Hilbert Uniqueness Method and regularity

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Outline of the talk

1. Introduction: The Hilbert Uniqueness Method

2. An alternate HUM type method
   - The $\eta$-weighted HUM
   - Main result
   - Regularity of the controlled trajectory

3. Applications
   - Distributed controls
   - Boundary control

4. Conclusion

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4 Conclusion
An abstract control problem

Let $\mathcal{A}$ be the generator of a group on a Hilbert space $\mathcal{X}$. Consider the following model:

$$y'(t) = \mathcal{A} y(t) + B v(t), \quad y(0) = y_0 \in \mathcal{X},$$

where $B \in \mathcal{L}(U, D(\mathcal{A})^*)$ and $v \in L^2(0, T; U)$.

**Assumption**

For all $v \in L^2(0, T; U)$, solutions can be defined in the sense of transposition in $C^0([0, T]; \mathcal{X})$.

**Goal : Exact controllability**

Fix a time $T > 0$ and $y_0 \in \mathcal{X}$. Can we find $v \in L^2(0, T; U)$ such that $y(T) = 0$?
Hypotheses

Main Assumption

\[ \mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{X} \] generates a group.

Consequences:
- one can solve the equations forward and backward
- the same holds for the adjoint equation.
Examples

- **Wave equation in a bounded domain** with distributed control
  \[
  \begin{cases}
  u'' - \Delta u + a(x) \nabla u + b(x) u + c(x) u' = \chi_\omega v, & (t, x) \in \mathbb{R} \times \Omega, \\
  u|_{\partial\Omega} = 0, \\
  (u(0), \dot{u}(0)) = (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega),
  \end{cases}
  \]
  \[
  \mathcal{A} = \begin{pmatrix}
  0 & \text{Id} \\
  \Delta - a(x) \nabla - b & -c(x)
  \end{pmatrix}, \quad \mathcal{X} = H^1_0(\Omega) \times L^2(\Omega),
  \]
  \[
  \mathcal{B} = \begin{pmatrix}
  0 \\
  \chi_\omega
  \end{pmatrix}, \quad \mathcal{U} = L^2(\omega).
  \]

- **Wave equation in a bounded domain** with boundary control

- **Schrödinger equation** $\mathcal{A} = -i\Delta + \text{BC}$, **Linearized KdV**
  \[
  \mathcal{A} = \partial_{xxx} + \text{BC}, \text{ Maxwell equation, }\ldots
  \]
Use the adjoint system to characterize the controls!

For all $z$ solution of

$$-z' = A^* z, \quad z(T) = z_T \in \mathcal{X},$$

we have

$$\langle y(T), z_T \rangle_{\mathcal{X}} - \langle y_0, z(0) \rangle_{\mathcal{X}} = \int_0^T \langle v(t), B^* z(t) \rangle_{\mathcal{U}} \, dt.$$

In particular, $v$ is a control if and only if $\forall z_T \in \mathcal{X}$

$$0 = \int_0^T \langle v(t), B^* z(t) \rangle_{\mathcal{U}} \, dt + \langle y_0, z_0 \rangle_{\mathcal{X}}.$$
Fundamental hypotheses

- $B^* : \mathcal{D}(A) \rightarrow U, B^* \in \mathcal{L}(\mathcal{D}(A), U)$.

**Definition**

$B^*$ is **admissible** if $\forall T > 0, \exists K_T > 0,$

$$\int_0^T \| B^* z(t) \|_U^2 \, dt \leq K_T \| z_T \|_{\mathcal{X}}^2, \quad \forall \, z^0 \in \mathcal{D}(A).$$

**Definition**

$B^*$ is **exactly observable** at time $T^* > 0$ if $\exists k_* > 0,$

$$k_* \| z(0) \|_{\mathcal{X}}^2 \leq \int_0^{T^*} \| B^* z(t) \|_U^2 \, dt, \quad \forall \, z^0 \in \mathcal{X}.$$
If $B^*$ is admissible and exactly observable in some time $T^*$, then the following norms are equivalent:

- $\|z_T\|_{\mathcal{X}}, \|z(0)\|_{\mathcal{X}}$

- $\|z_T\|_{\text{obs},1}^{2} = \int_0^{T^*} \|B^*z(t)\|_{\mathcal{U}}^2 \, dt$

- $\|z_T\|_{\text{obs},2}^{2} = \int_0^{T^*} \eta(t) \|B^*z(t)\|_{\mathcal{U}}^2 \, dt$,

for $\eta \geq 0, \eta \geq \alpha > 0$ on some interval of length $T^*$. 

