

Approximate Lagrangian controllability for the 2-D Euler equation. Application to the control of the shape of vortex patches.

O. Glass*, & T. Horsin†

Abstract.

In this paper, we consider the two-dimensional Euler equation in a bounded domain Ω , with a boundary control located on an arbitrary part of the boundary. We prove that, given two Jordan curves which are homotopic in Ω and which surround the same area, given an initial data and a positive time T , one can find a control such that the corresponding solution drives the first curve inside Ω arbitrarily close to the second one (in any C^k norm) at time T . We also prove that given two vortex patches satisfying the same conditions on their contour, one can approximately deform the first one into the second one.

Résumé.

Dans cet article, nous considérons l'équation d'Euler des fluides parfaits incompressibles dans un domaine borné bidimensionnel, avec un contrôle frontière localisé sur une partie arbitraire du bord. Nous montrons qu'étant donnés deux courbes de Jordan homotopes et encerclant la même aire, une donnée initiale et un temps T strictement positif, on peut trouver un contrôle tel que la solution correspondante de l'équation d'Euler mène la première courbe vers la seconde de manière arbitrairement proche (en toute norme C^k) au temps T . Nous montrons également que sous la même condition sur les contours, on peut déformer de manière approchée une poche de tourbillon sur une autre dans le domaine.

Keywords. Controllability, perfect incompressible fluids, vortex patches.

1 Introduction

1.1 Position of the problem

In this paper, we investigate the problem of Lagrangian controllability for the two-dimensional Euler equation. Let us introduce $\Omega \subset \mathbb{R}^2$ an open, bounded and regular domain. We consider the Euler equation of perfect incompressible fluids in Ω :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } [0, T] \times \Omega, \\ \operatorname{div} u = 0 & \text{in } [0, T] \times \Omega, \end{cases} \quad (1)$$

for some time $T > 0$. Here $u : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ denotes the velocity of the fluid and $p : [0, T] \times \Omega \rightarrow \mathbb{R}$ the pressure field. We also consider a non empty open part Σ of the boundary $\partial\Omega$ of Ω . We will use the following classical non-penetration boundary condition on $\partial\Omega \setminus \Sigma$:

$$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma), \quad (2)$$

*UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 4 place Jussieu, F-75005, Paris, France & CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

†Laboratoire de Mathématiques de Versailles, UMR 8100, Université de Versailles, 45, avenue des États-Unis, 78030 Versailles Cedex

and use the remaining of the boundary condition on Σ as a control, that is, a way to influence the system. The initial-boundary problem for equation (1) with non-homogeneous boundary conditions has been studied by Yudovich [26]. Given a divergence-free initial data:

$$u|_{t=0} = u_0 \text{ in } \Omega, \quad (3)$$

under suitable regularity and compatibility conditions, the system is well-posed provided that one imposes the following boundary conditions on Σ :

- the normal component of the velocity on Σ for any time

$$u(t, x) \cdot n(x) \text{ on } [0, T] \times \Sigma, \quad (4)$$

where $n(x)$ is the unit outward normal on $\partial\Omega$, and due to incompressibility this condition has of course to satisfy

$$\int_{\partial\Omega} u(t, x) \cdot n(x) dx = 0 \quad \forall t \in [0, T],$$

- the vorticity $\omega(t, x) := \text{curl } u(t, x)$ at entering points of $\partial\Omega$, that is,

$$\text{curl } u(t, x) \text{ on } \Sigma_T^- := \{(t, x) \in [0, T] \times \Sigma / u(t, x) \cdot n(x) < 0\}. \quad (5)$$

(Note however that the assumptions of [26] are not quite satisfied here).

With this boundary condition on Σ , the usual problems of exact controllability and approximate controllability have been studied by J.-M. Coron [5, 6] and the first author [13, 14, 15]. The question raised in those papers is the possibility of driving the state of the system from a given initial condition (3) to a given target at time T .

Here we are interested in another type of controllability: given a certain zone in the fluid, is it possible to use the control so as to move this zone from some place to another prescribed one when following the flow of the fluid velocity? If not exactly, at least in an approximate manner? One may think of the following application: assume that a zone of the fluid is polluted; is it possible to find a control driving the pollutant to some prescribed safe place?

In this paper, the two zones of fluid correspond to a Jordan domain, that is the interior of a Jordan curve contained in the open domain Ω . The problem becomes more precisely the following.

- *Exact Lagrangian controllability.* Given $T > 0$, two Jordan curves γ_1 and γ_2 in Ω and an initial state u_0 , does there exist a control such that the flow ψ of the solution u of (1)-(3) satisfies $\psi(T, \gamma_1) = \gamma_2$?
- *Approximate Lagrangian controllability in C^k .* Given $\varepsilon > 0$, $T > 0$, two Jordan curves γ_1 and γ_2 in Ω and an initial state u_0 , does there exist a control such that the flow ψ of the solution u of (1)-(3) satisfies that, up to reparameterization,

$$\|\psi(T, \gamma_1) - \gamma_2\|_{C^k} \leq \varepsilon?$$

In both cases, it seems moreover natural, in order to control the zone to be transported during the whole time interval, to ask that it should not leave the domain Ω :

$$\forall t \in [0, T], \psi(t, \gamma_0) \subset \Omega. \quad (6)$$

The denomination ‘‘Lagrangian’’ is related to the fact that, contrary to most other works on controllability of fluids, one does not try to impose the velocity of the fluid which is an Eulerian description of it but we want to prescribe the motion of a set of fluid particles.

Another way of expressing these controllability problems is to look for the solution u of the system (1)-(24)-(3), which is underdetermined since we do not specify the condition on $[0, T] \times \Sigma$, rather than for the control explicitly. Of course, in that case, one can recover the control from u by considering the relevant trace.

In this paper, we will prove two approximate Lagrangian controllability results for the bidimensional Euler equation. The first one concerns the case of regular solutions of (1), the second one the so-called vortex patch solutions. We will give some arguments proving that the exact Lagrangian controllability does not hold in general (see Remark 5 below), at least if we impose the natural condition (6). Our results are based on an approximate Lagrangian controllability property for the equation of potential flows, which can be seen as the central statement of this paper.

1.2 Results

Our first main result is the following one.

Theorem 1. *Consider γ_0 and γ_1 two C^∞ Jordan curves in Ω such that*

$$|\text{Int}(\gamma_0)| = |\text{Int}(\gamma_1)|. \quad (7)$$

$$\gamma_0 \text{ and } \gamma_1 \text{ are homotopic in } \Omega, \quad (8)$$

Let us consider $u_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying

$$\text{div}(u_0) = 0 \text{ in } \Omega \text{ and } u_0 \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma). \quad (9)$$

For any $T > 0$, any $k \in \mathbb{N}$, any $\varepsilon > 0$, there exists $u \in C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ satisfying (1), (2) and (3), and whose flow Φ^u satisfies

$$\forall t \in [0, T], \Phi^u(t, 0, \gamma_0) \subset \Omega, \quad (10)$$

and up to a reparameterization of the curve,

$$\|\gamma_1 - \Phi^u(T, 0, \gamma_0)\|_{C^k(\mathbb{S}^1)} \leq \varepsilon. \quad (11)$$

Remark 1. *Due to the incompressibility of the fluid, Property (7) is of course necessary. Property (8) is of course a necessary condition as well. The standard case of (8) is when neither γ_1 nor γ_2 contains a connected component of $\partial\Omega$, that is, are contractible in Ω .*

Theorem 1 considers the case of smooth data, namely when the initial data is in $C^\infty(\overline{\Omega})$. The celebrated result of Yudovich [25] proves the existence and uniqueness of a solution with initial vorticity in $L^\infty(\Omega)$. In particular, one can consider the case of the so-called vortex patches, that is, solutions for which the initial condition in vorticity is the characteristic function of a regular open set in Ω , typically the interior of a regular Jordan curve. An important result of Chemin [3, 4] is that the regularity of the boundary of a vortex patch propagates globally over time (actually, Chemin's result is much more general). An alternative proof of this fact was given by Bertozzi and Constantin [1]. The case of a vortex patch in a bounded domain, in a case including the one considered here, was investigated by Depauw [8] and Dutrifoy [9]. For a recent survey and new results concerning this topic, we refer to [23].

Our next result considers this particular type of solutions. Roughly speaking, it states that one can approximately transform the shape of a vortex patch corresponding to Jordan domain into another prescribed one, inside Ω , provided the target has the same area and is homotopic to the initial shape. Note that in order that a vortex patch solution remains a vortex patch solution, we should impose the control to satisfy

$$\text{curl } u = 0 \text{ on } \Sigma_T^-, \quad (12)$$

that is, we should do not add vorticity in the domain (and the patch should not leave the domain either as to satisfy (6)).

Theorem 2. *Consider γ_0 and γ_1 two C^∞ Jordan curves in Ω satisfying (7) and (8), the control zone Σ being outside these curves. Let us consider $u_0 \in \mathcal{Lip}(\overline{\Omega}; \mathbb{R}^2)$ with $u_0 \cdot n \in C^\infty(\partial\Omega)$ a "vortex patch" initial condition corresponding to γ_0 in Ω , namely, a solution of*

$$\begin{cases} \text{curl}(u_0) = \mathbf{1}_{\text{Int}(\gamma_0)} \text{ in } \Omega, \\ \text{div}(u_0) = 0 \text{ in } \Omega, \\ u_0 \cdot n = 0 \text{ on } \partial\Omega \setminus \Sigma. \end{cases} \quad (13)$$

For any $T > 0$, any $k \in \mathbb{N}$, any $\varepsilon > 0$, there exists $u \in L^\infty([0, T]; \mathcal{Lip}(\overline{\Omega}))$ satisfying (1), (2), (3) and (12), and whose flow Φ^u satisfies (10) and, up to a reparameterization,

$$\|\gamma_1 - \Phi^u(T, 0, \gamma_0)\|_{C^k(\mathbb{S}^1)} \leq \varepsilon. \quad (14)$$

Remark 2. The system (13) gives a unique solution up to a g -dimensional vector space (namely the first de Rham cohomology space), when $\partial\Omega$ has $g + 1$ connected components. The uniqueness can be retrieved by imposing the circulation of u_0 along g connected components of $\partial\Omega$.

Remark 3. In Theorems 1 and 2, the solution which we determine will be shown to be the unique solution of the initial-boundary problem in its class. More precisely, in the case of Theorem 1, the solution u will proved to be unique in $L^\infty(0, T; W^{2,\infty}(\Omega))$; this requires more regularity than for the homogeneous case: this is a consequence of the entering data on the boundary, see [26] where such a regularity is also required for the uniqueness. In Theorem 2, the solution u will be unique in the class $L^\infty(0, T; LL(\Omega))$ (where $LL(\Omega)$ is the space of log-Lipschitz vector fields – see Section 5) and provided that there is no vorticity near the connected components of the boundary containing the control zone.

