# Complementary Young regularity in Gaussian rough path theory

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Conférence annuelle du GDR TRAG
June 8, 2023

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# Summary

In Gaussian rough path theory, one is often led to make two assumptions :

1. Finite $\rho$-variation of the covariance (in order to get existence of a rough path lift for the considered process),
2. Complementary Young regularity (in order to get continuity of Cameron-Martin translation on rough path space).

The point of this talk is to show that in fact, roughly speaking, assuming only one of those is enough.

This is inspired by the works on singular SPDE of Schönbauer '18 and Linares, Otto, Tempelmayr, Tsatsoulis '21.

# Outline

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# Gaussian processes

Let $X = (X_t)_{t \in [0,T]}$ be a Gaussian process under the probability $\mathbb{P}$ (say with values in $\Omega = C([0,T])$).
A fundamental object is the associated **Cameron-Martin space** $\mathcal{H}$:

- $h = (h_t)_{0 \leqslant t \leqslant T}$ is in $\mathcal{H}$ if, it holds that, for some $Z$ in the $L^2$-closure of $span\{X_s, 0 \leqslant s \leqslant T\}$,

$$h_t = \mathbb{E}[Z X_t] \quad \forall 0 \leqslant t \leqslant T,$$

  We then define

$$\|h\|_{\mathcal{H}} = \|Z\|_{L^2(\mathbb{P})},$$

  under which $\mathcal{H}$ is a Hilbert space.

- For $h \in \Omega$, let $\mathbb{P}^h$ be the measure image of $\mathbb{P}$ under the translation $X \mapsto X + h$.Then
$$\mathbb{P}^h \sim \mathbb{P} \Leftrightarrow h \in \mathcal{H},$$
in which case
$$\frac{d\,\mathbb{P}^h}{d\,\mathbb{P}} = \exp\left(-\frac{1}{2}\|h\|_{\mathcal{H}}^2 + Z^h\right).$$
- If $\mathcal{H}$ has ONB $(e_n)_{n \geqslant 0}$, then
$$X_t = \sum_{n \geqslant 0} \gamma_n e_n(t), \qquad \text{with } \gamma_n \text{ i.i.d. } \mathcal{N}(0,1).$$
(Karhunen-Loeve type expansion).
- Examples : if $X$ is Brownian motion,
$\mathcal{H} = H^1 = \left\{ h = \int_0^\cdot \dot{h}(s)ds, \quad \int_0^T |\dot{h}(s)|^2 ds < \infty \right\}$
More generally, if $X_t = \int_0^t K(t,s)dB_s$, then
$$\mathcal{H} = \left\{ h : h_t = \int_0^t K(t,s)u(s)ds,\ u \in L^2([0,T] \right\},\ \text{with } \|h\|_{\mathcal{H}} = \|u\|_{L^2}.$$
$X$ fBm with Hurst index $H$ : $\mathcal{H} \approx H^{H+\frac{1}{2}}$.

Translation by Cameron-Martin elements is useful in many contexts, for instance :

- Concentration of measure :

$$F(X+h) - F(X) \leqslant C\|h\|_{\mathcal{H}} \quad \Rightarrow \quad F(X) \text{ has Gaussian tails}$$

- Large deviations (and Laplace asymptotics) :

$$\mathbb{E}[F(\epsilon X)] = e^{-\frac{\|h\|_{\mathcal{H}}^2}{2\epsilon^2}}\, \mathbb{E}[F(h+\epsilon X)e^{-\frac{Z^h}{\epsilon}}]$$

- Support theorems,
- Malliavin calculus :

$$\text{Consider } DF(X) := \frac{d}{dh}F(X+h) \in \mathcal{H},$$

$$\text{if it exists and is non-degenerate a.s.,} \quad \mathcal{L}^{F(X)} << Leb.$$

# Gaussian rough paths

Recall lifting **canonically** $X = (X^1, \dots, X^d)$ to a rough path, in the case of regularity $\alpha \in (1/3, 1/2)$ (for simplicity) means that one needs, (a.s.)

$$X_{s,t} \lesssim |t-s|^\alpha, \quad \mathbb{X}_{s,t} := \int_s^t X_{s,r} \otimes dX_r \lesssim |t-s|^{2\alpha}.$$

(the object in the r.h.s. is not defined a priori, but is obtained as a limit of any reasonable approximation $X^N \to X$.)