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Convergence rates of discrete controls
If $B^*$ is admissible and exactly observable in some time $T^*$, then the following norms are equivalent:

- $\|z_T\|_X, \|z(0)\|_X$
- $\|z_T\|_{obs,1}^2 = \int_0^{T^*} \|B^*z(t)\|_{U}^2 dt$
- $\|z_T\|_{obs,2}^2 = \int_0^{T^*} \eta(t) \|B^*z(t)\|_{U}^2 dt,$

for $\eta \geq 0$, $\eta \geq \alpha > 0$ on some interval of length $T^*$. 

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If $\mathcal{B}^*$ is admissible and exactly observable in some time $T^*$, then the following norms are equivalent:

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- $\|z_T\|_{\text{obs}, 1}^2 = \int_0^{T^*} \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt$
- $\|z_T\|_{\text{obs}, 2}^2 = \int_0^{T^*} \eta(t) \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt$

for $\eta \geq 0$, $\eta \geq \alpha > 0$ on some interval of length $T^*$. 
Let $T \geq T^*$. Define, for $z_T \in \mathcal{X}$,

$$J(z_T) = \frac{1}{2} \int_0^T \|B^* z(t)\|^2_U \, dt + \langle y_0, z(0) \rangle,$$ 

where $z$ satisfies $-z' = A^* z$, $z(T) = z_T$. Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z_T$. Then $V = B^* Z$ is such that the solution $y$ of

$$y' = Ay + B V, \quad y(0) = y_0,$$

satisfies $y(T) = 0$. Besides, $V$ is the control of minimal $L^2(0, T; \mathcal{U})$-norm.
Recall the following facts:

- the Hilbert Uniqueness Method does not prove observability/controllability, but prove their equivalence.
- Observability/controllability properties have to be proved first:
  - several possible methods:
    - multipliers, see Lions, Komornik, Alabau...
    - non-harmonic Fourier series, see Ingham, Haraux, Tucsnak...
    - Carleman estimates, see Puel, Osses, Zhang, etc
    - pseudo-differential calculus, see Bardos, Lebeau, Rauch, Burq, Gerard, etc
    - Resolvent estimates, see Burq, Zworski, Miller, Tucsnak...
On the regularity

If \( y_0 \in \mathcal{D}(A) \),
- Does the function \( Z_T \) computed that way belongs to \( \mathcal{D}(A) \)?
- Is the controlled solution \( (y, \nu) \) a strong solution?

Here, strong solutions means:

\[
y \in C^1([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X}_1),
\]

for some space \( \mathcal{X}_1 \) smoother than \( \mathcal{X} \) (for instance \( \simeq \mathcal{D}(A) \)).

General Answer: NO!
Consider the wave equation

$$\begin{cases}
y_{tt} - y_{xx} = 0, \\
y(0, t) = 0, \quad y(1, t) = v(t), \\
y(x, 0), y_t(x, 0)) = (y_0(x), y_1(x)) \in L^2(0, 1) \times H^{-1}(0, 1).
\end{cases}$$

0 < x < 1, 0 < t < T,$$

The adjoint problem is

$$z_{tt} - z_{xx} = 0, \quad z(0, t) = z(1, t) = 0, \quad (z_0, z_1) \in H_0^1(0, 1) \times L^2(0, 1),$$

and the solutions write

$$z(x, t) = \sqrt{2} \sum_{k \geq 1} \left( \hat{z}_0^k \cos(k \pi t) + \frac{\hat{z}_1^k}{k \pi} \sin(k \pi t) \right) \sin(k \pi x),$$

Controllability in time $T = 4$:

If $(y_0(x), y_1(x)) = \sqrt{2} \sum_{k \geq 1} (\hat{y}_0^k, \hat{y}_1^k) \sin(k \pi x),$$

$$\hat{z}_0^k = \frac{\hat{y}_1^k}{4k^2 \pi^2}, \quad \hat{z}_1^k = -\frac{\hat{y}_0^k}{4}. $$
In particular, the HUM control can be computed explicitly

\[ v(t) = Z_x(1, t) \]

\[ = \frac{1}{4} \sum_{k \geq 1} (-1)^k k \pi \left( \frac{\hat{y}_1^k}{k^2 \pi^2} \cos(k \pi t) - \frac{\hat{y}_0^k}{k \pi} \sin(k \pi t) \right). \]

\[ \implies v(0) = \frac{1}{4} \sum_{k \geq 1} (-1)^k \frac{\hat{y}_1^k}{k \pi} \neq 0 ! \]

\[ \implies \text{If } y_0 \in H^1_0(0, 1), \text{ the controlled solution is not a strong solution in general because of the failure of the compatibility conditions } y_0(1) = v(0) = 0. \]
Main question

Goal
To design a control method, and only one, which respects the regularity of the solutions.
If $y_0 \in D(A)$, we want

- $Z_T \in D(A^*)$,
- the controlled equation $y' = Ay + BV$ is satisfied in a strong sense.

Related result - Dehman Lebeau ’09 and Lebeau Nodet ’09: The wave equation with distributed control $B = \chi_\omega$ where $\chi_\omega$ is smooth, and where the HUM operator is modified by a weight function $\eta(t)$ vanishing at $t \in \{0, T\}$. 
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The modified HUM method: The $\eta$-weighted HUM

Let $y_0 \in \mathcal{X}$, and $\delta > 0$ such that $T - 2\delta \geq T^*$, where $T^*$ is the time of observability. Define, for $z_T \in \mathcal{X}$,

$$J(z_T) = \frac{1}{2} \int_0^T \eta(t) \|B^* z(t)\|_U^2 \, dt + \langle y_0, z(0) \rangle,$$

where $z$ satisfies $-z' = A^* z$, $z(T) = z_T$ and

$$\eta = \begin{cases} 
0 & \text{on } (-\infty, 0] \cup [T, \infty) \\
1 & \text{on } [\delta, T - \delta] 
\end{cases} \quad \eta \geq 0.$$

Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z_T$. Then $V = \eta B^* Z$ is such that the solution $y$ of

$$y' = Ay + BV, \quad y(0) = y_0,$$

satisfies $y(T) = 0$.

Besides, $V$ is the control of minimal $L^2((0, T), \frac{dt}{\eta}; U)$-norm.
Theorem (SE Zuazua ’09)

Assume that admissibility and observability property hold. Also assume that $\eta \in C^1(\mathbb{R})$.

If $y_0 \in D(A)$, then the minimizer $Z_T$ and the control function $V = \eta B^* Z$ computed by the $\eta$-weighted HUM are more regular:

- $Z_T \in D(A^*)$,
- $V \in H_0^1(0, T; U)$.

Moreover, there exists a constant $C$ such that

$$\|Z_T\|_{D(A^*)} + \|V\|_{H_0^1(0, T; U)} \leq C \|y_0\|_{D(A)}.$$
First remark that, due to the classical observability property,

$$\|Z_T\|_X + \|v\|_{L^2(0,T;\mathcal{U})} \leq C \|y_0\|_X.$$

Also remark that admissibility and observability properties yield

$$\tilde{k} \|Z_T\|_{D(A^*)} \leq k \|Z(0)\|_{D(A^*)} \leq \int_0^T \eta(t) \|B^*Z'(t)\|_{\mathcal{U}}^2 \, dt \leq K \|Z_T\|_{D(A^*)}$$

\[\rightarrow \] It is sufficient to prove that

$$\int_0^T \eta(t) \|B^*Z'(t)\|_{\mathcal{U}}^2 \, dt < \infty.$$

Indeed, this implies $Z_T \in D(A^*)$ and $v \in H^1_0(0,T;\mathcal{U})$. 

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Idea of the proof-I

Write the characterization of the control $V = \eta B^* Z$:

$$0 = \int_0^T \eta(t) \langle B^* Z(t), B^* z(t) \rangle_U \, dt + \langle y_0, z(0) \rangle_x,$$

for all $z$ solution of $-z' = A^* z$, $z(T) = z_T$.