The two above results are consequences of the following central one, which states that it is possible to approximately control the displacement of a Jordan curve via a potential flow (without letting the curve leave the domain).

Theorem 3. Let Ω a bounded regular nonempty connected open set in \mathbb{R}^2 . Let γ_0 and γ_1 two C^∞ Jordan curves in Ω satisfying (7) and (8). Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C_0^\infty([0, 1]; C^\infty(\bar{\Omega}; \mathbb{R}))$ such that

$$\Delta_x \theta(t, \cdot) = 0 \text{ in } \Omega, \text{ for all } t \in [0, 1], \quad (15)$$

$$\frac{\partial \theta}{\partial n} = 0 \text{ on } [0, 1] \times (\partial\Omega \setminus \Sigma), \quad (16)$$

$$\forall t \in [0, 1], \Phi^{\nabla \theta}(t, 0, \gamma_0) \subset \Omega, \quad (17)$$

and, up to a reparameterization,

$$\|\gamma_1 - \Phi^{\nabla \theta}(1, 0, \gamma_0)\|_{C^k(\mathbb{S}^1)} \leq \varepsilon. \quad (18)$$

Remark 4. Let us emphasize that potential solutions are solutions to the Euler equation (1) and to the Navier-Stokes system as well:

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0. \quad (19)$$

However, we only consider the boundary condition (2), which is insufficient for the Navier-Stokes equation for which either Dirichlet or Navier slip boundary conditions are used.

Remark 5. The exact Lagrangian controllability does not hold for the equation of potential flows: $u(t, x) = \nabla \theta(t, x)$ with θ satisfying (15). This is easily seen by considering γ_1 an analytic curve and γ_2 a C^∞ but not analytic curve. Since potential flows are analytic in space, it is clear that these flow cannot drive exactly γ_1 toward γ_2 . Since we impose the condition (10), the same argument can be extended to the case of the Euler equation: consider again γ_1 analytic and γ_2 of class C^∞ but not analytic, and impose that $\text{curl } u_0 \equiv 0$ in a neighborhood of γ_1 . Since the vorticity of the equation is transported by the flow, the vorticity will stay 0 in a neighborhood of the curve, which again yields that its analyticity is propagated over time.

Remark 6. Due to (2), even the approximate controllability would not hold in general if we authorized the curves γ_0 and γ_1 to meet $\partial\Omega$.

Remark 7. The only condition imposed on Σ for Theorems 1 is its non-emptiness. For the exact controllability in the usual sense to occur, it is necessary and sufficient that Σ meets all the connected components of the boundary, see [6].

Remark 8. Under the conditions that Σ meets all the connected components of the boundary, the results for exact controllability of the Euler equation [5, 6, 13] rely on the strategy consisting in making all the fluid go outside of the domain. Applying this strategy to the problem under view, in the particular case where the curves are contractible in Ω , we could make γ_0 leave the domain, and then (by a time-reversibility argument) let a curve enter in Ω and take the place of γ_1 . But this is not what we intend to do here, where we really want to control the trajectory of γ_0 inside the fluid domain, see in particular condition (17).

1.3 Structure of the proof

The largest part of the proof consists in establishing Theorem 3. The proof of Theorem 3 can be split in two pieces, which are the following propositions.

Proposition 1. *Let Ω a bounded regular nonempty connected open set in \mathbb{R}^2 . Let J_1 and J_2 two C^∞ Jordan curves in Ω such that*

$$J_1 \text{ and } J_2 \text{ are homotopic in } \Omega, \quad (20)$$

$$|\text{Int}(J_1)| = |\text{Int}(J_2)|. \quad (21)$$

Then there exists $v \in C_0^\infty((0, 1) \times \Omega; \mathbb{R}^2)$ such that

$$\text{div } v = 0 \text{ in } (0, 1) \times \Omega, \quad (22)$$

$$\Phi^v(1, 0, J_1) = J_2. \quad (23)$$

The second proposition is the following.

Proposition 2. *Let γ_0 be a smooth Jordan curve; let $X \in C^0([0, 1]; C^\infty(\overline{\Omega}))$ be a smooth divergence-free vector field satisfying*

$$X.n = 0 \text{ on } [0, 1] \times \partial\Omega. \quad (24)$$

Fix

$$\gamma_1 := \Phi^X(1, 0, \gamma_0). \quad (25)$$

Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C^\infty([0, 1] \times \overline{\Omega}; \mathbb{R})$ satisfying (15), (16), (17) and (18).

Remark 9. *Condition (24) is here only to make sure that $\Phi^X(t, 0, \gamma_0)$ does not quit the domain Ω .*

Once these two propositions proven, establishing Theorem 3 is immediate, since the compactness in time of θ is just a matter of reparameterization in time.

In Section 2, we introduce the main notations of the paper. Sections 3 and 4 are devoted to the proof of Theorem 3, to be specific to Propositions 1 and 2 respectively. Theorems 1 and 2 are finally proven in Section 5.

2 Notations

In this section, we fix several notations.

The domain Ω is smooth and we will denote by n the unit outward normal vector field on $\partial\Omega$. We will prefer to use the letter ν for the outward normal on some other curve. Call $\Gamma_0, \dots, \Gamma_g$ the connected components of $\partial\Omega$. In the sequel, we will suppose that Σ meets Γ_0 .

In the sequel, for $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ and a domain V , we denote by $C^{k,\alpha}(V)$ the space of functions on V having k derivatives in the Hölder space $C^\alpha(V)$ of index α . We denote by $\mathcal{Lip}(V)$ the space of Lipschitz functions on V . As usual, we add an index 0 to refer to compactly supported functions.

Given a vector field in $v \in L^1([0, T], \mathcal{Lip}(U))$ for some $T > 0$ and U a nonempty connected open set in \mathbb{R}^2 , we will denote by $(s, t, x) \in [0, T]^2 \times U \mapsto \Phi^v(t, s, x) \in U$ the flow of the vector field v , that is, the solution of

$$\frac{\partial}{\partial t} \Phi^v(t, s, x) = v(t, \Phi^v(t, s, x)) \text{ and } \Phi^v(s, s, x) = x,$$

whenever it is defined. Of course we include the case where v is time-independent.

We will systematically identify the complex plane \mathbb{C} to \mathbb{R}^2 ; \mathbb{S} will denote the unit circle in \mathbb{C} and \mathbb{B} the closed unit ball. For U an open set in \mathbb{C} , we will denote by $\mathcal{H}(U)$ the set of holomorphic functions on U ; when F is a closed set, $\mathcal{H}(F)$ denotes the set of holomorphic functions on some open neighborhood of F .

Given a Jordan curve $J \subset \mathbb{R}^2$, we will denote $\text{Int}(J)$ its interior, i.e. the bounded connected component of $\mathbb{R}^2 \setminus J$.

To a holomorphic function $f : \omega \subset \mathbb{C} \rightarrow \mathbb{C}$, we associate the corresponding vector field $Vf : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f = f_1 + if_2 \mapsto Vf = \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}.$$

Of course

$$f \text{ satisfies the Cauchy-Riemann equations } \iff \operatorname{curl} Vf = \operatorname{div} Vf = 0. \quad (26)$$

Let us recall the standard topology for the real-analytic functions (see for instance [19]). Let K be some compact set in \mathbb{R}^n . Introduce U_i a decreasing family of open neighborhoods of K in \mathbb{C}^n , such that $\bigcap_i U_i = K$. One considers for $j \leq k$ the natural mapping $\mathcal{H}(U_j) \rightarrow \mathcal{H}(U_k)$ given by the restriction. Then one defines $C^\omega(K)$ as the inductive limit (in the category of locally convex spaces)

$$C^\omega(K) = \lim_{i \in \mathbb{I}} \mathcal{H}(U_i).$$

As is classical, we extend the definition to $C^\omega(M, \mathbb{R})$ where M is a real-analytic manifold (some analytic Jordan curve in the sequel).

Given a set $F \subset \mathbb{R}^2$, we denote $\mathcal{V}_\varepsilon F$ its ε -neighborhood.

Finally we mention that in the sequel, by ‘‘smooth’’ we will systematically mean of class C^∞ ; by ‘‘analytic’’ we will mean ‘‘real-analytic’’.

3 Proof of Proposition 1

In this section we establish Proposition 1. This is done in several steps.

3.1 Reduction to a special case

Let us prove that it suffices to establish Proposition 1 in the particular case where

$$J_1 \cap J_2 \neq \emptyset \text{ and } J_1 \text{ and } J_2 \text{ intersect transversally.} \quad (27)$$

Then in Section 3.2 we will prove Proposition 1 under condition (27).

1. To prove that this is sufficient, let us introduce J_1 and J_2 as assumed in Proposition 1. Then it is enough to find $\bar{v} \in C_0^\infty((0, 1) \times \Omega; \mathbb{R}^2)$ such that (22) applies and that $\tilde{J}_1 := \Phi^{\bar{v}}(1, 0, J_1)$ satisfies (27). Indeed, once this is obtained, introducing $\hat{v} \in C_0^\infty((0, 1) \times \Omega; \mathbb{R}^2)$ satisfying (22) and leading \tilde{J}_1 to J_2 , it is just a matter of considering

$$v := \begin{cases} 2\bar{v}(2t, x) & \text{for } (t, x) \in [0, 1/2] \times \Omega, \\ 2\hat{v}(2(t - 1/2), x) & \text{for } (t, x) \in [1/2, 1] \times \Omega. \end{cases} \quad (28)$$

2. Let us now explain the construction of \bar{v} . By the same concatenation argument as (28), we first construct \bar{v} which ensures only that

$$\Phi^{\bar{v}}(1, 0, J_1) \cap J_2 \neq \emptyset, \quad (29)$$

and then show how to get the transversality in a second time. By connectedness of Ω , one can find a smooth (not self-intersecting) path γ from some point M in J_1 to some point N in $\operatorname{Int}(J_2)$. By reparameterizing the time, one can assume that $\gamma \in C^\infty([0, 1]; \Omega)$, with $\gamma(0) = M$ (resp. $\gamma(1) = N$) in some neighborhood of $t = 0$ (resp. of $t = 1$). Now consider the vector field defined on the graph of γ in $[0, 1] \times \Omega$ by $(t, \gamma(t)) \mapsto \dot{\gamma}(t)$. One can extend it to $v_1 \in C_0^\infty((0, 1) \times \Omega; \mathbb{R}^2)$ in a way that fulfills (22): for this one introduces some smooth ψ_1 defined on some neighborhood the graph of γ , compactly supported in time, such that

$$\nabla_x^\perp \psi_1(t, x) = \dot{\gamma}(t) \text{ on the graph of } \gamma. \quad (30)$$

Then one extends ψ_1 to $\hat{\psi}_1 \in C_0^\infty((0, 1) \times \Omega; \mathbb{R})$ and define

$$v_1 := \nabla^\perp \hat{\psi}_1. \quad (31)$$

Note that $\Phi^{v_1}(1, 0, J_1)$ still satisfies (20)-(21) with J_2 .

