}_k(\hat{y}_k - \overline{y})^3 + \frac{\left|\widehat{X}_k - \hat{x}'_k\right|^2}{\varepsilon_k} \\ &\to 0, \text{ as } k \to \infty, \end{aligned}$$

by (2.7.12) and (2.7.13). Moreover,

$$\begin{aligned} \tilde{U}(p_1) - \tilde{U}(p'_1) &= \tilde{U}\left(\frac{\hat{x}_k - \hat{x}'_k}{\varepsilon_k} + 4\hat{x}_k^3\right) - \tilde{U}\left(\frac{\hat{x}_k - \hat{x}'_k}{\varepsilon_k} - \frac{3}{x_k}\left(\frac{\hat{x}'_k}{x_k} - 1\right)_-^2\right) \\ &\leq 0, \end{aligned}$$

since $\tilde{U}$ is nonincreasing. Finally,

$$\begin{aligned} \hat{y}_k^2 a_{22} - (\hat{y}'_k)^2 a'_{22} &= \begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} \\ &\leq \begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} \left(D + \varepsilon_k D^2\right) \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} \end{aligned}$$

by (2.7.15). Since

$$D^2\psi_k(x, y, x', y') = \begin{pmatrix} 12x^2 & 0 & -\frac{1}{\varepsilon_k} & 0 \\ 0 & 12(y - \overline{y})^2 + \frac{1}{\varepsilon_k} & 0 & -\frac{1}{\varepsilon_k} \\ -\frac{1}{\varepsilon_k} & 0 & \frac{1}{\varepsilon_k} + \frac{6}{x_k^2}\left(\frac{x'}{x_k} - 1\right)_- & 0 \\ 0 & -\frac{1}{\varepsilon_k} & 0 & -\frac{1}{\varepsilon_k} \end{pmatrix},$$

a direct calculation gives

$$\begin{pmatrix} 0 & \hat{y}_k & 0 & \hat{y}'_k \end{pmatrix} \left(D + \varepsilon_k D^2\right) \begin{pmatrix} 0 \\ \hat{y}_k \\ 0 \\ \hat{y}'_k \end{pmatrix} = \frac{3}{\varepsilon_k}(\hat{y}_k - \hat{y}'_k)^2 - 12(\hat{y}_k - \overline{y})^2 \hat{y}_k \hat{y}'_k$$
$$+ \left(36(\hat{y}_k - \overline{y})^2 + \varepsilon_k \left(12(\hat{y}_k - \overline{y})^2\right)\right) \hat{y}_k^2$$
$$\to 0, \quad \text{as } k \to \infty,$$

where we used again (2.7.12) and (2.7.13), and the boundedness of $(\widehat{y}_k, \widehat{y'}_k)$.

Finally by letting $k$ go to infinity in (2.7.16) we obtain $\rho M \leq 0$, which is the required contradiction.

• Case 2 : $\overline{x} > 0$. This is the easier case, and we can obtain a contradiction similarly as in the first case, by considering for instance the function

$$\Phi_k(X, X') = V_i(X) - W_i^n(X') - (x - \overline{x})^4 - (y - \overline{y})^4 - k\frac{|X - X'|^2}{2}.$$

□

# Chapter 3

# Investment/consumption problem in a market with liquid and illiquid assets

Abstract: We consider a market consisting of a liquid asset and an illiquid asset. The liquid asset can be traded continuously, while the illiquid one can only be traded and observed at discrete random times. In this setting, we study the problem of an economic agent optimizing his expected utility from consumption under a non-bankruptcy constraint. This is a nonstandard, mixed discrete/continuous control problem. By a dynamic programming approach, we reduce this problem to a standard continuous time stochastic control problem, and we give an analytical characterization of the value function as a viscosity solution to a PDE. We present an iterative numerical scheme to compute it, and we finally illustrate the impact of illiquidity on the investor and his strategies by some numerical experiments.

**Key words :** liquidity, random trading times, portfolio/consumption problem, integrodifferential equations, viscosity solutions.

## 3.1 Introduction

Following the seminal works of Merton on portfolio management, a classical assumption in mathematical finance is to suppose that assets may be continuously traded by the agents operating on the market. However, this assumption is unrealistic in practice, especially in the case of less liquid markets, where investors cannot buy and sell assets immediately, and have to wait some time before being able to unwind a position.

In the recent years, several works have studied the impact of this type of illiquidity on the investors. Rogers and Zane [66], Matsumoto [53], Pham and Tankov [61] consider an investment model where the discrete trading times are given by the jump times of a Poisson process with constant intensity $\lambda > 0$. Bayraktar and Ludkovski [8] study a portfolio liquidation problem in a similar context.

The aforementioned works focus on an agent investing exclusively in an illiquid asset. However, in practice it is common to have several correlated tradable assets with different liquidity. For instance an index fund over some given financial market will be usually much more liquid than the individual tracked assets, while sharing a positive correlation with those assets. An investor in this market will then have the possibility of hedging his exposure in the less liquid assets by investing in the index and rebalancing his position frequently. Tebaldi and Schwartz [68], Longstaff [48] consider a market constituted of a liquid asset that can be traded continuously, and an illiquid asset that may only be traded at the initial time and is liquidated at a terminal date. Following the line of the latter papers, here we also consider a market composed by a liquid asset and an illiquid one, but we take a less restrictive approach assuming, as in [66, 53, 61], that the illiquid asset may be traded at discrete random times.

To this regard, we have to mention the recent paper by Ang, Papanikolaou and Westerfield [2] that studies a very similar problem to the one studied here. However, we stress that our results are different for two reasons. First, they consider utility functions of CRRA type with risk aversion parameter $R \geq 1$, while we study the problem for a different class of functions, not assumed of CRRA type. Second, they assume that the agent is able to observe the illiquid asset's price continuously, while in our case observation is restricted to the trading dates. We believe this is a more natural assumption, as in practice trading possibilities and observation of the price coincide via the arrival of buy/sell orders on the market.

We study a problem of optimal investment/consumption over an infinite horizon in a market consisting of a liquid and an illiquid asset. The liquid asset is observed and can be traded continuously, while the illiquid one can only be traded and observed at discrete random times corresponding to the jumps of a Poisson process with intensity $\lambda$. This makes the problem a nonstandard mixed discrete/continuous problem, which we solve by following the same approach as in [61]. By means of dynamic programming, we show that the stochastic control problem "between trading times" can be written as a standard continuous time-inhomogeneous problem. Then we apply the usual machinery of Dynamic Programming and characterize the value function as the unique (viscosity) solution of an HJB equation. This allows to perform a numerical analysis of the solution via a suitable numerical scheme that we describe in detail.

The plan of the paper is as follows. Section 2 describes our illiquid market model and formulates the investment/consumption problem for the investor. In Section 3 we show how, by a suitable Dynamic Programming Principle, the problem can be reduced to a standard continuous-time stochastic control problem. Section 4 presents some useful properties satisfied by our value functions. In Section 5, we first prove an analytical characterization of our value function by means of viscosity solutions to the associated HJB equation, and then show the special form taken by the HJB equation in the case of power utility. Finally, Section 6 introduces an iterative scheme to solve our problem numerically, and presents some numerical results.

## 3.2 The model

In this section we present the model and the optimization problem we deal with.

Let us consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, on which there are defined:

- A Poisson process $(N_t)_{t\geq 0}$, with intensity $\lambda > 0$; we denote by $(\mathcal{N}_t)_{t\geq 0}$ the filtration generated by this process and by $(\tau_n)_{n\geq 1}$ its jump times; moreover we set $\tau_0 = 0$.

- Two independent standard Brownian motions $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, independent also on the Poisson process $(N_t)_{t\geq 0}$; we denote by $(\mathcal{F}_t^B)_{t\geq 0}$ and $(\mathcal{F}_t^W)_{t\geq 0}$ the filtration generated by $B$ and $W$ respectively.

The market model we consider on this probability space consists of two risky assets with correlation $\rho \in (-1, 1)$:

- A liquid risky asset that can be traded continuously; it is described by a stochastic process denoted by $L_t$ whose dynamics is

$$dL_t = L_t(b_L dt + \sigma_L dW_t),$$

  where $b_L \in \mathbb{R}$ and $\sigma_L > 0$.

- An illiquid risky asset that can only be traded at the trading times $\tau_n$; it is described by a stochastic process denoted by $I_t$, whose dynamics is

$$dI_t = I_t(b_I dt + \sigma_I(\rho dW_t + \sqrt{1-\rho^2} dB_t)),$$

  where $b_I \in \mathbb{R}$ and $\sigma_I > 0$.

Without loss of generality we assume $L_0 = I_0 = 1$. We also suppose that on the market is present a riskless asset with deterministic dynamics. Without loss of generality we assume that the interest rate of such asset is constant and equal to 0.

Define the $\sigma$-algebra

$$\mathcal{I}_t = \sigma(I_{\tau_n} \mathbf{1}_{\{\tau_n \leq t\}}, n \geq 0), \quad t \geq 0.$$

Moreover define the filtration

$$\mathbb{G}^0 := (\mathcal{G}_t)_{t \geq 0}; \quad \mathcal{G}_t^0 = \mathcal{N}_t \vee \mathcal{I}_t \vee \mathcal{F}_t^W = \sigma(\tau_n, I_{\tau_n}; \tau_n \leq t) \vee \mathcal{F}_t^W.$$

The observation filtration we consider is

$$\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}; \quad \mathcal{G}_t = \mathcal{G}_t^0 \vee \sigma(\mathbb{P}\text{-null sets}).$$

This means that at time $t$ we know the past of the liquid asset up to time $t$, the trading dates of the illiquid assets occurred before $t$ and the values of the illiquid asset at such trading dates.

### 3.2.1 Trading/consumption strategies

In the setting above, we define the set of admissible trading/consumption strategies in the following way. Consider all the triplets of processes $(c_t, \pi_t, \alpha_k)$ such that

(h1) $c = (c_t)$ is a nonnegative locally integrable process $(\mathcal{G}_t)$-predictable; $c_t$ represents the consumption rate at time $t$;

(h2) $\pi = (\pi_t)$ is a locally square integrable process $(\mathcal{G}_t)$-predictable; $\pi_t$ represents the amount of money invested in the liquid asset at time $t$;

(h3) $\alpha = (\alpha_k)$, is a discrete process where $\alpha_k$ is $\mathcal{G}_{\tau_k}$-measurable; $\alpha_k$ represents the amount of money invested in the illiquid asset in the interval $(\tau_k, \tau_{k+1}]$.

Given a triplet $(c_t, \pi_t, \alpha_k)$ satisfying the requirements (h1)-(h3) above, we can consider the process $R_t$ representing the wealth associated to such strategy. Its dynamics can be defined by recursion on $k$ by

$$R_0 = r, \tag{3.2.1}$$
$$R_t = R_{\tau_k} + \int_{\tau_k}^{t} (-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)) + \alpha_k \left( \frac{I_t}{I_{\tau_k}} - 1 \right), \quad t \in (\tau_k, \tau_{k+1}]. \tag{3.2.2}$$

We observe that the process $R$ is not $\mathbb{G}$-predictable, as $I$ is not. At time $t \in [\tau_k, \tau_{k+1})$ the last information carried by the illiquid asset is given by the $\sigma$-algebra

$$\mathcal{I}_t = \mathcal{I}_{\tau_k} = \sigma(I_{\tau_h},\ h = 0, ..., k).$$

However we can split $R$ in a predictable part related to the observation $\mathcal{G}_t$ and a not-predictable one related to the unknown information on the illiquid asset in the interval $[\tau_k, t)$. Let $\tau$ be a generic $\mathbb{G}$-stopping time and consider the auxiliary processes on $[\tau, +\infty)$ $E, J$ whose dynamics are

$$\frac{dE_t^\tau}{E_t^\tau} = \frac{\rho\sigma_I}{\sigma_L}\frac{dL_t}{L_t} = (\rho b_L \frac{\sigma_I}{\sigma_L} dt + \rho\sigma_I dW_t), \quad E_\tau^\tau = 1; \tag{3.2.3}$$
$$\frac{dJ_t^\tau}{J_t^\tau} = (b_I - \rho b_L \frac{\sigma_I}{\sigma_L})dt + \sigma_I\sqrt{1-\rho^2} dB_t, \quad J_\tau^\tau = 1. \tag{3.2.4}$$

Recursively in the intervals $(\tau_k, \tau_{k+1}]$ define the processes

$$X_t = R_{\tau_k} - \alpha_k + \int_{\tau_k}^t (-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)), \quad Y_t = \alpha_k E_t^{\tau_k}, \quad Z_t = Y_t J_t^{\tau_k}. \tag{3.2.5}$$

In the interval $(\tau_k, \tau_{k+1}]$, the process $X_t$ represents the liquid part of the wealth $R_t$, while the process $Z_t$ represents the illiquid part; so we have

$$R_t = X_t + Z_t, \quad \forall t \geq 0.$$

As a class of admissible controls we consider the triplets of processes $(c_t, \pi_t, \alpha_k)$ satisfying the measurability and integrability conditions above and such that the corresponding wealth process $R_t$ is nonnegative (no-bankruptcy constraint). One can see without big difficulty that this requirement is equivalent to require that both the liquid and the illiquid wealth have to be nonnegative at each time, i.e. that $X_t \geq 0, Z_t \geq 0$ for every $t \geq 0$. So, the admissibility of a strategy $(c_t, \pi_t, \alpha_k)$ amounts to require

$$\begin{aligned} 0 &\leq \alpha_k \leq R_{\tau_k}, \quad \forall k \geq 0, \\ \int_{\tau_k}^t ((c_s - b_L \pi_s) ds - \sigma_L \pi_s dW_s) &\leq R_{\tau_k} - \alpha_k, \quad \forall t \in [\tau_k, \tau_{k+1}). \end{aligned}$$

The class of admissible controls depends on the initial wealth $R_0 = r$. We denote this class by $\mathcal{A}(r)$.

### 3.2.2 Optimization problem

Let $R_0 = r$. The optimization problem consists in maximizing over the set of admissible strategies $\mathcal{A}(r)$ the expected discounted utility from consumption over an infinite horizon. In other terms, chosen a utility function $U$ and a discount factor $\beta > 0$, the optimization problem we consider is the mixed discrete/continuous stochastic control problem

$$\text{Maximize} \quad \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right], \quad \text{over } (c, \pi, \alpha) \in \mathcal{A}(r).$$

The value function of such optimal stochastic control problem is denoted by $V$:

$$V(r) = \sup_{(c,\pi,\alpha) \in \mathcal{A}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right].$$

About the function $U$ we assume the following

**Assumption 3.2.1.** *The preference of the agent are described by a utility function*

$$U : [0, +\infty) \to \mathbb{R}$$

*which is continuous, increasing, concave, $C^1$ on $(0, +\infty)$, such that $U(0) = 0$ and satisfies the usual Inada's conditions:*

$$U'(0^+) = +\infty, \quad U'(\infty) = 0.$$

*Moreover we assume the following growth condition on $U$: there exist constants $K_U > 0$ and $p \in (0, 1)$ such that*

$$U(c) \le K_U\, c^p. \tag{3.2.6}$$

We observe that, due to the assumption on $U$, the Legendre tranform of $U$ on $[0, +\infty)$, i.e. the function

$$\tilde{U}(q) = \sup_{c \ge 0}\{U(c) - cq\}, \quad q > 0,$$

is finite, decreasing and convex. Moreover, the growth condition (3.2.6) yields the following growth condition for $\tilde{U}$: there exist $K_{\tilde{U}} > 0$ such that

$$\tilde{U}(q) \le K_{\tilde{U}}\, q^{-\frac{p}{1-p}}. \tag{3.2.7}$$

About the discount rate $\beta$ we assume the following.

**Assumption 3.2.2.** *We assume that*

$$\begin{aligned} \beta \; > \; k_M(p) \quad &:= \quad \sup_{\pi_L \in \mathbb{R}, \pi_I \in [0,1]} p(\pi_L b_L + \pi_I b_I) - \frac{p(1-p)}{2}(\pi_L^2 \sigma_L^2 + \pi_I^2 \sigma_I^2 + 2\rho \pi_L \pi_I \sigma_L \sigma_I) \\ &= \quad \frac{p}{2(1-p)} \frac{b_L^2}{\sigma_L^2} + k_J(p), \end{aligned} \tag{3.2.8}$$

*where*

$$k_J(p) \quad = \quad \sup_{\pi_I \in [0,1]} p(b_I - \frac{\rho b_L \sigma_I}{\sigma_L})\pi_I - \frac{p(1-p)}{2}\sigma_I^2(1-\rho^2)\pi_I^2.$$

**Remark 3.2.1.** This assumption on $\beta$ is related to the investment/consumption problem with the same assets but in a liquid market.

Indeed, consider an agent with initial wealth $r$, consuming at rate $c_t$ and investing in $L_t$ and $I_t$ continuously with respective proportions $\pi_t^L$ and $\pi_t^I$, with the constraint that $\pi_t^I \in [0, 1]$. If we denote $\mathcal{A}^M(r)$ the strategies s.t. wealth remains nonnegative and define the value function

$$V_M^{(p)}(r) = \sup_{(\pi^L,\pi^I,c)\in\mathcal{A}^M(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} c_t^p dt\right], \quad (3.2.9)$$

it is then easy to see (for instance by solving the HJB equation) that $V_M^{(p)}$ is finite if and only if (3.2.8) is satisfied, and that in this case

$$V_M^{(p)}(r) = \left(\frac{1-p}{\beta - k_M(p)}\right)^{1-p} r^p. \quad (3.2.10)$$

Further note that the liquid investment/consumption problem can always be reduced to the case where the two assets are independent, because

$$\pi_t^L \frac{dL_t}{L_t} + \pi_t^I \frac{dI_t}{I_t} = \left(\pi_t^L + \frac{\rho b_L \sigma_I}{\sigma_L}\pi_t^I\right)\frac{dL_t}{L_t} + \pi_t^I \frac{dJ_t}{J_t},$$

and the problem is equivalent to an agent investing in $L$ and $J$ (with the same constraint for the proportion invested in $J$).

However this reduction does not work for the illiquid problem that we consider : neither the observation constraint (the integrand in $L$ being $\mathbb{G}$-predictable) nor the trading constraint (the amount held in the illiquid asset being constant between $\tau_k$ and $\tau_{k+1}$) are preserved by this transformation.

## 3.3 Dynamic Programming

Following [61], we state a suitable Dynamic Programming Principle (DPP) to reduce our mixed discrete/continuous problem to a standard one between two trading times.

**Proposition 3.3.1** (DPP). *We have the following equality:*

$$V(r) = \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta\tau_1} V\left(R_{\tau_1}\right)\right]. \quad (3.3.11)$$

**Proof.** The proof is long and technical, but similar to the one in [62] and we omit it. Note however that unlike in [62], there is some additional random information between 0 and $\tau_1$ (brought by $W$), so that the "shifting" procedure is slightly more technical to achieve, see for instance Appendix B in [27] for details. □

Now we will use this DPP to rewrite our original problem into a standard continuous-time control problem. For each $x \geq 0$, let $\mathcal{A}_0(x)$ be the set of couples of stochastic processes $(c_s, \pi_s)_{s\geq 0}$ such that

- $(c_s)_{s\geq 0}$ is $(\mathcal{F}^W_s)$-predictable, nonnegative and locally integrable;
- $(\pi_s)_{s\geq 0}$ is $(\mathcal{F}^W_s)$-predictable and locally square-integrable;
- $x + \int_0^T (-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)) \geq 0, \quad \forall T \geq 0.$

**Lemma 3.3.1.** *Given $r \geq 0$, for any $(c, \pi, \alpha) \in \mathcal{A}(r)$, there exists $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$ such that*

$$(c, \pi)\mathbf{1}_{\{t\leq\tau_1\}} \quad = \quad (c^0, \pi^0)\mathbf{1}_{\{t\leq\tau_1\}}, \quad d\mathbb{P}\otimes ds \ a.e. \ . \tag{3.3.12}$$

**Proof.** First, using the definition of $\mathcal{G}$, by a simple monotone class argument, for any $(c, \pi)$ $\mathcal{G}$-predictable we may find $(c^0, \pi^0)$ $\mathcal{F}^W$-predictable satisfying (3.3.12). It is then easy to see that the admissibility constraint $(c, \pi, \alpha) \in \mathcal{A}(r)$ implies $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$. □

By the previous Lemma, (3.3.11) may actually be rewritten as

$$V(r) \quad = \quad \sup_{a\leq r} \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{3.3.13}$$

We want somehow to rewrite the inner optimization problem in (3.3.13). To this purpose, consider the linear operator ($\mathcal{M}$ denotes the space of measurable functions; also recall that $J^\tau$ has been defined in (3.2.3))

$$G : \mathcal{M}(\mathbb{R}_+; \mathbb{R}) \quad \longrightarrow \quad \mathcal{M}([0, +\infty) \times \mathbb{R}^2_+; \mathbb{R}) \tag{3.3.14}$$

$$\varphi \quad \longmapsto \quad G[\varphi](t, x, y) := \mathbb{E}\left[\varphi(x + yJ^0_t)\right]. \tag{3.3.15}$$

Let us see the properties of this operator.

**Proposition 3.3.2.**

*(i) G is well defined on the set of measurable functions with at most linear growth.*

*(ii)* *$G$ is linear and positive, in the sense that it maps positive functions in positive ones. As a consequence $G$ is increasing in the sense that*

$$\varphi \le \psi \implies G[\varphi] \le G[\psi].$$

*(iii)* *$G$ maps increasing functions in functions which are increasing with respect to both $x$ and $y$.*

*(iv)* *$G$ maps concave functions in functions which are concave with respect to $(x, y)$.*

*(v)* *If $\varphi(r) = r^p$, $p \in (0, 1)$, then*

$$G[\varphi](t, \alpha x, \alpha y) = \alpha^p G[\varphi](t, x, y), \quad \forall t \ge 0, (x, y) \in \mathbb{R}^2_+, \ \alpha \ge 0. \tag{3.3.16}$$

$$G[\varphi](t, x, y) \le e^{k_J(p)t}(x + y)^p, \quad \forall t \ge 0, (x, y) \in \mathbb{R}^2_+. \tag{3.3.17}$$

*(vi)* *Let $p \in (0, 1]$, and $v$ a $p$-Hölder continuous function on $\mathbb{R}^2_+$ with Hölder coefficient $C$. Then for all $t, x, x', y, y'$, and $0 \le h \le 1$, there exists some constant $C_1 \ge C$ s.t. the following are true :*

$$\begin{aligned}
|G[v](t, x, y) - G[v](t, x', y)| &\le C|x - x'|^p, && (3.3.18)\\
|G[v](t, x, y) - G[v](t, x, y')| &\le Ce^{k_J(p)t}|y - y'|^p, && (3.3.19)\\
|G[v](t, x, y) - G[v](t + h, x, y)| &\le C_1 e^{k_J(p)t} y^p h^{p/2}, && (3.3.20)
\end{aligned}$$

**Proof.** (i)-(iv) are straightforward.

We prove (v). (3.3.16) comes directly from the definition of $G$. Let us prove (3.3.17). If $x = y = 0$ the claim is obvious, so we assume $x + y > 0$. By Itô's formula,

$$\begin{aligned}
d(e^{-k_J(p)t}(x + yJ^0_t)^p) \;=\; & e^{-k_J(p)t}(x + yJ^0_t)^p \Bigg\{ \Bigg( -k_J(p) + p(b_I - \frac{\rho b_L \sigma_I}{\sigma_L}) \frac{yJ_t}{x + yJ^0_t} \\
& - \frac{1}{2} p(1 - p) \frac{(yJ^0_t)^2}{(x + yJ^0_t)^2} \sigma_I^2 (1 - \rho^2) \Bigg) dt \\
& + p \frac{yJ^0_t}{x + yJ^0_t} \sigma_I \sqrt{1 - \rho^2} dB_t \Bigg\}.
\end{aligned}$$

By definition of $k_J(p)$, the drift term above is nonpositive, so it follows that the process $(e^{-k_J(p)t}(x + yJ^0_t)^p)_{t \ge 0}$ is a supermartingale, implying (3.3.17).

Now we turn to (vi). (3.3.18) is obvious, and (3.3.19) follows directly from (v). To prove (3.3.20) we fix $t$, $x$, $y$ and $h$, and write

$$\begin{aligned} |G[v](t,x,y) - G[v](t+h,x,y)| &\leq Cy^p \mathbb{E}\left[|J_t^0 - J_{t+h}^0|^p\right] \\ &= Cy^p \mathbb{E}\left[|J_t^0|^p\right] \mathbb{E}\left[|1 - J_{t+h}^t|^p\right] \\ &\leq Ce^{k_J(p)t} y^p \mathbb{E}\left[|1 - J_h^0|^p\right] \end{aligned}$$

Now we write $J_h^0 = F(h, \sqrt{h}N)$ where $N \sim \mathcal{N}(0,1)$ and $F$ verifies $F(0,0) = 1$. Checking that the derivatives of $F$ satisfy reasonable growth conditions, it is then straightforward to obtain

$$\mathbb{E}\left[|1 - J_h^0|^p\right] \leq C_1 h^{p/2}, \qquad \text{for } 0 \leq h \leq 1.$$

□

Given $(c,\pi) \in \mathcal{A}_0(x)$, let $(\tilde{X}_s^{x,c,\pi})_{s\geq 0}, (\tilde{Y}_s^y)_{s\geq 0}$ be the solutions starting from $x, y$ to the SDEs

$$d\tilde{X}_s = -c_s ds + \pi_s(b_L ds + \sigma_L dW_s), \tag{3.3.21}$$

$$d\tilde{Y}_s = \tilde{Y}_s(\rho \frac{b_L \sigma_I}{\sigma_L} dt + \rho \sigma_I dW_s). \tag{3.3.22}$$

Given $w$ a nonnegative measurable function on $\mathbb{R}_+$ and $x, y \geq 0$, let us consider the functional on $\mathcal{A}_0(x)$

$$\mathcal{J}_w^0(x,y;c,\pi) = \mathbb{E}\int_0^\infty e^{-(\beta+\lambda)s}\left(U(c_s) + \lambda G[w]\left(s, \tilde{X}_s^{x,\pi,c}, \tilde{Y}_s^y\right)\right) ds.$$

The importance of the operator $G$ relies in the following result.