Then take formally $z = Z'' = (A^*)^2 Z$:

$$\int_0^T \eta(t) \| B^* Z'(t) \|^2_U \, dt = -\langle A y_0, A^* Z(0) \rangle_x$$

$$- \int_0^T \eta'(t) \langle B^* Z'(t), B^* Z(t) \rangle_U \, dt.$$

But

$$|\langle A y_0, A^* Z(0) \rangle_x| \leq C \| A y_0 \|_x \| A^* Z_T \|_x$$

and

$$\| A^* Z_T \|_x \leq C \int_0^T \eta(t) \| B^* Z'(t) \|^2_U \, dt.$$
Idea of the proof-II

But

\[
\left| \int_0^T \eta'(t) \langle B^* Z'(t), B^* Z(t) \rangle_U \, dt \right| \\
\leq C \left( \int_0^T \| B^* Z'(t) \|_U^2 \, dt \right)^{1/2} \left( \int_0^T \| B^* Z(t) \|_U^2 \, dt \right)^{1/2} \\
\leq C \| A^* Z_T \|_X \| Z_T \|_X \leq C \| A^* Z_T \|_X \| y_0 \|_X
\]

Using observability,

\[
\int_0^T \eta(t) \| B^* Z'(t) \|_U^2 \, dt \leq C \| y_0 \|_{D(A)}^2.
\]
Theorem (SE-Zuazua 09)

Let $s \in \mathbb{N}$ and assume that $\eta \in C^s(\mathbb{R})$. If the initial datum $y_0 \in D(A^s)$, then the minimizer $Z_T$ and the control function $V$ given by the $\eta$-weighted HUM satisfy:

- $Z_T \in D((A^*)^s)$
- $V \in H^s_0(0, T; \mathcal{U})$.

Besides, there exists a positive constant $C_s = C_s(\eta, k^*, K_T)$ independent of $y_0 \in D(A^s)$, such that:

$$\|Z_T\|_{D((A^*)^s)}^2 + \int_0^T \|V^{(s)}(t)\|_{\mathcal{U}}^2 \, dt \leq C_s \|y_0\|_{D(A^s)}^2.$$
If $\eta \in C^\infty(\mathbb{R})$, the $\eta$-weighted HUM map defined by

$$\Theta : \begin{cases} \mathcal{X} \rightarrow \mathcal{X} \times L^2(0, T; dt/\eta, \mathcal{U}) \\ y_0 \mapsto (Z_T, V) \end{cases}$$

defines by restriction a map on $\mathcal{D}(\mathcal{A}^s)$:

$$\Theta : \mathcal{D}(\mathcal{A}^s) \rightarrow \mathcal{D}((\mathcal{A}^*)^s) \times H_0^s(0, T; \mathcal{U}).$$

- **No smoothness assumption** on $\mathcal{B}$, or on $[\mathcal{A}, \mathcal{B}\mathcal{B}^*]$. 
  generalizes Dehman Lebeau ’09 to the cases $\mathcal{B} = \chi_\omega$ not smooth and boundary control cases, but less precise: pseudo-differential estimates on $\Theta$ for the wave equation with a distributed controlled.

- We have not make precise yet the regularity of the control trajectory !!!!

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Convergence rates of discrete controls
If $\eta \in C^\infty(\mathbb{R})$, the $\eta$-weighted HUM map defined by

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- **We have not make precise yet the regularity of the control trajectory !!!!**
And the regularity of the controlled trajectory?

**Corollary**

if the initial datum $y_0 \in \mathcal{D}(A^s)$, then the controlled trajectory $y$ with control function $V$ given by the $\eta$-weighted HUM satisfies

$$y(t) \in \bigcap_{k=0}^{s} C^k([0, T]; \mathcal{Z}_{s-k}),$$

where the spaces $\mathcal{Z}_j$ are defined by induction by

$$\mathcal{Z}_0 = X, \quad \mathcal{Z}_j = A^{-1}(\mathcal{Z}_{j-1} + BB^*\mathcal{D}((A^*)^j)).$$