3. Now let us explain how we can get the transversality property: consider J_1 and J_2 satisfying (20), (21) and (29) and let us find some $v_2 \in C_0^\infty((0, 1) \times \Omega; \mathbb{R}^2)$ satisfying (22) and such that $\Phi^{v_2}(1, 0, J_1)$ satisfies (27). For this, we notice that by the openness of Ω and the compactness of a Jordan curve, a small translation of J_1 in \mathbb{R}^2 still lies in Ω . But it follows from the parametric version of Thom's transversality theorem [17, Theorem 2.7] that the set of vectors by the translation of which J_1 is sent to a curve transversal to J_2 is dense. Hence there exists $r \in \mathbb{R}^2$ such that

$$\begin{aligned} \forall t \in [0, 1], J_1 + tr \subset \Omega, \\ (J_1 + r) \cap J_2 \neq \emptyset \text{ and } J_1 + r \text{ is transverse to } J_2. \end{aligned}$$

Introduce ψ_2 defined for each $t \in [0, 1]$ in some neighborhood in Ω of $J_1 + tr$ and satisfying

$$\nabla^\perp \psi_2 = r. \quad (32)$$

Note that since

$$\int_{J_1+tr} r \cdot \nu = r \cdot \int_{J_1+tr} \nu = 0,$$

where ν is the unit outward normal on $J_1 + tr$, this is defined without trouble.

Now extend as previously ψ_2 to $\hat{\psi}_2 \in C_0^\infty([0, 1] \times \Omega; \mathbb{R})$ to get the result; that one can obtain a vector field which is compactly supported in time is again just of matter of reparameterization. Again, $\Phi^{v_2}(1, 0, J_1)$ still satisfies (20)-(21).

3.2 Proof in the reduced case: if $\text{Int}(J_1) \cap \text{Int}(J_2)$ is connected

Let us now suppose that J_1 and J_2 satisfy (20), (21) and (27), and moreover that $\text{Int}(J_1)$ and $\text{Int}(J_2)$ have a connected intersection. We will explain how to deduce the general case in the next paragraph.

1. Since the intersection between J_1 and J_2 is transverse, it is finite and composed of an even number of points P_1, \dots, P_{2n} such that the part of J_1 (resp. J_2) between P_{2i} and P_{2i+1} (resp. P_{2i+1} and P_{2i+2}) — with the convention that the indices are considered in $\mathbb{Z}/2n\mathbb{Z}$ — is contained in $\text{Int}(J_2)$ (resp. $\text{Int}(J_1)$).

For each i , define J_1^i (resp. J_2^i) the portion of J_1 (resp. J_2) between P_i and P_{i+1} , contained in $\text{Int}(J_2)$ (resp. ${}^c\text{Int}(J_1)$) for i even, and in ${}^c\text{Int}(J_2)$ (resp. $\text{Int}(J_1)$) for i odd. Then $J_1^i \cup J_2^i$ form a Jordan curve; define D_i its closed interior. It follows from the construction that the D_i for i even are the connected components of $\overline{\text{Int}(J_2)} \setminus \text{Int}(J_1)$, and for i odd, they are the connected components of $\overline{\text{Int}(J_1)} \setminus \text{Int}(J_2)$. Note that due to (20), D_i does not contain any connected component of $\partial\Omega$.

The situation is described in Figure 1.

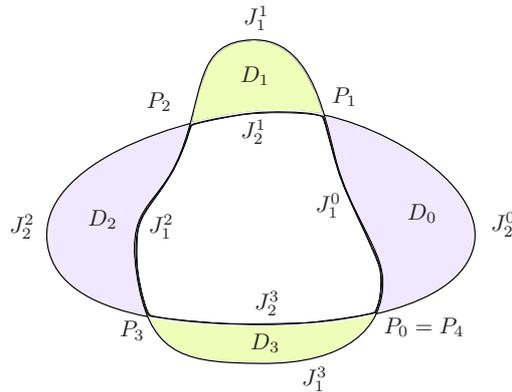


Figure 1: The reduced case

Now we are going to construct some vector field \hat{v} defined in some ε -neighborhood of the symmetric difference of $\text{Int}(J_1)$ and $\text{Int}(J_2)$, which we will denote $\text{Int}(J_1) \triangle \text{Int}(J_2)$. Due to the transversality of

J_1 and J_2 and to the inverse mapping theorem, one can find for each $i \in \{1, \dots, 2n\}$ an orientation-preserving diffeomorphism φ_i from some neighborhood \mathcal{N}_i of P_i to some closed square $\tilde{\mathcal{N}}_i$ centered at $(0,0)$ in \mathbb{R}^2 , and which moreover transforms J_1 (resp. J_2) into the x -axis (resp. y -axis) for i odd, and transforms J_1 (resp. J_2) into the y -axis (resp. x -axis) for i even. As a consequence, we assume that $\mathcal{N}_i \cap [\text{Int}(J_1) \triangle \text{Int}(J_2)]$ is sent by φ_i to the quarter planes $\{x \geq 0, y \geq 0\}$ and $\{x \leq 0, y \leq 0\}$ in \mathbb{R}^2 . Up to a rotation of angle π , the positive coordinates correspond to the interval $[P_i, P_{i+1}]$, while the negative ones correspond to $[P_{i-1}, P_i]$. Finally, reducing \mathcal{N}_i and $\tilde{\mathcal{N}}_i$ if necessary, we assume that the various neighborhoods \mathcal{N}_i are disjoint, and do not meet $\partial\Omega$. The transformation φ_i is described in Figure 2 (in this case i is odd).

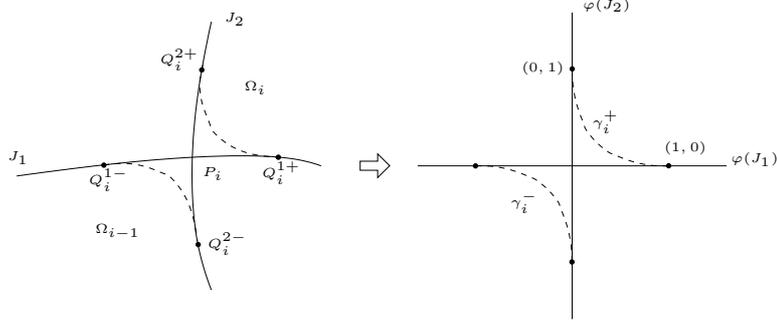


Figure 2: At an intersection

Now we introduce for each i and for each of the first and the third quarter planes, a curve “connecting” together J_1 and J_2 in a smooth way. Let us explain this in the case of the first quarter plane and when i is odd. Pick some points inside $\varphi_i(J_1)$ and $\varphi_i(J_2)$ in the intersection of the first quarter plane and of $\tilde{\mathcal{N}}_i \setminus \{(0,0)\}$; call them $(1,0)$ and $(0,1)$ (scaling φ_i if necessary). Call $Q_i^{1+} := \varphi_i^{-1}((1,0))$ and $Q_i^{2+} := \varphi_i^{-1}((0,1))$ (and in the same way Q_i^{1-} and Q_i^{2-} are the points picked in J_1 and J_2 respectively when applying the same process on the third quarter plane). Now consider $\rho \in C^\infty([0,1]; [0,1])$ such that $\rho \equiv 0$ (resp. $\rho \equiv 1$) in a neighborhood of 0 (resp. of 1), non-decreasing. Then $\tilde{\gamma}$ is the curve parameterized on $[0,1]$ by

$$\tilde{\gamma} : t \mapsto ((1-t)(1-\rho(t)), t\rho(t)). \quad (33)$$

Observe that $\tilde{\gamma}$ is non-singular since

$$\langle \dot{\tilde{\gamma}}, (-1,1) \rangle < 0 \text{ in } [0,1].$$

One can define the curve $\gamma := \varphi_i^{-1}(\tilde{\gamma})$ on each side of each P_i . Call γ_i^+ (resp γ_i^-) the curve constructed in this way in \mathcal{N}_i , connecting the curves J_1 and J_2 between Q_i^{1+} and Q_i^{2+} (resp. Q_i^{1-} and Q_i^{2-}). Now gluing γ_i^+ , γ_{i+1}^- , J_1^i and J_2^i , we obtain a smooth Jordan curve g_i . Call Ω_i the interior of this Jordan curve. Of course, Ω_i is a part of D_i , and hence does not meet $\partial\Omega$.

For $\lambda > 0$ and $i \in \{1, \dots, 2n\}$ we introduce the sets

$$\tilde{\mathcal{O}}_i^\lambda := \left\{ (x_1, x_2) \in \tilde{\mathcal{N}}_i / |x_1 + x_2| \leq \lambda \right\} \text{ and } \mathcal{O}_i^\lambda := \varphi_i^{-1}(\tilde{\mathcal{O}}_i^\lambda). \quad (34)$$

We introduce $\varepsilon > 0$ such that

$$\text{for all } i \in \{1, \dots, 2n\}, \mathcal{O}_i^\varepsilon \cap \bigcup_j \overline{\Omega_j} = \emptyset. \quad (35)$$

Now our goal is to construct a smooth vector field \hat{v} in a neighborhood of $\text{Int}(J_1) \triangle \text{Int}(J_2)$. The construction follows several steps.

2. Introduce the notations for the push-forward and the pull-back of a vector field by φ_i :

$$(\varphi_{i*}X)(x) := (d\varphi_i)_{\varphi_i^{-1}(x)}(X) \text{ and } \varphi_i^*(Y) := (\varphi_i^{-1})_*(Y).$$

Let us prove the following.

Lemma 1. *There exists \check{v} a smooth vector field defined in a neighborhood \mathcal{V} of $\overline{\text{Int}(J_1) \triangle \text{Int}(J_2)}$ and satisfying that:*

$$\text{for all } x \in \mathcal{V}, |\check{v}(x)| > 0, \quad (36)$$

$$\forall i \in \{1, \dots, 2n\}, \forall x \in \mathcal{O}_i^{\varepsilon/2}, \varphi_{i*}(\check{v}) = (-1)^{i+1}(x_1 + x_2)(-1, 1) \text{ where } (x_1, x_2) = \varphi_i(x), \quad (37)$$

$$\text{for all } x \in J_2 \setminus J_1, \check{v} \text{ is transverse to } J_2 \text{ and for all } x \in J_1 \setminus J_2, \check{v} \text{ is transverse to } J_1, \quad (38)$$

and moreover that the flow of \check{v} starting from x which is defined on some time interval $[0, T_x]$, satisfies

$$\text{for all } x \in J_1, \exists t_x \in [0, T_x], \Phi^{\check{v}}(t_x, 0, x) \in J_2 \text{ and } x \in J_1 \mapsto \Phi^{\check{v}}(t_x, 0, x) \in J_2 \text{ is one-to-one.} \quad (39)$$

Recall that ε was introduced in (35).