$\mathrm{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha$ rough path space.

Friz-Victoir '10 : finite 2D $\rho$-variation of the covariance, $\rho \in [1,2)$ implies existence of such a canonical lift.

(Interest : continuity of Itô-Lyons map $\mathrm{X} \mapsto Y$ solution to $dY = V(Y)dX$...)

# Cameron-Martin translation of a rough path

Assume we have $X$ with a rough path lift $\mathbf{X}$, $h$ in the Cameron-Martin space $\mathcal{H}$, and we want to understand $X + h$ as a rough path, namely we want to construct

$$T^h\mathbf{X} = \left(X + h, \int (X + h) \otimes d(X + h)\right)$$

with suitable bounds.
Note that

$$\int (X + h) \otimes d(X + h) = \mathbb{X} + \int X \otimes dh + \int h \otimes dX + \int h \otimes dh.$$

Need to make sense of cross-integrals. In general, $\mathcal{H} \subset C^\alpha$ (not better).
But typically better regularity in other scales !
e.g. B.M., $\mathcal{H} = H^1 \subset C^{1/2}$, but also $H^1 \subset C^{1-var}$.

# Complementary Young regularity

A natural assumption is Complementary Young regularity (CYR)

$$\mathcal{H} \subset C^{q-var}, \qquad \text{for some } \frac{1}{q} + \alpha > 1.$$

By Young integration, this means that the cross terms in $T^h X$ are well-defined !

A slight strengthening is controlled CYR , with $q$ as above,

$$\forall s < t, \quad \|h\|_{q-var;[s,t]} \leqslant C(t-s)^{\alpha} \|h\|_{\mathcal{H}}. \qquad \text{(cCYR)}$$

Under this assumption, it then holds that the translation map

$$(\mathrm{X}, h) \in \mathcal{C}^{\alpha} \times \mathcal{H} \mapsto T^h \mathrm{X}$$

is continuous.

So far, typical works on Gaussian rough paths, for the above mentioned applications, typically made two assumptions :

1. An assumption ensuring existence of a r.p. lift (e.g. finite $\rho$-variation of covariance, $\rho < 2$ )
2. An assumption ensuring continuity of C.M. translation (e.g. (CYR) or (cCYR)).

cf. Friz-Victoir '10 (paper or book), Inahama '14, Cass-Hairer-Litterer-Tindel '16, Friz-Hairer book...

Rest of the talk : only one of those is necessary !

# Outline

- In this part, we assume that (cCYR) holds. We show that this implies existence of a canonical rough path lift.
- The key idea is to use the **Poincaré inequality** from Malliavin calculus :
$$\mathbb{E}\left[(F-\mathbb{E}[F])^2\right] \lesssim \mathbb{E}\left[\|DF\|_{\mathcal{H}}^2\right].$$
- This is inspired by the work of Linares, Otto, Tempelmayr, Tsatsoulis '21 on singular SPDE. (See also Bailleul-Bruned '23, Hairer-Steele '23).

# Main result

**Theorem**

*Let $X = (X^1, \dots, X^d)$ be a Gaussian process with independent components. Assume that*

$$\forall s < t, \quad \|h\|_{q-var;[s,t]} \leqslant C(t-s)^{\alpha} \|h\|_{\mathcal{H}}.$$

*for some $\alpha, q$ satisfying*

$$\frac{1}{q} + \alpha > 1, \quad \alpha > \frac{1}{4}.$$

*Then $X$ admits a canonical rough path lift in $\mathcal{C}^{\alpha'}$, for any $\alpha' < \alpha$.*