**Important Remark**

In several situations, these spaces can be computed.
Idea of the proof

\( Z_T \in \mathcal{D}(\mathcal{A}^s) \) and \( Z \) solves \( -Z' = \mathcal{A}^s Z \)

\[ \implies Z \in \bigcap_{k=0}^{s} C^k([0, T]; \mathcal{D}(\mathcal{A}^{s-k})) \]

\[ \implies V = \eta \mathcal{B}^* Z \in \bigcap_{k=0}^{s} C^k([0, T]; \mathcal{B}^* D(\mathcal{A}^{s-k})) \]

\[ \implies \mathcal{B} V = \eta \mathcal{B} \mathcal{B}^* Z \in \bigcap_{k=0}^{s} C^k([0, T]; \mathcal{B} \mathcal{B}^* D(\mathcal{A}^{s-k})) \]

Besides, \( V \in H^s_0(0, T; \mathcal{U}) \) since \( \mathcal{A}^s Z_T \in \mathcal{X} \).

Hence the result by standard regularity results.
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4 Conclusion
Let $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$.

Wave equation:

\[
\begin{cases}
  y'' - \Delta y = v\chi_\omega, & \text{in } \Omega \times (0, \infty), \\
  y = 0, & \text{on } \partial\Omega \times (0, \infty), \\
  (y(0), y'(0)) = (y_0, y_1) & \in H^1_0(\Omega) \times L^2(\Omega),
\end{cases}
\]

- $v \in L^2((0, T) \times \omega)$ is the control function
- $\chi_\omega(x)$ supported in $\omega$, strictly positive on some $\overline{\omega_0} \subset \omega$ (no smoothness assumption)
Assume GCC in time $T^*$:

(Bardos Lebeau Rauch ’92 & Burq Gerard ’96)

All the rays of Geometric Optics enter in the control region $\omega$ before the time $T^* > 0$.

Take $T - 2\delta > T^*$, and $\eta = \begin{cases} 0 & \text{on } (-\infty, 0) \cup [T, \infty) \\ 1 & \text{on } [\delta, T - \delta] \end{cases}$ $\eta \geq 0$. 

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The $\eta$-weighted HUM

Our functional reads:

$$J(z_0, z_1) = \frac{1}{2} \int_0^T \int_\omega \eta(t) \chi_\omega^2(x) |z(x, t)|^2 \, dx \, dt$$

$$+ \langle y_0, z'(\cdot, 0) \rangle_{H^1_0(\Omega) \times H^{-1}(\Omega)} - \int_\Omega y_1(x) z(x, 0) \, dx$$

where $z$ is the solution of

$$\begin{cases}
  z'' - \Delta z = 0, & \text{in } \Omega \times (0, \infty), \\
  z = 0, & \text{on } \partial\Omega \times (0, \infty), \\
  (z(T), z'(T)) = (z_0, z_1) & \in L^2(\Omega) \times H^{-1}(\Omega).
\end{cases}$$
Theorem-I

Given any \((y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)\), there exists a unique minimizer \((Z_0, Z_1)\) of \(J\) over \(L^2(\Omega) \times H^{-1}(\Omega)\). The function

\[ V(x, t) = \eta(t) \chi_\omega(x) Z(x, t) \]

is the control function of minimal \(L^2(0, T; dt/\eta; L^2(\omega))\)-norm, defined by

\[ \|v\|_{L^2(0,T;dt/\eta;L^2(\omega))}^2 = \int_0^T \int_\omega |v(x, t)|^2 dx \frac{dt}{\eta(t)}. \]

If \((y_0, y_1) \in \mathcal{D}(A^s)\) for some \(s \in \mathbb{N}\), 
\((Z_0, Z_1) \in \mathcal{D}((A^*)^s) = \mathcal{D}(A^{s-1})\) and \(V \in H_0^s(0, T; L^2(\omega))\).
### Results

#### Theorem-II

If \((y_0, y_1) \in \mathcal{D}(A^s)\) for some \(s \in \mathbb{N}\) and \(\chi_\omega\) is smooth, the \(\eta\)-weighted HUM yields

\[
V \in H^s_0(0, T; L^2(\omega)) \cap \bigcap_{k=0}^{s} C^k([0, T]; H^{s-k}_0(\omega))
\]

and the controlled trajectory satisfies

\[
y \in \bigcap_{k=0}^{s} C^k([0, T]; H^{s+1-k}(\Omega) \times H^{s-k}(\Omega)).
\]

\(\Rightarrow\) Yields another proof of the one in Dehman Lebeau 09.