Proof of Lemma 1.

a. Consider for each $i = 1, \dots, 2n$ the following smooth function defined on $g_i = \partial\Omega_i$:

$$b_i = 1 \text{ on } \gamma_{i+1}^-, \quad (40)$$

$$b_i = 0 \text{ on } \gamma_i^+, \quad (41)$$

$$b_i \text{ is decreasing between } Q_i^{1+} \text{ and } Q_{i+1}^{1-} \text{ (resp. } Q_i^{2+} \text{ and } Q_{i+1}^{2-}). \quad (42)$$

As in [5], define θ_i on $\overline{\Omega}_i$ by

$$\begin{cases} \Delta\theta_i = 0 \text{ in } \Omega_i, \\ \theta_i = b_i \text{ on } g_i. \end{cases} \quad (43)$$

Due to (40)-(42) and the strict maximum principle one has $\partial_\nu\theta_i > 0$ on γ_{i+1}^- and $\partial_\nu\theta_i < 0$ on γ_i^+ (where ν is the unit outward normal on g_i). With (42) we infer that

$$\deg(\nabla\theta_i; g_i) = 0.$$

But $\nabla\theta_i$ is the gradient of a harmonic function, hence $V^{-1}[\nabla\theta_i]$ is holomorphic, hence its degree along b_i counts its zeros. It follows that

$$\nabla\theta_i(x) \neq 0 \text{ in } \overline{\Omega}_i. \quad (44)$$

Define \hat{v}_i by

$$\hat{v}^i := (-1)^i \nabla^\perp \theta_i \text{ in } \overline{\Omega}_i. \quad (45)$$

Define the function μ_i by

$$\hat{v}^i = \nabla\mu_i \text{ in } \overline{\Omega}_i. \quad (46)$$

This is possible thanks to the simple connectedness of Ω_i and to (43).

b. Now our goal is to extend and modify the vector field given by \hat{v}_i in Ω_i . Call \mathcal{V} a small neighborhood of $\bigcup_i [\mathcal{N}_i \cup \Omega_i]$. First, we define \bar{v} in $\bigcup_i (\overline{\Omega}_i \cup \overline{\mathcal{O}_i^\varepsilon})$ by

$$\bar{v}(x) = \begin{cases} \hat{v}_i(x) \text{ for } x \in \overline{\Omega}_i, \\ (-1)^{i+1} \varphi_i^*[(x_1 + x_2)(-1, 1)] \text{ for } x \in \overline{\mathcal{O}_i^\varepsilon} \text{ where } \varphi_i(x) = (x_1, x_2). \end{cases}$$

Next, we extend arbitrarily but smoothly the vector field \bar{v} in \mathcal{V} .

Now let us modify the field \bar{v} in \mathcal{N}_i . We introduce an odd function $\Lambda \in C_0^\infty(\mathbb{R})$ such that $0 \leq \Lambda \leq 1$ in \mathbb{R}^+ , $\text{Supp}(\Lambda) \subset [-1 - \varepsilon, -\varepsilon/2] \cup [\varepsilon/2, 1 + \varepsilon]$ and $\Lambda \equiv 1$ in $[\varepsilon, 1]$. Then define \hat{v} in \mathcal{N}_i by

$$\hat{v}(x) := \bar{v}(x) + C\varphi_i^*[\Lambda(x_1 + x_2)(-1, 1)] \text{ in } \mathcal{N}_i. \quad (47)$$

Taking $C > 0$ large enough, one may ensure that

$$\begin{cases} (-1)^{i+1} \langle \varphi_{i*} \hat{v}, (-1, 1) \rangle > 0 \text{ on } \{(x_1, x_2) \in \tilde{\mathcal{N}}_i / x_1 + x_2 > 0\}, \\ (-1)^{i+1} \langle \varphi_{i*} \hat{v}, (-1, 1) \rangle < 0 \text{ on } \{(x_1, x_2) \in \tilde{\mathcal{N}}_i / x_1 + x_2 < 0\}, \end{cases} \quad (48)$$

and

$$\begin{cases} (-1)^{i+1}x_1\langle\varphi_{i*}\hat{v},(0,1)\rangle > 0 \text{ on } \{(x_1,x_2)\in\tilde{\mathcal{N}}_i / x_1\neq 0 \text{ and } x_2=0\}, \\ (-1)^{i+1}x_2\langle\varphi_{i*}\hat{v},(0,1)\rangle > 0 \text{ on } \{(x_1,x_2)\in\tilde{\mathcal{N}}_i / x_2\neq 0 \text{ and } x_1=0\}. \end{cases} \quad (49)$$

Note that considering the support of Λ , the vector field \hat{v} is then defined in some neighborhood of $\text{Int}(J_1)\triangle\text{Int}J_2$, and coincides with \bar{v} outside of $\cup_i\mathcal{O}_i^\varepsilon$.

c. Let us show that \hat{v} satisfies (36)-(39). First let us consider (36): outside $\cup_i\mathcal{N}_i$ this is a consequence of (44)-(46), while inside $\cup_i\mathcal{N}_i$, this is a consequence of (48) which is valid there. Now (37) is a trivial consequence of the construction, and (38) follows from (42) and (46) outside $\cup_i\mathcal{N}_i$, from (49) inside $\cup_i\mathcal{N}_i$. To obtain (39), we observe that we can deduce the characteristics $\Phi^{\varphi_{i*}\hat{v}}(\cdot,0,x)$ associated to $\varphi_{i*}\hat{v}$ from the ones of \hat{v} by

$$\Phi^{\varphi_{i*}\hat{v}}(\cdot,0,x) = \varphi_i[\Phi^{\hat{v}}(\cdot,0,\varphi_i^{-1}(x))].$$

Hence the existence of t_x follows easily for $x\in J_1\cap(\cup_i\mathcal{N}_i)$; for $x\in J_1\setminus\cup_i\mathcal{N}_i$, this follows from (40)-(42), (43) and (44), since the flow of \hat{v}_i satisfies

$$\frac{d}{dt}\mu_i(\Phi) = |\nabla\mu_i|^2(\Phi).$$

The proof that any point of J_2 can be obtained (in a unique way) as $\Phi^{-\hat{v}}(\tilde{t}_x,0,x)$ is similar, which gives the second property of (39). This ends the proof of Lemma 1.

3. Now we have to modify this vector field in order that (22) applies and that property (23) is obtained for the uniform time $t=1$. Due to the fact that the characteristics of v are not closed, one can introduce \check{v} by

$$\check{v}(x) := \alpha(x)\hat{v}(x),$$

for some smooth and positive α solution to

$$\hat{v}\cdot\nabla\alpha = -\alpha\text{div}(\hat{v}),$$

in order to ensure that

$$\text{div}(\check{v}) = 0.$$

Now we have the following.

Lemma 2. *There exists some smooth and positive $\beta = \beta(t,x)$ such that*

$$\beta(t,x) = 1 \text{ for all } t\in[0,1] \text{ and } x \text{ in a neighborhood of } P_i, \quad (50)$$

$$\nabla\beta(t,x)\cdot\check{v}(x) = 0 \text{ for all } (t,x), \quad (51)$$

the flow of $\beta\check{v}$ is well defined for $(t,x)\in[0,1]\times J_1$ and

$$\Phi^{\beta\check{v}}(1,0,J_1) = J_2, \quad (52)$$

$$\int_{\Phi^{\beta\check{v}}(t,0,J_1)} \beta(t,x)\check{v}(x)\cdot\nu(t,x) dx = 0, \quad \forall t\in[0,1], \quad (53)$$

where $\nu(t,x)$ is the unit outward normal on $\Phi^{\beta\check{v}}(t,0,J_1)$.

Proof of Lemma 2. Note that condition (51) means that for fixed t , β has to be constant on each characteristic associated to $x\mapsto\check{v}(x)$. We will describe the construction of β inside each $[P_i,P_{i+1}]$ (in a fiberwise constant way). In that case, we will replace condition (53) with the following one:

$$\int_{\Phi^{\beta\check{v}}(t,0,J_1^i)} \beta(t,x)\check{v}(x)\cdot\nu(t,x) dx = (-1)^i a_i, \quad \forall t\in[0,1], \quad (54)$$

where a_i is the area of D_i , i.e. the zone delimited by J_1^i and J_2^i . Note that

$$\sum_{i=1}^{2n} (-1)^i a_i = 0, \quad (55)$$

as follows from (21). Next we will see that β is globally and smoothly defined, and satisfies (52)-(53).

a. To get (52), we reparameterize the time. Precisely, for each $x \in J_1$ between P_i and P_{i+1} , we make the change of variable $t \rightarrow t/t_x$ where t_x was defined in Lemma 1. Precisely we define

$$\check{v}(x) := \beta_1(x)\hat{v}(x) := \frac{1}{t_x}\hat{v}(x). \quad (56)$$

That t_x is C^∞ as a function of $x \in J_1$ is a consequence of the inverse mapping theorem and (38). Note that due to (37), we have $t_x = 1$ in a neighborhood of P_i . Hence β_1 and \check{v} satisfy (50), (51) and (52).

b. Now we have to modify \check{v} in order to obtain (54) while keeping the other properties. Call $A_i(t)$ the area delimited by J_1^i and $\Phi^{\beta\check{v}}(t, 0, J_1^i)$. It follows from the construction that A_i is a smooth increasing function, and it follows from the coarea formula (see for instance [11]) that

$$A_i'(t) = \int_{\Phi^{\beta\check{v}}(t, 0, J_1^i)} \beta_1(x)\check{v}(x).\nu(t, x) dx. \quad (57)$$

We set

$$\tau(t) := \frac{A_i(t)}{a_i}. \quad (58)$$

We note that due to the construction, $A_i'(t) \geq c > 0$ and consequently, there exists some $\kappa > 0$ such that

$$\kappa^{-1}(1 - \tau(t)) \leq 1 - t \leq \kappa(1 - \tau(t)). \quad (59)$$

Now if we define

$$\underline{v}(t, x) := \beta_2(t)\check{v}(t, x) := \frac{A_i'(t)}{a_i}\check{v}(t, x), \quad (60)$$

we have (54), and we have kept (51) and (52). Moreover we have the following relation between the flows of \check{v} and \underline{v} :

$$\Phi^{\underline{v}}(\tau(t), 0, x) = \Phi^{\check{v}}(t, 0, x). \quad (61)$$

However, (50) is no longer necessarily satisfied.

c. Now we finally modify \underline{v} in $\mathcal{O}_i^{\varepsilon/2}$ in order to obtain all the properties. The advantage of the preceding procedure is that now we modify the flow of J_1 in a zone where it is explicit. Precisely, under the flow of $\varphi_{i*}(\underline{v})$, the image of $\varphi_i(J_1)$ at time t in \mathcal{N}_i is given by

$$y = \frac{\tau(t)}{1 - \tau(t)}x. \quad (62)$$

In fact, rather than modifying \underline{v} directly, we will modify (in $\tilde{\mathcal{O}}_i^{\varepsilon/2}$ only) the image $\mathcal{J}(t)$ of $\varphi_i(J_1)$ at time t under the flow of $\varphi_{i*}(\underline{v})$ and obtain the modified version \hat{v} of the vector field as a byproduct. We parameterize the curve $\mathcal{J}(t)$ by x_1 in $\varphi(\mathcal{O}_i^{\varepsilon/2})$. Then we will deduce \hat{v} by the fact that it should be constant along lines $x_1 + x_2 = \text{constant}$ and

$$[\varphi_{i*}\hat{v}](t, \mathcal{J}(t, x_1 + x_2)) = \frac{\partial}{\partial t}\mathcal{J}(t, x_1 + x_2). \quad (63)$$

This is done according to Figure 3.