**Lemma 3.3.2.** *Let $x, y > 0$, $(c,\pi) \in \mathcal{A}_0(x)$, and $w$ nonnegative measurable on $\mathbb{R}_+$. Then denoting $R_{\tau_1} := \tilde{X}_{\tau_1}^{x,c,\pi} + \tilde{Y}_{\tau_1}^{x,c,\pi}$,*

$$\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta \tau_1} w(R_{\tau_1})\right] = \mathcal{J}_w^0(x,y;c,\pi),$$

**Proof.** Let $(c,\pi)\in\mathcal{A}(x)$ and set $(\tilde{X}_s,\tilde{Y}_s)=(\tilde{X}_s^{x,\pi,c},\tilde{Y}_s^{y})$, $\tilde{Z}_s=\tilde{Y}_sJ_s^0$. Since $\tau_1$ is independent from $\mathcal{F}^W$ and $\mathcal{F}^B$ with distribution $\mathcal{E}(\lambda)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}w(R_{\tau_1})\Big|\mathcal{F}^W,\mathcal{F}^B\right]\\
=\;&\int_0^\infty \lambda e^{-\lambda t}\left(\int_0^t e^{-\beta s}U(c_s)ds+e^{-\beta t}w(\tilde{X}_t+\tilde{Z}_t)\right)dt\\
=\;&\int_0^\infty e^{-\beta s}U(c_s)\int_s^\infty \lambda e^{-\lambda t}dtds+\int_0^\infty \lambda e^{-(\lambda+\beta)t}w(\tilde{X}_t+\tilde{Z}_t)dt\\
=\;&\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda w(\tilde{X}_t+\tilde{Z}_t)\right)dt,
\end{aligned}$$

where in the second inequality we have used Fubini's theorem. Now since $\tilde{Z}_t=\tilde{Y}_tJ_t^0$, with $J_t^0$ independent from $\mathcal{F}^W$, we have for all $t\ge 0$

$$\begin{aligned}
\mathbb{E}\left[w(\tilde{X}_t+\tilde{Z}_t))\Big|\mathcal{F}^W\right] &= \mathbb{E}\left[w(x+yJ_t^0)\right]\Big|_{\substack{x=\tilde{X}_t\\ y=\tilde{Y}_t}}\\
&= G[w](t,\tilde{X}_t,\tilde{Y}_t),
\end{aligned}$$

and we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}w(R_{\tau_1})\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}w(R_{\tau_1})\;\Big|\;\mathcal{F}^W\right]\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda G[w](t,\tilde{X}_t,\tilde{Y}_t)\right)dt\right]\\
&= \mathcal{J}_w^0\left(x,y;c,\pi\right).
\end{aligned}$$

□

Due to Lemma 3.3.2, from (3.3.13) we obtain

$$V(r)=\sup_{a\le r}\sup_{(c,\pi)\in\mathcal{A}_0(r-a)}\mathcal{J}_V^0\left(r-a,a;c,\pi\right). \tag{3.3.23}$$

In order to solve the inner optimization problem in (3.3.23), we define a dynamic version of it. Given $t\ge 0$ we define the process $(W_s^t)_{s\ge t}=(W_s-W_t)_{s\ge t}$ and the filtration $(\mathcal{F}^{W^t})s\ge t$ generated by this process. For each $t,x\ge 0$, let $\mathcal{A}_t(x)$ be the set of couples of stochastic processes $(c_s,\pi_s)_{s\ge t}$ such that

- $(c_s)_{s\geq t}$ is $(\mathcal{F}_s^{W^t})$-predictable, nonnegative and locally integrable;
- $(\pi_s)_{s\geq t}$ is $(\mathcal{F}_s^{W^t})$-predictable and locally square-integrable;
- $x + \int_t^T(-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)) \geq 0, \quad \forall T \geq t.$

Given $(c,\pi) \in \mathcal{A}_t(x)$, let $(\tilde{X}_s^{t,x,c,\pi})_{s\geq 0}, (\tilde{Y}_s^{t,y})_{s\geq 0}$ be the solutions starting at time $t$ from $x, y$ to the SDEs

$$d\tilde{X}_s = -c_s ds + \pi_s(b_L ds + \sigma_L dW_s), \tag{3.3.24}$$

$$d\tilde{Y}_s = \tilde{Y}_s(\rho \frac{b_L \sigma_I}{\sigma_L} dt + \rho\sigma_I dW_s). \tag{3.3.25}$$

Given $w$ a nonnegative measurable function on $\mathbb{R}_+$ and $t, x, y \geq 0$, let us consider the functional on $\mathcal{A}_t(x)$

$$\mathcal{J}_w(t,x,y;c,\pi) = \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + \lambda G[w]\left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right)\right) ds.$$

Finally consider the optimization problem

$$\hat{V}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathcal{J}_V(t,x,y;c,\pi), \tag{3.3.26}$$

For any locally bounded function $\hat{v}$ on $[0,+\infty) \times \mathbb{R}_+^2$, we associate the function $\mathcal{H}\hat{v}$ defined on $\mathbb{R}_+$ by :

$$[\mathcal{H}\hat{v}](r) = \sup_{0\leq a\leq r} \hat{v}(0,a,r-a).$$

By (3.3.23), we can rewrite the original problem in terms of $\hat{V}$ as

$$V = \mathcal{H}\hat{V}. \tag{3.3.27}$$

## 3.4 Properties of the value functions

Let us see some first properties of the functions $V, \hat{V}$.

**Proposition 3.4.1.** *$V$ is concave, $p$-Hölder continuous and nondecreasing. Moreover*

$$V(r) \leq Kr^p, \quad \textit{for some } K > 0. \tag{3.4.1}$$

**Proof.** First of all we note that we clearly have $0 \leq V \leq K_U V_M^{(p)}$, where $V_M^{(p)}$ is the value function of the problem when $I$ is also liquid for utility $U^{(p)}(c) = c^p$. By (3.2.10), we have the estimate

$$V_M^{(p)}(r) \leq K r^p, \quad \text{for some } K > 0,$$

which in turn yields (3.4.1).

Concavity of $V$ comes from concavity of $U$ and linearity of the state equation by standard arguments. Also monotonicity is consequence of standard arguments due to the monotonicity of $U$. $p$-Hölder continuity follows from concavity and monotonicity of $V$ and from (3.4.1). □

Before examining the properties of $\hat{V}$ we state a lemma that will be needed repeatedly.

**Lemma 3.4.1.** *For $(t,x,y) \in [0,+\infty)$, $(\pi, c) \in \mathcal{A}_t(x)$, $p \in (0,1)$,*

$$\mathbb{E}\left[(\tilde{X}_s^{t,x,c,\pi} + \tilde{Y}_s^{t,y})^p\right] \quad \leq \quad e^{\frac{p}{1-p}\frac{b_L^2}{2\sigma_L^2}(s-t)}(x+y)^p, \tag{3.4.2}$$

*for all $s \geq t$. In particular, combining with Proposition 3.3.2(v), denoting $\varphi(r) = r^p$,*

$$\mathbb{E}\left[G[\varphi](s, \tilde{X}_s^{t,x,c,\pi}, \tilde{Y}_s^{t,y})\right] \quad \leq \quad e^{k_J(p)t} e^{k_M(p)(s-t)}(x+y)^p. \tag{3.4.3}$$

**Proof.** The proof follows the same arguments as that of Proposition 3.3.2(v), noticing that

$$\frac{p}{1-p}\frac{b_L^2}{2\sigma_L^2} \quad = \quad \sup_{\pi \in \mathbb{R}} \left\{ p b_L \pi - \frac{p(1-p)}{2}\sigma_L^2 \pi^2 \right\}.$$

□

**Proposition 3.4.2.** *$\hat{V}(t,\cdot)$ is concave and nondecreasing with respect to both $x, y$ for every $t \geq 0$. Moreover,*

$$\hat{V}(t,0,y) = \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G[V](s,0,\tilde{Y}_s^{t,y}) ds; \tag{3.4.4}$$

*In particular, due to Proposition 3.3.2-(v),*

$$\hat{V}(t,0,0) = 0. \tag{3.4.5}$$

*Furthermore, $\hat{V}$ is continuous on $[0,+\infty) \times \mathbb{R}_+^2$, and for some $K_{\hat{V}} > 0$,*

$$\hat{V}(t,x,y) \quad \leq \quad K_{\hat{V}} e^{k_J(p)t}(x+y)^p \tag{3.4.6}$$

*for every $(t,x,y) \in [0,+\infty) \times \mathbb{R}_+^2$.*

**Proof.** Since $V$ is concave and nondecreasing, by Proposition 3.3.2-(iii, iv), $G[V](t,\cdot)$ is concave and nondecreasing in both $x, y$ on $\mathbb{R}_+^2$. Then concavity and monotonicity properties of $\hat{V}$ follow by standard arguments, considering also the linearity of the SDE's (3.3.24)-(3.3.25).

Equality (3.4.4) is due to the fact that $\mathcal{A}_t(0) = \{(0,0)\}$, so

$$\hat{V}(t,0,y) = \mathcal{J}_V(t,0,y;0,0) = \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G[V](s,0,\tilde{Y}_s^{t,y})ds. \tag{3.4.7}$$

We prove the continuity of $\hat{V}$ in several steps.

1) Continuity of $\hat{V}(t,\cdot)$ in $(0,+\infty)^2$ follows from its concavity.

2) Here we prove the continuity of $\hat{V}(t,\cdot,y)$ at $x = 0^+$. First of all notice that (3.4.7) holds at $x = 0$, so using monotonicity of $V$ and 3.3.2-(iii) we get

$$0 \le \mathcal{J}_V(t,x,y;0,0) - \mathcal{J}_V(t,x,0;0,0) \le \hat{V}(t,x,y) - \hat{V}(t,x,0). \tag{3.4.8}$$

On the other hand, using Proposition 3.3.2-(vi) and Hölder continuity of $V$, we have for some $K > 0$ and all $(c,\pi) \in \mathcal{A}_t(x)$

$$\begin{aligned}
&\mathcal{J}_V(t,x,y;c,\pi) - V(t,0,y)\\
&\quad\le \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left\{U(c_s) + \lambda\left|G[V](s,\tilde{X}_s^{t,x,c,\pi},\tilde{Y}_s^{t,y}) - G[V](s,0,\tilde{Y}_s^{t,y})\right|\right\}ds\right]\\
&\qquad\qquad\le \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + K|\tilde{X}_s^{t,x,c,\pi}|^p\right)ds\right].
\end{aligned}$$

Taking the supremum over $(c,\pi) \in \mathcal{A}_t(x)$ and combining with (3.4.8) we get

$$\begin{aligned}
0 &\le \hat{V}(t,x,y) - \hat{V}(t,0,y)\\
&\le \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + K|\tilde{X}_s^{t,x,c,\pi}|^p\right)ds\right].
\end{aligned} \tag{3.4.9}$$

We have to estimate the right handside of (3.4.9). By definition of $\mathcal{A}_t(x)$, we have

$$0 \le \tilde{X}_s^{t,x,c,\pi} = x + \int_t^s \pi_u \frac{dL_u}{L_u} - \int_t^s c_u du. \tag{3.4.10}$$

Denoting by $\mathbb{Q}^L$ the probability with density process given by $Z_t = \exp\left(-\frac{b_L^2}{2\sigma_L^2}t - \frac{b_L}{\sigma_L}W_t\right)$, $L$ is a $\mathbb{Q}^L$-martingale. The process $\tilde{X}^{t,x,c,\pi}$ is then a $\mathbb{Q}^L$-local supermartingale and, being bounded from below, it is a true $\mathbb{Q}^L$-supermartingale. Hence, we have $\mathbb{E}[Z_s\tilde{X}_s^{t,x,c,\pi}] \le x$. Now, writing

$|\tilde{X}_s^{t,x,c,\pi}|^p = |Z_s \tilde{X}_s^{t,x,c,\pi}|^p Z_s^{-p}$, by Hölder's inequality we get

$$\mathbb{E}[|\tilde{X}_s^{t,x,c,\pi}|^p] \leq \mathbb{E}[Z_s \tilde{X}_s^{t,x,c,\pi}]^p \ \mathbb{E}[Z_s^{-\frac{p}{1-p}}]^{1-p} \leq x^p \exp\left(\left(\frac{p}{1-p}\frac{b_L^2}{2\sigma_L^2}\right)s\right). \tag{3.4.11}$$

Note also that since $\int_t^\infty e^{-(\beta+\lambda)(s-t)}U(c_s)ds$ is the utility obtained by the agent trading only in $L$,

$$\begin{aligned} \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}U(c_s)ds\right] &\leq V(x) \\ &\leq Kx^p, \end{aligned} \tag{3.4.12}$$

by (3.4.1). Combining (3.4.9), (3.4.11), (3.4.12), and using (3.2.8), we get for some $K > 0$

$$0 \leq \hat{V}(t,x,y) - \hat{V}(t,0,y) \leq Kx^p, \tag{3.4.13}$$

and we conclude.

3) Here we prove the continuity of $\hat{V}(t,x,\cdot)$ at $y = 0^+$.

Using monotonicity of $V$ and 3.3.2(iii) (in the first inequality below), Proposition 3.3.2-(vi) (in the second inequality below) and (3.4.2) (in the third inequality below), we have for some $K > 0$ and for all $(c,\pi) \in \mathcal{A}_t(x)$

$$\begin{aligned} 0 \leq \mathcal{J}_V(t,x,y;c,\pi) - \mathcal{J}_V(t,x,0;c,\pi) &\leq Ke^{K_J(p)t}\int_t^\infty e^{-(\beta+\lambda-k_J(p))(s-t)}\lambda\mathbb{E}[(Y_s^{t,y})^p]ds \\ &\leq Ke^{K_J(p)t}y^p\int_t^\infty e^{-(\beta+\lambda-k_M(p))(s-t)}ds = K\frac{\lambda}{\beta+\lambda-k_M(p)}e^{K_J(p)t}y^p. \end{aligned}$$

Therefore, taking the supremum over $(c,\pi) \in \mathcal{A}_t(x)$ we get

$$0 \leq \hat{V}(t,x,y) - \hat{V}(t,x,0) \leq \frac{K\lambda}{\beta+\lambda-k_M(p)}e^{k_J(p)t}y^p. \tag{3.4.14}$$

Letting $y \to 0$ we have the claim.

4) Since (3.4.14) and (3.4.13) are uniform estimates in $x, y$ respectively, also using the continuity on the lines provided by (2)-(3), we get the joint continuity of $\hat{V}$ with repect to $(x,y)$ at the boundary $\{(x,y) \in \mathbb{R}_+^2 \mid x = 0 \text{ or } y = 0\}$.

5) Here we prove the continuity of $\hat{V}(\cdot,x,y)$. Let $t, t' \geq 0$ and suppose that $t' = t + h$ for some $0 \leq h \leq 1$. There is a one-to-one correspondence between $\mathcal{A}_t(x)$ and $\mathcal{A}_{t'}(x)$ associating to a control $(c_s^t, \pi_s^t)_{s\geq t} \in \mathcal{A}_t(x)$ a control $(c_s^{t'}, \pi_s^{t'})_{s\geq t'} \in \mathcal{A}_{t'}(x)$ with the same law (see [71, Th. 2.10,

Ch. 1]). So, let $(c_s^t, \pi_s^t)_{s\geq t} \in \mathcal{A}_t(x)$ and let $(c_s^{t'}, \pi_s^{t'})_{s\geq t'} \in \mathcal{A}_{t'}(x)$ be the associated control by this one-to-one correspondence. We have

$$\begin{aligned}
& |\mathcal{J}_V(t,x,y,c^t,\pi^t) - \mathcal{J}_V(t',x,y,c^{t'},\pi^{t'})| \\
\leq\ & \left|\mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s^t) + \lambda G[V](s, \tilde{X}^{t,x,c^t,\pi^t}, \tilde{Y}_s^{t,y})\right) ds\right.\right. \\
& \left.\left. - \int_{t'}^\infty e^{-(\beta+\lambda)(s-t')}\left(U(c_s^{t'}) + \lambda G[V](s, \tilde{X}^{t',x,c^{t'},\pi^{t'}}, \tilde{Y}_s^{t',y})\right) ds\right]\right| \\
\leq\ & \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda\left|G[V](s, \tilde{X}^{t,x,c^t,\pi^t}, \tilde{Y}_s^{t,y})ds - G[V](s+h, \tilde{X}^{t,x,c^t,\pi^t}, \tilde{Y}_s^{t,y})\right| ds \\
\leq\ & K_1\mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda h^{p/2}e^{k_J(p)s}|Y_s^{t,y}|^p ds \\
\leq\ & K_1\frac{\lambda}{\beta+\lambda-k_M(p)}e^{k_J(p)t}h^{p/2},
\end{aligned}$$

where like in 3) we have used Proposition 3.3.2(vi) and (3.4.2). Passing to the supremum over $(c_s^t, \pi_s^t)_{s\geq t} \in \mathcal{A}_t(x)$ and taking into account Proposition 3.3.2-(iii) and Proposition 3.4.1, we get for some $K > 0$

$$|\hat{V}(t,x,y) - \hat{V}(t+h,x,y)| \leq K e^{k_J(p)t} y^p h^{p/2}. \tag{3.4.15}$$

Hence $\hat{V}$ is locally $p/2$-Hölder with respect to $t$.

6) Putting together all the information collected we get continuity of $V$ on $[0,+\infty)\times\mathbb{R}_+^2$.

Finally, the growth condition (3.4.6) is proved by combining (3.4.13),(3.4.14) and (3.4.5). □

## 3.5 The HJB equation

By standard arguments we can associate to $\hat{V}$ a HJB equation. It reads as

$$-\hat{v}_t + (\beta+\lambda)\hat{v} - \lambda G[\mathcal{H}\hat{v}](t,x,y) - \sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{v}, D^2_{(x,y)}\hat{v} : c,\pi) = 0, \tag{3.5.1}$$

where for $(y,p,A) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathcal{S}_2$ ($\mathcal{S}_2$ is the space of symmetric $2\times 2$ matrices), $c \geq 0, \pi \in \mathbb{R}$, $H_{cv}$ is defined by

$$H_{cv}(y,p,A;c,\pi) = \left[U(c) + (\pi b_L - c)p_1 + \frac{\rho b_L \sigma_I}{\sigma_L} y p_2 + \frac{\sigma_L^2\pi^2}{2}A_{11} + \pi\rho\sigma_I\sigma_L y A_{12} + \rho^2\frac{\sigma_I^2}{2}y^2 A_{22}\right].$$

Note that $\sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y,p,A;c,\pi)$ is finite if $p_1 > 0$, $A_{11} < 0$, in which case we have

$$\sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y,p,A;c,\pi) \;=\; \tilde{U}(p_1) - \frac{(b_L p_1 + \rho\sigma_L\sigma_I y A_{12})^2}{2\sigma_L^2 A_{11}} + \frac{\rho b_L \sigma_I}{\sigma_L} y p_2 + \rho^2 \frac{\sigma_I^2}{2} y^2 A_{22}.$$

A Dirichlet type boundary condition can be associated to equation (3.5.1) above. It is provided by (3.4.4) which leads to impose

$$\hat{v}(t,0,y) = \mathbb{E}\int_t^{\infty} e^{-(\beta+\lambda)(s-t)}\lambda G[\mathcal{H}\hat{v}](s,0,\tilde{Y}_s^{t,y})ds. \tag{3.5.2}$$

### 3.5.1 Viscosity solutions

Let us denote by $X = (x,y)$ vectors in $\mathbb{R}_+^2$. We are going to prove that $\hat{V}$ is the unique constrained viscosity solution to (3.5.1) according to the following definition.

**Definition 3.5.1.** **(1)** Given $w$ a continuous function on $[0,+\infty)\times\mathbb{R}_+^2$, the parabolic *superjet* of $w$ at $(t,X)\in[0,+\infty)\times\mathbb{R}_+^2$ is defined by:

$$\begin{aligned}\mathcal{P}^{1,2,+}w(t,X) \;=\; \Big\{&(q,p,A)\in\mathbb{R}\times\mathbb{R}^2\times\mathcal{S}^2 \text{ s.t. } w(s,X')\leq w(t,X)+q(s-t)+\langle p, X'-X\rangle \\ &+\frac{1}{2}\langle A(X'-X), X'-X\rangle \;+o\left(|X'-X|^2\right) \text{ as } X'\to X\Big\},\end{aligned}$$

and its closure $\overline{\mathcal{P}}^{1,2,+}w(t,X)$ as the set of elements $(q,p,A)\in\mathbb{R}\times\mathbb{R}^2\times\mathcal{S}^2$ for which there exists a sequence $(t_m,X_m,q_m,p_m,A_m)_m$ of $[0,+\infty)\times\mathbb{R}_+^2\times\mathcal{P}^{1,2,+}w(t_m,X_m)$ satisfying $(t_m,X_m,q_m,p_m,A_m)$ $\to(t,X,q,p,A)$. We also define the subjets $\mathcal{P}^{1,2,-}w_i(t,X) = -\mathcal{P}^{1,2,+}(-w_i)(t,X)$ and $\overline{\mathcal{P}}^{1,2,-}w(t,X)$ $= -\overline{\mathcal{P}}^{1,2,+}(-w)(t,X)$.

**(2)** We say that $w$ is a viscosity subsolution (resp. supersolution) to (3.5.1) at $(t,X)\in[0,+\infty)\times\mathbb{R}_+^2$ if

$$-q+(\beta+\lambda)w(t,x,y)-\lambda G[\mathcal{H}w](t,x,y)-\sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y,p,A;c,\pi) \;\leq\; 0,$$

for all $(q,p,A)\in\overline{\mathcal{P}}^{1,2,+}w(t,X)$ (resp. $\geq$, $\overline{\mathcal{P}}^{1,2,-}w(t,X)$).

**(3)** $w$ is a *constrained viscosity solution* to (3.5.1) if it is a subsolution on $[0,+\infty)\times\mathbb{R}_+^2$, a supersolution on $[0,+\infty)\times(0,+\infty)\times\mathbb{R}_+$ and satisfies boundary condition (3.5.2).

**Remark 3.5.1.** The concept of constrained viscosity solution we use comes naturally from the stochastic control problem. The boundaries $\{x=0\}$ and $\{y=0\}$ are both absorbing for the control problem (in the sense that starting from these boundaries, the trajectories of the control problem remain therein), but they have different features. Indeed starting from the boundary $\{y=0\}$ the control problem degenerates in a one dimensional control problem; the associated HJB equation is nothing else but our HJB equation restricted to this boundary and this is why we require viscosity sub- and supersolution properties to the value function at this boundary. Instead starting from at the boundary $\{x = 0\}$ there is no control problem (since $\mathcal{A}_t(0) = \{(0, 0)\}$) and the natural condition to impose is a Dirichlet boundary condition.

**Theorem 3.5.1.** *$\hat{V}$ is the unique constrained viscosity solution to* (3.5.1) *satisfying the growth condition* (3.4.6)*.*

**Proof.** The fact that $\hat{v}$ is a viscosity subsolution on $[0, +\infty) \times \mathbb{R}_+^2$ and a viscosity supersolution on $[0, +\infty) \times (0, +\infty)^2$ is standard (see, e.g., [71, Ch. 4]). The Dirichlet boundary condition (3.5.2) is verified due to (3.4.4) and the growth condition

Therefore, it remains to show that $\hat{V}$ is a supersolution when $y = 0$. In this case, as noticed in Remark (3.5.1) the control problem degenerates in a one dimensional one and again standard arguments applied to this control problem give the viscosity supersolution property.

Uniqueness is consequence of the comparison principle Proposition 3.5.1 below. □

**Proposition 3.5.1.** *Let $w_1$ (resp. $w_2$) be a viscosity subsolution (resp. supersolution) to* (3.5.1) *on* $[0, +\infty) \times (0, \infty) \times \mathbb{R}_+$. *Assume that $w_1$, $w_2$ satisfy the growth condition* (3.4.6)*, and the boundary condition*

$$w_1(t, 0, y) \quad \leq \quad \mathbb{E} \int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G[\mathcal{H} w_1](s, 0, Y_s^{t,y}) ds \tag{3.5.3}$$

*(resp. $\geq$ for $w_2$). Then $w_1 \leq w_2$ on* $[0, +\infty) \times \mathbb{R}_+^2$.

**Proof.** The argument are quite standard in viscosity PDE's theory, but we provide the proof here for sake of completeness.

The proof is done in two steps :

**Step 1.** Fix some $q > p$ such that

$$\beta \quad \geq \quad k_M(q) = \frac{q}{1-q} \frac{b_L^2}{2\sigma_L^2} + k_J(q) \tag{3.5.4}$$

(this is possible by (3.2.8) and the fact that $k_M$ is continuous in $q$), and define

$$f^q(t, x, y) \quad = \quad e^{k_J(q)t}(x + y)^q.$$

First we claim that on $[0, +\infty) \times (0, \infty) \times \mathbb{R}_+$

$$\begin{aligned} & -f_t^q + (\beta + \lambda) f^q - \lambda G[\mathcal{H} f^q] \\ - \sup_{\pi \in \mathbb{R}} & \left[ \pi b_L f_x^q + y \frac{\rho b_L \sigma_I}{\sigma_L} f^q y + \frac{\sigma_L^2 \pi^2}{2} f_{xx}^q + \pi \rho \sigma_I \sigma_L y f_{xy}^q + \rho^2 \frac{\sigma_I^2}{2} y^2 f_{yy}^q \right] \quad \geq \quad 0. \end{aligned} \tag{3.5.5}$$

Indeed, we first have

$$G[\mathcal{H}f^q] \quad \leq \quad f^q$$

by Proposition 3.3.2(v), and furthermore by straightforward computations we check that

$$\begin{aligned}
&\sup_{\pi\in\mathbb{R}}\left[\pi b_L f^q_x + y\frac{\rho b_L\sigma_I}{\sigma_L}f^q + \frac{\sigma_L^2\pi^2}{2}f^q_{xx} + \pi\rho\sigma_I\sigma_L y f^q_{xy} + \rho^2\frac{\sigma_I^2}{2}y^2\hat{f}^q_{yy}\right] \\
=\quad & f^q\left[\frac{\left(qb_L - q(1-q)\rho\sigma_I\sigma_L\frac{y}{x+y}\right)^2}{2\sigma_L^2 q(1-q)} + q\frac{\rho b_L\sigma_I}{\sigma_L}\frac{y}{x+y} - \frac{q(1-q)}{2}\rho^2\sigma_I^2\frac{y^2}{(x+y)^2}\right] \\
=\quad & f^q\frac{q}{1-q}\frac{b_L^2}{2\sigma_L^2},
\end{aligned}$$

and by (3.5.4) we obtain (3.5.5).

Now given an integer $n \geq 1$, consider $w_{2,n} := w_2 + \frac{1}{n}f^q$.

Let us check that for any $(t,x,y)$ with $x > 0$, $w_{2,n}$ is a supersolution to (3.5.1) at $(t,x,y)$.

Notice that $\mathcal{P}^{1,2,-}\hat{w}_{2,n}(t,x,y) = \mathcal{P}^{1,2,-}\hat{w}_2(t,x,y) + \frac{1}{n}(\partial_t\hat{f}^{l,r}, D_{x,y}\hat{f}^{l,r}, D^2_{x,y}\hat{f}^{l,r})(t,x,y)$, and we have for all $(q,p,A) \in \mathcal{P}^{1,2,-}w_2(t,x,y)$ :

$$\begin{aligned}
& -(q+\frac{1}{n}f^q_t) + (\beta+\lambda)(w_2+\frac{1}{n}f^q) - \tilde{U}\left(p+\frac{1}{n}f^q_x\right) - \lambda G[\mathcal{H}(w_2+\frac{1}{n}f^q)](t,x,y) \\
& -\sup_{\pi\in\mathbb{R}}\Bigg[\pi b_L(p_1+\frac{1}{n}f^q_x) + y\frac{\rho b_L\sigma_I}{\sigma_L}(p_2+\frac{1}{n}f^q_y) + \frac{\sigma_L^2\pi^2}{2}(A_{11}+\frac{1}{n}f^q_{xx}) \\
& \qquad\qquad + \pi\rho\sigma_I\sigma_L y(A_{12}+\frac{1}{n}f^q_{xy}) + \rho^2\frac{\sigma_I^2}{2}y^2(A_{22}+\frac{1}{n}f^q_{yy})\Bigg] \\
\geq\quad & -q + (\beta+\lambda)w_2 - \tilde{U}(p) - \lambda G[\mathcal{H}w_2](t,x,y) \\
& -\sup_{\pi\in\mathbb{R}}\left[\pi b_L p_1 + y\frac{\rho b_L\sigma_I}{\sigma_L}p_2 + \frac{\sigma_L^2\pi^2}{2}A_{11} + \pi\rho\sigma_I\sigma_L y A_{12} + \rho^2\frac{\sigma_I^2}{2}y^2 A_{22}\right] \\
& +\frac{1}{n}\Bigg\{-f^q_t + (\beta+\lambda)f^q - \lambda G[\mathcal{H}f^q](t,x,y) \\
& -\sup_{\pi\in\mathbb{R}}\left[\pi b_L f^q_x + y\frac{\rho b_L\sigma_I}{\sigma_L}f_y + \frac{\sigma_L^2\pi^2}{2}f^q_{xx} + \pi\rho\sigma_I\sigma_L y f^q_{xy} + \rho^2\frac{\sigma_I^2}{2}y^2\hat{f}^q_{yy}\right]\Bigg\} \\
\geq\quad & 0,
\end{aligned}$$

where we have used the fact that $\tilde{U}$ is nonincreasing and $f^q_x \geq 0$, and (3.5.5).

Moreover,

$$\begin{aligned}\lambda\mathbb{E}\int_t^{\infty} e^{-(\beta+\lambda)(s-t)} G[\mathcal{H}f^q](s,0,Y_s^{t,y})ds &\leq e^{k_J(q)t}y^q\lambda\mathbb{E}\int_t^{\infty} e^{(-\beta-\lambda+k_J(q))(s-t)}(Y_s^{t,1})^q ds\\ &\leq f^q(t,0,y)\lambda\int_t^{\infty} e^{(-\beta-\lambda+k_M(q))(s-t)}ds\\ &= \frac{\lambda}{\beta-k_M(q)+\lambda}f^q(t,0,y)\\ &\leq f^q(t,0,y),\end{aligned}$$

where in the second inequality we have used (3.4.2). By the subadditivity of $\mathcal{H}$ and linearity of $G$, it follows that $w_{2,n}$ also satisfies the boundary condition at $(t,0,y)$.

Finally, notice that by the growth condition on $\hat{w}_1$ and $\hat{w}_2$ we have

$$\lim_{|(t,x,y)|\to\infty}(\hat{w}_1-\hat{w}_{2,n})(t,x,y) = -\infty. \tag{3.5.6}$$

**Step 2.** We show that for all $n\geq 1$, $w_1\leq w_{2,n}$ on $[0,+\infty)\times\mathbb{R}_+^2$, and thus $w_1\leq w_2$. Fix $n\geq 1$ and define

$$M := \sup_{[0,+\infty)\times\mathbb{R}_+^2} w_1-w_{2,n} > 0.$$

By (3.5.6) and continuity of $w_1, w_{2,n}$,

$$M = \max_{[0,T_0]\times\mathcal{C}^2} w_1-w_{2,n} = (w_1-w_{2,n})(\bar{t},\bar{x},\bar{y}),$$

where $T_0>0$ and $\mathcal{C}$ is a compact subset of $\mathbb{R}_+^2$. We now distinguish between two cases.