# Idea of proof

(case $\alpha > \frac{1}{3}$). First : (cCYR) implies $X \in C^{\alpha-}$ a.s. by Kolmogrov's criterion.
Then : note that for $F = \int X^1 dX^2$, it holds that $\langle DF, h\rangle_{\mathcal{H}} = \int X^1 dh^2 + \int X^1 dh^2$.
Using Poincaré's inequality,

$$\mathbb{E}\left[\left(\int_s^t X^1_{s,r} dX^2_r\right)^2\right]^{1/2} \lesssim \underbrace{\mathbb{E}\left[\int_s^t X^1_{s,r} dX^2_r\right]}_{=0} + \mathbb{E}\left[\sup_{\|h\|_{\mathcal{H}} \leqslant 1} \left(\int_s^t X^1_{s,r} dh^2_r + h^1_{s,r} dX^2_r\right)^2\right]^{1/2}.$$

On the other hand, by Young integration and (cCYR), it holds that e.g.

$$\int_s^t X^1_{s,r} dh^2_r \lesssim \|h\|_{q-var;[s,t]} \|X\|_{C^{\alpha-}} (t-s)^{\alpha-} \lesssim_{L^p(\mathbb{P})} (t-s)^{2\alpha-}$$

and we conclude again with Kolmogorov.

# Remarks

- The case of $\alpha < \frac{1}{3}$ uses a similar induction argument.
- We also have an estimate on rough path distances between $X$ and $X'$ based on some assumption on the Cameron-Martin space of $(X, X')$.
- The assumptions that $\alpha > \frac{1}{4}$ and independence of components are only needed to control the $\mathbb{E}[\ldots]$ terms (the argument for the variance always holds).
- If the components of $X$ are i.i.d., we can show that $\alpha > \frac{1}{6}$ is enough (based on an algebraic identity).
- If the components are dependent, but we somehow know that the expected signature is of the right order, then the argument goes through (potentially useful ?).
- Arguably simpler than the 2D Young integration arguments from Friz-Victoir (especially in the case $\alpha \leqslant \frac{1}{3}$ !).

# Remarks 2

- How to check (cCYR) ? Friz-Victoir Besov-variation embedding :

$$\mathcal{H} \subset W^{\delta,p} \Rightarrow \text{ (cCYR) with } \alpha = \delta - \frac{1}{p}, \quad \frac{1}{q} = \delta.$$

- The case of fBm corresponds to $\mathcal{H} \approx W^{H+\frac{1}{2},2}$, i.e. $\alpha = H$ and $\frac{1}{q} = H + \frac{1}{2}$. ( (cCYR) holds iff $H > \frac{1}{4}$, as expected).
- Under the above assumption on $\mathcal{H}$, we can prove that piecewise-linear approximations converge at rate

$$\rho_{\beta-}(X, X^D) \lesssim |D|^{\alpha-\beta}$$

as long as $\alpha + \beta + \frac{1}{p} > 1$.
This recovers the optimal rate of almost $2H - \frac{1}{2}$ for fBm from Friz-Riedel '14.

# Application : random Fourier series

## Proposition

*Let*

$$X(t) = \sum_k x_k \gamma_k e_k(t), \quad Y(t) = \sum_k y_k \gamma_k e_k(t),$$

*where* $e_k$ *is the usual* $L^2$ *trigonometric basis,* $\gamma_k$ *are i.i.d.* $\mathcal{N}(0,1)$, *and* $(x_k)$, $(y_k)$ *are deterministic such that for some* $\alpha > 1/4$,

$$\sup_k (|x_k| + |y_k|)\, k^{1/2+\alpha} := K < \infty.$$

*Then* $X$, $Y$ *both admit a* $\alpha^-$*-Hölder rough path lift, and it holds that, for any* $\beta' < \beta \leqslant \alpha$, *with* $\beta + \alpha > \frac{1}{2}$,

$$|\rho_{\beta'}(X,Y)|_{L^r} \leqslant C \sup_k \left( |x_y - y_k| k^{1/2+\beta} \right).$$

(Previously known results required more assumptions on coefficients, cf. Friz-Gess-Gulisashvili-Riedel '16).