We introduce $\alpha > 0$ small and some $K_1, K_2 > 0$ with $K_2 \gg K_1 \gg 1$ and $K_2\alpha \leq \varepsilon/2$. Introduce $\eta \in C^\infty(\mathbb{R}; \mathbb{R})$ such that

$$0 \leq \eta \leq 1 \text{ and } \eta' \geq 0 \text{ on } \mathbb{R}, \eta = 0 \text{ on } \mathbb{R}^- \text{ and } \eta = 1 \text{ on } [1, +\infty]. \quad (64)$$

We modify this curve in $[\alpha, K_1\alpha]$ by

$$\mathcal{C}_1 : x \mapsto \left(x, \frac{(1 - \eta(\frac{x-\alpha}{(K_1-1)\alpha}))t + \eta(\frac{x-\alpha}{(K_1-1)\alpha})\tau(t)}{1 - [(1 - \eta(\frac{x-\alpha}{(K_1-1)\alpha}))t + \eta(\frac{x-\alpha}{(K_1-1)\alpha})\tau(t)]} x \right). \quad (65)$$

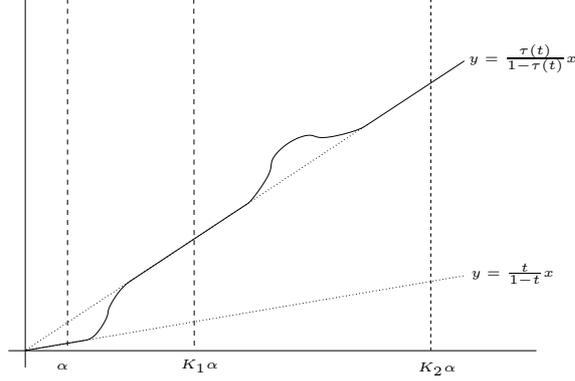


Figure 3: Modifying \underline{v} in a neighborhood of P_i

In other words, we reconnect in a smooth manner the two half-lines of slope $t/(1-t)$ and $\tau(t)/(1-\tau(t))$ in the interval $[\alpha, K_1\alpha]$.

In the interval $[K_1\alpha, K_2\alpha]$, we modify this curve in order that the area enclosed by J_1 and $\varphi_i^*(\mathcal{J}(t))$ is equal to $A_i(t)$. For that, we introduce $\xi \in C^\infty(\mathbb{R}; \mathbb{R})$ such that

$$\xi \geq 0 \text{ on } \mathbb{R}, \quad \xi = 0 \text{ on } \mathbb{R}^- \cup [1, +\infty], \quad \text{and} \quad \int_{\mathbb{R}} \xi = 1. \quad (66)$$

We modify the curve in $[K_1\alpha, K_2\alpha]$ by

$$\mathcal{C}_2 : x \mapsto \left(x, \frac{\tau(t)}{1-\tau(t)}x + \lambda(t)\xi\left(\frac{x - K_1\alpha}{(K_2 - K_1)\alpha}\right) \right), \quad (67)$$

where λ is chosen so that the area enclosed by J_1 and $\varphi_i^*(\mathcal{J}(t))$ is equal to $A_i(t)$. Note that it suffices that the area of the preimage via φ_i of the region determined by $y = \tau(t)/(1-\tau(t))x$ and \mathcal{C}_2 equals the one of the preimage via φ_i of the region determined by $y = \tau(t)/(1-\tau(t))x$ and \mathcal{C}_1 . It follows then (using that φ_i is a smooth diffeomorphism) that λ is a smooth function of the time and that

$$|\lambda(t)| \lesssim \frac{|\tau(t) - t|}{K_2 - K_1}. \quad (68)$$

Now it remains to check that the curve cuts each line $x_1 + x_2 = \text{constant}$ in a unique transverse way. For this it is sufficient to check that

$$\left\langle \left(1, \frac{\tau(t)}{1-\tau(t)} + \frac{\lambda(t)}{(K_2 - K_1)\alpha} \xi'\left(\frac{x - K_1\alpha}{(K_2 - K_1)\alpha}\right)\right), (1, 1) \right\rangle > 0.$$

This is easily obtained provided K_2 is large enough.

d. Now that we have constructed \hat{v} “inside each” $[P_i, P_{i+1}]$, we can observe that due to (50), the function that we constructed is defined smoothly on a whole neighborhood of $\text{Int}(J_1) \triangle \text{Int}(J_2)$. Note also that due to (55), (54) and (57), we deduce (53). This ends the proof of Lemma 2.

End of the proof of Proposition 1. It remains to explain how this can be extended on Ω as a global divergence-free vector field. For each part D_i of $\overline{\text{Int}(J_1)} \triangle \overline{\text{Int}(J_2)}$, the smooth divergence-free vector field \hat{v} can be written in the form $\nabla^\perp \psi$ for some smooth scalar function ψ . Since in each \mathcal{N}_i the connection between these pieces is smooth, we can define $\nabla^\perp \psi$ on this neighborhood of $\text{Int}(J_1) \triangle \text{Int}(J_2)$. That ψ can be globally defined on a whole neighborhood of $\text{Int}(J_1) \triangle \text{Int}(J_2)$ is due to (53). Now we extend ψ as a function in $C_0^\infty(\Omega)$ (we recall that $\partial\Omega$ does not meet $\overline{\text{Int}(J_1)} \triangle \overline{\text{Int}(J_2)}$); we reparameterize the time in order to have a compact support in time and we have finished the proof when we suppose that $\text{Int}(J_1) \cap \text{Int}(J_2)$ is connected.

3.3 Reduction to the case when $\text{Int}(J_1) \cap \text{Int}(J_2)$ is connected

Now let us explain how we deal with the case when $\text{Int}(J_1) \cap \text{Int}(J_2)$ has several connected components. We divide J_1 and J_2 into successive intervals bounded by points of $J_1 \cap J_2$ as previously. Again, we denote these intervals inside J_1 and J_2 as real intervals, in a way which should not be ambiguous. Let us call *simple* the bounded connected components of $\mathbb{R}^2 \setminus [J_1 \cup J_2]$ whose boundary is composed only of one interval of J_1 and one interval of J_2 . Let us explain how, if $\text{Int}(J_1) \cap \text{Int}(J_2)$ has several connected components, and provided that these components do not contain connected components of $\partial\Omega$, then we can reduce the number of intersection points between these two curves, by simple area-preserving movements. This will allow to conclude.

There are several different types of simple components; we describe two of them in Figure 4.

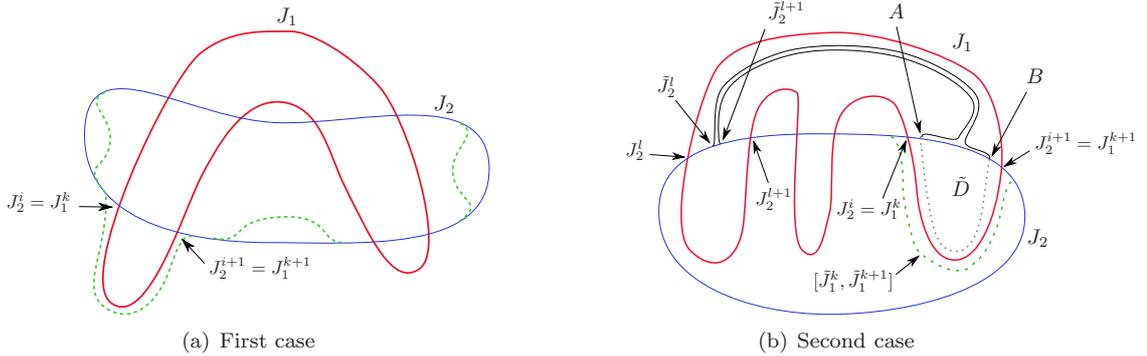


Figure 4: Simple components

First case. The first case concerns a simple component of $\text{Int}(J_1) \triangle \text{Int}(J_2)$. Let us say, this component which we will call \mathcal{C} is in $\text{Int}(J_1) \setminus \text{Int}(J_2)$ and bounded by the “intervals” $[J_2^i, J_2^{i+1}]$ and $[J_1^k, J_1^{k+1}]$, with $J_2^i = J_1^k$ and $J_2^{i+1} = J_1^{k+1}$, see Figure 4(a). Now we construct a new curve \tilde{J}_2 (represented in dotted lines in Figure 4(a)) in the following way. In the curve J_2 , replace the “interval” $[J_2^i, J_2^{i+1}]$ with the interval $[J_1^k, J_1^{k+1}]$. Smoothen the connection between the new interval and the rest of J_2 , in a manner that the interval $[J_1^k, J_1^{k+1}]$ is inside the interior of the new curve, and that the additional area (let us say ε) is arbitrarily small. This is easily done in a tubular neighborhood of $[J_1^k, J_1^{k+1}]$. Now the interior of the new curve has $|\mathcal{C}| + \varepsilon$ additional area with respect to $\text{Int}(J_2)$. But since $\mathcal{C} \subset \text{Int}(J_1) \setminus \text{Int}(J_2)$, we see that $|\text{Int}(J_2) \setminus \text{Int}(J_1)| > |\mathcal{C}|$. Hence we can construct smooth curves inside $\text{Int}(J_2) \setminus \text{Int}(J_1)$ starting from points of J_2 as describe in Figure 4 in such a way that the resulting curve \tilde{J}_2 encloses the same area as J_1 . Now in order to find a solenoidal vector field making J_2 reach \tilde{J}_2 , we can reason as in the previous paragraph. The resulting situation has strictly less intersection points between the two curves.