*Case 1* : $\bar{x}=0$. First note that $\mathcal{H}w_1-\mathcal{H}w_{2,n}\leq M$. Using the boundary condition (3.5.3), we then have

$$\begin{aligned}M &= (\hat{w}_1-\hat{w}_{2,n})(\bar{t},0,\bar{y})\\ &\leq \mathbb{E}\int_{\bar{t}}^{\infty} e^{-(\beta+\lambda)(s-t)}\lambda G[\mathcal{H}w_1-\mathcal{H}w_{2,n}](s,0,Y_s^{\bar{t},\bar{y}})ds\\ &\leq \int_{\bar{t}}^{\infty} e^{-(\beta+\lambda)(s-t)}\lambda M ds\\ &= \frac{\lambda}{\beta+\lambda}M,\end{aligned}$$

and it follows that $M\leq 0$.

*Case 2* : $\bar{x} > 0$. Define on $[0, T_0] \times (\mathcal{C})^2$

$$\Phi_\varepsilon(t, X, X') = \hat{w}_1(t, X) - \hat{w}_{2,n}(t, X') - \frac{|X - X'|^2}{2\varepsilon}.$$

Since $\Phi_\varepsilon$ is continuous on the compact set $[0, T_0] \times (\mathcal{C})^2$, there exists $(t_\varepsilon, X_\varepsilon, X'_\varepsilon)$ s.t.

$$M_\varepsilon := \sup_{[0,T_0]\times(\mathcal{C})^2} \Phi_\varepsilon = \Phi_\varepsilon(t_\varepsilon, X_\varepsilon, X'_\varepsilon),$$

and a subsequence, still denoted $(t_\varepsilon, X_\varepsilon, X'_\varepsilon)$ converging to some $(\hat{t}, \hat{X}, \hat{X}')$. By standard arguments (see e.g. Lemma 3.1 in [16]), $\hat{X} = \hat{X}'$ and $(\hat{t}, \hat{X})$ is a maximum point of $(\hat{w}_1 - \hat{w}_{2,n}$, hence w.l.o.g. $(\hat{t}, \hat{X}) = (\bar{t}, \bar{X})$, and we further have

$$\lim_{\varepsilon \to 0} \frac{|X_\varepsilon - X'_\varepsilon|^2}{2\varepsilon} \quad = \quad 0. \tag{3.5.7}$$

Now we apply the parabolic Ishii lemma (Theorem 8.3 in [16]) to obtain $q, q' \in \mathbb{R}$, $A, A'$ in $\mathcal{S}^2$ such that

$$(q, \frac{X_\varepsilon - X'_\varepsilon}{\varepsilon}, A) \in \bar{\mathcal{P}}^{1,2,+}\hat{w}_1(t_\varepsilon, X_\varepsilon), \qquad (q', \frac{X_\varepsilon - X'_\varepsilon}{\varepsilon}, A') \in \bar{\mathcal{P}}^{1,2,-}\hat{w}_{2,n}(t_\varepsilon, X'_\varepsilon), \tag{3.5.8}$$

$$\begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \quad \leq \quad \frac{3}{\varepsilon} \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}, \tag{3.5.9}$$

$$q + q' \quad = \quad 0. \tag{3.5.10}$$

Since $X_\varepsilon$ converges to $\bar{X}$, $x_\varepsilon > 0$ for $\varepsilon$ small enough, and we can use the viscosity subsolution property of $\hat{w}_1$ to obtain

$$\begin{aligned} & -q + (\beta + \lambda) w_1(t_\varepsilon, X_\varepsilon) - \tilde{U}\left(\frac{x_\varepsilon - x'_\varepsilon}{\varepsilon}\right) - \lambda G[\mathcal{H} w_1](t_\varepsilon, x_\varepsilon, y_\varepsilon) \qquad (3.5.11) \\ & - \sup_{\pi \in \mathbb{R}} \left[\pi \frac{x_\varepsilon - x'_\varepsilon}{\varepsilon} + \frac{\rho b_L \sigma_I}{\sigma_L} y_\varepsilon \frac{y_\varepsilon - y'_\varepsilon}{\varepsilon} + \frac{\sigma_L^2 \pi^2}{2} A_{11} + \pi \rho \sigma_I \sigma_L y_\varepsilon A_{12} + \rho^2 \frac{\sigma_I^2}{2} y_\varepsilon^2 A_{22}\right] \quad \leq \quad 0, \end{aligned}$$

and the supersolution property of $\hat{w}_{2,n}$ to get

$$\begin{aligned} & -q' + (\beta + \lambda) w_{2,n}(t_\varepsilon, X'_\varepsilon) - \tilde{U}\left(\frac{x_\varepsilon - x'_\varepsilon}{\varepsilon}\right) - \lambda G[\mathcal{H} w_{2,n}](t_\varepsilon, x'_\varepsilon, y'_\varepsilon) \qquad (3.5.12) \\ & - \sup_{\pi \in \mathbb{R}} \left[\pi \frac{x_\varepsilon - x'_\varepsilon}{\varepsilon} + \frac{\rho b_L \sigma_I}{\sigma_L} y'_\varepsilon \frac{y_\varepsilon - y'_\varepsilon}{\varepsilon} + \frac{\sigma_L^2 \pi^2}{2} A'_{11} + \pi \rho \sigma_I \sigma_L y'_\varepsilon A'_{12} + \rho^2 \frac{\sigma_I^2}{2} (y'_\varepsilon)^2 A'_{22}\right] \quad \geq \quad 0. \end{aligned}$$

Substracting (3.5.11) by (3.5.12), and using the fact that the difference of the supremum is less than the supremum of the difference, and (3.5.10), we obtain

$$
\begin{aligned}
& (\beta+\lambda)(w_1(t_\varepsilon, X_\varepsilon) - w_{2,n}(t_\varepsilon, X'_\varepsilon)) \\
\le\ & \sup_{\pi\in\mathbb{R}} \left[ \frac{\sigma_L^2\pi^2}{2}(A_{11}-A'_{11}) + \pi\rho\sigma_I\sigma_L(y_\varepsilon A_{12} - y'_\varepsilon A'_{12}) + \rho^2\frac{\sigma_I^2}{2}(y_\varepsilon^2 A_{22} - (y'_\varepsilon)^2 A'_{22}) \right] \\
& + \frac{\rho b_L \sigma_I}{\sigma_L}\frac{(y_\varepsilon - y'_\varepsilon)^2}{\varepsilon} + \lambda\left(G[\mathcal{H}w_1](t_\varepsilon, X_\varepsilon) - G[\mathcal{H}w_{2,n}](t_\varepsilon, X'_\varepsilon)\right).
\end{aligned} \tag{3.5.13}
$$

First notice that

$$
\begin{aligned}
\lim_{\varepsilon\to 0} G[\mathcal{H}w_1](t_\varepsilon, X_\varepsilon) - G[\mathcal{H}w_{2,n}](t_\varepsilon, X'_\varepsilon) &= G[\mathcal{H}w_1](\bar t, \bar X) - G[\mathcal{H}w_{2,n}](\bar t, \bar X) \\
&\le \sup_{\mathbb{R}_+}(\mathcal{H}w_1 - \mathcal{H}w_{2,n}) \\
&\le \sup_{x,y\in\mathbb{R}_+}(\hat w_1 - \hat w_{2,n})(0,x,y) \\
&\le M.
\end{aligned} \tag{3.5.14}
$$

Furthermore, for all $\pi \in \mathbb{R}$,

$$
\begin{aligned}
& \frac{\sigma_L^2\pi^2}{2}(A_{11}-A'_{11}) + \pi\rho\sigma_I\sigma_L(y_\varepsilon A_{12} - y'_\varepsilon A'_{12}) + \rho^2\frac{\sigma_I^2}{2}(y_\varepsilon^2 A_{22} - (y'_\varepsilon)^2 A'_{22}) \\
=\ & \frac{1}{2}\begin{pmatrix} \sigma_L\pi & \rho\sigma_I y_\varepsilon & \sigma_L\pi & \rho\sigma_I y'_\varepsilon \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \begin{pmatrix} \sigma_L\pi \\ \rho\sigma_I y_\varepsilon \\ \sigma_L\pi \\ \rho\sigma_I y'_\varepsilon \end{pmatrix} \\
\le\ & \frac{1}{2}\begin{pmatrix} \sigma_L\pi & \rho\sigma_I y_\varepsilon & \sigma_L\pi & \rho\sigma_I y'_\varepsilon \end{pmatrix} \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix} \begin{pmatrix} \sigma_L\pi \\ \rho\sigma_I y_\varepsilon \\ \sigma_L\pi \\ \rho\sigma_I y'_\varepsilon \end{pmatrix} \\
=\ & (\rho\sigma_I)^2\frac{3}{2\varepsilon}|y_\varepsilon - y'_\varepsilon|^2.
\end{aligned} \tag{3.5.15}
$$

where we have used (3.5.9).

Recall that by (3.5.7),

$$
\frac{(y_\varepsilon - y'_\varepsilon)^2}{\varepsilon} \quad \to \quad 0 \qquad \text{when } \varepsilon \to 0. \tag{3.5.16}
$$

Letting $\varepsilon$ go to 0 in (3.5.13), and combining (3.5.14),(3.5.15) and (3.5.16), we finally obtain

$$(\beta + \lambda)M \quad \leq \quad \lambda M,$$

and $M \leq 0$. □

### 3.5.2 Power utility

In this subsection we consider the problem in the case $U(c) = \frac{c^p}{p}$, for some $p$ in $(0, 1)$.

In this case, due to the fact that the utility function is homothetic, the HJB equation can be reduced to a PDE involving just a one-dimensional state variable. Indeed let $x, y \geq 0$, $x + y > 0$ and set

$$r = x + y \geq 0, \quad z = \frac{y}{x+y} \in [0, 1], \tag{3.5.17}$$

Since (with a slight but clear abuse of notation) $\mathcal{A}(\alpha r) = \alpha \mathcal{A}(r)$ for all $\alpha \geq 0$, taking into account the fact that $U$ is homothetic of degree $p$, it is straightforward to show that

$$V(r) = \Phi_0 \frac{r^p}{p}, \quad \forall \alpha \geq 0. \tag{3.5.18}$$

for some $\Phi_0 \geq 0$. We have also $\mathcal{A}_t(\alpha x) = \alpha \mathcal{A}_t(x)$ for all $\alpha \geq 0$, so it is straightforward to show that also

$$\hat{V}(t, \alpha x, \alpha y) = \alpha^p \; \hat{V}(t, x, y), \quad \forall \alpha \geq 0. \tag{3.5.19}$$

Therefore, by the change of variables (3.5.17), we can rewrite $\hat{V}$ in separated form as

$$\hat{V}(t, x, y) = \frac{r^p}{p} \; \hat{\Phi}\left(t, z\right), \tag{3.5.20}$$

where

$$\hat{\Phi} : \bar{\mathcal{Q}} \to \mathbb{R}^+, \quad \mathcal{Q} := [0, +\infty) \times (0, 1)$$

and

$$\hat{\Phi}(t, z) = \hat{V}(t, z, 1 - z). \tag{3.5.21}$$

On the other hand, we have also (3.3.27), so

$$\Phi_0 = \sup_{z\in[0,1]} \hat{\Phi}(0,z). \tag{3.5.22}$$

Plugging the expression above for $\hat{V}$ (3.5.20) into (3.5.1), we can get rid of the terms involving the variable $r$, remaining with an equation for $\hat{\Phi}$. Set

$$\tilde{c} = \frac{c}{x+y}, \qquad \tilde{\pi} = \frac{\pi}{x+y}.$$

The variables $\tilde{c}, \tilde{\pi}$ express the consumption and the investment in the liquid asset in terms of the variable $x+y$, which is in our case the counterpart of the total wealth in [2] (indeed, due to the lack of full information, the variable $X_t + Y_t$ is somehow what we know about the total wealth, which is only partially observable).

Taking into account (3.3.16) and dividing everything by $(x+y)^p/p$ it becomes

$$-\hat{\Phi}_t + (\beta+\lambda)\hat{\Phi} - \lambda p \Phi_0 G[U](t, 1-z, z) - \sup_{\tilde{c}\geq 0, \tilde{\pi}\in\mathbb{R}} \tilde{H}_{cv}(z, \hat{\Phi}, \hat{\Phi}_z, \hat{\Phi}_{zz}; \tilde{c}, \tilde{\pi}) = 0, \tag{3.5.23}$$

where

$$\begin{aligned}
&\tilde{H}_{cv}(z, \hat{\Phi}, \hat{\Phi}_z, \hat{\Phi}_{zz}; \tilde{c}, \tilde{\pi}) = pU(\tilde{c}) - \tilde{c}\,(p\hat{\Phi} - z\hat{\Phi}_z) \\
&\quad + \rho\frac{b_L\sigma_I}{\sigma_L}\Big(pz\hat{\Phi} - z(1-z)\hat{\Phi}_z\Big) + \rho^2\frac{\sigma_I^2}{2}z^2\Big(p(p-1)\hat{\Phi} - 2(1-z)(1-p)\hat{\Phi}_z + (1-z)^2\hat{\Phi}_{zz}\Big) \\
&\qquad + \tilde{\pi}\left(b_L(p\hat{\Phi} - z\hat{\Phi}_z) + \rho\sigma_L\sigma_I z\left(p(p-1)\hat{\Phi} - (1-2z)(1-p)\hat{\Phi}_z - z(1-z)\hat{\Phi}_{zz}\right)\right) \\
&\qquad\qquad + \tilde{\pi}^2\frac{\sigma_L^2}{2}\left(p(p-1)\hat{\Phi} + 2z(1-p)\hat{\Phi}_z + z^2\hat{\Phi}_{zz}\right).
\end{aligned} \tag{3.5.24}$$

Moreover (3.4.4) becomes

$$\hat{\Phi}(t,1) = \lambda p \Phi_0\, \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)} G[U](s, 0, rE_s^t)ds\right], \tag{3.5.25}$$

and (3.4.6) becomes

$$\hat{\Phi}(t,z) \leq K_{\hat{V}} e^{k_J(p)t}, \quad (t,z)\in\mathcal{Q}. \tag{3.5.26}$$

Due to the results of the previous section and to the argument above, we get the following (with standard meaning of viscosity solution in the interior region $\mathcal{Q}$).

**Proposition 3.5.2.** *The function* $\hat{\hat{\Phi}}$ *is the unique viscosity solution over* $\mathcal{Q}$ *of* (3.5.23) *fulfilling boundary condition* (3.5.25) *and growth condition* (3.5.26).

## 3.6 Numerical analysis

In this section we present an iterative scheme to approximate the value functions $V$ and $\hat{V}$, and present some numerical results.

### 3.6.1 Iterative procedure

Because of the nonlocal term $G[\mathcal{H}\hat{V}]$ in (3.5.1), we have to couple the standard numerical scheme with an iterative procedure as we are going to describe.

We start with

$$V^0 \quad = \quad 0 \tag{3.6.1}$$

and inductively :

• Given $V^n$ we define $\hat{V}^{n+1}$ on $[0, +\infty) \times R_+^2$ as the unique (constrained viscosity) solution to

$$\begin{aligned} -\hat{V}_t^{n+1} + (\beta+\lambda)\hat{V}^{n+1} - \lambda G[V^n](t,x,y) & \\ - \sup_{c \geq 0, \pi \in \mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{V}^{n+1}, D^2_{(x,y)}\hat{V}^{n+1}; c, \pi) &= 0, \end{aligned} \tag{3.6.2}$$

with boundary condition

$$\hat{V}^{n+1}(t,0,y) = \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G[V^n](s,0,\tilde{Y}_s^{t,y})ds. \tag{3.6.3}$$

and growth condition

$$|\hat{V}^{n+1}(t,x,y)| \quad \leq \quad Ke^{k_J(p)t}(x+y)^p. \tag{3.6.4}$$

• Given $\hat{V}^{n+1}$, $V^{n+1}$ is defined by

$$V^{n+1} \quad = \quad \mathcal{H}\hat{V}^{n+1}. \tag{3.6.5}$$

We provide a stochastic control representation for $(\hat{V}^n, V^n)_n$ in the proposition below.

**Proposition 3.6.1.** *For $n \geq 0$, define*

$$V^n(r) \quad = \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\int_0^{\tau_n} e^{-\beta t} U(c_s)ds, \tag{3.6.6}$$

*and*

$$\hat{V}^{n+1}(t,x,y) \quad = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\int_t^{\infty} e^{-(\beta+\lambda)(s-t)} \left(U(c_s) + \lambda G[V^n]\left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right)\right) ds. \tag{3.6.7}$$

*Then $(V^n, \hat{V}^n)_{n\geq 0}$ is the unique solution to* (3.6.1) *and* (3.6.2)-(3.6.3)-(3.6.4)*, $n \geq 0$.*

**Proof.** $V^0 = 0$ is obvious, and by induction we assume that $(3.6.2) - (3.6.3) - (3.6.4)$ has a unique solution for $k \leq n$, given by (3.6.6)-(3.6.7).

The fact that $\hat{V}^{n+1}$ is a constrained viscosity solution to (3.6.2) follows from the same arguments as for $\hat{V}$ in Theorem 3.5.1, (3.6.3)is satisfied by definition. Since $V^n(r) \leq V(r) \leq Kr^p$, the growth condition (3.6.4) is also verified. The uniqueness of this solution is proved by a comparison principle as in the proof of Proposition 3.5.1 (actually even easier since there is no nonlocal term).

Furthermore, in the same way as Proposition 3.3.1 and Lemma 3.3.2, we have the following DPP for the $V^n$ :

$$\begin{aligned} V^{n+1}(r) &= \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s)ds + e^{-\beta\tau_1} V^n(R_{\tau_1})\right] \\ &= \sup_{0\leq a\leq r}\ \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\int_0^{\infty} e^{-(\beta+\lambda)(s-t)} \left(U(c_s) + \lambda G[V^n]\left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right)\right) ds \\ &= (\mathcal{H}\hat{V}^{n+1})(r), \end{aligned}$$

which proves (3.6.5). □

We now prove that $V^n$ converges to $V$ at an exponential rate.

**Proposition 3.6.2.** *For some $K > 0$, we have*

$$0 \leq (V - V^n)(r) \quad \leq \quad Kr^p\delta^n, \tag{3.6.8}$$

$$0 \leq (\hat{V} - \hat{V}^n)(t,x,y) \quad \leq \quad Ke^{k_J(p)t}(x+y)^p\delta^n, \tag{3.6.9}$$

*where*

$$\delta \quad := \quad \frac{\lambda}{\lambda + \beta - k_M(p)} \quad < \quad 1.$$

**Proof.** We will prove the claim by induction. The case $n = 0$ follows from (3.4.1) and (3.4.6), and we assume that (3.6.8)-(3.6.9) are true for some $n$. Fixing $t, x, y$ and $(c, \pi) \in \mathcal{A}_t(x)$, we have

$$\begin{aligned} 0 \leq \mathcal{J}_V(t,x,y;c,\pi) - \mathcal{J}_{V^n}(t,x,y;c,\pi) &\leq \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G[(V - V^n)] \left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right) ds \\ &\leq \mathbb{E}\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda K r^p \delta^n e^{k_J(p)t} e^{k_M(p)(s-t)} (x+y)^p \\ &= K e^{k_J(p)t} \delta^{n+1} (x+y)^p, \end{aligned}$$

where we have used the induction hypothesis and Lemma 3.4.1. Taking the supremum over $(c, \pi) \in \mathcal{A}_t(x)$ we obtain (3.6.9) for $n + 1$, and (3.6.8) follows by (3.6.6).

### 3.6.2 Finite horizon problem

To solve the PDE (3.6.2) we approximate it by a finite horizon PDE. We fix some time $T > 0$, and we consider the functions $\hat{V}^{n,T}$, $V^{n,T}$ defined recursively by $V^{0,T} = \hat{V}^{0,T} = 0$,

$$\begin{aligned} \hat{V}^{n+1,T}(t,x,y) \quad = \quad \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\Bigg[\int_t^T e^{-(\beta+\lambda)(s-t)} \left(U(c_s) + \lambda G[V^{n,T}]\left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right)\right) ds \\ + e^{-(\beta+\lambda)(T-t)} \phi^{n+1,T}(\tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y})\Bigg], \end{aligned}$$

$(t, x, y) \in [0, T] \times \mathbb{R}_+^2$ where $\phi^{n+1,T}$ is some given terminal condition, and

$$V^{n,T} \quad = \quad \mathcal{H}\hat{V}^{n,T} \tag{3.6.10}$$

By the same methods as above it is then straightforward to check that $\hat{V}^{n+1,T}$ is a constrained viscosity solution to

$$\begin{aligned} -\hat{V}_t^{n+1,T} + (\beta+\lambda)\hat{V}^{n+1} - \lambda G[V^{n,T}](t,x,y) \\ - \sup_{c\geq 0, \pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{V}^{n+1,T}, D^2_{(x,y)}\hat{V}^{n+1,T}; c, \pi) \quad = \quad 0, \end{aligned} \tag{3.6.11}$$

on $[0,T) \times \mathbb{R}_+^2$, with boundary conditions

$$\begin{aligned}
\hat{V}^{n+1,T}(T,x,y) &= \phi^{n+1,T}(x,y), \\
\hat{V}^{n+1,T}(t,0,y) &= \mathbb{E}\left[\int_t^T e^{-(\beta+\lambda)(s-t)}\lambda G[V^{n,T}](s,0,\tilde{Y}_s^{t,y})ds + e^{-(\beta+\lambda)(T-t)}\phi^{n+1,T}(0,\tilde{Y}_s^{t,x,\pi,c})\right].
\end{aligned}$$

We assume that the terminal condition $\phi^{n+1,T}$ satisfies :

$$\left|\phi^{n+1,T}(x,y) - \hat{V}^{n+1}(T,x,y)\right| \leq \mathcal{E}e^{k_J(p)T}(x+y)^p, \tag{3.6.12}$$

for some $\mathcal{E}$ not depending on $n$. Note that this assumption is not restrictive since $0 \leq \hat{V}^{n+1} \leq \hat{V}$, and by (3.4.6), (3.6.12) is satisfied by taking $\phi^{n+1,T} = 0$.

We then have the following estimate for the numerical error induced by the finite horizon approximation :

**Proposition 3.6.3.** *For all $n \geq 0$, for all $r \in \mathbb{R}_+$ :*

$$|(V^{n,T} - V^n)(r)| \leq \frac{\mathcal{E}}{1-\delta}e^{-(\beta+\lambda-k_M(p))T}r^p,$$

**Proof.** We prove by induction that

$$|(V^{n,T} - V^n)(r)| \leq (1+\ldots+\delta^{n-1})\mathcal{E}e^{-(\beta+\lambda-k_M(p))T}r^p. \tag{3.6.13}$$

$V^0 = V^{0,T} = 0$, so the claim is satisfied for $n = 0$. Assume that it is satisfied by some $n$.

By the dynamic programming principle applied to $\hat{V}^{n+1}$,

$$\begin{aligned}
\hat{V}^{n+1}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\Bigg[&\int_t^T e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + \lambda G[V^n]\left(s,\tilde{X}_s^{t,x,\pi,c},\tilde{Y}_s^{t,y}\right)\right)ds \\
&+e^{-(\beta+\lambda)(T-t)}\hat{V}^{n+1}(\tilde{X}_s^{t,x,\pi,c},\tilde{Y}_s^{t,y})\Bigg],
\end{aligned}$$

so that

$$
\begin{aligned}
& \left|(\hat{V}^{n+1,T} - \hat{V}^{n+1})(t,x,y)\right| \\
\leq \; & \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^T e^{-(\beta+\lambda)(s-t)}\lambda G[|V^{n,T} - V^n|]\left(s, \tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y}\right) ds \right. \\
& \qquad \left. + e^{-(\beta+\lambda)(T-t)}\left|\hat{V}^{n+1,T} - \hat{V}^{n+1}\right|(\tilde{X}_s^{t,x,\pi,c}, \tilde{Y}_s^{t,y})\right] \\
\leq \; & e^{k_J(p)t}\lambda\frac{1-e^{-(\beta-k_M(p)+\lambda)(T-t)}}{\beta - k_M(p)+\lambda}(1+\ldots+\delta^{n-1})\mathcal{E}e^{-(\beta+\lambda-k_M(p))T}(x+y)^p \\
& + \mathcal{E}e^{-(\beta+\lambda-k_M(p))(T-t)}e^{k_J(p)t}(x+y)^p \\
\leq \; & e^{k_J(p)t}e^{-(\beta+\lambda-k_M(p))T}(x+y)^p\mathcal{E}\left((\delta+\ldots+\delta^n) + e^{-(\beta+\lambda-k_M(p))(t)}\right),
\end{aligned}
$$

where we have used Lemma 3.4.1, (3.6.12) and (3.6.13). Taking $t = 0$, we obtain

$$
\begin{aligned}
\left|V^{n+1,T} - V^{n+1}(r)\right| &= \left|\mathcal{H}\hat{V}^{n+1,T} - \mathcal{H}\hat{V}^{n+1}(r)\right| \\
&\leq \mathcal{H}\left|\hat{V}^{n+1,T} - \hat{V}^{n+1}\right|(r) \\
&\leq e^{-(\beta+\lambda-k_M(p))T}r^p\mathcal{E}(1+\ldots+\delta^n).
\end{aligned}
$$

□

By combining Propositions 3.6.2 and 3.6.3, we can choose $n$ and $T$ large enough to compute $V$ with any required precision. The choice of $n$ and $T$ will mainly depend on $\lambda$ :

- When $\lambda$ is large, $\delta$ is close to 1 so that the number of iterations $n$ must be chosen large.
- The finite horizon error is roughly of order $(\lambda+1)e^{-(1+\lambda)T}$ so that $T$ may be chosen small for large $\lambda$, and must be reasonably large for small $\lambda$.

### 3.6.3 Numerical results

We now focus on the numerical resolution in the case of power utility. Recall that in that case with the change of variables described in subsection 3.5.2 we are reduced to one space variable $z \in [0,1]$. (3.6.11)-(3.6.10) then take the form

$$
\begin{aligned}
& -\hat{\Phi}_t^{n+1,T} + (\beta+\lambda)\hat{\Phi}^{n+1,T} - \lambda p \Phi_0^{n,T} G[U](t, 1-z, z) \\
& \qquad - \sup_{\tilde{c}\geq 0, \tilde{\pi}\in\mathbb{R}} \tilde{H}_{cv}(z, \hat{\Phi}^{n+1,T}, \hat{\Phi}_z^{n+1,T}, \hat{\Phi}_{zz}^{n+1,T}; \tilde{c}, \tilde{\pi}) = 0,
\end{aligned}
$$

where $\tilde{H}_{cv}$ is the hamiltonian defined in (3.5.24), and

$$\Phi_0^{n,T} = \sup_{z\in[0,1]} \hat{\Phi}^{n,T}(0,z).$$

We solved these PDEs using an explicit finite-difference scheme for parabolic viscosity solutions (see e.g. chapter IX in [24]). We have taken $T$ between 1 and 5 (depending on $\lambda$) and used a uniform grid on $[0,T]\times[0,1]$ with time step $5.10^{-4}$ and space step 0.02. The numbers $G[U](t,z,1-z)$ were computed beforehand at each point of the grid using an $L^2$-optimal quantization grid for the gaussian law with $N = 5000$ points.

We have taken for value of the parameters

$$\beta = 0.2, \qquad p = 0.5, \qquad b_L = 0.15, \qquad \sigma_L = 1, \qquad b_I = 0.2, \qquad \sigma_I = 1,$$

and make vary $\lambda$ and $\rho$.

In Figures 3.1 and 3.2, we look at the value function $\Phi_0$ and the optimal proportion in the illiquid asset $z^*$. We also include the values for the Merton problem (i.e. the fully liquid problem as defined in Remark 3.2.1). For both of these values, we observe convergence to the Merton values, the optimal investment in $I$ increasing with $\lambda$. However, when the optimal Merton proportion is close to 1, this convergence is much slower. This is intuitive : in the illiquid case the agent consumes from his liquid wealth, and must ensure that it remains positive; so, having most of his wealth invested in the illiquid asset is costly for the agent, as he may find himself in a position where he has "nothing more to consume" (or, more rigorously, very little) before the next trading time.