# Outline

# Goal

Recall that assuming (CYR) gives continuity of

$$(\mathrm{X}, h) \in \mathcal{C}^\alpha \times \mathcal{H} \mapsto T^h \mathrm{X} \in \mathcal{C}^\alpha,$$

which is very useful.
Without (CYR) ?
Consider the topology $\mathcal{C}^{\alpha,\mathcal{H}}$ defined ($\alpha > \frac{1}{3}$) by

$$\|\mathrm{X}\|_{\alpha,\mathcal{H}} = \|\mathrm{X}\|_\alpha + \sup_{s<t} \sup_{\|h\|_\mathcal{H} \leqslant 1} \frac{\left|\int_s^t X_{s,r} \otimes dh_r\right| + \left|\int_s^t h_{s,r} \otimes dX_r\right|}{|t-s|^{2\alpha}}.$$

(In principle : stronger topology than $\mathcal{C}^\alpha$, equivalent under (cCYR)).
More or less by construction, it holds that

$$(\mathrm{X}, h) \in \mathcal{C}^{\alpha,\mathcal{H}} \times \mathcal{H} \mapsto T^h \mathrm{X} \in \mathcal{C}^{\alpha,\mathcal{H}}$$

is continuous.

# Main result : existence of canonical $(\alpha, \mathcal{H})$-lift.

It turns out that a simple condition guarantees $X$ lifts to $\mathbf{X} \in \mathcal{C}^{\alpha,\mathcal{H}}$.

**Theorem**

*Let $\widetilde{X}$ be an independent copy of $X$ and assume that $(X, \widetilde{X})$ admits a canonical $\alpha^-$-rough path lift.*
*Then $X$ admits a canonical $(\alpha^-, \mathcal{H})$-rough path lift.*

# Idea of proof

Based on an idea of Schönbauer '18 (in the context of singular SDPE), see also Inahama '14.

Key bound : if $F = F(X)$ is polynomial in $X$, then

$$\mathbb{E}\left[\|DF\|_{\mathcal{H}}^2\right] \lesssim \mathbb{E}\left[|F|^2\right].$$

If $F = \int X \otimes d\widetilde{X}$, then $\left\langle DF, \widetilde{h}\right\rangle_{\mathcal{H}} = \int X \otimes d\widetilde{h}$ .

It follows immediately that

$$\mathbb{E}\left[\sup_{\|h\|_{\mathcal{H}} \leqslant 1}\left|\int_s^t X_{s,r} \otimes dh_r\right|^2\right]^{1/2} \lesssim \mathbb{E}\left[\left|\int_s^t X_{s,r} \otimes d\widetilde{X}_r\right|^2\right]^{1/2} \lesssim |t-s|^{2\alpha^-}.$$

# Remarks

- In most cases of interest, $(X, \widetilde{X})$ has r.p. lift $\Leftrightarrow$ $X$ has r.p. lift, e.g. $X = (X^1, \dots, X^d)$, $X^i$ i.i.d., $d \geqslant 2$.
- The consequence of that result, is that anything that requires rough path bounds + C.M. translation can be done without explicitely assuming (CYR). This includes :
  - concentration of measure
  - Malliavin differentiability and existence of densities
  - Large deviations
  - Support theorem

  for RDE driven by Gaussian rough paths.

  Simply needs to work in $\mathcal{C}^{\alpha,\mathcal{H}}$ instead of $\mathcal{C}^{\alpha}$.

# Conclusion

When considering Gaussian processes, tricks based around

$$\text{stochastic noise} \;\leftrightarrow\; \text{Cameron-Martin element}$$

allow to prove some new estimates.

This was observed in the singular SPDE literature by Schönbauer '18 and Linares, Otto, Tempelmayr, Tsatsoulis '21.

In particular, in a rough path context, this yields

$$\text{Existence of a rough path lift} \;\underset{\approx}{\overset{\approx}{\Leftrightarrow}}\; \text{Complementary Young regularity} .$$

This simplifies earlier methods and gives new results.

Potentially useful in other contexts !