Second case. The second case concerns a simple component of $\text{Int}(J_1) \cap \text{Int}(J_2)$. We use the same notations \mathcal{C} , $[J_1^k, J_1^{k+1}]$ and $[J_2^i, J_2^{i+1}]$ as previously. Here we construct the new curve \tilde{J}_2 as follows. As previously, we begin by modifying J_2 : we cut the interval $[J_2^i, J_2^{i+1}]$ and replace it with the interval $[J_1^k, J_1^{k+1}]$. Then we smoothen the connection in such a way that the interval $[J_1^k, J_1^{k+1}]$ is outside the interior of the new curve, that it does not add intersections with J_1 and that the difference of area (let us say again ε) is arbitrarily small. Call $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$ the new interval, $[\tilde{J}_2^i, \tilde{J}_2^{i+1}]$ the corresponding interval of J_2 , and \tilde{D} the domain delimited by these two curves. Call V a divergence-free vector field constructed as previously (see \underline{v} in (60)), whose flow Φ^V between times 0 and 1 sends the interval $[\tilde{J}_2^i, \tilde{J}_2^{i+1}]$ to $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$, and such that the flux of $V(t, \cdot)$ across $\Phi^V(t, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$ is constant.

Since we know that there are several connected components of $\text{Int}(J_1) \cap \text{Int}(J_2)$, we know that we can find a smooth simple path \mathcal{H} in $\text{Int}(J_1) \setminus \text{Int}(J_2)$ from a point in $[J_2^i, J_2^{i+1}]$ to some point in another interval of J_2 , let us say $[J_2^l, J_2^{l+1}]$ (see Figure 4(b)). We expand this path into a pipe (two smooth simple non-intersecting curves g_1 and g_2 joining $[J_2^i, J_2^{i+1}]$ to $[J_2^l, J_2^{l+1}]$, on each side of \mathcal{H}). This can be easily done in a tubular neighborhood of \mathcal{H} . We smoothen the connection of this pipe to $[J_2^l, J_2^{l+1}]$ as previously; call \tilde{J}_2^l and \tilde{J}_2^{l+1} the points in $[J_2^l, J_2^{l+1}]$ to which g_1 and g_2 connect. Now for what concerns the other side of the pipe, we fix two points A and B in $[J_1^k, J_1^{k+1}]$ such that $\hat{D} := \{\Phi^V(t, 0, [A, B]), t \in [0, 1]\}$

has measure at least $|\mathcal{C}| - \varepsilon$. Now in a tubular neighborhood of $[J_2^i, J_2^{i+1}]$ we modify the curves g_1 and g_2 in order that they join A and B , in such a way that they reconnect smoothly with the orbits of V at A and B . Moreover we manage in order that the area of the pipe (the zone \mathcal{P} delimited by $[A, B]$, $[J_2^i, J_2^{i+1}]$, g_1 and g_2) is larger than 3ε , provided that ε is small enough. The curve \tilde{J}_2 is obtained by gluing $J_2 \setminus ([\tilde{J}_2^i, \tilde{J}_2^{i+1}] \cup [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ with $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$, g_1 , g_2 , $\Phi([0, 1], 0, A)$, $\Phi([0, 1], 0, B)$ and $\Phi(1, 0, [A, B])$.

Now it remains to explain by which divergence-free vector field we send J_2 not exactly to but merely *inside* \tilde{J}_2 . We construct a (time-dependent) vector field W by imposing first that it coincides with V on $\Phi^V(t, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$, and that it is tangent to $J_2 \setminus ([\tilde{J}_2^i, \tilde{J}_2^{i+1}] \cup [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ and to g_1 , g_2 . Next we impose its value on $\Phi^W(t, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ in such a way that the flux of W across the closed curve \mathcal{J} composed by $\Phi^V(t, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$, $\Phi^W(t, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ and g_1 , g_2 is zero. To do so, we find a vector field W inside \mathcal{P} as in Lemma 2 in order that for some $\tau > 0$, $\Phi^W(\tau, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}]) = [A, B]$. The condition on the flux allows to extend W as a global solenoidal vector field. After time τ , we require that W coincides with $\alpha(t)V$ on $\Phi^V(t, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ when $\Phi^V(t, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ enters the domain \tilde{D} . The value of α is determined to allow the condition on the flux. Moreover one can manage (reparameterizing in time if necessary) that W smoothly reconnects at time τ .

For the consistency of the definition, it remains to explain why the curve obtained by considering $\Phi^W(\cdot, 0, J_2)$ does not self-intersect in \tilde{D} . It is enough to see that $\Phi^W(t, \tau, [A, B])$ does not cut $\Phi^W(t, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$. Now for $t \geq \tau$, α is the ratio of the flux of V across $\Phi^V(t, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$ and the flux of W across $\Phi^W(\tau, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$, so clearly $\alpha \geq 1$. It follows that, should the two curves cross for some time t , they would cross at the final time 1. Hence $\{\Phi^W(1, \tau, [A, B])\}$ would contain \tilde{D} . But $\Phi^V(1, 0, [\tilde{J}_2^i, \tilde{J}_2^{i+1}])$ covers an area less than $|\mathcal{C}| + \varepsilon$, this contradicts the fact that the area covered by $\Phi^W(1, 0, [\tilde{J}_2^l, \tilde{J}_2^{l+1}])$ contains both \mathcal{P} and \tilde{D} . Note that the process does not add any component to $\Omega \setminus (J_1 \cup J_2)$; but it decreases the number of intersections between the two curves.

Third case. We consider the case of a simple component \mathcal{C} of $\mathbb{R}^2 \setminus [\text{Int}(J_1) \cap \text{Int}(J_2)]$ (as again in Figure 4(a)). Let us again say that \mathcal{C} is surrounded by $[\tilde{J}_2^i, \tilde{J}_2^{i+1}]$ and $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$ with $\tilde{J}_1^k = \tilde{J}_2^i$ and $\tilde{J}_1^{k+1} = \tilde{J}_2^{i+1}$. In the case where $|\mathcal{C}| < |\text{Int}(J_1) \setminus \text{Int}(J_2)|$, one can proceed as for the first case, that is, introducing \tilde{J}_2 by cutting inside J_2 the interval $[\tilde{J}_2^i, \tilde{J}_2^{i+1}]$, replacing it by $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$, smoothening the resulting curve inside $\text{Int}(J_1)$, and modifying other components in order to get $|\text{Int}(\tilde{J}_2)| = |\text{Int}(J_2)|$ (without adding any intersection). If we do not have $|\mathcal{C}| < |\text{Int}(J_1) \setminus \text{Int}(J_2)|$, it is easily be seen that one can proceed by a finite number of steps, by introducing intermediate curves between $[\tilde{J}_2^i, \tilde{J}_2^{i+1}]$ and $[\tilde{J}_1^k, \tilde{J}_1^{k+1}]$.

Conclusion. We first consider the case when no connected component of $\partial\Omega$ is inside a bounded component of $\mathbb{R}^2 \setminus (\overline{\text{Int}(J_1)} \cup \overline{\text{Int}(J_2)})$. In that case, using the above steps, either we have met the situation of a single intersection between $\text{Int}(J_1)$ and $\text{Int}(J_2)$ (and this case was treated in the previous paragraph), or we are in a situation where all the simple components contain a connected component of $\partial\Omega$. Since the two curves J_1 and J_2 are homotopic, these components cannot be components of $\text{Int}(J_1) \triangle \text{Int}(J_2)$. They cannot be components of $\mathbb{R}^2 \setminus [\text{Int}(J_1) \cup \text{Int}(J_2)]$ either, because of our assumption and because the above steps do not add components of $\mathbb{R}^2 \setminus [\text{Int}(J_1) \cup \text{Int}(J_2)]$. But one easily sees that if the two curves intersect transversally and that their intersection is not connected, there must be simple components in $\Omega \setminus (\overline{\text{Int}(J_1)} \cup \overline{\text{Int}(J_2)})$; hence we must have met the case when $\text{Int}(J_1)$ and $\text{Int}(J_2)$ have only one intersection.

It remains to explain how we can reduce to the case when no connected component of $\partial\Omega$ is inside a bounded component of $\mathbb{R}^2 \setminus (\overline{\text{Int}(J_1)} \cup \overline{\text{Int}(J_2)})$. We use the fact that J_1 and J_2 being homotopic, they are isotopic (see Epstein [10]). Hence we can find a finite number of isotopic embeddings $\mathbb{S}^1 \rightarrow \Omega$: $j_0 = J_1$, \dots , $j_N = J_2$, with j_i arbitrarily close to j_{i+1} for the C^0 topology. In particular we can manage in order that no connected component of $\partial\Omega$ is inside a bounded component of $\mathbb{R}^2 \setminus (\overline{\text{Int}(j_i)} \cup \overline{\text{Int}(j_{i+1})})$. Now j_i and j_{i+1} can be approximated by piecewise linear embeddings (see again [10]), and hence by smooth embeddings. Finally we modify j_i by adding/subtracting a part as before (obtaining \tilde{j}_i) in order that $|\text{Int}(\tilde{j}_i)| = |\text{Int}(J_1)|$ for all i , in such a way that the added part does not cross \tilde{j}_{i-1} and does not change the topological situation. We apply successively the previous steps between \tilde{j}_{i-1} and \tilde{j}_i and we are done.

4 Proof of Proposition 2

The goal of this section is to prove Proposition 2. This is done in several steps of growing generality.

4.1 The analytic case

The goal of this section is to prove the following proposition, which is Proposition 2 in the particular case where the curve and the vector field are both analytic.

Proposition 3. *Let γ_0 be an analytic Jordan curve; let $X \in C^0([0, 1]; C^\omega(\Omega; \mathbb{R}^2))$ be an analytic divergence-free vector field satisfying (24). Fix γ_1 by (25). Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\theta \in C^\infty([0, 1] \times \overline{\Omega}; \mathbb{R})$ satisfying (15), (16), (17) and (18) up to reparameterization.*

We will suppose that Σ meets a component Γ_0 of $\partial\Omega$ which does not belong to $\text{Int}(\gamma_0)$, for instance the outer connected component of $\partial\Omega$. Without loss of generality, we may assume that $\Sigma \cap (\Gamma_1 \cup \dots \cup \Gamma_g) = \emptyset$ (reducing Σ if necessary). At the end of the proof, we will explain the few modifications of the construction needed if Σ meets $\partial\Omega$ inside $\text{Int}(\gamma_0)$ only.

In order to prove Proposition 3, let us start with a lemma.