We also look at the optimal consumption and investment in the liquid asset. Even if we do not have proved theoretically sufficient regularity of the value function to justify the structure of the optimal strategies as feedbacks, we can still look formally at $\tilde{c}^*(t,z), \tilde{\pi}^*(t,z)$ defined by

$$(\tilde{c}^*(t,z), \tilde{\pi}^*(t,z)) = \arg\max_{\tilde{c}\geq 0, \tilde{\pi}\in\mathbb{R}} \tilde{H}_{cv}(z, \hat{\Phi}(t,z), \Delta^1\hat{\Phi}(t,z), \Delta^2\hat{\Phi}(t,z); \tilde{c}, \tilde{\pi}),$$

where $\Delta^1\hat{\Phi}$, $\Delta^2\hat{\Phi}$ are the finite difference approximations to $\hat{\Phi}_z$, $\hat{\Phi}_{zz}$ corresponding to our discretization grid, and guess that they are good approximations for the optimal policies.

First in Figure 3.3 we have plotted the optimal investment proportion $\tilde{\pi}^*$ in the illiquid asset at time 0 for an agent investing the optimal proportion $z^*$ in the illiquid asset. Again, we observe convergence to the Merton value for large $\lambda$. At a rough look, it seems that $\tilde{\pi}^*$ is higher than the Merton proportion for positive $\rho$, and lower for negative $\rho$. This may be interpreted in the following way : for an agent forced to keep a invested proportion $z$ in the asset $I$, the optimal investment in $L$ may be written as

$$\pi^*(z) \quad = \quad \frac{b_L}{(1-p)\sigma_L^2} - \frac{\sigma_I}{\sigma_L}\rho z. \tag{3.6.14}$$

Hence the sign of the dependence in $z$ depends on the sign of $\rho$. Since we have seen in Figure 3.2 that in illiquid markets the optimal investment $z^*_\lambda$ will be smaller than $z^*_M$ this explains the behavior of $\tilde{\pi}^*$ (see also Figures 3.5-3.8).

It is also interesting to look at the functions $\tilde{c}^*$, $\tilde{\pi}^*$ in function of the repartition $z$ of the wealth. We plot these functions at time $t = 0$ (later dates give qualitatively similar profiles).

In Figure 3.4 we have plotted the optimal proportional consumption rate $\tilde{c}^*$ for $\rho = 0$ (different correlations give similar results). As in [27] we observe that the influence of $\lambda$ on the optimal consumption rate depends on $z$ : when $z$ is close to 1 i.e. most of the portfolio is constituted of illiquid wealth, the investor faces the risk of "having nothing more to consume" and the further away the next trading date is the smaller the consumption rate should be, i.e. $c^*$ is increasing in $\lambda$. When $z$ is far from 1 it is the opposite : when $\lambda$ is smaller the investor will not be able to invest optimally to maximize future income and should consume more quickly.

In Figures 3.5-3.6-3.7-3.8 we have plotted the optimal investment $\tilde{\pi}^*$ in the liquid asset for $\rho = 0$,$\rho = -0.5$ and $\rho = 0.5$. The "Merton" line corresponds to (3.6.14). We have also plotted the optimal consumption for $\lambda = 0$. In that case, the proportion $z$ invested in $I$ is actually lost, and the optimal investment in the liquid asset is then

$$\tilde{\pi}^*_0(z) = \frac{b_L}{(1-p)\sigma_L^2}(1-z).$$

Notice that when $\lambda$ increases, $\tilde{\pi}^*$ goes increasingly or decreasingly depending on the value of $\rho$ from $\tilde{\pi}^*_0(z)$ to $\tilde{\pi}^*_M(z)$. More precisely it seems that, when $\rho < 0.3$ the convergence is monotone increasing, while when $\rho > 0.3$ it is monotone decreasing. The value $\rho = 0.3$ is therefore a critical value for the allocation in the liquid asset: in this case $\tilde{\pi}^*_\lambda(z) = \tilde{\pi}^*_0(z) = \tilde{\pi}^*_M(z)$ for all values of $\lambda$ and $z$.

Finally, we compare our investment strategy in the illiquid asset with the case where no liquid asset is present in the market. This case as well as its numerical resolution are studied in Pham-Tankov[61]. In Table 3.1 we present the optimal investment proportion $z^*_{PT}$ in $I$ in function of $\lambda$. Numerically, this strategy coincides with the one we obtain in the uncorrelated case $\rho = 0$. Hence this model (which is numerically easier to solve than ours) can be used without much loss when the liquid and illiquid assets are only weakly correlated.

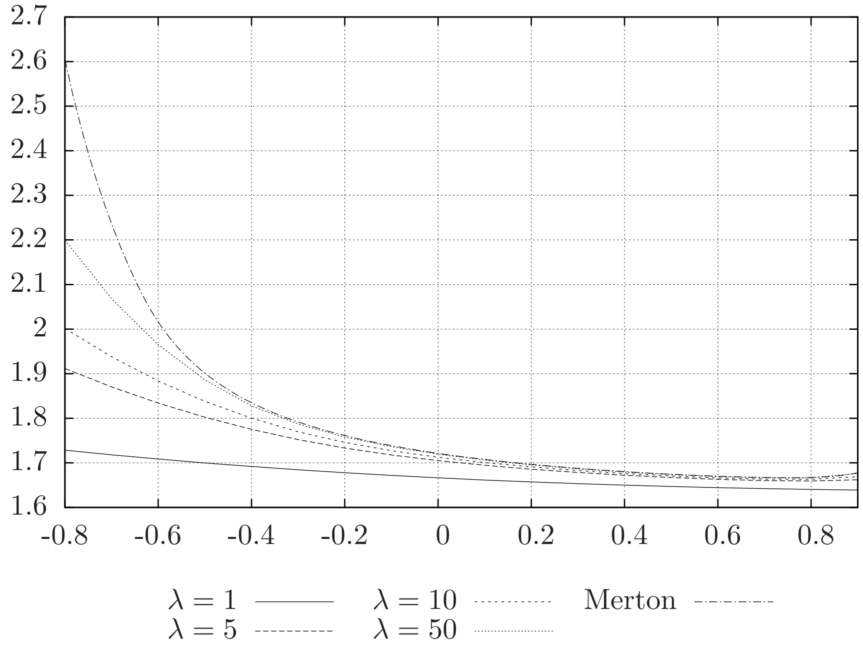


Figure 3.1: Value function $\Phi_0$ as a function of $\rho$

| $\lambda$ | 1 | 5 | 10 | 50 | Merton |
|---|---|---|---|---|---|
| $z^*$ | 0.18 | 0.34 | 0.36 | 0.4 | 0.4 |

Table 3.1: Optimal investment proportion $z^*_{PT}$ in the illiquid asset on a market with no liquid asset

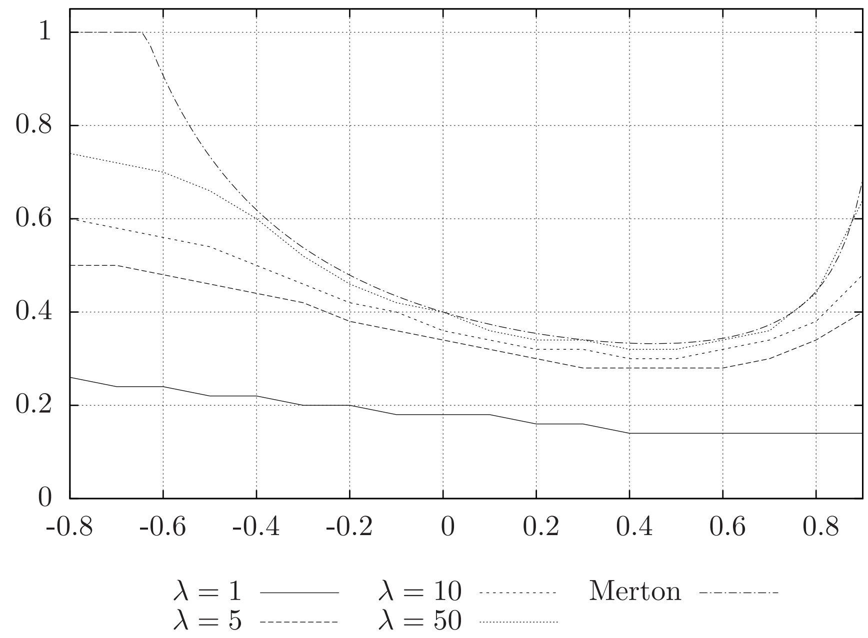


Figure 3.2: Optimal proportion in the illiquid asset in function of $\rho$

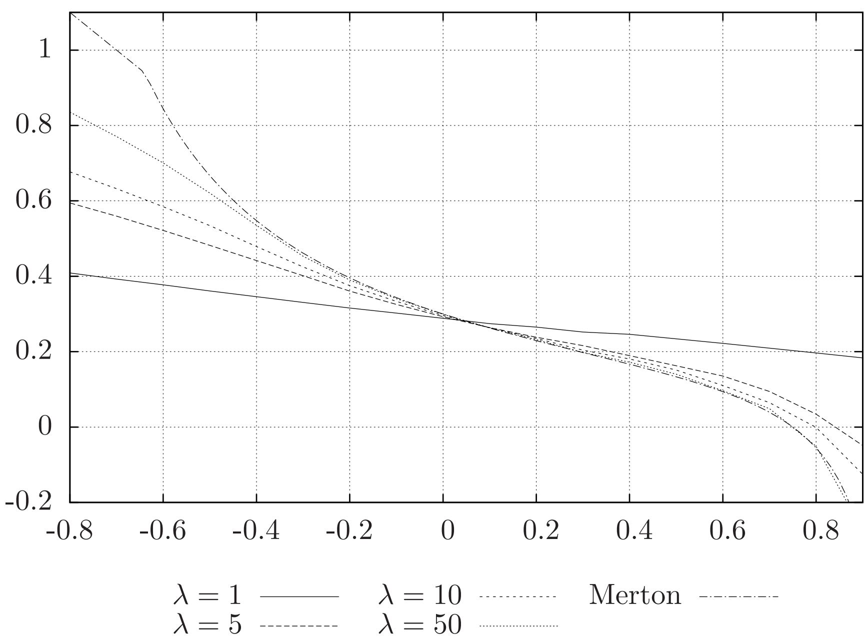


Figure 3.3: Optimal proportion in the liquid asset in function of $\rho$

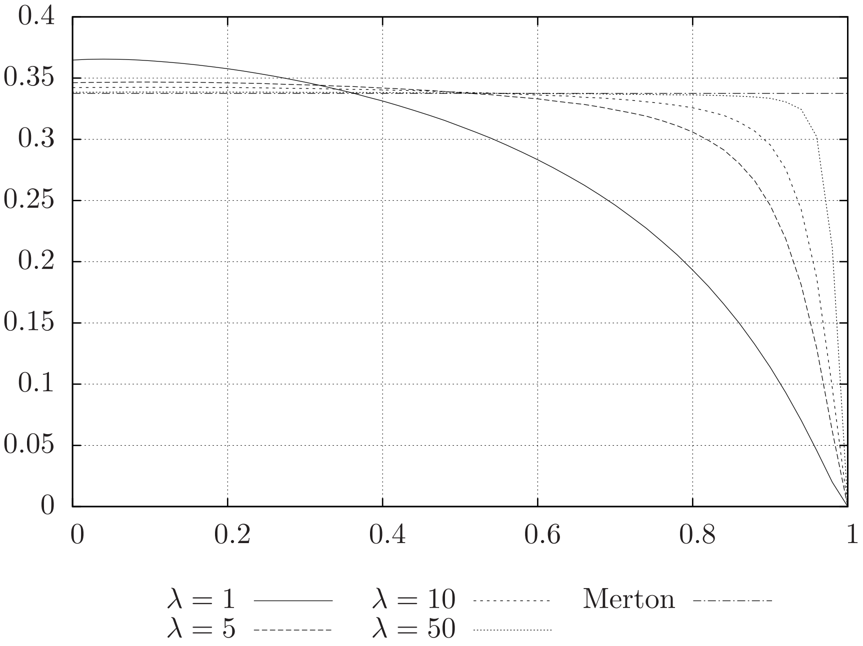


Figure 3.4: Optimal consumption rate $\tilde{c}^*(0, \cdot)$ in function of $z$ for $\rho = 0$

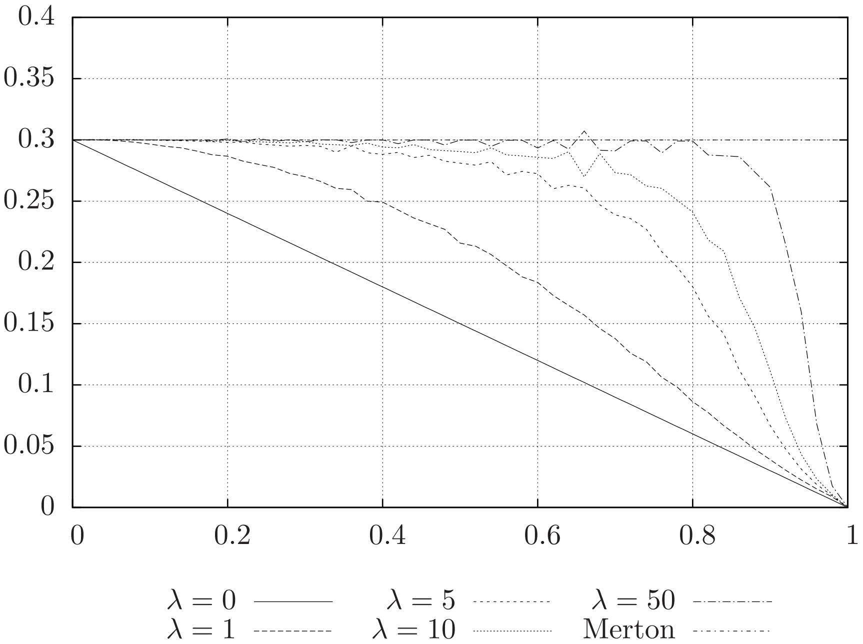


Figure 3.5: Optimal proportion in the liquid asset $\tilde{\pi}^*(0, \cdot)$ in function of $z$ for $\rho = 0$

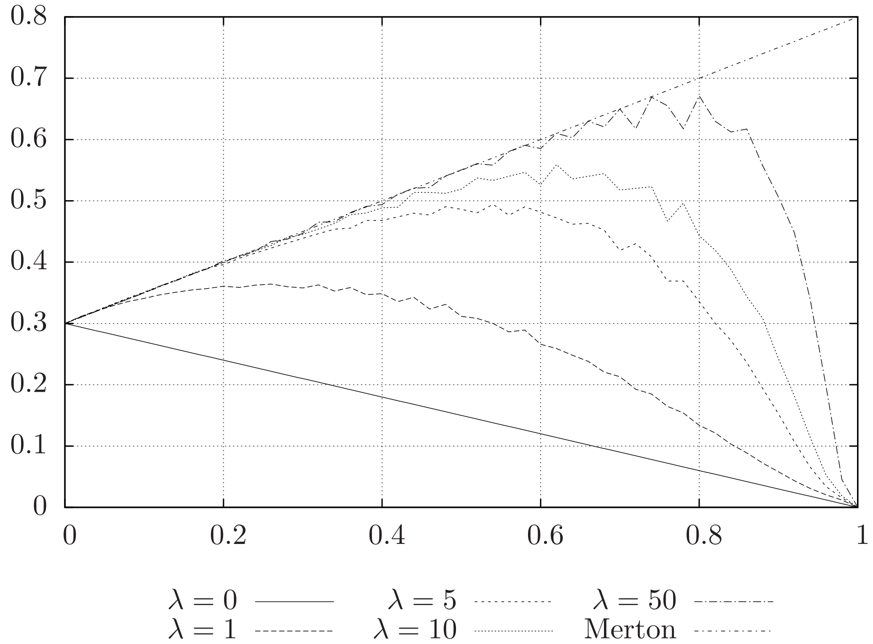


Figure 3.6: Optimal proportion in the liquid asset $\tilde{\pi}^*(0, \cdot)$ in function of $z$ for $\rho = -0.5$

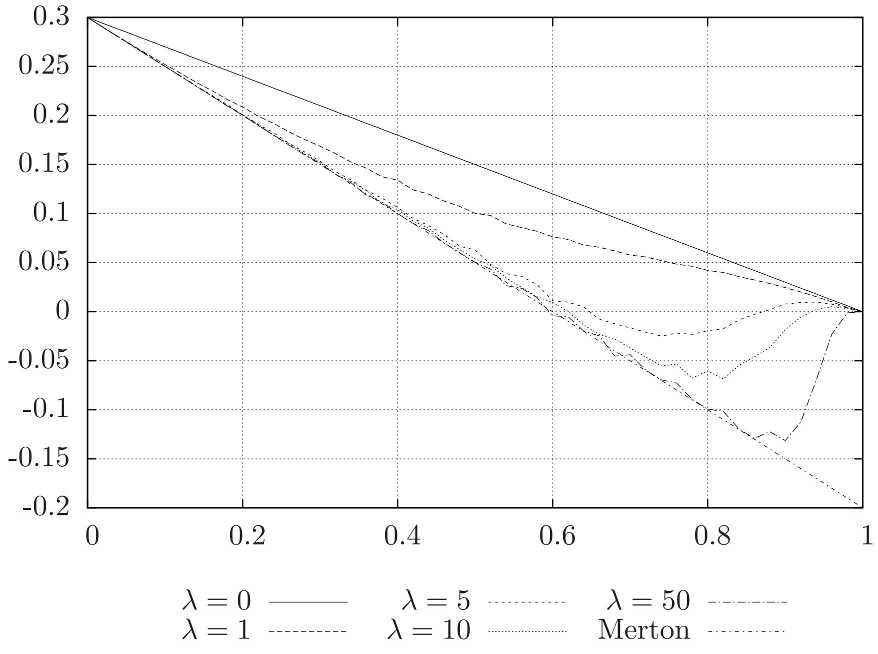


Figure 3.7: Optimal proportion in the liquid asset $\tilde{\pi}^*(0, \cdot)$ in function of $z$ for $\rho = 0.5$

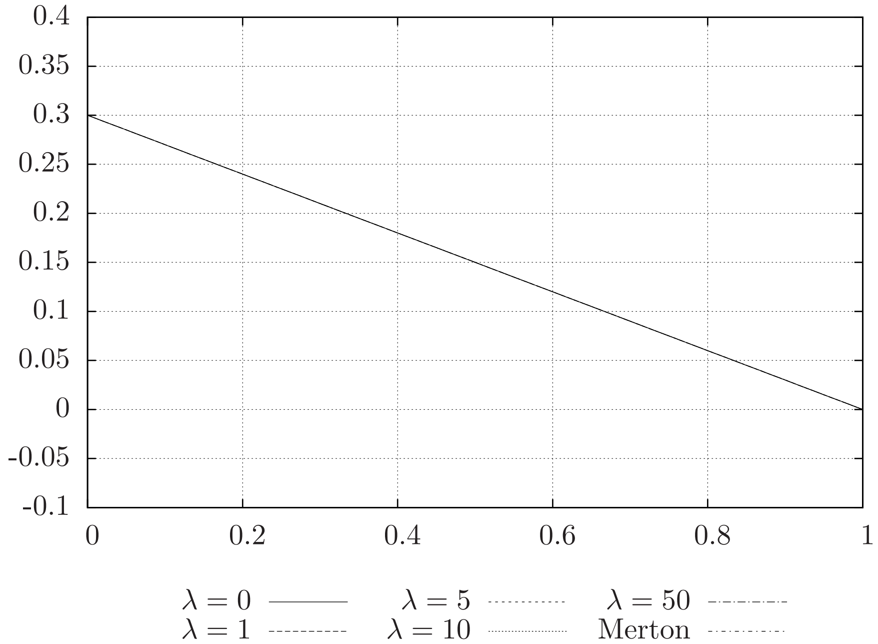


Figure 3.8: Optimal proportion in the liquid asset $\tilde{\pi}^*(0, \cdot)$ in function of $z$ for $\rho = 0.3$

# Part II

# Time discretization and quantization methods for optimal multiple switching problem

# Chapter 4

# Time discretization and quantization methods for optimal multiple switching problem

Abstract : In this paper, we study probabilistic numerical methods based on optimal quantization algorithms for computing the solution to optimal multiple switching problems with regime-dependent state process. We first consider a discrete-time approximation of the optimal switching problem, and analyze its rate of convergence. The error is of order $\frac{1}{2}-\varepsilon$, $\varepsilon > 0$, and of order $\frac{1}{2}$ when the switching costs do not depend on the state process. We next propose quantization numerical schemes for the space discretization of the discrete-time Euler state process. A Markovian quantization approach relying on the optimal quantization of the normal distribution arising in the Euler scheme is analyzed. In the particular case of uncontrolled state process, we describe an alternative marginal quantization method, which extends the recursive algorithm for optimal stopping problems as in [5]. A priori $L^p$-error estimates are stated in terms of quantization errors. Finally, some numerical tests are performed for an optimal switching problem with two regimes.

**Key words:** Optimal switching, quantization of random variables, discrete-time approximation, Markov chains, numerical probability.

## 4.1 Introduction

On some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let us introduce the controlled regime-switching diffusion in $\mathbb{R}^d$ governed by

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

where $W$ is a standard $d$-dimensional Brownian motion, $\alpha = (\tau_n, \iota_n)_n \in \mathcal{A}$ is the switching control represented by a nondecreasing sequence of stopping times $(\tau_n)$ together with a sequence $(\iota_n)$ of $\mathcal{F}_{\tau_n}$-measurable random variables valued in a finite set $\{1, \ldots, q\}$, and $\alpha_t$ is the current regime process, i.e. $\alpha_t = \iota_n$ for $\tau_n \leq t < \tau_{n+1}$. We then consider the optimal switching problem over a finite horizon:

$$V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[ \int_0^T f(X_t, \alpha_t)dt + g(X_T, \alpha_T) - \sum_{\tau_n \leq T} c(X_{\tau_n}, \iota_{n-1}, \iota_n) \Big]. \tag{4.1.1}$$

Optimal switching problems can be seen as sequential optimal stopping problems belonging to the class of impulse control problems, and arise in many applied fields, for example in real option pricing in economics and finance. It has attracted a lot of interest during the past decades, and we refer to Chapter 5 in the book [60] and the references therein for a survey of some applications and results in this topic. It is well-known that optimal switching problems are related via the dynamic programming approach to a system of variational inequalities with inter-connected obstacles in the form:

$$\min \Big[ -\frac{\partial v_i}{\partial t} - b(x,i).D_x v_i - \frac{1}{2}\mathrm{tr}(\sigma(x,i)\sigma(x,i)' D_x^2 v_i) - f(x,i) \;, \tag{4.1.2}$$
$$v_i - \max_{j \neq i}(v_j - c(x,i,j)) \Big] = 0 \quad \text{on } [0,T) \times \mathbb{R}^d,$$

together with the terminal condition $v_i(T,x) = g(x,i)$, for any $i = 1, \ldots, q$. Here $v_i(t,x)$ is the value function to the optimal switching problem starting at time $t \in [0,T]$ from the state $X_t$ $= x \in \mathbb{R}^d$ and the regime $\alpha_t = i \in \{1, \ldots, q\}$, and the solution to the system (4.1.2) has to be understood in the weak sense, e.g. viscosity sense.

The purpose of this paper is to solve numerically the optimal switching problem (4.1.1), and consequently the system of variational inequalities (4.1.2). These equations can be solved by analytical methods (finite differences, finite elements, etc ...), see e.g. [52], but are known to require heavy computations, especially in high dimension. Alternatively, when the state process is uncontrolled, i.e. regime-independent, optimal switching problems are connected to multi-dimensional reflected Backward Stochastic Differential Equations (BSDEs) with oblique reflections, as shown in [34] and [35], and the recent paper [13] introduced a discretely obliquely reflected numerical scheme to solve such BSDEs. From a computational viewpoint, there are rather few papers dealing with numerical experiments for optimal switching problems. The special case of two regimes for switching problems can be reduced to the resolution of a single BSDE with two reflecting barriers when considering the difference value process, and is exploited numerically in [33]. We mention also the paper [12], which solves an optimal switching problem with three regimes by considering a cascade of reflected BSDEs with one reflecting barrier derived from an iteration on the number of switches.

We propose probabilistic numerical methods based on dynamic programming and optimal quantization methods combined with a suitable time discretization procedure for computing the solution to optimal multiple switching problem. Quantization methods were introduced in [5] for solving variational inequality with given obstacle associated to optimal stopping problem of some diffusion process $(X_t)$. The basic idea is the following. One first approximates the (continuous-time) optimal stopping problem by the Snell envelope for the Markov chain $(\bar{X}_{t_k})$ defined as the Euler scheme of the (uncontrolled) diffusion $X$, and then spatially discretize each random vector $\bar{X}_{t_k}$ by a random vector taking finite values through a quantization procedure. More precisely, $(\bar{X}_{t_k})_k$ is approximated by $(\hat{X}_k)_k$ where $\hat{X}_k$ is the projection of $\bar{X}_{t_k}$ on a finite grid in the state space following the closest neighbor rule. The induced $L^p$-quantization error, $\|\bar{X}_{t_k} - \hat{X}_k\|_p$, depends only on the distribution of $\bar{X}_{t_k}$ and the grid, which may be chosen in order to minimize the quantization error. Such an optimal choice, called optimal quantization, is achieved by the competitive learning vector quantization algorithm (or Kohonen algorithm) developed in full details in [5]. One finally computes the approximation of the optimal stopping problem by a quantization tree algorithm, which mimics the backward dynamic programming of the Snell envelope. In this paper, we develop quantization methods to our general framework of optimal switching problem. With respect to standard optimal stopping problems, some new features arise on one hand from the regime-dependent state process, and on the other hand from the multiple switching times, and the discrete sum for the cumulated switching costs.

We first study a time discretization of the optimal switching problem by considering an

Euler-type scheme with step $h = T/m$ for the regime-dependent state process $(X_t)$ controlled by the switching strategy $\alpha$:

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(\bar{X}_{t_k}, \alpha_{t_k})h + \sigma(\bar{X}_{t_k}, \alpha_{t_k})\sqrt{h}\,\vartheta_{k+1}, \quad t_k = kh, \ k = 0, \ldots, m, \tag{4.1.3}$$

where $\vartheta_k$, $k = 1, \ldots, m$, are iid, and $\mathcal{N}(0, I_d)$-distributed. We then introduce the optimal switching problem for the discrete-time process $(\bar{X}_{t_k})$ controlled by switching strategies with stopping times valued in the discrete time grid $\{t_k, k = 0, \ldots, m\}$. The convergence of this discrete-time problem is analyzed, and we prove that the error is in general of order $h^{\frac{1}{2}-\varepsilon}$, and this estimate holds true with $\varepsilon = 0$, as for optimal stopping problems, when the switching costs $c(i,j)$ do not depend on the state process. Arguments of the proof rely on a regularity result of the controlled diffusion with respect to the switching strategy, and moment estimates on the number of switches. This extends the convergence rate result in [13] derived in the case where $X$ is regime-independent.

Next, we propose approximation schemes by quantization for computing explicitly the solution to the discrete-time optimal switching problem. Since the controlled Markov chain $(\bar{X}_{t_k})_k$ cannot be directly quantized as in standard optimal stopping problems, we adopt a Markovian quantization approach in the spirit of [57], by considering an optimal quantization of the Gaussian random vector $\vartheta_{k+1}$ arising in the Euler scheme (4.1.3). A quantization tree algorithm is then designed for computing the approximating value function, and we provide error estimates in terms of the quantization errors $\|\vartheta_k - \hat{\vartheta}_k\|_p$ and state space grid parameters. Alternatively, in the case of regime-independent state process, we propose a quantization algorithm in the vein of [5] based on marginal quantization of the uncontrolled Markov chain $(\bar{X}_{t_k})_k$. A priori $L^p$-error estimates are also established in terms of quantization errors $\|\bar{X}_{t_k} - \hat{X}_k\|_p$. Finally, some numerical tests on the two quantization algorithms are performed for an optimal switching problem with two regimes.

The plan of this paper is organized as follows. Section 2 formulates the optimal switching problem and sets the standing assumptions. We also show some preliminary results about moment estimates on the number of switches. We describe in Section 3 the time discretization procedure, and study the rate of convergence of the discrete-time approximation for the optimal switching problem. Section 4 is devoted to the approximation schemes by quantization for the explicit computation of the value function to the discrete-time optimal switching problem, and to the error analysis. Finally, we illustrate our results with some numerical tests in Section 5.