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Convergence rates of discrete controls
Let $\Omega \subset \mathbb{R}^N$, $\Gamma = \partial \Omega$, $\Gamma_0 \subset \Gamma$ satisifying GCC in time $T^*$. 
Let $\xi_{\Gamma_0}$ be defined on $\partial \Omega$, non-vanishing on $\Gamma_0$. 
We now consider the following wave equation:

$$
\begin{align*}
\left\{
\begin{array}{ll}
y'' - \Delta y = 0, & \text{in } \Omega \times (0, \infty), \\
y = \xi_{\Gamma_0} v, & \text{on } \Gamma \times (0, \infty), \\
(y(0), y'(0)) = (y_0, y_1) & \in L^2(\Omega) \times H^{-1}(\Omega).
\end{array}
\right.
\end{align*}
$$

- $\eta(t)$ as before.
The functional $J$ reads:

$$J(z_0, z_1) = \frac{1}{2} \int_0^T \int_{\Gamma} \eta(t) \xi_0(x)^2 |\partial_n z(x, t)|^2 \, d\Gamma \, dt$$

$$+ \int_0^1 y_0(x) z'(x, 0) \, dx - \langle y_1, z(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)},$$

where $z$ is the solution of

$$\begin{cases} 
  z'' - \Delta z = 0, & \text{in } \Omega \times (0, \infty), \\
  z = 0, & \text{on } \partial \Omega \times (0, \infty), \\
  (z(T), z'(T)) = (z_0, z_1) & \in H^1_0(\Omega) \times L^2(\Omega).
\end{cases}$$
Theorem

Assume that $\Gamma_0$, $T^*$ satisfies GCC.
Given any $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique minimizer $(Z_0, Z_1)$ of $J$ over $H_0^1(\Omega) \times L^2(\Omega)$. The function

$$V(x, t) = \eta(t) \xi_{\Gamma_0}(x) \partial_n Y(x, t)|_{\Gamma_1}$$

is the control method of minimal $L^2(0, T; dt/\eta; L^2(\Gamma))$-norm, defined by

$$\|V\|_{L^2(0,T;dt/\eta;L^2(\Gamma))}^2 = \int_0^T \int_{\Gamma} |V(x, t)|^2 d\Gamma \frac{dt}{\eta(t)}.$$ 

If $(y_0, y_1) \in \mathcal{D}(A^s)$ for some $s \in \mathbb{N}$, then $(Z^0, Z^1) \in \mathcal{D}((A^*)^s) = \mathcal{D}(A^{s+1})$ and $V \in H^s_0(0, T; L^2(\Gamma))$. 

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Convergence rates of discrete controls
Assume that the function $\xi_{\Gamma_0}$ is smooth. If $(y_0, y_1) \in D(A^s)$ for some $s \in \mathbb{N}$, the $\eta$-weighted HUM yields:

$$V \in H_0^s(0, T; L^2(\Gamma_1)) \cap \bigcap_{k=0}^s C^k([0, T]; H^{s-k-1/2}(\Gamma_1))$$

and $(Z_0, Z_1) \in D((A^*)^s) = D(A^{s+1})$. In particular, the controlled trajectory $y$ satisfies

$$y \in \bigcap_{k=0}^s C^k([0, T]; H^{s-k}(\Omega) \times H^{s-1-k}(\Omega)).$$
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Conclusion
Further applications

★ Orders of convergence of discrete controls
Joint work with Enrique Zuazua
The exact controls computed with the $\eta$-weighted HUM converge at a rate $h^{2/3}$ to the continuous $\eta$-HUM control.

★ Controllability of semi-linear wave equations
See Dehman-Lebeau ’09 for such examples.
Open problem-I

Exact controllability

What happens for the exact controllability when $\mathcal{A}$ does not generate a group?

Typical example: Linearized KdV equation

$\Rightarrow$ Can we design a HUM type method which yields smooth controlled trajectory for smooth initial data?

$\Rightarrow$ Do we have such regularity properties for the non linear KdV equation?

Work in progress...
Open problem-II

Exact controllability to trajectories

What happens when $A$ does not generate a group?

Typical example: The heat equation.

$\rightsquigarrow$ Can we design a HUM type method which yields smooth controlled trajectory for smooth initial data?

Widely open.
Thank you for your attention!

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