Lemma 3. *Consider $\gamma : t \in [0, 1] \mapsto \gamma(t) \in C^\omega(\mathbb{S}^1; \mathbb{R}^2)$ a continuous time-dependent family of analytic Jordan curves included in $\Omega \subset \mathbb{R}^2$. Let $X \in C^0([0, 1]; C^\omega(\overline{\Omega}; \mathbb{R}^2))$ a time-dependent real-analytic in space vector field satisfying*

$$\int_{\gamma_t} X \cdot \nu = 0. \quad (69)$$

Then there exist $\eta > 0$ and $\psi \in C^0([0, 1]; C^\infty([\text{Int}(\gamma(t)) \cap \overline{\Omega}] \cup \mathcal{V}_\eta(\gamma(t)); \mathbb{R}))$ such that

$$\forall t \in [0, 1], \Delta_x \psi(t, x) = 0 \text{ in } [\text{Int}(\gamma(t)) \cap \Omega] \cup \mathcal{V}_\eta(\gamma(t)), \quad (70)$$

and

$$\nabla_x \psi \cdot \nu = X \cdot \nu \text{ on } \gamma(t), \text{ for each } t, \quad (71)$$

$$\nabla_x \psi \cdot n = 0 \text{ on each connected component of } \partial\Omega \text{ inside } \gamma(t). \quad (72)$$

Remark 10. *In particular, if $\gamma(t)$ is given by $\Phi^X(t, 0, \gamma_0)$ for some analytic Jordan curve γ_0 , then the flow of γ_0 by $\nabla\psi$ is the same as the one by X (up to reparameterization).*

Proof of Lemma 3. Our strategy is to describe the function ψ for each time t , and then to prove that the construction is indeed continuous in the variable t .

Call $\Gamma_1, \dots, \Gamma_l$ the connected components of $\partial\Omega$ inside $\text{Int}(\gamma(t))$. These of course are independent of t . For each t , introduce $\psi(t, \cdot) \in C^\infty(\overline{\text{Int}(\gamma(t)) \cap \overline{\Omega}}; \mathbb{R})$ as the solution of

$$\begin{cases} \Delta_x \psi(t, \cdot) = 0 \text{ in } \text{Int}(\gamma(t)) \cap \Omega, \\ \frac{\partial \psi}{\partial \nu}(t, \cdot) = X(t, \cdot) \cdot \nu(\cdot) \text{ on } \gamma(t), \\ \frac{\partial \psi}{\partial n}(t, \cdot) = 0 \text{ on } \bigcup_{i=1}^l \Gamma_i, \\ \int_{\gamma(t)} \psi(t, \cdot) d\sigma = 0. \end{cases} \quad (73)$$

We will temporarily drop the dependence of ψ and γ on t (and consider t as fixed) to simplify the notations.

That ψ is C^∞ up to the boundary follows from standard elliptic regularity theory (see for instance [12]); all the C^k norms can be bounded (for fixed k) by some norm of X . Let us explain why ψ is analytic up to the boundary, that is, can be analytically extended across it.

In some neighborhood U_x in \mathbb{R}^2 of $x \in \gamma$, one can extend the normal ν analytically and define an analytic local diffeomorphism $\varphi_x : U_x \rightarrow V_x \subset \mathbb{R}^2$ by which $\gamma \cap U_x$ is transformed into $x_2 = 0$ and the characteristics of ν into $x_1 = \text{constant}$. Now the equation $\Delta\psi = 0$ is transported by φ_x to an elliptic equation with analytic coefficients satisfied by $\psi \circ \varphi_x^{-1}$. As a consequence, $g := \partial_{x_2}(\psi \circ \varphi_x^{-1})$ also satisfies an elliptic equation with analytic coefficients, let us say

$$a \cdot \nabla^2 g + b \cdot \nabla g + cg = 0, \quad (74)$$

and moreover satisfies an analytic condition at $x_1 = 0$: $\partial_{x_2} g = (X \circ \varphi_x^{-1}) \cdot (\nu \circ \varphi_x^{-1})$. Note that $(X \circ \varphi_x^{-1}) \cdot (\nu \circ \varphi_x^{-1})$ is analytically defined in $\varphi(U_x)$. Now let us recall the following result of analytic continuation across the boundary for solutions of analytic elliptic equations, see [21, Theorem 5.7.1]:

Theorem 4. *Suppose that a, b, c and f are analytic in $\overline{G}_R := \overline{B}_{\mathbb{R}^N}(0, R) \cap \{(x_1, \dots, x_N) \in \mathbb{R}^N / x_N \geq 0\}$ and satisfy for some positive constants A and L uniformly on \overline{G}_R*

$$|\nabla^p a(x)|, |\nabla^p b(x)|, |\nabla^p c(x)|, |\nabla^p f(x)| \leq LA^{|p|}, \text{ for any multi-index } p.$$

Let $u \in H^2(G_R)$ with $u = 0$ on $\overline{B}_{\mathbb{R}^N}(0, R) \cap \{x_N = 0\}$ satisfying

$$a \cdot \nabla^2 u + b \cdot \nabla u + cu = f.$$

Then there exists an $R' < R$ depending only on N, A, L and R such that u can be extended to be analytic in $B(0, r)$ for any $r < R'$.

Consequently one deduces that $\nabla \psi \cdot \nu$ can be extended across the boundary $\gamma \cap U_x$ as an analytic function, locally around x . Hence ψ can be extended in the same open set: this is obtained by integration in the direction ν since, as is classical, ψ is analytic inside $\text{Int}(\gamma) \cap \Omega$ (see for instance [21, Theorem 5.7.1]). Call W_x such an open domain containing x , say that it contains some ball $B(x, 2R_{x,t})$ for instance. Of course, the extension of ψ in W_x is harmonic by the unique continuation principle since $\Delta \psi$ is real analytic.

Now by compactness of $\gamma(t)$ and the unique continuation principle for real-analytic functions, we obtain a harmonic extension of $\psi(t, \cdot)$ on a some η -neighborhood of $\gamma(t)$.

Now let us underline that the neighborhoods found above can be taken locally constant in t . In the analytic inverse mapping theorem, one can use the same neighborhood as in the usual (differentiable) inverse mapping theorem; see the proof of the former in [19]. Considering the dependence of R' given in Theorem 4 and due to the fact that both the data γ and X are continuous in time with values in the space of analytic functions, one can find a lower bound for $R_{x,t}$ locally in t . Hence a compactness argument shows that one can define ψ in an analytical way in some η -neighborhood of $\gamma(t)$ for each t , for some η which is uniform in t .

Now that we have a uniform size of the neighborhood of $\gamma(t)$, the continuity of $t \mapsto \psi(t, \cdot)$ follows from a compactness argument. Inside $\gamma(t)$, we have bounds on ψ in arbitrary norm; hence up to extraction, $\psi(\tau, \cdot)$ converges as $\tau \rightarrow t$ in arbitrary C^k norm. But due to the uniqueness of the solution of (73), one deduces that $\psi(\tau, \cdot)$ converges towards $\psi(t, \cdot)$ as $\tau \rightarrow t$ in arbitrary norm. For what concerns the behavior on the neighborhood of $\gamma(t)$, it is again a consequence of the proof of [21, Theorem 5.7.1] that we have the following bounds in a neighborhood of $x \in \gamma(t)$

$$|\partial^\alpha \psi| \leq CM^{|\alpha|},$$

with constants C and M that can be chosen locally constant around (t, x) . Hence the same compactness argument applies. This concludes the proof of Lemma 3.

Proof of Proposition 3. Given X , introduce the family of curves

$$\gamma(t, \cdot) = \Phi^X(t, 0, \gamma_0(\cdot)).$$

Since X is continuous in time with values in the space of real-analytic vector fields and since γ_0 is analytic, it follows that γ is a time-continuous family of analytic curves. Applying Lemma 3, we deduce $\eta > 0$ and ψ . Reducing η if necessary, one may assume that $\mathcal{V}_\eta(\gamma(t))$ does not meet $\mathcal{V}_\eta(\partial\Omega)$, that $\overline{\mathcal{V}_\eta(\Gamma_i)} \cap \overline{\mathcal{V}_\eta(\Gamma_j)} = \emptyset$ for $i \neq j$ and that $\Sigma \not\subset \overline{\mathcal{V}_\eta(\partial\Omega \setminus \Sigma)}$. In particular, some non-trivial open part in Σ lies outside $\overline{\mathcal{V}_{\eta/2}(\partial\Omega \setminus \Sigma)}$.

By compactness of $[0, 1]$, there exist $0 \leq t_1 < \dots < t_N \leq 1$ and $\delta_1, \dots, \delta_N > 0$ such that $[0, 1] \subset \cup_i (t_i - \delta_i, t_i + \delta_i)$ and such that for all $t \in [t_i - \delta_i, t_i + \delta_i] \cap [0, 1]$, the curve $\gamma(t)$ belongs to the $\eta/2$ neighborhood of $\gamma(t_i)$ and for all $s, t \in [t_i - \delta_i, t_i + \delta_i] \cap [0, 1]$,

$$\|X(s, \cdot) - X(t, \cdot)\|_{C^k(\mathcal{V}_{\eta/2}(\gamma(t_i)))} \leq \varepsilon. \quad (75)$$

As in [15], we use Runge's theorem and the correspondence between gradients of harmonic functions and holomorphic functions. For each i , we choose as a compact K_i the union of $[\text{Int}(\gamma(t_i)) \cap \overline{\Omega}] \cup \overline{\mathcal{V}_{\eta/2}(\gamma(t_i))}$

and of $\overline{V}_{\eta/2}(\partial\Omega \setminus \Sigma)$. Note that thanks to our assumption on η , each connected component of $\mathbb{C} \setminus K_i$ meets $\mathbb{C} \setminus \overline{\Omega}$, so that we may place a point $Z_k \in \mathbb{C} \setminus \overline{\Omega}$ in each of these components (call Z_0, \dots, Z_g these point, with $Z_0 \in \mathbb{C} \setminus \overline{\Omega}$). By Runge's approximation theorem, for any $\varepsilon > 0$, there exists a holomorphic function $f_i \in \mathcal{H}(\overline{\Omega})$ such that

$$\|f_i(z)\|_{C^{k+1}} \leq \varepsilon \text{ in } \overline{V}_{\eta/2}(\partial\Omega \setminus \Sigma), \quad (76)$$

$$\|f_i - V^{-1}(\nabla\psi(t_i, \cdot))\|_{C^{k+1}} \leq \varepsilon \text{ in } [\text{Int}(\gamma(t_i)) \cap \overline{\Omega}] \cup \overline{V}_{\eta/2}(\gamma(t_i)). \quad (77)$$

It suffices indeed to choose f_i as a rational function having poles at the Z_k only. Recall moreover that for holomorphic functions, the uniform convergence determines the C^k one on interior compact subsets.