## 4.2 Optimal switching problem

### 4.2.1 Formulation and assumptions

We formulate the finite horizon multiple switching problem. Let us fix a finite time $T \in (0,\infty)$, and some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $\mathbb{I}_q = \{1,\ldots,q\}$ be the set of all possible regimes (or activity modes). A switching control is a double sequence $\alpha = (\tau_n, \iota_n)_{n\geq 0}$, where $(\tau_n)$ is a nondecreasing sequence of stopping times, and $\iota_n$ are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{I}_q$. The switching control $\alpha = (\tau_n, \iota_n)$ is said to be admissible, and denoted by $\alpha \in \mathcal{A}$, if there exists an integer-valued random variable $N$ with $\tau_N > T$ a.s. Given $\alpha = (\tau_n, \iota_n)_{n\geq 0} \in \mathcal{A}$, we may then associate the indicator of the regime value defined at any time $t \in [0,T]$ by

$$I_t \quad = \quad \iota_0 \mathbf{1}_{\{0\leq t<\tau_0\}} + \sum_{n\geq 0} \iota_n \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}},$$

which we shall sometimes identify with the switching control $\alpha$, and we introduce $N(\alpha)$ the (random) number of switches before $T$:

$$N(\alpha) \quad = \quad \#\{n \geq 1 \ : \tau_n \leq T\} \ .$$

For $\alpha \in \mathcal{A}$, we consider the controlled regime-switching diffusion process valued in $\mathbb{R}^d$, governed by the dynamics

$$dX_s \quad = \quad b(X_s, I_s)ds + \sigma(X_s, I_s)dW_s, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{4.2.1}$$

where $W$ is a standard $d$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. We shall assume that the coefficients $b_i = b(.,i)$: $\mathbb{R}^d \to \mathbb{R}^d$, and $\sigma_i(.) = \sigma(.,i) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $i \in \mathbb{I}_q$, satisfy the usual Lipschitz conditions.

We are given a running reward, terminal gain functions $f, g : \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$, and a cost function $c : \mathbb{R}^d \times \mathbb{I}_q \times \mathbb{I}_q \to \mathbb{R}$, and we set $f_i(.) = f(.,i)$, $g_i(.) = g(.,i)$, $c_{ij}(.) = c(.,i,j)$, $i,j \in \mathbb{I}_q$. We shall assume the Lipschitz condition:

**(Hl)** The coefficients $f_i$, $g_i$ and $c_{ij}$, $i,j \in \mathbb{I}_q$ are Lipschitz continuous on $\mathbb{R}^d$.

We also make the natural triangular condition on the functions $c_{ij}$ representing the instantaneous cost for switching from regime $i$ to $j$:

**(Hc)**

$$\begin{aligned} c_{ii}(.) &= 0, \quad i \in \mathbb{I}_q, \\ \inf_{x\in\mathbb{R}^d} c_{ij}(x) &> 0, \quad \text{for } i,j \in \mathbb{I}_q,\ j \neq i, \\ \inf_{x\in\mathbb{R}^d} [c_{ij}(x) + c_{jk}(x) - c_{ik}(x)] &> 0, \quad \text{for } i,j,k \in \mathbb{I}_q,\ j \neq i,k. \end{aligned}$$

The triangular condition on the switching costs $c_{ij}$ in **(Hc)** means that when one changes from regime $i$ to some regime $j$, then it is not optimal to switch again immediately to another regime, since it would induce a higher total cost, and so one should stay for a while in the regime $j$.

The expected total profit over $[0, T]$ for running the system with the admissible switching control $\alpha = (\tau_n, \iota_n) \in \mathcal{A}$ is given by:

$$J_0(\alpha) = \mathbb{E}\Big[\int_0^T f(X_t, I_t)dt + g(X_T, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\Big].$$

The maximal profit is then defined by

$$V_0 = \sup_{\alpha\in\mathcal{A}} J_0(\alpha). \tag{4.2.2}$$

The dynamic version of this optimal switching problem is formulated as follows. For $(t, i) \in [0, T] \times \mathbb{I}_q$, we denote by $\mathcal{A}_{t,i}$ the set of admissible switching controls $\alpha = (\tau_n, \iota_n)$ starting from $i$ at time $t$, i.e. $\tau_0 = t$, $\iota_0 = i$. Given $\alpha \in \mathcal{A}_{t,i}$, and $x \in \mathbb{R}^d$, and under the Lipschitz conditions on $b$, $\sigma$, there exists a unique strong solution to (4.2.1) starting from $x$ at time $t$, and denoted by $\{X_s^{t,x,\alpha}, t \leq s \leq T\}$. It is then given by

$$X_s^{t,x,\alpha} = x + \sum_{\tau_n\leq s} \int_{\tau_n}^{\tau_{n+1}\wedge s} b_{\iota_n}(X_u^{t,x,\alpha})du + \int_{\tau_n}^{\tau_{n+1}\wedge s} \sigma_{\iota_n}(X_u^{t,x,\alpha})dW_u,\ t \leq s \leq T. \tag{4.2.3}$$

The value function of the optimal switching problem is defined by

$$v_i(t, x) = \sup_{\alpha\in\mathcal{A}_{t,i}} \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big], \tag{4.2.4}$$

for any $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_q$, so that $V_0 = \max_{i\in\mathbb{I}_q} v_i(0, x_0)$.

For simplicity, we shall also make the assumption

$$g_i(x) \quad \geq \quad \max_{j\in\mathbb{I}_q}[g_j(x) - c_{ij}(x)], \quad \forall (x,i) \in \mathbb{R}^d \times \mathbb{I}_q. \tag{4.2.5}$$

This means that any switching decision at horizon $T$ induces a terminal profit, which is smaller than a no-decision at this time, and is thus suboptimal. Therefore, the terminal condition for the value function is given by:

$$v_i(T,x) \quad = \quad g_i(x), \quad (x,i) \in \mathbb{R}^d \times \mathbb{I}_q.$$

Otherwise, it is given in general by $v_i(T,x) = \max_{j\in\mathbb{I}_q}[g_j(x) - c_{ij}(x)]$.

**Notations.** $|.|$ will denote the canonical Euclidian norm on $\mathbb{R}^d$, and $(.|.)$ the corresponding inner product. For any $p \geq 1$, and $Y$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\|Y\|_p = (\mathbb{E}|Y|^p)^{\frac{1}{p}}$.

### 4.2.2 Preliminaries

We first show that one can restrict the optimal switching problem to controls $\alpha$ with bounded moments of $N(\alpha)$. More precisely, let us associate to a strategy $\alpha \in \mathcal{A}_{t,i}$, the cumulated cost process $C^{t,x,\alpha}$ defined by

$$C_u^{t,x,\alpha} \quad = \quad \sum_{n\geq 1} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\mathbf{1}_{\tau_n \leq u}, \qquad t \leq u \leq T.$$

We then consider for $x \in \mathbb{R}^d$ and a positive sequence $K = (K_p)_{p\in\mathbb{N}}$ the subset $\mathcal{A}_{t,i}^K(x)$ of $\mathcal{A}_{t,i}$ defined by

$$\mathcal{A}_{t,i}^K(x) \quad = \quad \Big\{\alpha \in \mathcal{A}_{t,i} : \mathbb{E}\big|C_T^{t,x,\alpha}\big|^p \ \leq \ K_p(1+|x|^p), \quad \forall p \geq 1\Big\}.$$

In the sequel, we shall assume that for each $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, the optimal switching problem $v_i(t,x)$ admits an optimal strategy $\alpha^*$ satisfying $\mathbb{E}\big[|C_T^{t,x,\alpha^*}|^2\big] < \infty$. The existence of an optimal strategy $\alpha^*$ with $\mathbb{E}|C_T^{t,x,\alpha^*}|^2 < \infty$ is a wide assumption that is valid under **(Hl)** and **(Hg)** in the case where the diffusion $X$ is not controlled i.e. the functions $b$ and $\sigma$ do not depend on the variable $i$ and the function $c$ does not depend on the variable $x$, as shown in Theorem 3.1 of [35].

**Proposition 4.2.1.** *Assume that* **(Hl)** *and* **(Hc)** *holds. Then there exists a positive sequence*

$\bar{K} = (\bar{K}_p)_p$ *such that*

$$v_i(t,x) \quad = \quad \sup_{\alpha \in \mathcal{A}^{\bar{K}}_{t,i}(x)} \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big] \tag{4.2.6}$$

*for any* $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$.

**Remark 4.2.1.** Under the uniformly strict positive condition on the switching costs in **(Hc)**, there exists some positive constant $\eta > 0$ s.t. $N(\alpha) \leq \eta C_T^{t,x,\alpha}$ for any $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}_{t,i}$. Thus, for any $\alpha \in \mathcal{A}^K_{t,i}(x)$, we have

$$\mathbb{E}|N(\alpha)|^p \ \leq \ \eta K_p(1+|x|^p),$$

which means that in the value functions $v_i(t,x)$ of optimal switching problems, one can restrict to controls $\alpha$ for which the moments of $N(\alpha)$ are bounded by a constant depending on $x$.

Before proving Proposition 4.2.1, we need the following Lemmata.

**Lemma 4.2.1.** *For all* $p \geq 1$, *there exists a positive constant* $K_p$ *such that*

$$\sup_{\alpha \in \mathcal{A}_{t,i}} \Big\| \sup_{s \in [t,T]} |X_s^{t,x,\alpha}| \Big\|_p \quad \leq \quad K_p(1+|x|) \ ,$$

*for all* $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$.

**Proof.** Fix $p \geq 1$. Then, we have from the definition of $X_s^{t,x,\alpha}$ in(4.2.3), for $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}_{t,i}$:

$$\begin{aligned} \mathbb{E}\Big[\sup_{s\in[t,r]} |X_s^{t,x,\alpha}|^p\Big] \quad &\leq \quad K_p\Big(|x|^p + \mathbb{E}\Big[\sum_{\tau_n \leq r} \int_{\tau_n}^{\tau_{n+1}\wedge r} |b_{\iota_n}(X_u^{t,x,\alpha})|^p du\Big] \\ &\quad + \mathbb{E}\Big[\sup_{s\in[t,r]} \Big|\sum_{\tau_n \leq s} \int_{\tau_n}^{\tau_{n+1}\wedge s} \sigma_{\iota_n}(X_u^{t,x,\alpha}) dW_u\Big|^p\Big]\Big) \ , \end{aligned}$$

for all $r \in [t,T]$. From the linear growth conditions on $b_i$ and $\sigma_i$, for $i \in \mathbb{I}_q$, and Burkholder-Davis-Gundy's (BDG) inequality, we then get by Hölder inequality when $p \geq 2$:

$$\mathbb{E}\Big[\sup_{s\in[t,r]} |X_s^{t,x,\alpha}|^p\Big] \quad \leq \quad K_p\Big(1 + |x|^p + \int_t^r \mathbb{E}\Big[\sup_{s\in[t,u]} |X_s^{t,x,\alpha}|^p du\Big]\Big) \ ,$$

for all $r \in [t,T]$. By applying Gronwall's Lemma, we obtain the required estimate for $p \geq 2$ , and then also for $p \geq 1$ by Hölder inequality. □

**Lemma 4.2.2.** *Under* **(Hl)** *and* **(Hc)**, *the functions* $v_i$, $i \in \mathbb{I}_q$, *satisfy a linear growth condition, i.e. there exists a constant* $K$ *such that*

$$|v_i(t,x)| \quad \leq \quad K(1+|x|) \ ,$$

*for all* $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$.

**Proof.** Under the linear growth condition on $f_i$, $g_i$ in **(Hl)**, and the nonnegativity of the switching costs in **(Hc)**, there exists some positive constant $K$ s.t.

$$\begin{aligned} & \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big] \\ \leq \quad & K\Big(1 + \mathbb{E}\Big[\sup_{u\in[0,T]} |X_u^{t,x,\alpha}|\Big]\Big), \end{aligned}$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}_{t,i}$. By combining with the estimate in Lemma 4.2.1, this shows that

$$v_i(t,x) \quad \leq \quad K(1+|x|) \ .$$

Moreover, by considering the strategy $\alpha^0$ with no intervention i.e. $N(\alpha^0) = 0$, we have

$$\begin{aligned} v_i(t,x) \quad & \geq \quad \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha^0}, i)ds + g(X_T^{t,x,\alpha^0}, i)\Big] \\ & \geq \quad -K\Big(1 + \mathbb{E}\Big[\sup_{u\in[0,T]} |X_u^{t,x,\alpha}|\Big]\Big). \end{aligned}$$

Again, by the estimate in Lemma 4.2.1, this proves that

$$v_i(t,x) \quad \geq \quad -K(1+|x|) \ ,$$

and therefore the required linear growth condition on $v_i$. □

We now turn to the proof of the Proposition.

**Proof of Proposition 4.2.1.** Fix $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$. Denote by $\alpha^* = (\tau_n^*, \zeta_n^*)_{n\geq 0}$ an optimal strategy associated to $v_i(t,x)$:

$$v_i(t,x) \quad = \quad \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha^*}, I_s^*)ds + g(X_T^{t,x,\alpha^*}, I_T^*) - \sum_{n=1}^{N(\alpha^*)} c(X_{\tau_n}^{t,x,\alpha^*}, \iota_{n-1}^*, \iota_n^*)\Big]. \quad (4.2.7)$$

where $I^*$ is the indicator regime associated to $\alpha^*$. Consider the process $(Y^{t,x,\alpha^*}, Z^{t,x,\alpha^*})$ solution to the following Backward Stochastic Differential Equation (BSDE)

$$\begin{aligned} Y_u^{t,x,\alpha^*} &= g(X_T^{t,x,\alpha^*}, I_T^*) + \int_u^T f(X_s^{t,x,\alpha^*}, I_s^*) ds \\ &\quad - \int_u^T Z_s^{t,x,\alpha^*} dW_s - C_T^{t,x,\alpha^*} + C_u^{t,x,\alpha^*}, \qquad t \le u \le T \end{aligned} \tag{4.2.8}$$

and satisfying the condition

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|^2\Big] + \mathbb{E}\Big[\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds\Big] \;<\; \infty.$$

Such a solution exists under **(Hl)**, Lemma 4.2.1 and $\mathbb{E}\big[|C_T^{t,x,\alpha^*}|^2\big] < \infty$. Moreover, by taking expectation in (4.2.8) and from the dynamic programming principle for the value function in (4.2.7), we have

$$Y_u^{t,x,\alpha^*} \;=\; v_{I_u^*}(u, X_u^{t,x,\alpha^*}), \quad t \le u \le T.$$

From Lemma 4.2.1 and 4.2.2, there exists for each $p \ge 1$ a constant $K_p$ such that

$$\mathbb{E}\Big[\sup_{u\in[t,T]} |Y_u^{t,x,\alpha^*}|^p\Big] \;\le\; K_p(1 + |x|^p). \tag{4.2.9}$$

We now prove that there exists a sequence $\bar{K} = (\bar{K}_p)_p$ which does not depend on $(t, x, i)$ such that

$$\mathbb{E}\big[|C_T^{t,x,\alpha^*}|^p\big] \;\le\; \bar{K}_p(1 + |x|^p). \tag{4.2.10}$$

Applying Itô's formula to $|Y^{t,x,\alpha^*}|^2$ in (4.2.8), we have

$$\begin{aligned} |Y_t^{t,x,\alpha^*}|^2 + \int_t^T |Z_s^{t,x,\alpha^*}|^2 ds &= |g(X_T^{t,x,\alpha^*}, I_T^*)|^2 + 2\int_t^T Y_s^{t,x,\alpha^*} f(X_s^{t,x,\alpha^*}, I_s^*) ds \\ &\quad - 2\int_t^T Y_s^{t,x,\alpha^*} Z_s^{t,x,\alpha^*} dW_s - 2\int_t^T Y_s^{t,x,\alpha^*} dC_s^{t,x,\alpha^*}. \end{aligned}$$

Using **(Hl)** and the inequality $2ab \le a^2 + b^2$ for $a, b \in \mathbb{R}$, we get

$$\begin{aligned} \int_t^T |Z_s^{t,x,\alpha^*}|^2 ds &\le K\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha^*}|^2 + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|^2 + |C_T^{t,x,\alpha^*} - C_t^{t,x,\alpha^*}| \sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|\Big) \\ &\quad -2\int_t^T Y_s^{t,x,\alpha^*} Z_s^{t,x,\alpha^*} dW_s. \end{aligned} \tag{4.2.11}$$

Moreover, from (4.2.8), we have

$$|C_T^{t,x,\alpha^*} - C_t^{t,x,\alpha^*}|^2 \leq K\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha^*}|^2 + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|^2 + \Big|\int_t^T Z_s^{t,x,\alpha^*} dW_s\Big|^2\Big) \tag{4.2.12}$$

Combining (4.2.11) and (4.2.12) and using the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, for $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we obtain

$$\begin{aligned}\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds \leq\ & K\Big((1+\varepsilon)\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha^*}|^2\Big) + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|^2 (\varepsilon + \frac{1}{\varepsilon}) \\ & + \varepsilon\Big|\int_t^T Z_s^{t,x,\alpha^*} dW_s\Big|^2\Big) - 2\int_t^T Y_s^{t,x,\alpha^*} Z_s^{t,x,\alpha^*} dW_s .\end{aligned}$$

Elevating the previous estimate to the power $p/2$ and taking expectation, it follows from BDG inequality, Lemma 4.2.1 and (4.2.9) that

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds\Big)^{\frac{p}{2}}\Big] \leq\ & K_p\Big((1+\varepsilon^{\frac{p}{2}})\Big(1 + \mathbb{E}\sup_{s\in[t,T]} |X_s^{t,x,\alpha^*}|^p\Big) + (\varepsilon^{\frac{p}{2}} + \frac{1}{\varepsilon^{\frac{p}{2}}})\mathbb{E}\sup_{s\in[t,T]} |Y_s^{t,x,\alpha^*}|^p \\ & + \varepsilon^{\frac{p}{2}}\mathbb{E}\Big|\int_t^T Z_s^{t,x,\alpha^*} dW_s\Big|^p + \mathbb{E}\Big|\int_t^T Y_s^{t,x,\alpha^*} Z_s^{t,x,\alpha^*} dW_s\Big|^{\frac{p}{2}}\Big) \\ \leq\ & K_p\Big((1+|x|^p)(1+\varepsilon^{\frac{p}{2}} + \frac{1}{\varepsilon^{\frac{p}{2}}}) + \varepsilon^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds\Big)^{\frac{p}{2}}\Big] \\ & + \mathbb{E}\Big[\Big(\int_t^T |Y_s^{t,x,\alpha^*} Z_s^{t,x,\alpha^*}|^2 ds\Big)^{\frac{p}{4}}\Big]\Big) \qquad (4.2.13) \\ \leq\ & K_p\Big((1+|x|^p)(1+\varepsilon^{\frac{p}{2}} + \frac{1}{\varepsilon^{\frac{p}{2}}}) + \varepsilon^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds\Big)^{\frac{p}{2}}\Big]\Big) ,\end{aligned}$$

where we used again the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ for the last term in the r.h.s of (4.2.13). Taking $\varepsilon$ small enough, this yields

$$\mathbb{E}\Big[\Big(\int_t^T |Z_s^{t,x,\alpha^*}|^2 ds\Big)^{\frac{p}{2}}\Big] \leq K_p(1+|x|^p) ,$$

Elevating now inequality (4.2.12) to the power $p/2$, and using the previous inequality together with BDG inequality, we get with the estimate of Lemma 4.2.1 and (4.2.9):

$$\mathbb{E}|C_T^{t,x,\alpha^*} - C_t^{t,x,\alpha^*}|^p \leq \bar{K}_p(1+|x|^p), \tag{4.2.14}$$

for some positive constant $\bar{K}_p$. Since $\alpha^*$ is optimal, and from the triangular condition in **(Hc)**, we know that at the initial time $t$, there is at most one decision time $\tau_1^*$. Thus, from the linear growth condition on the switching cost, $\mathbb{E}[|C_t^{t,x,\alpha^*}|^p] \le \bar{K}_p(1+|x|^p)$, which implies with (4.2.14) that $\alpha^* \in \mathcal{A}_{t,i}^{\bar{K}}$, and proves the required result. □

In the sequel of this paper, we shall assume that **(Hl)** and **(Hc)** stand in force.

## 4.3 Time discretization

We first consider a time discretization of $[0,T]$ with time step $h = T/m \le 1$, and partition $\mathbb{T}_h = \{t_k = kh, k = 0,\ldots,m\}$. For $(t_k,i) \in \mathbb{T}_h \times \mathbb{I}_q$, we denote by $\mathcal{A}_{t_k,i}^h$ the set of admissible switching controls $\alpha = (\tau_n,\iota_n)_n$ in $\mathcal{A}_{t_k,i}$, such that $\tau_n$ are valued in $\{\ell h, \ell = k,\ldots,m\}$, and we consider the value functions for the discretized optimal switching problem:

$$v_i^h(t_k,x) = \sup_{\alpha\in\mathcal{A}_{t_k,i}^h} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})h + g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n)\Big], \tag{4.3.1}$$

for $(t_k,i,x) \in \mathbb{T}_h \times \mathbb{I}_q \times \mathbb{R}^d$.

The next result provides an error analysis between the continuous-time optimal switching problem and its discrete-time version.

**Theorem 4.3.1.** *For any $\varepsilon > 0$, there exists a positive constant $K_\varepsilon$ (not depending on $h$) such that*

$$|v_i(t_k,x) - v_i^h(t_k,x)| \le K_\varepsilon(1+|x|)h^{\frac{1}{2}-\varepsilon},$$

*for all $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$. Moreover if the cost functions $c_{ij}$, $i,i \in \mathbb{I}_q$, do not depend on $x$, then the previous inequality also holds for $\varepsilon = 0$.*

**Remark 4.3.1.** For optimal stopping problems, it is known that the approximation by the discrete-time version gives an error of order $h^{\frac{1}{2}}$, see e.g. [45] and [4]. We recover this rate of convergence for multiple switching problems when the switching costs do not depend on the state process. However, in the general case, the error is of order $h^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$. Such feature was showed in [13] in the case of uncontrolled state process $X$, and is extended here when $X$ may be influenced through its drift and diffusion coefficient by the switching control.

Before proving this Theorem, we need the two following lemmata. The first one deals with the regularity in time of the controlled diffusion uniformly in the control, and the second one deals with the regularity of the controlled diffusion with respect to the control.

**Lemma 4.3.1.** *For any $p \geq 1$, there exists a constant $K_p$ such that*

$$\sup_{\alpha\in\mathcal{A}_{t_k,i}} \max_{k\leq\ell\leq m-1} \Big\| \sup_{s\in[t_\ell,t_{\ell+1}]} |X_s^{t_k,x,\alpha} - X_{t_\ell}^{t_k,x,\alpha}| \Big\|_p \;\leq\; K_p(1+|x|)h^{\frac{1}{2}} ,$$

*for all $x \in \mathbb{R}^d$, $i \in \mathbb{I}_q$, $k = 0, \ldots, n$.*

**Proof.** Fix $p \geq 1$. From the definition of $X^{t,x,\alpha}$ in (4.2.3), we have for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$ and $\alpha \in \mathcal{A}_{t_k,i}$,

$$\mathbb{E}\Big[\sup_{u\in[t_\ell,s]} |X_u^{t,x,\alpha} - X_{t_\ell}^{t,x,\alpha}|^p\Big] \;\leq\; K_p\Big(\mathbb{E}\Big[\Big(\int_{t_\ell}^s |b_{I_u}(X_u^{t,x,\alpha})|du\Big)^p\Big] + \mathbb{E}\Big[\sup_{u\in[t_\ell,s]} \Big|\int_{t_\ell}^u \sigma_{I_r}(X_r^{t,x,\alpha})dW_r\Big|^p\Big]\Big) ,$$

for all $s \in [t_\ell, t_{\ell+1}]$. From BDG and Jensen inequalities for $p \geq 2$, we then have

$$\mathbb{E}\Big[\sup_{u\in[t_\ell,s]} |X_u^{t,x,\alpha} - X_{t_\ell}^{t,x,\alpha}|^p\Big] \;\leq\; K_p h^{\frac{p}{2}-1}\Big(\mathbb{E}\Big[\int_{t_\ell}^s |b_{I_u}(X_u^{t,x,\alpha})|^p du\Big] + \mathbb{E}\Big[\int_{t_\ell}^s |\sigma_{I_u}(X_u^{t,x,\alpha})|^p du\Big]\Big) ,$$

From the linear growth conditions on $b_i$ and $\sigma_i$, for $i \in \mathbb{I}_q$, and Lemma 4.2.1, we conclude that the following inequality

$$\mathbb{E}\Big[\sup_{s\in[t_\ell,t_{\ell+1}]} |X_s^{t,x,\alpha} - X_{t_\ell}^{t,x,\alpha}|^p\Big] \;\leq\; K_p(1+|x|^p)h^{\frac{p}{2}},$$

holds for $p \geq 2$, and then also for $p \geq 1$ by Hölder inequality. □

For a strategy $\alpha = (\tau_n, \iota_n)_n \in \mathcal{A}_{t_k,i}$ we denote by $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_n$ the strategy of $\mathcal{A}^h_{t_k,i}$ defined by

$$\tilde{\tau}_n \;=\; \min\{t_\ell \in \mathbb{T}_h \;:\; t_\ell \geq \tau_n\} , \qquad \tilde{\iota}_n \;=\; \iota_n, \qquad n \in \mathbb{N}.$$

The strategy $\tilde{\alpha}$ can be seen as the approximation of the strategy $\alpha$ by an element of $\mathcal{A}^h_{t_k,i}$. We then have the following regularity result of the diffusion in the control $\alpha$.

**Lemma 4.3.2.** *There exists a constant $K$ such that*

$$\Big\| \sup_{s\in[t_k,T]} |X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}| \Big\|_2 \;\leq\; K\Big(\mathbb{E}[N(\alpha)^2]\Big)^{\frac{1}{4}}(1+|x|)h^{\frac{1}{2}} ,$$

*for all* $x \in \mathbb{R}^d$, $i \in \mathbb{I}_q$, $k = 0, \ldots, n$ *and* $\alpha \in \mathcal{A}_{t_k,i}$.

**Proof.** From the definition of $X^{t,x,\alpha}$ and $X^{t,x,\tilde{\alpha}}$, for $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{R}_d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}^K_{t_k,i}$, we have by BDG inequality:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{u\in[t_k,s]} |X_u^{t,x,\alpha} - X_u^{t,x,\tilde{\alpha}}|^2\Big] \;\; &\leq \;\; K\Big(\mathbb{E}\Big[\int_{t_k}^s \big|b(X_u^{t,x,\alpha}, I_u) - b(X_u^{t,x,\tilde{\alpha}}, \tilde{I}_u)\big|^2 du\Big] \\
&\quad + \mathbb{E}\Big[\int_{t_k}^s \big|\sigma(X_u^{t,x,\alpha}, I_u) - \sigma(X_u^{t,x,\tilde{\alpha}}, \tilde{I}_u)\big|^2 du\Big]\Big) \;,
\end{aligned}$$

for all $s \in [t_k, T]$. Then using Lipschitz property of $b_i$ and $\sigma_i$ for $i \in \mathbb{I}_q$ we get:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{u\in[t_k,s]} |X_s^{t,x,\alpha} - X_s^{t,x,\tilde{\alpha}}|^2\Big] \;\; &\leq \;\; K\Big(\mathbb{E}\Big[\int_{t_k}^s |X_u^{t,x,\alpha} - X_u^{t,x,\tilde{\alpha}}|^2 du\Big] \\
&\quad + \mathbb{E}\Big[\int_{t_k}^s \big|b(X_u^{t,x,\alpha}, I_u) - b(X_u^{t,x,\alpha}, \tilde{I}_u)\big|^2 du\Big] \\
&\quad + \mathbb{E}\Big[\int_{t_k}^s \big|\sigma(X_u^{t,x,\alpha}, I_u) - \sigma(X_u^{t,x,\alpha}, \tilde{I}_u)\big|^2 du\Big]\Big) \\
&\leq \;\; K\Big(\mathbb{E}\Big[\int_{t_k}^s \sup_{r\in[t_k,u]} |X_r^{t,x,\alpha} - X_r^{t,x,\tilde{\alpha}}|^2 du\Big] \qquad (4.3.2) \\
&\quad + \mathbb{E}\Big[\big(\sup_{u\in[t_k,T]} |X_u^{t,x,\alpha}|^2 + 1\big) \int_{t_k}^s \mathbf{1}_{I_s\neq\tilde{I}_s} ds\Big]\Big) \;,
\end{aligned}$$

for all $s \in [t_k, T]$. From the definition of $\tilde{\alpha}$ we have

$$\int_{t_k}^s \mathbf{1}_{I_s\neq\tilde{I}_s} ds \;\; \leq \;\; N(\alpha)h \;,$$

which gives with (4.3.2), Lemma 4.2.1, Remark 4.2.1 and Hölder inequality:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{u\in[t_k,s]} |X_u^{t,x,\alpha} - X_u^{t,x,\tilde{\alpha}}|^2\Big] \;\; &\leq \;\; K\Big(\mathbb{E}\Big[\int_{t_k}^s \sup_{r\in[t_k,u]} |X_r^{t,x,\alpha} - X_r^{t,x,\tilde{\alpha}}|^2 du\Big] \\
&\quad + \big(\mathbb{E}[N(\alpha)^2]\big)^{\frac{1}{2}}(1 + |x|^2)h\Big),
\end{aligned}$$

for all $s \in [t_k, T]$. We conclude with Gronwall's Lemma. □

We are now ready to prove the convergence result for the time discretization of the optimal switching problem.

**Proof of Theorem 4.3.1.** We introduce the auxiliary function $\tilde{v}_i^h$ defined by

$$\tilde{v}_i^h(t_k,x) = \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\alpha},I_s)ds + g(X_T^{t_k,x,\alpha},I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n)\Big],$$

for all $(t_k,x)\in\mathbb{T}_h\times\mathbb{R}^d$. We then write

$$|v_i(t_k,x)-v_i^h(t_k,x)| \leq |v_i(t_k,x)-\tilde{v}_i^h(t_k,x)| + |\tilde{v}_i^h(t_k,x)-v_i^h(t_k,x)|,$$

and study each of the two terms in the right-hand side.

• Let us investigate the first term. By definition of the approximating strategy $\tilde{\alpha}=(\tilde{\tau}_n,\tilde{\iota}_n)_n\in\mathcal{A}^h_{t_k,i}$ of $\alpha\in\mathcal{A}_{t_k,i}$, we see that the auxiliary value function $\tilde{v}_i^h$ may be written as

$$\tilde{v}_i^h(t_k,x) = \sup_{\alpha\in\mathcal{A}_{t_k,i}} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\tilde{\alpha}},\tilde{I}_s)ds + g(X_T^{t_k,x,\tilde{\alpha}},\tilde{I}_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tilde{\tau}_n}^{t_k,x,\tilde{\alpha}},\tilde{\iota}_{n-1},\tilde{\iota}_n)\Big],$$

where $\tilde{I}$ is the indicator of the regime value associated to $\tilde{\alpha}$. Fix now a positive sequence $\bar{K}=(\bar{K}_p)_p$ s.t. relation (4.2.6) in Proposition 4.2.1 holds, and observe that

$$\begin{aligned}
&\sup_{\alpha\in\mathcal{A}^{\bar{K}}_{t_k,i}(x)} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\tilde{\alpha}},\tilde{I}_s)ds + g(X_T^{t_k,x,\tilde{\alpha}},\tilde{I}_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tilde{\tau}_n}^{t_k,x,\tilde{\alpha}},\tilde{\iota}_{n-1},\tilde{\iota}_n)\Big]\\
\leq\ & \tilde{v}_i^h(t_k,x) \ \leq\ v_i(t_k,x)\\
=\ &\sup_{\alpha\in\mathcal{A}^{\bar{K}}_{t_k,i}(x)} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\alpha},I_s)ds + g(X_T^{t_k,x,\alpha},I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n)\Big].
\end{aligned}$$

We then have

$$|v_i(t_k,x)-\tilde{v}_i^h(t_k,x)| \leq \sup_{\alpha\in\mathcal{A}^{\bar{K}}_{t_k,i}(x)} \Big[\Delta^1_{t_k,x}(\alpha)+\Delta^2_{t_k,x}(\alpha)\Big], \tag{4.3.3}$$

with

$$\begin{aligned}
\Delta^1_{t_k,x}(\alpha) &= \mathbb{E}\Big[\int_{t_k}^T \big|f(X_s^{t_k,x,\alpha},I_s)-f(X_s^{t_k,x,\tilde{\alpha}},\tilde{I}_s)\big|ds + \big|g(X_T^{t_k,x,\alpha},I_T)-g(X_T^{t,x,\tilde{\alpha}},\tilde{I}_T)\big|\Big],\\
\Delta^2_{t_k,x}(\alpha) &= \mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n)-c(X_{\tilde{\tau}_n}^{t_k,x,\tilde{\alpha}},\tilde{\iota}_{n-1},\tilde{\iota}_n)\big|\Big].
\end{aligned}$$

Under **(Hl)**, and by definition of $\tilde{\alpha}$, there exists some positive constant $K$ s.t.

$$\begin{aligned}
\Delta^1_{t_k,x}(\alpha) &\leq K\Big( \sup_{s\in[t_k,T]} \mathbb{E}\Big[\big|X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}\big|\Big] + \mathbb{E}\Big[\big( \sup_{s\in[t_k,T]} |X_s^{t_k,x,\alpha}| + 1\big) \int_{t_k}^T \mathbf{1}_{I_s\neq \tilde{I}_s} ds\Big]\Big) . \\
&\leq K\Big( \sup_{s\in[t_k,T]} \mathbb{E}\Big[\big|X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}\big|\Big] \qquad (4.3.4)\\
&\quad + \Big(1 + \Big\| \sup_{s\in[t_k,T]} |X_s^{t_k,x,\alpha}|\Big\|_2\Big)\Big(\mathbb{E}\Big[\int_{t_k}^T \mathbf{1}_{I_s\neq\tilde{I}_s} ds\Big]\Big)^{\frac{1}{2}}\Big),
\end{aligned}$$

by Cauchy-Schwarz inequality. For $\alpha \in \mathcal{A}^{\bar{K}}_{t_k,i}(x)$, we have by Remark 4.2.1

$$\mathbb{E}\Big[\int_{t_k}^T \mathbf{1}_{I_s\neq\tilde{I}_s} ds\Big] \;\leq\; h\mathbb{E}\Big[N(\alpha)\Big] \;\leq\; \eta\bar{K}_1(1+|x|)h,$$

for some positive constant $\eta > 0$. By using this last estimate together with Lemmata 4.2.1 and 4.3.2 into (4.3.4), we obtain the existence of some constant $K$ s.t.

$$\sup_{\alpha\in\mathcal{A}^{\bar{K}}_{t_k,i}(x)} \Delta^1_{t_k,x}(\alpha) \;\leq\; K(1+|x|)h^{\frac{1}{2}}, \tag{4.3.5}$$

for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$.

We now turn to the term $\Delta^2_{t,x}(\alpha)$. Under **(Hl)**, and by definition of $\tilde{\alpha}$, there exists some positive constant $K$ s.t.

$$\begin{aligned}
\Delta^2_{t_k,x}(\alpha) &\leq K\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} |X_{\tau_n}^{t_k,x,\alpha} - X_{\tilde{\tau}_n}^{t_k,x,\tilde{\alpha}}|\Big] \\
&\leq K\Big(\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} |X_{\tau_n}^{t_k,x,\alpha} - X_{\tilde{\tau}_n}^{t_k,x,\alpha}|\Big] + \mathbb{E}\Big[N(\alpha) \sup_{s\in[t_k,T]} \big|X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}\big|\Big]\Big) \\
&\leq K\Big(\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} |X_{\tau_n}^{t_k,x,\alpha} - X_{\tilde{\tau}_n}^{t_k,x,\alpha}|\Big] \\
&\quad + \Big\|N(\alpha)\Big\|_2 \Big\| \sup_{s\in[t_k,T]} |X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}|\Big\|_2\Big), \qquad (4.3.6)
\end{aligned}$$

by Cauchy-Schwarz inequality. For $\alpha \in \mathcal{A}^{\bar{K}}_{t_k,i}(x)$ with Remark 4.2.1, and from Lemma 4.3.2, we get the existence of some positive constant $K$ s.t.

$$\Big\|N(\alpha)\Big\|_2 \Big\| \sup_{s\in[t_k,T]} |X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde{\alpha}}|\Big\|_2 \;\leq\; K(1+|x|)h^{\frac{1}{2}} . \tag{4.3.7}$$

On the other hand, for any $\varepsilon \in (0, 1]$, we have from Hölder inequality applied to expectation and Jensen's inequality applied to the summation:

$$
\begin{aligned}
\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)}|X_{\tau_n}^{t_k,x,\alpha}-X_{\tilde{\tau}_n}^{t_k,x,\alpha}|\Big] &\leq \Big(\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)}|X_{\tau_n}^{t_k,x,\alpha}-X_{\tilde{\tau}_n}^{t_k,x,\alpha}|\Big]^{\frac{1}{\varepsilon}}\Big)^{\varepsilon}\\
&\leq \Big(\mathbb{E}\Big[|N(\alpha)|^{\frac{1}{\varepsilon}-1}\sum_{n=1}^{N(\alpha)}|X_{\tau_n}^{t_k,x,\alpha}-X_{\tilde{\tau}_n}^{t_k,x,\alpha}|^{\frac{1}{\varepsilon}}\Big]\Big)^{\varepsilon}\\
&\leq 2\Big(\sum_{\ell=k}^{n-1}\mathbb{E}\Big[|N(\alpha)|^{\frac{1}{\varepsilon}}\sup_{s\in[t_\ell,t_{\ell+1}]}\big|X_s^{t,x,\alpha}-X_{t_\ell}^{t,x,\alpha}\big|^{\frac{1}{\varepsilon}}\Big]\Big)^{\varepsilon}\\
&\leq \frac{2}{h^{\varepsilon}}\big\|N(\alpha)\big\|_{\frac{2}{\varepsilon}}\max_{k\leq\ell\leq m-1}\Big\|\sup_{s\in[t_\ell,t_{\ell+1}]}\big|X_s^{t,x,\alpha}-X_{t_\ell}^{t,x,\alpha}\big|\Big\|_{\frac{2}{\varepsilon}}
\end{aligned}
$$

by Cauchy-Schwarz inequality. By Lemma 4.3.1, this yields the existence of some positive constant $K_\varepsilon$ s.t.

$$
\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)}|X_{\tau_n}^{t_k,x,\alpha}-X_{\tilde{\tau}_n}^{t_k,x,\alpha}|\Big] \leq K_\varepsilon(1+|x|)h^{\frac{1}{2}-\varepsilon}. \tag{4.3.8}
$$

By plugging (4.3.7) and (4.3.8) into (4.3.6), we then get

$$
\Delta_{t,x}^2(\alpha) \leq K_\varepsilon(1+|x|)h^{\frac{1}{2}-\varepsilon}\ . \tag{4.3.9}
$$

Combining (4.3.5) and (4.3.9), we obtain with (4.3.3)

$$
|v_i(t_k,x)-\tilde{v}_i^h(t_k,x)| \leq K_\varepsilon(1+|x|)h^{\frac{1}{2}-\varepsilon}\ .
$$

In the case where $c$ does not depend on the variable $x$, we have $\Delta_{t,x}^2(\alpha)=0$, and so by (4.3.3), (4.3.5):

$$
|v_i(t_k,x)-\tilde{v}_i^h(t_k,x)| \leq K(1+|x|)h^{\frac{1}{2}}\ .
$$

• For the second term, we have by definition of $v_i^h$ and $\tilde{v}_i^h$:

$$
|\tilde{v}_i^h(t_k,x)-v_i^h(t_k,x)| \leq \sup_{\alpha\in\mathcal{A}_{t_k,i}^h}\mathbb{E}\Big[\sum_{\ell=k}^{m-1}\int_{t_\ell}^{t_{\ell+1}}|f(X_s^{t,x,\alpha},I_s)-f(X_{t_\ell}^{t,x,\alpha},I_s)|ds\Big],
$$

since $I_s = I_{t_\ell}$ on $[t_\ell, t_{\ell+1})$. Under **(Hl)**, we get

$$|\tilde{v}_i^h(t_k,x) - v_i^h(t_k,x)| \;\; \leq \;\; K \sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \max_{k \leq \ell \leq m-1} \sup_{s \in [t_\ell, t_{\ell+1}]} \mathbb{E}\Big[\big|X_s^{t,x,\alpha} - X_{t_\ell}^{t,x,\alpha}\big|\Big],$$

for some positive constant $K$, and by Lemma 4.3.1, this shows that

$$|\tilde{v}_i^h(t_k,x) - v_i^h(t_k,x)| \;\; \leq \;\; K(1+|x|)h^{\frac{1}{2}} \; .$$

□

In a second step, we approximate the continuous-time (controlled) diffusion by a discrete-time (controlled) Markov chain following an Euler type scheme. For any $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}^h_{t_k,i}$, we introduce $(\bar{X}_{t_\ell}^{h,t_k,x,\alpha})_{k \leq \ell \leq m}$ defined by:

$$\bar{X}_{t_k}^{h,t_k,x,\alpha} \;=\; x, \qquad \bar{X}_{t_{\ell+1}}^{h,t_k,x,\alpha} \;=\; F^h_{I_{t_\ell}}(\bar{X}_{t_\ell}^{h,t_k,x,\alpha}, \vartheta_{\ell+1}), \quad k \leq \ell \leq m-1,$$

where

$$F_i^h(x, \vartheta_{k+1}) \;=\; x + b_i(x)h + \sigma_i(x)\sqrt{h}\,\vartheta_{k+1},$$

and $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$, $k = 0, \ldots, m-1$, are iid, $\mathcal{N}(0, I_d)$-distributed, independent of $\mathcal{F}_{t_k}$. Similarly as in Lemma 4.2.1, we have the $L^p$-estimate:

$$\sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \Big\| \max_{\ell = k,\ldots,m} |\bar{X}_{t_\ell}^{h,t_k,x,\alpha}| \Big\|_p \;\; \leq \;\; K_p(1+|x|), \tag{4.3.10}$$

for some positive constant $K_p$, not depending on $(h, t_k, x, i)$. Moreover, one can also derive the standard estimate for the Euler scheme, as e.g. in section 10.2 of [43]:

$$\sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \Big\| \max_{\ell = k,\ldots,m} |X_{t_\ell}^{t_k,x,\alpha} - \bar{X}_{t_\ell}^{h,t_k,x,\alpha}| \Big\|_p \;\; \leq \;\; K_p(1+|x|)\sqrt{h}. \tag{4.3.11}$$

We then associate to the Euler controlled Markov chain, the value functions $\bar{v}_i^h$, $i \in \mathbb{I}_q$, for the optimal switching problem:

$$\bar{v}_i^h(t_k,x) = \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\bar{X}^{h,t_k,x,\alpha}_{t_\ell}, I_{t_\ell})h + g(\bar{X}^{h,t_k,x,\alpha}_{t_m}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\bar{X}^{h,t_k,x,\alpha}_{\tau_n}, \iota_{n-1}, \iota_n)\Big]. \tag{4.3.12}$$

The next result provides the error analysis between $v_i^h$ by $\bar{v}_i^h$, and thus of the continuous time optimal switching problem $v_i$ by its Euler discrete-time approximation $\bar{v}_i^h$.

**Theorem 4.3.2.** *There exists a constant $K$ (not depending on $h$) such that*

$$\big|v_i^h(t_k,x) - \bar{v}_i^h(t_k,x)\big| \leq K(1+|x|)\sqrt{h}, \tag{4.3.13}$$

*for all $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}^d\times\mathbb{I}_q$.*

**Remark 4.3.2.** The above theorem combined with Theorem 4.3.1 gives the rate of convergence for the approximation of the continuous time optimal switching problem by its Euler discrete-time version: For any $\varepsilon>0$, there exists a positive constant $K_\varepsilon$ s.t.

$$|v_i(t_k,x) - \bar{v}_i^h(t_k,x)| \leq K_\varepsilon(1+|x|)h^{\frac{1}{2}-\varepsilon}, \tag{4.3.14}$$

for all $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}^d\times\mathbb{I}_q$. Moreover if the cost functions $c_{ij}$, $i,i\in\mathbb{I}_q$, do not depend on $x$, then the previous inequality also holds for $\varepsilon=0$.

**Proof of Theorem 4.3.2.**

• *Step 1.* For $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}^d\times\mathbb{I}_q$, denote by $\alpha^{h,*}$ (resp. $\bar{\alpha}^{h,*}$) the optimal switching strategy corresponding to $v_i^h(t_k,x)$ (resp. $\bar{v}_i^h(t_k,x)$). Let us prove that there exists some constant $K$, not depending on $(t_k,x,i,h)$, such that

$$\mathbb{E}\big|N(\alpha^{h,*})\big|^2 + \mathbb{E}\big|N(\bar{\alpha}^{h,*})\big|^2 \leq K(1+|x|^2). \tag{4.3.15}$$

We use discrete-time arguments, which are analog to the continuous-time case in the proof of Proposition 4.2.1. For $\alpha^{h,*}$ optimal strategy to $v_i^h(t_k,x)$ with corresponding indicator regime $I^{h,*}$ , and to alleviate notations, we denote by $Y_\ell = v^h_{I^{h,*}_{t_\ell}}(t_k, X^{t_k,x,\alpha^{h,*}}_{t_\ell})$, $F_\ell = f(X^{t_k,x,\alpha^{h,*}}_{t_\ell}, I^{h,*}_{t_\ell})$, $c_\ell = c(X^{t_k,x,\alpha^{h,*}}_{t_\ell}, I^{h,*}_{t_{\ell-1}}, I^{h,*}_{t_\ell})$, for $\ell=k,\ldots,m$. From the estimates on $X^{t_k,x,\alpha}_{t_\ell}$ in Lemma 4.2.1, we know that

$$\mathbb{E}\Big[\sup_{k\leq\ell\leq m}\big(|Y_\ell|^2 + |F_\ell|^2 + |c_\ell|^2\big)\Big] \leq K(1+|x|^2), \tag{4.3.16}$$

for some positive constant $K$. Moreover, by the DPP for the value function $v_i^h$, we have :

$$Y_\ell \;=\; \mathbb{E}\left[Y_{\ell+1}|\mathcal{F}_{t_\ell}\right] + hF_\ell - c_\ell, \quad \ell = k, \ldots, m-1.$$

Letting $\Delta M_{\ell+1} := Y_{\ell+1} - \mathbb{E}[Y_{\ell+1}|\mathcal{F}_{t_\ell}]$, we obtain in particular

$$\sum_{\ell=k}^{m-1} c_\ell \;=\; h\sum_{\ell=k}^{m-1} F_\ell - \sum_{\ell=k}^{m-1} \Delta M_{\ell+1} + (Y_m - Y_k),$$

and so by (4.3.16)

$$\begin{aligned}
\mathbb{E}\Big|\sum_{\ell=k}^{m} c_\ell\Big|^2 \;&\leq\; K(1+|x|^2) + 3\,\mathbb{E}\left[\left(\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}\right)^2\right] \\
&=\; K(1+|x|^2) + 3\,\mathbb{E}\left[\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}^2\right]. \qquad (4.3.17)
\end{aligned}$$

Now by writing that

$$\begin{aligned}
Y_m^2 - Y_0^2 \;=\; \sum_{\ell=k}^{m-1}\left(Y_{\ell+1}^2 - Y_\ell^2\right) \;&=\; \sum_{\ell=k}^{m-1}(Y_{\ell+1}-Y_\ell)(Y_{\ell+1}+Y_\ell) \\
&=\; \sum_{\ell=k}^{m-1}(\Delta M_{\ell+1} - hF_\ell + c_\ell)(2Y_\ell + \Delta M_{\ell+1} - hF_\ell + c_\ell),
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}^2 \;=\; & Y_m^2 - Y_0^2 - \sum_{\ell=0}^{m-1} hF_\ell(hF_\ell - 2Y_\ell - 2c_l) - 2\sum_{\ell=0}^{m-1} c_l Y_l \\
& - \sum_{\ell=0}^{m-1}\Delta M_{\ell+1}(2Y_\ell - 2hF_\ell + 2c_\ell) - \sum_{\ell=0}^{m-1} c_\ell^2.
\end{aligned}$$

Since $\mathbb{E}\Big[\Delta M_{\ell+1}|\mathcal{F}_{t_\ell}\Big] = 0$, this shows that

$$\begin{aligned}
\mathbb{E}\Big[\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}^2\Big] \;&\leq\; \mathbb{E}\Big[Y_m^2 - \sum_{\ell=0}^{m-1} hF_\ell(hF_\ell - 2Y_\ell - 2c_\ell) - 2\sum_{\ell=0}^{m-1} c_\ell Y_\ell\Big] \\
&\leq\; K(1+|x|^2) + 2\mathbb{E}\Big[\Big|\sum_{\ell=0}^{m-1} c_\ell Y_\ell\Big|\Big], \qquad (4.3.18)
\end{aligned}$$

where we used again (4.3.16). Now since $c_\ell \geq 0$,

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{\ell=0}^{m-1} c_\ell Y_\ell\Big|\Big] &\leq \mathbb{E}\Big[\Big(\sum_{\ell=0}^{m-1} c_\ell\Big)\sup_{k\leq\ell\leq m-1}|Y_\ell|\Big] \\ &\leq \varepsilon\mathbb{E}\Big[\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}^2\Big] + K(1+\frac{1}{\varepsilon})(1+|x|^2),\end{aligned}$$

for all $\varepsilon > 0$, by (4.3.16), (4.3.17) and Cauchy-Schwarz inequality. Hence taking $\varepsilon$ small enough and plugging this estimate into (4.3.18), we obtain

$$\mathbb{E}\Big[\sum_{\ell=k}^{m-1}\Delta M_{\ell+1}^2\Big] \leq K(1+|x|^2).$$

Using (4.3.17) one more time and recalling that $N(\alpha^{h,*}) \leq \eta\sum_\ell c_\ell$ for some $\eta > 0$ under the uniformly lower bound condition in **(Hc)**, we thus obtain

$$\mathbb{E}\big|N(\alpha^{h,*})\big|^2 \leq K(1+|x|^2).$$

The proof for $N(\bar{\alpha}^{h,*})$ is the same, by using estimate (4.3.10) on $\|\bar{X}_{t_\ell}^{h,t_k,x,\alpha}\|_2$.

• *Step 2.* By Step 1, the supremum in the definitions (4.3.1) and (4.3.12) of $v_i^h(t_k,x)$ and $\bar{v}_i^h(t_k,x)$ can be taken over $\mathcal{A}_{t_k,i}^{h,K}(x) = \{\alpha \in \mathcal{A}_{t_k,i}^h \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K(1+|x|^2)\}$. Now, for any $\alpha \in \mathcal{A}_{t_k,i}^{h,K}(x)$, we have under **(Hl)** and by Cauchy-Schwarz inequality

$$\begin{aligned}&\mathbb{E}\Big[\sum_{\ell=k}^{m-1} h|f(X_{t_\ell}^{t_k,x,\alpha},I_{t_\ell}) - f(\bar{X}_{t_\ell}^{h,t_k,x,\alpha},I_{t_\ell})| + |g(X_{t_m}^{t_k,x,\alpha},I_{t_m}) - g(\bar{X}_{t_m}^{h,t_k,x,\alpha},I_{t_m})| \\ &\quad + \sum_{n=1}^{N(\alpha)}|c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n) - c(\bar{X}_{\tau_n}^{h,t_k,x,\alpha},\iota_{n-1},\iota_n)|\Big] \\ &\leq K\mathbb{E}\Big[(1+N(\alpha))\big(\sup_{k\leq\ell\leq m}|X_{t_\ell}^{t_k,x,\alpha} - \bar{X}_{t_\ell}^{h,t_k,x,\alpha}|\big)\Big] \\ &\leq K(1+|x|)\Big\|\sup_{k\leq\ell\leq m}|X_{t_\ell}^{t_k,x,\alpha} - \bar{X}_{t_\ell}^{h,t_k,x,\alpha}|\Big\|_2 \\ &\leq K(1+|x|^2)\sqrt{h}, \end{aligned} \tag{4.3.19}$$

by (4.3.11). Taking the supremum over $\alpha \in \mathcal{A}_{t_k,i}^{h,K}(x)$ into (4.3.19), this shows that

$$\big|v_i^h(t_k,x) - \bar{v}_i^h(t_k,x)\big| \leq K(1+|x|^2)\sqrt{h}.$$

□

## 4.4 Approximation schemes by optimal quantization

In this section, for a fixed time discretization step $h$, we focus on a computational approximation for the value functions $\bar{v}_i^h$, $i \in \mathbb{I}_q$, defined in (4.3.12). To alleviate notations, we shall often omit the dependence on $h$ in the superscripts, and write e.g. $\bar{v}_i = \bar{v}_i^h$. The corresponding dynamic programming relation for $\bar{v}_i$ is written in the backward induction:

$$
\begin{aligned}
\bar{v}_i(t_m, x) &= g_i(x), \\
\bar{v}_i(t_k, x) &= \max\Big\{\mathbb{E}[\bar{v}_i(t_{k+1}, \bar{X}_{t_{k+1}}^{t_k,x,i})] + f_i(x)h \ , \ \max_{j \neq i}[\bar{v}_j(t_k, x) - c_{ij}(x)]\Big\},
\end{aligned}
$$

for $k = 0, \ldots, m-1$, $(i, x) \in \mathbb{I}_q \times \mathbb{R}^d$, where $\bar{X}^{t_k,x,i}$ is the solution to the Euler scheme:

$$
\bar{X}_{t_{k+1}}^{t_k,x,i} = F_i^h(x, \vartheta_{k+1}) := x + b_i(x)h + \sigma_i(x)\sqrt{h}\, \vartheta_{k+1}.
$$

Observe that under the triangular condition on the switching costs $c_{ij}$ in **(Hc)**, these backward relations can be written as an explicit discrete-time scheme. Indeed, if $\bar{v}_i(t_k, x) = \bar{v}_j(t_k, x) - c_{ij}(x)$ for some $j \neq i$, for $l \neq i, j$, we have

$$
\begin{aligned}
\bar{v}_j(t_k, x) - c_{ij}(x) &\geq \bar{v}_l(t_k, x) - c_{il}(x) \\
&> \bar{v}_l(t_k, x) - c_{ij}(x) - c_{jl}(x),
\end{aligned}
$$

so that $\bar{v}_j(t_k, x) > \bar{v}_l(t_k, x) - c_{jl}(x)$. By positivity of the switching costs, we also have

$$
\bar{v}_j(t_k, x) = \bar{v}_i(t_k, x) + c_{ij}(x) > \bar{v}_i(t_k, x) - c_{ji}(x).
$$

It follows that

$$
\bar{v}_j(t_k, x) = \mathbb{E}[\bar{v}_j(t_{k+1}, \bar{X}_{t_{k+1}}^{t_k,x,j})] + f_j(x)h,
$$

and (recalling that $c_{ii}(\cdot) = 0$), the backward induction may be rewritten as

$$
\begin{aligned}
\bar{v}_i(t_m, x) &= g_i(x) && (4.4.1) \\
\bar{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big\{\mathbb{E}[\bar{v}_j(t_{k+1}, \bar{X}_{t_{k+1}}^{t_k,x,j})] + f_j(x)h - c_{ij}(x)\Big\}, && (4.4.2)
\end{aligned}
$$

for $k = 0, \ldots, m-1$, $(i, x) \in \mathbb{I}_q \times \mathbb{R}^d$. Next, the practical implementation for this scheme requires a computational approximation of the expectations arising in the above dynamic programming formulae, and a space discretization for the state process $X$ valued in $\mathbb{R}^d$. We shall propose two numerical approximations schemes by optimal quantization methods, the second one in the particular case where the state process $X$ is not controlled by the switching control.

### 4.4.1 A Markovian quantization method

Let $\mathbb{X}$ be a bounded lattice grid on $\mathbb{R}^d$ with step $\delta/d$ and size $R$, namely $\mathbb{X} = (\delta/d)\mathbb{Z}^d \cap B(0, R) = \{x \in \mathbb{R}^d : x = (\delta/d)z$ for some $z \in \mathbb{Z}^d$, and $|x| \leq \mathbb{R}\}$. We then denote by $\mathrm{Proj}_{\mathbb{X}}$ the projection on the grid $\mathbb{X}$ according to the closest neighbour rule, which satisfies

$$|x - \mathrm{Proj}_{\mathbb{X}}(x)| \quad \leq \quad \max(|x| - R, 0) + \delta, \quad \forall x \in \mathbb{R}^d. \tag{4.4.3}$$

At each time step $t_k \in \mathbb{T}_h$, and point space-grid $x \in \mathbb{X}$, we have to compute in (4.4.2) expectations in the form $\mathbb{E}\Big[\varphi(\bar{X}^{t_k,x,i}_{t_{k+1}})\Big]$, for $\varphi(.) = \bar{v}^h_i(t_{k+1}, .)$, $i \in \mathbb{I}_q$. We shall then use an optimal quantization for the Gaussian random variable $\vartheta_{k+1}$, which consists in approximating the distribution of $\vartheta \rightsquigarrow \mathcal{N}(0, I_d)$ by the discrete law of a random variable $\hat{\vartheta}$ of support $N$ points $w_l$, $l = 1, \ldots, N$, in $\mathbb{R}^d$, and defined as the projection of $\vartheta$ on the grid $\{w_1, \ldots, w_N\}$ following the closest neighbor rule. The grid $\{w_1, \ldots, w_N\}$ is optimized in order to minimize the distorsion error, i.e. the quadratic $L^2$-norm $\|\vartheta - \hat{\vartheta}\|_2$. This optimal grid and the associated weights $\{\pi_1, \ldots, \pi_N\}$ are downloaded from the website: "http://www.quantize.maths-fi.com/downloads". We refer to the survey article [57] for more details on the theoretical and computational aspects of optimal quantization methods. In the vein of [58], we introduce the quantized Euler scheme:

$$\hat{X}^{t_k,x,i}_{t_{k+1}} \quad = \quad \mathrm{Proj}_{\mathbb{X}}(F^h_i(x, \hat{\vartheta})),$$

and define the value functions $\hat{v}_i$ on $\mathbb{T}_m \times \mathbb{X}$, $i \in \mathbb{I}_q$ in backward induction by

$$\begin{aligned}
\hat{v}_i(t_m, x) &= g_i(x) \\
\hat{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big\{ \mathbb{E}\big[\hat{v}_j(t_{k+1}, \hat{X}^{t_k,x,j}_{t_{k+1}})\big] + f_j(x)h - c_{ij}(x) \Big\}, \ k = 0, \ldots, m-1.
\end{aligned}$$

This numerical scheme can be computed explicitly according to the following recursive algorithm:

$$\begin{aligned}
\hat{v}_i(t_m, x) &= g_i(x), \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q \\
\hat{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big[ \sum_{l=1}^{N} \pi_l \, \hat{v}_j\big(t_{k+1}, \mathrm{Proj}_{\mathbb{X}}(F_j^h(x, w_l))\big) + f_j(x)h - c_{ij}(x) \Big], \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q,
\end{aligned}$$

for $k = 0, \ldots, m-1$. At each time step, we need to make $O(N)$ computations for each point of the grid $\mathbb{X}$. Therefore, the global complexity of the algorithm is of order $O(mN(R/\delta)^d)$.

The main result of this paragraph is to provide an error analysis and rate of convergence for the approximation of $\bar{v}_i$ by $\hat{v}_i$.

**Theorem 4.4.1.** *There exists a constant $K$ (not depending on $h$) such that*

$$\begin{aligned}
\big|\bar{v}_i(t_k, x) - \hat{v}_i(t_k, x)\big| \leq \; & K \exp\big(Kh^{-1} \|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1 + |x| + \frac{\delta}{h}\Big) \\
& \Big[\frac{\delta}{h} + h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \Big(1 + |x| + \frac{\delta}{h}\Big) \\
& + \frac{1}{Rh} \exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big],
\end{aligned}$$

*for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$. In the case where the switching costs $c_{ij}$ do not depend on $x$, the above estimation is stengthened into:*

$$\begin{aligned}
\big|\bar{v}_i(t_k, x) - \hat{v}_i(t_k, x)\big| \leq \; & K\Big[h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1 + |x| + \frac{\delta}{h}\Big) \\
& + \frac{\delta}{h} + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big].
\end{aligned}$$

**Remark 4.4.1.** The estimation in Theorem 4.4.1 consists of error terms related to

- the space discretization parameters $\delta$, $R$, which have to be chosen s.t. $\delta/h$ and $1/Rh$ go to zero.
- the quantization error $\|\vartheta - \hat{\vartheta}\|_p$ of the normal distribution $\mathcal{N}(0, I_d)$, which converges to zero at a rate $N^{\frac{1}{d}}$, where $N$ is the number of grid points chosen s.t. $h^{\frac{-1}{2}} N^{\frac{-1}{d}}$ goes to zero.

By combining with the discrete-time approximation error (4.3.14), and by choosing grid parameters $\delta$, $1/R$ of order $h^{\frac{3}{2}}$, and a number of points $N$ of order $1/h^d$, we see that the error estimate between the value function of the continuous-time optimal switching problem and its approximation by Markovian quantization is of order $h^{\frac{1}{2}}$. With these values of the parameters, we then see that the complexity of this Markovian quantization algorithm is of order $O(1/h^{4d+1})$.

Let us now focus on the proof of Theorem 4.4.1. First, notice from the dynamic programming principle that the value functions $\hat{v}_i$, $i \in \mathbb{I}_q$, admit the Markov control problem representation:

$$\hat{v}_i(t_k, x) = \sup_{\alpha \in \mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[ \sum_{\ell=k}^{m-1} f(\hat{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell})h + g(\hat{X}^{t_k,x,\alpha}_{t_m}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\hat{X}^{t_k,x,\alpha}_{\tau_n}, \iota_{n-1}, \iota_n)\Big], \quad (4.4.4)$$

where $\hat{X}^{t_k,x,\alpha}$ is defined by

$$\hat{X}^{t_k,x,\alpha}_{t_k} = x, \qquad \hat{X}^{t_k,x,\alpha}_{t_{\ell+1}} = \mathrm{Proj}_{\mathbb{X}}(F^h_{I_{t_\ell}}(\hat{X}^{t_k,x,\alpha}_{t_\ell}, \hat{\vartheta}_{\ell+1})), \quad k \leq \ell \leq m-1,$$

for $\alpha \in \mathcal{A}^h_{t_k,i}$, and $\hat{\vartheta}_{k+1}$, $k = 0, \ldots, m-1$, are iid, $\hat{\vartheta}$-distributed, and independent of $\mathcal{F}_{t_k}$. We first prove several estimates on $\hat{X}^{t_k,x,\alpha}$.

**Lemma 4.4.1.** *For each $p \geq 1$ there exists a constant $K_p$ (not depending on $h$) such that*

$$\sup_{\alpha \in \mathcal{A}^h_{t_k,i}, k \leq \ell \leq m} \left\| \hat{X}^{t_k,x,\alpha}_{t_\ell} \right\|_p + \sup_{\alpha \in \mathcal{A}^h_{t_k,i}, k \leq \ell \leq m-1} \left\| F^h_{I_{t_\ell}}(\hat{X}^{t_k,x,\alpha}_{t_\ell}, \hat{\vartheta}_{k+1}) \right\|_p \leq K_p \exp\left(K_p h^{-p/2} \|\vartheta - \hat{\vartheta}\|^p_p\right) \left(1 + |x| + \frac{\delta}{h}\right), \quad (4.4.5)$$

*for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$.*

**Proof.** We fix $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$, $\alpha \in \mathcal{A}^h_{t_k,i}$, and denote $\hat{X}_{t_\ell} = \hat{X}^{t_k,x,\alpha}_{t_\ell}$, $k \leq \ell \leq m$. Denoting by $\mathbb{E}_l$ the conditional expectation w.r.t. $\mathcal{F}_{t_\ell}$, by a standard use of Gronwall's lemma and linear growth of $b_i$, $\sigma_i$, we have

$$\mathbb{E}_\ell \left| F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \vartheta_{\ell+1}) \right|^p \leq e^{K_p h} \left| \hat{X}_{t_\ell} \right|^p + K_p h. \quad (4.4.6)$$

We will use the following convexity inequality : for $a$, $b \in \mathbb{R}_+$, $h \in [0, 1]$,

$$(a + hb)^p \leq (1 + K_p h)a^p + K_p h b^p. \quad (4.4.7)$$

By definition of $F^h$, and the fact that $|\mathrm{Proj}_{\mathbb{X}}(y)| \leq |y| + \delta$ for all $y \in \mathbb{R}^d$,

$$\begin{aligned} \left| \hat{X}_{t_{\ell+1}} \right| &\leq \left| F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \vartheta_{\ell+1}) \right| + h^{1/2} \sigma_{I_{t_\ell}}(\hat{X}_{t_\ell}) |\hat{\vartheta}_{\ell+1} - \vartheta_{\ell+1}| + \delta \\ &= \left| F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \vartheta_{\ell+1}) \right| + h \left( \frac{\sigma_{I_{t_\ell}}(\hat{X}_{t_\ell}) |\hat{\vartheta}_{\ell+1} - \vartheta_{\ell+1}|}{h^{1/2}} + \frac{\delta}{h} \right) \end{aligned}$$

Combining this last inequality with (4.4.6), (4.4.7), linear growth of $\sigma_i$ and the fact that $\hat{\vartheta}_{\ell+1}, \vartheta_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$, we obtain

$$\begin{aligned}\mathbb{E}_\ell\Big|\hat{X}_{t_{\ell+1}}\Big|^p &\leq (1+K_ph)(e^{K_ph}|\hat{X}_{t_\ell}|^p + K_ph) + K_ph\left(\frac{\sigma_{I_{t_\ell}}(\hat{X}_{t_\ell})\|\vartheta-\hat{\vartheta}\|_p^p}{h^{p/2}} + \frac{\delta^p}{h^p}\right)\\ &\leq \Big(1+K_ph+K_ph^{1-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)|\hat{X}_{t_\ell}|^p + K_ph\Big(1+\|\vartheta-\hat{\vartheta}\|_p^p h^{-p/2} + \frac{\delta^p}{h^p}\Big).\end{aligned}$$

By induction, taking the expectation, recalling that $h=\frac{T}{m}$, and since $(1+\frac{y}{m})^m \leq e^y$ for all $y \geq 0$, we obtain

$$\begin{aligned}\mathbb{E}\Big|\hat{X}_{t_{\ell+1}}\Big|^p &\leq K_p\exp\Big(K_ph^{-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)\Big(1+|x|^p+\frac{\delta^p}{h^p}+h^{-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)\\ &\leq K_p\exp\Big(K_p'h^{-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)\Big(1+|x|^p+\frac{\delta^p}{h^p}\Big),\end{aligned}$$

for all $k \leq \ell \leq m$. The estimate for $F^h(\hat{X}_{t_\ell}, \vartheta_{\ell+1})$ then follows from (4.4.6). □

**Lemma 4.4.2.** *There exists some constant $K$ (not depending on $h$) such that*

$$\begin{aligned}&\sup_{\alpha\in\mathcal{A}^h_{t_k,i}}\Big\|\sup_{k\leq\ell\leq m}|\hat{X}^{t_k,x,\alpha}_{t_\ell}-\bar{X}^{t_k,x,\alpha}_{t_\ell}|\Big\|_2\\ &\leq K\Big[h^{-1/2}\|\vartheta-\hat{\vartheta}\|_2\exp\big(Kh^{-1/2}\|\vartheta-\hat{\vartheta}\|_2\big)\Big(1+|x|+\frac{\delta}{h}\Big)\\ &\qquad+\frac{\delta}{h}+\frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta-\hat{\vartheta}\|_4^4\big)\Big(1+|x|^2+(\frac{\delta}{h})^2\Big)\Big],\end{aligned} \tag{4.4.8}$$

*for all $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$.*

**Proof.** As before we fix $(t_k,x,i)$, $\alpha$ and omit the dependence on $(t_k,x,i,\alpha)$ in $\hat{X}_{t_\ell}$. Let us first show an estimate on $\Big\|\hat{X}_{t_{\ell+1}}-\bar{X}_{t_{\ell+1}}\Big\|_2$. For $k \leq \ell \leq m-1$, we get

$$\begin{aligned}\Big\|\hat{X}_{t_{\ell+1}}-\bar{X}_{t_{\ell+1}}\Big\|_2 &\leq \Big\|\hat{X}_{t_{\ell+1}}-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_2+\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})\Big\|_2\\ &\quad+\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})-F^h_{I_{t_\ell}}(\bar{X}_{t_\ell},\vartheta_{\ell+1})\Big\|_2.\end{aligned} \tag{4.4.9}$$

On the other hand, since

$$|y-\mathrm{Proj}_{\mathbb{X}}(y)| \;\leq\; \delta+|y|\mathbf{1}_{\{|y|\geq R\}} \;\leq\; \delta+\frac{|y|^2}{R},$$

by inequality (4.4.3), we have

$$\left\|\hat{X}_{t_{\ell+1}} - F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\right\|_2 \;\leq\; \delta + \frac{\left\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\right\|_4^2}{R}. \tag{4.4.10}$$

Furthermore by standard estimates for the Euler scheme (see e.g. Lemma A.1 in [58]), we have

$$\left\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \vartheta_{\ell+1}) - F^h_{I_{t_\ell}}(\bar{X}_{t_\ell}, \vartheta_{\ell+1})\right\|_2 \;\leq\; (1+Kh)\left\|\hat{X}_{t_\ell} - \bar{X}_{t_\ell}\right\|_2,$$

and by the linear growth property of $\sigma$ and the fact that $\hat{\vartheta}_{\ell+1}, \vartheta_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$,

$$\left\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \vartheta_{\ell+1}) - F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\right\|_2 \;\leq\; Kh^{1/2}\left(1 + \left\|\hat{X}_{t_\ell}\right\|_2\right)\|\vartheta - \hat{\vartheta}\|_2. \tag{4.4.11}$$

Plugging these three inequalities into (4.4.9), we get :

$$\begin{aligned}\left\|\hat{X}_{t_{\ell+1}} - \bar{X}_{t_{\ell+1}}\right\|_2 \;&\leq\; (1+Kh)\left\|\hat{X}_{t_\ell} - \bar{X}_{t_\ell}\right\|_2 + Kh^{1/2}\left(\left\|\hat{X}_{t_\ell}\right\|_2 + 1\right)\|\vartheta - \hat{\vartheta}\|_2 \\ &\quad + \delta \;+\; \frac{\left\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\right\|_4^2}{R}.\end{aligned}$$

Finally since $\hat{X}_{t_k} = \bar{X}_{t_k} = x$, we obtain by induction, and using the estimates (4.4.5) on $\left\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\right\|_4$:

$$\begin{aligned}\left\|\hat{X}_{t_\ell} - \bar{X}_{t_\ell}\right\|_2 \;&\leq\; K\Big[h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \exp\left(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\right)\Big(1 + |x| + \frac{\delta}{h}\Big) + \frac{\delta}{h} \\ &\quad + \frac{1}{Rh}\exp\left(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\right)\Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big],\end{aligned} \tag{4.4.12}$$

for all $k \leq \ell \leq m$. Now by definition of $\hat{X}_{t_k}$, $\bar{X}_{t_k}$, we may write for $k \leq \ell \leq m-1$:

$$\begin{aligned}\hat{X}_{t_{\ell+1}} - \bar{X}_{t_{\ell+1}} \;&=\; (\hat{X}_{t_\ell} - \bar{X}_{t_\ell}) + h\big(b(\hat{X}_{t_\ell}, I_{t_\ell}) - b(\bar{X}_{t_\ell}, I_{t_\ell})\big) \\ &\quad + \sqrt{h}\big(\sigma(\hat{X}_{t_\ell}, I_{t_\ell})\hat{\vartheta}_{\ell+1} - \sigma(\bar{X}_{t_\ell}, I_{t_\ell})\vartheta_{\ell+1}\big) \\ &\quad + \mathrm{Proj}_{\mathbb{X}}\big(F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})\big) - F^h_{I_{t_\ell}}(\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1}),\end{aligned}$$

Since $\hat{X}_{t_k} = \bar{X}_{t_k}$ $(= x)$, we obtain by induction:

$$\begin{aligned} \Big\| \sup_{k\leq \ell\leq m} \big| \hat{X}_{t_\ell} - \bar{X}_{t_\ell} \big| \Big\|_2 \;\; & \leq \;\; h \sum_{\ell=k}^{m-1} \Big\| b(\hat{X}_{t_\ell}, I_{t_\ell}) - b(\bar{X}_{t_\ell}, I_{t_\ell}) \Big\|_2 \\ & \quad + \sqrt{h} \Big\| \sup_{k\leq \ell\leq m} \big| \sum_{r\leq \ell} \sigma(\hat{X}_{t_r}, I_{t_r}) \hat{\vartheta}_{r+1} - \sigma(\bar{X}_{t_r}, I_{t_r}) \vartheta_{r+1} \big| \Big\|_2 \\ & \quad + \sum_{\ell=k}^{m-1} \Big\| \mathrm{Proj}_{\mathbb{X}} (F^h_{I_{t_\ell}} (\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1})) - F^h_{I_{t_\ell}} (\hat{X}_{t_\ell}, \hat{\vartheta}_{\ell+1}) \Big\|_2 . \end{aligned} \tag{4.4.13}$$

We now bound each of the three terms in the right hand side of (4.4.13). First, by the Lipschitz property of $b$ and (4.4.12), we have

$$\begin{aligned} & h \sum_{\ell=k}^{m-1} \big\| b(\hat{X}_{t_\ell}, I_{t_\ell}) - b(\bar{X}_{t_\ell}, I_{t_\ell}) \big\|_2 \\ \leq \;\; & K \Big[ h^{-1/2} \|\vartheta - \hat{\vartheta}\|_2 \exp\big( K h^{-1} \|\vartheta - \hat{\vartheta}\|_2^2 \big) \Big( 1 + |x| + \frac{\delta}{h} \Big) \\ & \quad + \frac{\delta}{h} + \frac{1}{Rh} \exp\big( K h^{-2} \|\vartheta - \hat{\vartheta}\|_4^4 \big) \Big( 1 + |x|^2 + (\frac{\delta}{h})^2 \Big) \Big]. \end{aligned}$$

Next, recalling that $\hat{\vartheta}_{\ell+1}$ is independent of $\mathcal{F}_{t_\ell}$, with distribution law $\hat{\vartheta}$, and since $\hat{\vartheta}$ is an optimal $L^2$-quantizer of $\vartheta$, it follows that $\mathbb{E}[\hat{\vartheta}_{\ell+1} | \mathcal{F}_{t_\ell}] = \mathbb{E}[\hat{\vartheta}] = \mathbb{E}[\vartheta] = 0$. Thus, the process $(\sum_{r\leq \ell} \sigma(\hat{X}_{t_r}, I_{t_r}) \hat{\vartheta}_{r+1} - \sigma(\bar{X}_{t_r}, I_{t_r}) \vartheta_{r+1})_\ell$ is a $\mathcal{F}_{t_\ell}$-martingale, and from Doob's inequality, we have:

$$\begin{aligned} & \Big\| \sup_{k\leq \ell\leq m} \big| \sum_{r\leq \ell} \sigma(\hat{X}_{t_r}, I_{t_r}) \hat{\vartheta}_{r+1} - \sigma(\bar{X}_{t_r}, I_{t_r}) \vartheta_{r+1} \big| \Big\|_2 \\ \leq \;\; & K \Big( \mathbb{E} \Big[ \sum_{\ell=k}^{m-1} \big| \sigma(\hat{X}_{t_\ell}, I_{t_\ell}) \hat{\vartheta}_{\ell+1} - \sigma(\bar{X}_{t_\ell}, I_{t_\ell}) \vartheta_{\ell+1} \big|^2 \Big] \Big)^{\frac{1}{2}}. \end{aligned}$$

By writing from the Lipschitz condition on $\sigma_i$ that

$$\begin{aligned} \big| \sigma(\hat{X}_{t_\ell}, I_{t_\ell}) \hat{\vartheta}_{\ell+1} - \sigma(\bar{X}_{t_\ell}, I_{t_\ell}) \vartheta_{\ell+1} \big|^2 \;\; & \leq \;\; K \Big( \big| \hat{X}_{t_\ell} - \bar{X}_{t_\ell} \big|^2 \big| \vartheta_{\ell+1} \big|^2 \\ & \quad + (1 + \big| \hat{X}_{t_\ell} \big|^2) \big| \vartheta_{\ell+1} - \hat{\vartheta}_{\ell+1} \big|^2 \Big), \end{aligned}$$

and since $\vartheta_{\ell+1}, \hat{\vartheta}_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$, we then obtain

$$\begin{aligned}
& \sqrt{h}\Big\| \sup_{k\leq \ell\leq m} \big|\sum_{r\leq \ell} \sigma(\hat{X}_{t_r}, I_{t_r})\hat{\vartheta}_{r+1} - \sigma(\bar{X}_{t_r}, I_{t_r})\vartheta_{r+1}\big|\Big\|_2 \\
\leq \;& K \sup_{k\leq \ell\leq m-1} \Big[\big\|\hat{X}_{t_\ell} - \bar{X}_{t_\ell}\big\|_2 + (1+\|\hat{X}_{t_\ell}\|_2)\|\vartheta - \hat{\vartheta}\|_2\Big] \\
\leq \;& K\Big[h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1+|x|+\frac{\delta}{h}\Big) \\
& \quad + \frac{\delta}{h} + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1+|x|^2+(\frac{\delta}{h})^2\Big)\Big],
\end{aligned}$$

where we used the estimates (4.4.5) and (4.4.12). Finally the third term in (4.4.13) is bounded as before by (4.4.10). □

**Proof of Theorem 4.4.1.** For $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$, denote by $\hat{\alpha}^*$ the optimal switching strategy corresponding to $\hat{v}_i(t_k, x)$. Then, similarly as in the derivation of (4.3.15), by using the estimation (4.4.5) for $\|\hat{X}_{t_\ell}^{t_k,x,\alpha}\|_2$, we get the existence of some constant $K$, not depending on $(t_k, x, i, h)$, such that

$$\mathbb{E}\big|N(\hat{\alpha}^*)\big|^2 \;\leq\; K\exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1+|x|^2+\frac{\delta^2}{h^2}\Big).$$

Therefore, the supremum in the representation (4.3.1) of $\hat{v}_i(t_k, x)$ can be taken over the subset $\hat{\mathcal{A}}_{t_k,i}^{h,K}(x) = \Big\{\alpha \in \mathcal{A}_{t_k,i}^h \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K\exp\Big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\Big)\Big(1+|x|^2+\frac{\delta^2}{h^2}\Big)\Big\}$. Then, for $\alpha \in \hat{\mathcal{A}}_{t_k,i}^{h,K}(x)$, we have under **(Hl)** and by Cauchy-Schwarz inequality

$$\begin{aligned}
& \mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar{X}_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell}) - f(\hat{X}_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})\big| + \big|g(\bar{X}_{t_m}^{t_k,x,\alpha}, I_{t_m}) - g(\hat{X}_{t_m}^{t_k,x,\alpha}, I_{t_m})\big| \\
& \quad + \sum_{n=1}^{N(\alpha)} \big|c(\bar{X}_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1}, \iota_n) - c(\hat{X}_{\tau_n}^{h,t_k,x,\alpha}, \iota_{n-1}, \iota_n)\big|\Big] \\
\leq \;& K\mathbb{E}\Big[(1+N(\alpha))\big(\sup_{k\leq \ell\leq m} \big|\bar{X}_{t_\ell}^{t_k,x,\alpha} - \hat{X}_{t_\ell}^{t_k,x,\alpha}\big|\big)\Big] \\
\leq \;& K\exp\Big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\Big)\,(1+|x|+\frac{\delta}{h})\Big\|\sup_{k\leq \ell\leq m} \big|\bar{X}_{t_\ell}^{t_k,x,\alpha} - \hat{X}_{t_\ell}^{t_k,x,\alpha}\big|\Big\|_2 \\
\leq \;& K\exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1+|x|+\frac{\delta}{h}\Big)\Big[\frac{\delta}{h} + h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2\Big(1+|x|+\frac{\delta}{h}\Big) \\
& \quad + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1+|x|^2+(\frac{\delta}{h})^2\Big)\Big], \qquad (4.4.14)
\end{aligned}$$

by Lemma 4.4.2. Taking the supremum over $\alpha \in \hat{\mathcal{A}}_{t_k,i}^{h,K}(x)$ in the above inequality, we obtain an estimate for $|\bar{v}_i(t_k,x) - \hat{v}_i(t_k,x)|$ with an upper bound given by the r.h.s. of (4.4.14), which gives the required result.

Finally, notice that in the special case where the switching cost functions $c_{ij}$ do not depend on $x$, we have

$$\begin{aligned}
|\bar{v}_i(t_k,x) - \hat{v}_i(t_k,x)| &\leq \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell}) - f(\hat{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell})\big| \\
&\qquad + \big|g(\bar{X}^{t_k,x,\alpha}_{t_m}, I_{t_m}) - g(\hat{X}^{t_k,x,\alpha}_{t_m}, I_{t_m})\big|\Big] \\
&\leq K \sup_{\alpha\in\mathcal{A}^h_{t_k,i}, k\leq\ell\leq m} \mathbb{E}\big|\bar{X}^{t_k,x,\alpha}_{t_\ell} - \hat{X}^{t_k,x,\alpha}_{t_\ell}\big| \\
&\leq K\Big[h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1 + |x| + \frac{\delta}{h}\Big) \\
&\qquad + \frac{\delta}{h} + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big],
\end{aligned}$$

by the estimate in Lemma 4.4.2. □

### 4.4.2 Marginal quantization in the uncontrolled diffusion case

In this paragraph, we consider the special case where the diffusion $X$ is not controlled, i.e. $b_i = b$, $\sigma_i = \sigma$. The Euler scheme for $X$, denoted by $\bar{X}$, is given by:

$$\begin{aligned}
\bar{X}_0 = X_0, \quad \bar{X}_{t_{k+1}} &= F^h(\bar{X}_{t_k}, \vartheta_{k+1}) \\
&:= \bar{X}_{t_k} + b(\bar{X}_{t_k})h + \sigma(\bar{X}_{t_k})\sqrt{h}\,\vartheta_{k+1}, \quad k = 0,\ldots,m-1,
\end{aligned}$$

where $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$, $k = 0,\ldots,m-1$, are iid, $\mathcal{N}(0,I_d)$-distributed, independent of $\mathcal{F}_{t_k}$. Let us recall the well-known estimate: for any $p \geq 1$, there exists some $K_p$ s.t.

$$\|\bar{X}_{t_k}\|_p \leq K_p(1 + \|X_0\|_p). \tag{4.4.15}$$

Notice that the backward dynamic programming formulae (4.4.1)-(4.4.2) for $\bar{v}_i$ can be written in this case as:

$$\begin{aligned}
\bar{v}_i(t_m,.) &= g_i(.), \quad i \in \mathbb{I}_q \\
\bar{v}_i(t_k,.) &= \max_{j\in\mathbb{I}_q}[P^h\bar{v}_j(t_{k+1},.) + hf_j - c_{ij}].
\end{aligned} \tag{4.4.16}$$

Here $P^h$ is the probability transition kernel of the Markov chain $\bar{X}$, given by:

$$P^h\varphi(x) \;=\; \mathbb{E}[\varphi(\bar{X}_{t_{k+1}})|\bar{X}_{t_k}=x] \;=\; \mathbb{E}[\varphi(F^h(x,\vartheta))], \tag{4.4.17}$$

where $\vartheta$ is $\mathcal{N}(0,I_d)$-distributed. Let us next consider the family of discrete-time processes $(\bar{Y}^i_{t_k})_{k=0,\ldots,m}$, $i\in\mathbb{I}_q$, defined by:

$$\bar{Y}^i_{t_k} \;=\; \bar{v}_i(t_k,\bar{X}_{t_k}), \quad k=0,\ldots,m,\ i\in\mathbb{I}_q.$$

**Remark 4.4.2.** By the Markov property of the Euler scheme $\bar{X}$ w.r.t. $(\mathcal{F}_{t_k})_k$, we see that $(\bar{Y}^i_{t_k})_{k=0,\ldots,m}$, $i\in\mathbb{I}_q$, satisfy the backward induction:

$$\begin{aligned}
\bar{Y}^i_{t_m} &= g_i(\bar{X}_{t_m}) \;=\; g_i(\bar{X}_T), \quad i\in\mathbb{I}_q\\
\bar{Y}^i_{t_k} &= \max_{j\in\mathbb{I}_q}\Big\{\mathbb{E}[\bar{Y}^j_{t_{k+1}}|\mathcal{F}_{t_k}] + hf_j(\bar{X}_{t_k}) - c_{ij}(\bar{X}_{t_k})\Big\}, \quad k=0,\ldots,m-1,
\end{aligned}$$

and is represented as

$$\bar{Y}^i_{t_k} \;=\; \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\bar{X}_{t_\ell},I_{t_\ell})h + g(\bar{X}_{t_m},I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\bar{X}_{\tau_n},\iota_{n-1},\iota_n)\Big|\mathcal{F}_{t_k}\Big].$$

On the other hand, the continuous-time optimal switching problem (4.2.4) admits a representation in terms of the following reflected Backward Stochastic Differential Equations (BSDE):

$$\begin{aligned}
Y^i_t &= g_i(X_T) + \int_t^T f(X_s)ds - \int_t^T Z^i_s dW_s + K^i_T - K^i_t, \quad i\in\mathbb{I}_q,\ 0\le t\le T,\\
Y^i_t &\ge \max_{j\ne i}[Y^j_t - c_{ij}(X_t)] \quad \text{and} \quad \int_0^T (Y^i_t - \max_{j\ne i}[Y^j_t - c_{ij}(X_t)])dK^i_t \;=\; 0.
\end{aligned} \tag{4.4.18}$$

We know from [22], [35] or [34] that there exists a unique solution $(Y,Z,K)=(Y^i,Z^i,K^i)_{i\in\mathbb{I}_q}$ solution to (4.4.18) with $Y\in\mathcal{S}^2(\mathbb{R}^q)$, the set of adapted continuous processes valued in $\mathbb{R}^q$ s.t. $\mathbb{E}[\sup_{0\le t\le T}|Y_t|^2]<\infty$, $Z\in\mathcal{M}^2(\mathbb{R}^q)$, the set of predictable processes valued in $\mathbb{R}^q$ s.t. $\mathbb{E}[\int_0^T|Z_t|^2dt]<\infty$, and $K^i\in\mathcal{S}^2(\mathbb{R})$, $K^i_0=0$, $K^i$ is nondecreasing. Moreover, we have

$$\begin{aligned}
Y^i_t &= v_i(t,X_t), \quad i\in\mathbb{I}_q,\\
&= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,i}} \mathbb{E}\Big[\int_t^T f(X_s,I_s)ds + g(X_T,I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n},\iota_{n-1},\iota_n)\Big|\mathcal{F}_t\Big], \quad 0\le t\le T.
\end{aligned}$$

We recall from [13] the error estimation: for any $\varepsilon>0$, there exists some constant $K_\varepsilon$ s.t.

$$\max_{k=0,\ldots,m}\Big\|Y^i_{t_k} - \bar{Y}^i_{t_k}\Big\|_2 \;\le\; K_\varepsilon(1+\|X_0\|_2)h^{\frac{1}{2}-\varepsilon},$$

for all $i \in \mathbb{I}_q$, and $\varepsilon$ can be chosen equal to zero when the switching costs $c_{ij}$ do not depend on $x$.

We propose now an optimal quantization method in the vein of [4] for optimal stopping problems, for a computational approximation of $(\bar{Y}^i_{t_k})_{k=0,\ldots,m}$. This is based on results about optimal quantization of each marginal distribution of the Markov chain $(\bar{X}_{t_k})_{0\leq k\leq m}$. Let us recall the construction. For each time step $k = 0, \ldots, m$, we are given a grid $\Gamma_k = \{x^1_k, \ldots, x^{N_k}_k\}$ of $N_k$ points in $\mathbb{R}^d$, and we define the quantizer $\hat{X}_k = \mathrm{Proj}_k(\bar{X}_{t_k})$ of $\bar{X}_{t_k}$ where $\mathrm{Proj}_k$ denotes a closest neighbour projection on $\Gamma_k$. For $N_k$ being fixed, the grid $\Gamma_k$ is said to be $L^p$-optimal if it minimizes the $L^p$-quantization error: $\|\bar{X}_{t_k} - \mathrm{Proj}_k(\bar{X}_{t_k})\|_p$. Optimal grids $\Gamma_k$ are produced by a stochastic recursive algorithm, called Competitive Learning Vector Quantization (or also Kohonen Algorithm), and relying on Monte-Carlo simulations of $\bar{X}_{t_k}$, $k = 0, \ldots, m$. We refer to [57] for details about the CLVQ algorithm. We also compute the transition weights

$$\pi^{ll'}_k \;=\; \mathbb{P}[\hat{X}_{k+1} = x^{l'}_{k+1} | \hat{X}_k = x^l_k] \;=\; \frac{\mathbb{P}[(\bar{X}_{t_{k+1}}, \bar{X}_{t_k}) \in C_{l'}(\Gamma_{k+1}) \times C_l(\Gamma_k)]}{\mathbb{P}[\bar{X}_{t_k} \in C_l(\Gamma_k)]},$$

where $C_l(\Gamma_k) \subset \{x \in \mathbb{R}^d : |x - x^l_k| = \min_{y\in\Gamma_k} |x - y|\}$, $l = 1, \ldots, N_k$, is a Voronoi tesselation of $\Gamma_k$. These weights can be computed either during the CLVQ phase, or by a regular Monte-Carlo simulation once the grids $\Gamma_k$ are settled. The associated discrete probability transition $\hat{P}_k$ from $\hat{X}_k$ to $\hat{X}_{k+1}$, $k = 0, \ldots, m-1$, is given by:

$$\hat{P}_k\varphi(x^l_k) \;:=\; \sum_{l'=1}^{N_{k+1}} \pi^{ll'}_k \varphi(x^{l'}_{k+1}) \;=\; \mathbb{E}[\varphi(\hat{X}_{k+1}) | \hat{X}_k = x^l_k].$$

One then defines by backward induction the sequence of $\mathbb{R}^q$-valued functions $\hat{v}_k = (\hat{v}^i_k)_{i\in\mathbb{I}_q}$ computed explicitly on $\Gamma_k$, $k = 0, \ldots, m$, by the quantization tree algorithm:

$$\begin{aligned} \hat{v}^i_m &= g_i, \quad i \in \mathbb{I}_q, \\ \hat{v}^i_k &= \max_{j\in\mathbb{I}_q} \big[\hat{P}_k\hat{v}^j_{k+1} + hf_j - c_{ij}\big], \quad k = 0, \ldots, m-1. \end{aligned} \tag{4.4.19}$$

The discrete-time processes $(\bar{Y}^i_{t_k})_{k=0,\ldots,m}$, $i \in \mathbb{I}_q$, are then approximated by the quantized processes $(\hat{Y}^i_k)_{k=0,\ldots,m}$, $i \in \mathbb{I}_q$ defined by

$$\hat{Y}^i_k \;=\; \hat{v}^i_k(\hat{X}_k), \quad k = 0, \ldots, m,\; i \in \mathbb{I}_q.$$

The rest of this section is devoted to the error analysis between $\bar{Y}^i$ and $\hat{Y}^i$. The analysis follows arguments as in [5] for optimal stopping problems, but has to be slightly modified since the functions $\bar{v}_i(t_k,.)$ are not Lipschitz in general when the switching costs depend on $x$. Let us introduce the subset $LLip(\mathbb{R}^d)$ of measurable functions $\varphi$ on $\mathbb{R}^d$ satisfying:

$$|\varphi(x) - \varphi(y)| \leq K(1 + |x| + |y|)|x - y|, \qquad \forall x, y \in \mathbb{R}^d,$$

for some positive constant $K$, and denote by

$$[\varphi]_{LLip} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{(1 + |x| + |y|)|x - y|}.$$

**Lemma 4.4.3.** *The functions $\bar{v}_i(t_k,.)$, $k = 0,\ldots,m$, $i \in \mathbb{I}_q$, lie in $LLip(\mathbb{R}^d)$, and $[\bar{v}_i(t_k,.)]_{LLip}$ is bounded by a constant not depending on $(k,i,h)$.*

**Proof.** We set $\bar{v}_k^i = \bar{v}_i(t_k,.)$. From the representation (4.3.12), we have

$$\bar{v}_k^i(x) = \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\bar{X}^{t_k,x}_{t_\ell}, I_{t_\ell})h + g(\bar{X}^{t_k,x}_{t_m}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\bar{X}^{t_k,x}_{\tau_n}, \iota_{n-1}, \iota_n)\Big],$$

where $\bar{X}^{t_k,x}$ is the solution to the Euler scheme starting from $x$ at time $t_k$. From (4.3.15), notice that in the above representation for $\bar{v}_k^i(x)$, one can restrict the supremum to $\mathcal{A}^{h,K}_{t_k,i}(x) = \{\alpha \in \mathcal{A}^h_{t_k,i} \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K(1 + |x|^2)\}$ for some positive constant $K$ not depending on $(t_k, x, i, h)$. Then, as in the proof of Theorem 4.4.1, we have for any $x, y \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}^{h,K}_{t_k,i}(x) \cup \mathcal{A}^{h,K}_{t_k,i}(y)$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar{X}^{t_k,x}_{t_\ell}, I_{t_\ell}) - f(\bar{X}^{t_k,y}_{t_\ell}, I_{t_\ell})\big| + \big|g(\bar{X}^{t_k,x}_{t_m}, I_{t_m}) - g(\bar{X}^{t_k,y}_{t_m}, I_{t_m})\big| \\
&\qquad + \sum_{n=1}^{N(\alpha)} \big|c(\bar{X}^{t_k,x}_{\tau_n}, \iota_{n-1}, \iota_n) - c(\bar{X}^{t_k,x}_{\tau_n}, \iota_{n-1}, \iota_n)\big|\Big] \\
&\leq K(1 + \|N(\alpha)\|_2)\Big\|\sup_{k\leq\ell\leq m} |\bar{X}^{t_k,x}_{t_\ell} - \bar{X}^{t_k,y}_{t_\ell}|\Big\|_2 \\
&\leq K(1 + |x| + |y|)|x - y|,
\end{aligned}$$

by standard Lipschitz estimates on the Euler scheme. By taking the supremum over $\mathcal{A}^{h,K}_{t_k,i}(x) \cup \mathcal{A}^{h,K}_{t_k,i}(y)$ in the above inequality, this shows that

$$|\bar{v}_k^i(x) - \bar{v}_k^i(y)| \leq K(1 + |x| + |y|)|x - y|,$$

i.e. $\bar{v}_k^i \in LLip(\mathbb{R}^d)$ with $[\bar{v}_k^i]_{LLip} \leq K$. □

The next Lemma shows that the probability transition kernel of the Euler scheme preserves the growth linear Lipschitz property.

**Lemma 4.4.4.** *For any $\varphi \in LLip(\mathbb{R}^d)$, the function $P^h\varphi$ also lies in $LLip(\mathbb{R}^d)$, and there exists some constant $K$, not depending on $h$, such that*

$$[P^h\varphi]_{LLip} \quad \leq \quad \sqrt{3}(1 + O(h))[\varphi]_{LLip},$$

*where $O(h)$ denotes any function s.t. $O(h)/h$ is bounded when $h$ goes to zero.*

**Proof.** From (4.4.17) and Cauchy-Schwarz inequality, we have for any $x, y \in \mathbb{R}^d$:

$$\begin{aligned}
& |P^h\varphi(x) - P^h\varphi(y)| \\
\leq \quad & \Big(\mathbb{E}\big|\varphi(F^h(x,\vartheta)) - \varphi(F^h(y,\vartheta))\big|^2\Big)^{1/2} \\
\leq \quad & [\varphi]_{LLip}\Big(\mathbb{E}\big|(1 + |F^h(x,\vartheta)| + |F^h(y,\vartheta)|)^2\big|F^h(x,\vartheta) - F^h(y,\vartheta)\big|^2\Big)^{1/2} \\
\leq \quad & \sqrt{3}[\varphi]_{LLip}\Big(\mathbb{E}\big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2\big]\Big)^{\frac{1}{2}}, \quad (4.4.20)
\end{aligned}$$

where we used the relation $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Since $\vartheta$ has a symmetric distribution, we have

$$\begin{aligned}
& \mathbb{E}\Big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2\Big] \\
= \quad & \frac{1}{2}\mathbb{E}\Big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2 \\
& \quad + (1 + |F^h(x,-\vartheta)|^2 + |F^h(y,-\vartheta)|^2)|F^h(x,-\vartheta) - F^h(y,-\vartheta)|^2\Big]
\end{aligned}$$

A straightforward calculation gives

$$\begin{aligned}
& \frac{1}{2}\Big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2 \\
& \quad + (1 + |F^h(x,-\vartheta)|^2 + |F^h(y,-\vartheta)|^2)|F^h(x,-\vartheta) - F^h(y,-\vartheta)|^2\Big] \\
= \quad & (1 + |x + hb(x)|^2 + |y + hb(y)|^2 + h|\sigma(x)\vartheta|^2 + h|\sigma(y)\vartheta|^2)\big|x - y + h(b(x) - b(y))\big|^2 \\
& + h|(\sigma(x) - \sigma(y))\vartheta|^2(|x + hb(x)|^2 + |y + hb(y)|^2) \\
& + 4h\Big[(x + hb(x)|\sigma(x)\vartheta) + (y + hb(y)|\sigma(y)\vartheta)\Big](x - y + h(b(x) - b(y))|(\sigma(x) - \sigma(y))\vartheta) \\
& + h^2(|\sigma(x)\vartheta|^2 + |\sigma(y)\vartheta|^2)|(\sigma(x) - \sigma(y))\vartheta|^2.
\end{aligned}$$

By Lipschitz continuity of $b$ and $\sigma$, and the fact that $\mathbb{E}|\vartheta|^4 < \infty$, we deduce that

$$\begin{aligned} & \mathbb{E}\Big[(1+|F^h(x,\vartheta)|^2+|F^h(y,\vartheta)|^2)|F^h(x,\vartheta)-F^h(y,\vartheta)|^2\Big] \\ \leq \ & (1+O(h))(1+|x|^2+|y|^2)|x-y|^2. \end{aligned}$$

Plugging this last inequality into (4.4.20) shows the required result. □

We now pass to the main result of this section by providing some a priori estimates for $\|\bar{Y}_{t_k} - \hat{Y}_k\|$ in terms of the quantization error $\|\bar{X}_{t_k} - \hat{X}_k\|$.

**Theorem 4.4.2.** *There exists some positive constant $K$, not depending on $h$, such that*

$$\max_{i\in\mathbb{I}_q} \big\|\bar{Y}^i_{t_k} - \hat{Y}^i_k\big\|_p \ \leq \ K\sum_{\ell=k}^{m}(1+\|X_0\|_r + \|\hat{X}_\ell\|_r)\big\|\bar{X}_{t_\ell} - \hat{X}_\ell\big\|_s, \tag{4.4.21}$$

*for any $k = 0,\ldots,m$, and $(p,r,s) \in (1,\infty)$ s.t. $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.*

**Proof.** We set $\bar{v}^i_k = \bar{v}_i(t_k,.)$, and by misuse of notations, we also set $\bar{Y}^i_k = \bar{Y}^i_{t_k} = \bar{v}^i_k(\bar{X}_k)$. From the recursive induction (4.4.16) (resp. (4.4.19)) on $\bar{v}^i_k$ (resp. $\hat{v}^i_k$), and the trivial inequality $|\max_j \bar{a}_j - \max_j \hat{a}_j| \leq \max_j |\bar{a}_j - \hat{a}_j|$, we have for all $i \in \mathbb{I}_q$:

$$\begin{aligned} |\bar{Y}^i_k - \hat{Y}^i_k| \ &= \ |\bar{v}^i_k(\bar{X}_{t_k}) - \hat{v}^i_k(\hat{X}_k)| \\ &\leq \ \max_{j\in\mathbb{I}_q}\big|[P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) + hf_j(\bar{X}_{t_k}) - c_{ij}(\bar{X}_{t_k})] - [\hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k) + hf_j(\hat{X}_k) - c_{ij}(\hat{X}_k)]\big| \\ &\leq \ \max_{j\in\mathbb{I}_q}\Big[\big|P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k)\big| + h\big|f_j(\bar{X}_{t_k}) - f_j(\hat{X}_k)\big| + \big|c_{ij}(\bar{X}_{t_k}) - c_{ij}(\hat{X}_k)\big|\Big] \\ &\leq \ K\big|\bar{X}_{t_k} - \hat{X}_k\big| + \max_{j\in\mathbb{I}_q}\big|P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k)\big| \end{aligned}$$

by the Lipschitz property of $f_j$ and $c_{ij}$, and so

$$\max_{i\in\mathbb{I}_q}\Big\|\bar{Y}^i_k - \hat{Y}^i_k\Big\|_p \ \leq \ K\Big\|\bar{X}_{t_k} - \hat{X}_k\Big\|_p + \max_{i\in\mathbb{I}_q}\Big\|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^i_{k+1}(\hat{X}_k)\Big\|_p \tag{4.4.22}$$

Writing $\hat{\mathbb{E}}_k$ for the conditional expectation w.r.t. $\hat{X}_k$, we have for any $i \in \mathbb{I}_q$

$$
\begin{aligned}
& |P^h \bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k \hat{v}^i_{k+1}(\hat{X}_k)| \\
\leq \; & |P^h \bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h \bar{v}^i_{k+1}(\hat{X}_k)| + |P^h \bar{v}^i_{k+1}(\hat{X}_k) - \hat{\mathbb{E}}_k[P^h v^i_{k+1}(\bar{X}_{t_k})]| \\
& \quad + |\hat{\mathbb{E}}_k[P^h \bar{v}^i_{k+1}(\bar{X}_{t_k})] - \hat{P}_k \hat{v}^i_{k+1}(\hat{X}_k)| \\
= \; & |P^h \bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h \bar{v}^i_{k+1}(\hat{X}_k)| + |\hat{\mathbb{E}}_k[P^h \bar{v}^i_{k+1}(\hat{X}_k) - P^h \bar{v}^i_{k+1}(\bar{X}_{t_k})]| \\
& \quad + |\hat{\mathbb{E}}_k[\bar{Y}^i_{k+1} - \hat{Y}^i_{k+1}]|.
\end{aligned}
$$

Since $\hat{\mathbb{E}}_k$ is a $L^p$-contraction, we then obtain

$$
\begin{aligned}
& \Big\| P^h \bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k \hat{v}^i_{k+1}(\hat{X}_k) \Big\|_p \\
\leq \; & 2 \Big\| P^h \bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h \bar{v}^i_{k+1}(\hat{X}_k) \Big\|_p + \Big\| \bar{Y}^i_{k+1} - \hat{Y}^i_{k+1} \Big\|_p \\
\leq \; & K(1 + O(h)) \Big\| (1 + |\bar{X}_{t_k}| + |\hat{X}_k|) |\bar{X}_{t_k} - \hat{X}_k| \Big\|_p + \Big\| \bar{Y}^i_{k+1} - \hat{Y}^i_{k+1} \Big\|_p \\
\leq \; & K(1 + O(h))(1 + \|X_0\|_r + \|\hat{X}_k\|_r) \Big\| \bar{X}_{t_k} - \hat{X}_k \Big\|_s + \Big\| \bar{Y}^i_{k+1} - \hat{Y}^i_{k+1} \Big\|_p,
\end{aligned}
\tag{4.4.23}
$$

where we used Lemmata 4.4.4 and 4.4.3, Hölder's inequality and (4.4.15). Substituting (4.4.23) into (4.4.22), we get

$$
\begin{aligned}
& \max_{i \in \mathbb{I}_q} \Big\| \bar{Y}^i_k - \hat{Y}^i_k \Big\|_p \\
\leq \; & K(1 + O(h)) \Big( 1 + \|X_0\|_r + \|\hat{X}_k\|_r \Big) \Big\| \bar{X}_{t_k} - \hat{X}_k \Big\|_s + \max_{i \in \mathbb{I}_q} \Big\| \bar{Y}^i_{k+1} - \hat{Y}^i_{k+1} \Big\|_p,
\end{aligned}
$$

for all $k = 0, \ldots, m-1$. Since $\max_{i \in \mathbb{I}_q} \|\bar{Y}^i_m - \hat{Y}^i_m\|_p = \max_{i \in \mathbb{I}_q} \|g_i(\bar{X}_{t_m}) - g(\hat{X}_m)\|_p \leq K \|\bar{X}_{t_m} - \hat{X}_m\|_p$ by the Lipschitz condition on $g_i$, we conclude by induction. □

**Remark 4.4.3.** Assume that $\hat{X}_k$ is chosen to be an $L^2$-optimal quantizer of $\bar{X}_{t_k}$ for each $k = 0, \ldots, m$. It is in particular a stationary quantizer in the sense that $\mathbb{E}[\bar{X}_{t_k} | \hat{X}_k] = \hat{X}_k$ (see [57]), and by Jensen's inequality, we deduce that $\|\hat{X}_k\|_2 \leq \|\bar{X}_{t_k}\|_2$. Recalling (4.4.15), the inequality (4.4.21) in Theorem 4.4.2 gives

$$
\max_{i \in \mathbb{I}_q} \|\bar{Y}^i_{t_k} - \hat{Y}^i_k\|_1 \;\; \leq \;\; K(1 + \|X_0\|_2) \sum_{\ell = k}^{m} \|\bar{X}_{t_\ell} - \hat{X}_\ell\|_2,
$$

for all $k = 0, \ldots, m$. In particular, if $X_0 = x_0$ is deterministic, then $\hat{X}_0 = x_0$, and we have an error estimation by quantization of the value function function for the discrete-time optimal switching problem at the initial date measured by:

$$\max_{i\in\mathbb{I}_q} \left|\bar{v}_i(0,x_0) - \hat{v}_0^i(x_0)\right| \quad \leq \quad K(1+|x_0|)\sum_{k=1}^m \left\|\bar{X}_{t_k} - \hat{X}_k\right\|_2 \tag{4.4.24}$$

Suppose that one has at hand a global stack of $\bar{N}$ points for the whole space-time grid, to be dispatched with $N_k$ points for each $k$th-time step, i.e. $\sum_{k=1}^m N_k = \bar{N}$. Then, as in [5], in the case of uniformly elliptic diffusion with bounded Lipschitz coefficients $b$ and $\sigma$, one can optimize over the $N_k$'s by using the rate of convergence for the miminal $L^2$-quantization error given by Zador's theorem:

$$\left\|\bar{X}_{t_k} - \hat{X}_k\right\|_2 \quad \sim \quad \frac{J_{2,d}\left\|\varphi_k\right\|_{\frac{d}{d+2}}^{\frac{1}{2}}}{N_k^{\frac{1}{d}}} \quad \text{as } N_k \to \infty,$$

where $\varphi_k$ is the probability density function of $\bar{X}_{t_k}$, and $\left\|\varphi\right\|_r = (\int |\varphi(u)|^r du)^{\frac{1}{r}}$. From [6], we have the bound $\left\|\varphi_k\right\|_{\frac{d}{d+2}}^{\frac{1}{2}} \leq K\sqrt{t_k}$, for some constant $K$ depending only on $b$, $\sigma$, $T$, $d$. Substituting into (4.4.24) with Zador's theorem, we obtain

$$\max_{i\in\mathbb{I}_q} \left|\bar{v}_i(0,x_0) - \hat{v}_0^i(x_0)\right| \quad \leq \quad K(1+|x_0|)\sum_{k=1}^m \frac{\sqrt{t_k}}{N_k^{\frac{1}{d}}}.$$

For fixed $h = T/m$ and $\bar{N}$, the sum in the upper bound of the above inequality is minimized over the size of the grids $\Gamma_k$, $k = 1, \ldots, m$ with

$$N_k \quad = \quad \left\lceil \frac{t_k^{\frac{d}{2(d+1)}}\bar{N}}{\sum_{k=1}^m t_k^{\frac{d}{2(d+1)}}} \right\rceil,$$

where $\lceil x \rceil := \min\{k \in \mathbb{N},\ k \geq x\}$, and we have a global rate of convergence given by:

$$\max_{i\in\mathbb{I}_q} \left|\bar{v}_i(0,x_0) - \hat{v}_0^i(x_0)\right| \quad \leq \quad \frac{K(1+|x_0|)}{h(\bar{N}h)^{\frac{1}{d}}}.$$

By combining with the estimate (4.3.14), we obtain an error bound between the value function of the continuous-time optimal switching problem and its approximation by marginal quantization of order $h^{\frac{1}{2}}$ when choosing a number of points by grid $\bar{N}h$ of order $1/h^{\frac{3d}{2}}$. This has to be compared with the number of points $N$ of lower order $1/h^d$ in the Markovian quantization approach, see Remark 4.4.1. The complexity of this marginal quantization algorithm is of order $O\left(\sum_{k=1}^m N_k N_{k+1}\right)$. In terms of $h$, if we take $N_k = \bar{N}h = 1/h^{\frac{3d}{2}}$, we then need $O(1/h^{3d+1})$ operations to compute the value function. Recall that the Markovian quantization method requires a complexity of higher order $O(1/h^{4d+1})$, but provides in compensation an approximation of the value function in the whole space grid $\mathbb{X}$.

## 4.5 Numerical tests

We test our quantization algorithms by comparison results with explicit formulae for optimal switching problems derived from chapter 5 in [60]. The formulae are obtained for infinite horizon problems, that we adapt to our case by taking as the final gain the (discounted) value function for the infinite horizon problem.

We consider a two-regime switching problem where the diffusion is independent of the regime and follows a geometric Brownian motion, i.e. $b(x,i) = bx$, $\sigma(x,i) = \sigma x$, and the switching costs are constant $c(x,i,j) = c_{ij}$ ,$i,j = 1,2$. The profit functions are in the form $f_i(t,x) = e^{-\beta t} k_i x^{\gamma_i}$, $i = 1,2$. From Theorem 5.3.5 in [60]), the value functions are given by:

$$v_1(0,x) = \begin{cases} A_1 x^{m^+} + K_1 k_1 x^{\gamma_1}, & x < \underline{x}_1^* \\ B_2 x^{m^-} + K_2 k_2 x^{\gamma_2} - c_{12}, & x \geq \underline{x}_1^* \end{cases}$$

$$v_2(0,x) = \begin{cases} A_2 x^{m^+} + K_2 k_2 x^{\gamma_2}, & x < \underline{x}_2^* \\ A_1 x^{m^+} + K_1 k_1 x^{\gamma_1} - c_{21} & \underline{x}_2^* \leq x \leq \overline{x}_2^* \\ B_2 x^{m^-} + K_2 k_2 x^{\gamma_2}, & x > \overline{x}_2^* \end{cases},$$

where $A_i$, $B_i$, $K_i$, $\underline{x}_2^*$ and $\overline{x}_2^*$ depend explicitly on the parameters. In the sequel, we take for value of the parameters:

$$b = 0,\ \sigma = 1,\ c_{01} = c_{10} = 0.5,\ k_1 = 2, k_2 = 1,\ \gamma_1 = 1/3,\ \gamma_2 = 2/3,\ \beta = 1.$$

We compute the value function in regime 2 taken at $X_0 = 3.0$ by means of the first algorithm (Markovian quantization). We take $R = 10X_0$ and vary $m, \delta$ and $N$. The results are compared with the exact value in Table 1. Notice that the algorithm seems to be quite robust and provides good results even when $\delta m$ and $\frac{m}{R}$ do not satisfy the constraints given by our theoretical estimates in Remark 4.4.1.

In Table 2, we have computed the value with the marginal quantization algorithm. We make vary the number of time steps $m$ and the total number of grid points $\bar{N}$ (dispatched between the different time steps as described in Remark 4.4.3). We have used optimal quantization of the Brownian motion, and the transition probabilities $\pi_k^{ll'}$ were computed by Monte-Carlo simulations with $10^6$ sample paths (for an analysis of the error induced by this Monte-Carlo approximation, see Section 4 in [4]). We have also indicated the time spent for these computations. Actually, almost all of this time comes from the Monte-Carlo computations, as the tree descent algorithm is very fast (less than 1s for all the tested parameters).

For the two methods, we look at the impact of the quantization number for each time step (resp. $N$ and $\bar{N}h$) on the precision of the results. As our theoretical estimates showed (see Remarks 4.4.1 and 4.4.3), for the first method, increasing $N$ higher than $h^{-1}$ does not seem to improve the precision, whereas for the second method, we can see for several values of $h$ that changing $\bar{N}h$ from $h^{-1}$ to $h^{-2}$ or $h^{-3}$ improves the precision.

Comparing the two tables, the first method seems to provide precise estimates with slightly faster computation times, and it has the further advantage of computing simultaneously the value functions at any points of the space discretization grid $\mathbb{X}$. However, since most of the time spent by our second algorithm was devoted to the calculation of the transition probabilities $\pi_k^{ll'}$, if these were computed beforehand and stored offline, the marginal quantization method becomes more competitive.

| $(m, 1/\delta, N)$ | $\hat{v}_2(0, 3.0)$ | Numerical error (%) | Algorithm time (s) |
|---|---|---|---|
| (10,10,10) | 2.1925 | 3.0 | 0.2 |
| (10,10,100) | 2.1863 | 2.7 | 0.5 |
| (10,10,1000) | 2.1852 | 2.7 | 1.4 |
| (10,100,1000) | 2.1882 | 2.8 | 8.5 |
| (10,100,5000) | 2.1882 | 2.8 | 40 |
| (100,10,100) | 2.1218 | 0.31 | 1.0 |
| (100,10,1000) | 2.1213 | 0.33 | 8.0 |
| (100,10,5000) | 2.1213 | 0.33 | 39 |
| (100,100,100) | 2.1250 | 0.16 | 8.6 |
| (100,100,1000) | 2.1250 | 0.16 | 82 |
| Exact value | 2.1285 | | |

Table 4.1: Results obtained by Markovian quantization

| $(m, \bar{N})$ | $\hat{Y}_0^2$ | Numerical error (%) | Algorithm time (s) |
|---|---|---|---|
| (10,100) | 2.2080 | 3.7 | 4.4 |
| (10,1000) | 2.2174 | 4.2 | 4.9 |
| (10,10000) | 2.1276 | 0.04 | 5.8 |
| (100,1000) | 2.1233 | 0.24 | 36 |
| (100,10000) | 2.1316 | 0.15 | 48 |
| (100,50000) | 2.1301 | 0.07 | 65 |
| (1000,10000) | 2.1161 | 0.58 | 353 |
| (1000,50000) | 2.1213 | 0.34 | 498 |

Table 4.2: Results obtained by marginal quantization

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