Now the resulting vector field Vf_i has to be slightly modified, since it does not necessarily exactly satisfy $Vf_i \cdot n = 0$ on $\partial\Omega \setminus \Sigma$ and $\int_{\Gamma_j} Vf_i \cdot d\tau = 0$ for $j = 1 \dots g$ (in order to be a gradient); but these equalities are true up to an error of order ε , as follows from (76)-(77). For what concerns the circulations of Vf_i along the components Γ_j , one can define

$$\tilde{f}_i := f_i + \sum_{j=1}^g \lambda_j / (z - Z_j), \quad (78)$$

for λ_j chosen so that

$$\int_{\Gamma_j} V\tilde{f}_i \cdot \tau = 0 \text{ for } j \in \{1, \dots, g\}. \quad (79)$$

As a consequence, $V\tilde{f}_i$ is now a the gradient of a harmonic function, say

$$V\tilde{f}_i = \nabla\xi_i \text{ with } \Delta\xi_i = 0 \text{ in } \Omega. \quad (80)$$

Note in passing that due to (76)-(77)

$$|\lambda_j| \lesssim \varepsilon. \quad (81)$$

Concerning the condition on $\partial\Omega \setminus \Sigma$, introduce some smooth functions $k_i : \partial\Omega \rightarrow \mathbb{R}$ such that

$$k_i|_{\partial\Omega \setminus \Sigma} = [V\tilde{f}_i(t_i, \cdot) \cdot n]|_{\partial\Omega \setminus \Sigma}, \quad \|k_i\|_{C^{k+1}(\partial\Omega)} \leq C \left\| [V\tilde{f}_i(t_i, \cdot) \cdot n]|_{\partial\Omega \setminus \Sigma} \right\|_{C^{k+1}(\partial\Omega \setminus \Sigma)} \text{ and } \int_{\partial\Omega} k_i \, d\sigma = 0. \quad (82)$$

Now we introduce the solutions to

$$\begin{cases} \Delta\zeta_i = 0 \text{ in } \Omega \\ \frac{\partial\zeta_i}{\partial n} = k_i \text{ on } \partial\Omega. \end{cases} \quad (83)$$

It follows from standard elliptic estimates that

$$\|\zeta_i\|_{C^{k+1}(\overline{\Omega})} \lesssim \|k_i\|_{C^{k+\alpha}(\partial\Omega)} \lesssim \varepsilon. \quad (84)$$

As a consequence,

$$\Upsilon_i := \xi_i - \zeta_i, \quad (85)$$

now satisfies the required conditions.

Finally, introduce a partition of unity χ_i associated to the intervals $(t_i - \delta_i, t_i + \delta_i)$. We define

$$\theta(t, x) := \sum_{i=1}^N \chi_i(t) \Upsilon_i(x) \text{ in } [0, 1] \times \overline{\Omega}. \quad (86)$$

Then (15) follows from (80), (83) and (85); (16) follows from (82)-(83). Using (75), (76), (77), (81) and (84), we deduce that for some $C > 0$, we have for each $t \in [0, 1]$

$$\|\nabla\theta(t, \cdot) - \nabla\psi(t, \cdot)\|_{C^k(\overline{V}_{\eta/3}(\gamma(t)))} \leq C\varepsilon. \quad (87)$$

Hence (18) follows from Gronwall's lemma: as long as $\Phi^{\nabla\theta}(t, 0, \gamma_0) \in \mathcal{V}_{\eta/3}(\gamma(t))$ (so that the flows are defined and so that one can apply the above estimates), one has

$$\|\Phi^{\nabla\theta}(t, 0, \gamma_0) - \Phi^{\nabla\psi}(t, 0, \gamma_0)\|_{\infty} \leq \|\nabla\theta - \nabla\psi\|_{C^0([0,1]; C^0(\overline{\mathcal{V}}_{\eta/3}(\gamma(t))))} \exp(\|\nabla\psi\|_{L^1(0,1; \mathcal{L}ip[\mathcal{V}_{\eta/3}(\gamma(t))])}). \quad (88)$$

Hence, reducing ε if necessary, using (87) and since η is independent of ε , we are sure that $\Phi^{\nabla\theta}(t, 0, \gamma_0)$ stays in $\mathcal{V}_{\eta/3}(\gamma(t))$ for all $t \in [0, 1]$ (considering the maximal time for which this occurs). In particular, (17) is valid. Moreover one can apply (87) for all time and deduce classically from Gronwall's lemma that

$$\|\Phi^{\nabla\theta}(t, 0, \gamma_0) - \Phi^{\nabla\psi}(t, 0, \gamma_0)\|_{C^k} \lesssim \|\nabla\theta - \nabla\psi\|_{C^0([0,1]; C^k(\overline{\mathcal{V}}_{\eta/3}(\gamma(t))))} \exp(\|\nabla\psi\|_{L^1(0,1; W^{k+1,\infty}[\mathcal{V}_{\eta/3}(\gamma(t))])}), \quad (89)$$

which gives (18). Note that by (87)-(89) and taking ε small enough, we can ensure that

$$\|\nabla\theta(t, \cdot)\|_{C^k(\text{Int}[\Phi^{\nabla\theta}(t, 0, \gamma_0)])} \leq \|\nabla\psi(t, \cdot)\|_{C^k(\text{Int}[\Phi^X(t, 0, \gamma_0)])} + 1 \text{ uniformly in } t. \quad (90)$$

It remains to explain how the construction is modified when Σ meets connected components of $\partial\Omega$ inside $\text{Int}(\gamma_0)$ only. Call Γ_0 such a connected component. Again, we assume that $\Sigma \cap (\Gamma_1 \cup \dots \cup \Gamma_g) = \emptyset$ (but here $\Gamma_0 \subset \text{Int}(\gamma_0)$). In that case, we cannot use the same construction for ψ in Lemma 3, because we would no longer be able to use Runge's theorem without putting a pole in $\Omega \setminus \text{Int}(\gamma_0)$. Instead, we replace the ψ defined in Lemma 3 by the equivalent one defined on $\Omega \setminus \text{Int}(\gamma(t))$: by the same construction we deduce that there exists $\eta > 0$ and $\psi \in C^0([0, 1]; C^\infty(\overline{\Omega} \setminus \text{Int}(\gamma(t)) \cup \mathcal{V}_\eta(\gamma(t))))$ such that

$$\forall t \in [0, 1], \Delta_x \psi(t, x) = 0 \text{ in } [\Omega \setminus \text{Int}(\gamma(t))] \cup \mathcal{V}_\eta(\gamma(t)),$$

and

$$\begin{aligned} \nabla_x \psi \cdot \nu &= X \cdot \nu \text{ on } \gamma(t), \text{ for each } t, \\ \nabla_x \psi \cdot n &= 0 \text{ on each connected component of } \partial\Omega \text{ outside } \gamma(t). \end{aligned}$$

Then one can introduce the times t_i as previously, use the union of $[\overline{\Omega} \setminus \text{Int}(\gamma(t))] \cup \overline{\mathcal{V}}_{\eta/2}(\gamma(t))$ and of $\overline{\mathcal{V}}_{\eta/2}(\partial\Omega \setminus \Sigma)$ as compact K_i when applying Runge's theorem, and the remaining of the proof is the same.

4.2 A smooth Jordan curve transported by an analytic flow

The goal of this section is to prove the following result.

Proposition 4. *Proposition 3 is valid if we suppose γ_0 of class C^∞ only.*

Proof of Proposition 4. Let us consider γ_0 a smooth Jordan curve, and $X \in C^0([0, 1]; C^\omega(\overline{\Omega}; \mathbb{R}^2))$. The complement A of $\text{Int}(\gamma_0)$ in the Riemann sphere is then a connected and simply connected smooth domain. By Riemann's conformal mapping theorem, there exists a one-to-one conformal map $\Phi : \mathbb{B} \rightarrow A$. By Kellogg-Warschawski's theorem (see [22]), this map is C^∞ up to the boundary.

Now consider the image γ_κ of the inner circle $S(0, 1 - \kappa)$ as $\kappa \rightarrow 0^+$. As the image of a circle by a holomorphic mapping, this curve γ_κ is an analytical Jordan curve. Moreover, its interior contains γ_0 , and it converges in all C^k spaces toward γ_0 as $\kappa \rightarrow 0^+$. Hence one can apply Proposition 3 with γ_κ as initial curve: given $k \in \mathbb{N}$ and $\varepsilon > 0$, one deduces θ^ε such that (15), (16), (17) and (18) apply at order $k + 2$; we will also suppose that (90) is valid. Suppose as previously that $\Sigma \setminus \text{Int}(\gamma_0) \neq \emptyset$. Introduce

$$\Omega_t^{\kappa, \varepsilon} := \text{Int}[\Phi^{\nabla\theta^\varepsilon}(t, 0, \gamma_\kappa)] \cap \overline{\Omega}. \quad (91)$$

Then by Gronwall's lemma one has

$$\|\Phi^{\nabla\theta^\varepsilon}(t, 0, \gamma_0) - \Phi^{\nabla\theta^\varepsilon}(t, 0, \gamma_\kappa)\|_{C^k(\mathbb{S}^1)} \lesssim \|\gamma_0 - \gamma_\kappa\|_{C^k(\mathbb{S}^1)} \exp\left(\int_0^1 \|\nabla\theta^\varepsilon(t, \cdot)\|_{C^{k+1}(\overline{\Omega}_t^{\kappa, \varepsilon})} dt\right).$$

Now the main point is that, in $\overline{\Omega}_t^{\kappa, \varepsilon}$, $\|\nabla\theta^\varepsilon\|_{C^{k+1}}$ is bounded independently of ε . From (90), we see that it suffices to prove that $\|\nabla\psi(t, \cdot)\|_{C^k(\text{Int}[\Phi^X(t, 0, \gamma_\kappa)])}$ (which does not depend on ε) is bounded independently of κ . Of course, the domain $\text{Int}[\Phi^X(t, 0, \gamma_\kappa)]$ depends on κ . But since by Kellogg-Warschawski's

theorem one has $\gamma_\kappa \rightarrow \gamma$ in $C^\infty(\mathbb{S}^1)$ as $\kappa \rightarrow 0^+$, one deduces that $\Phi^X(t, 0, \gamma_\kappa) \rightarrow \Phi^X(t, 0, \gamma)$ in $C^\infty(\mathbb{S}^1)$ uniformly in t . Then it is standard that the constant in the elliptic estimate in the time-varying domains $\text{Int}[\Phi^X(t, 0, \gamma_\kappa)]$ can be bounded independently of t and κ for κ small enough (see for instance [12, Theorem 6.30] and the proof of [12, Lemma 6.5]). Then the conclusion follows from (73).

The case when $\Sigma \subset \text{Int}(\gamma_0)$ is analogous, again replacing $\overline{\Omega} \cap \overline{\text{Int}(\gamma(t))}$ by $\overline{\Omega} \setminus \text{Int}(\gamma(t))$ in the previous considerations. This concludes the proof of Proposition 4.

4.3 A smooth Jordan curve transported by a C^∞ flow

The goal of this section is to prove the following improvement of Proposition 4, which concludes the proof of Proposition 2.

Proposition 5. *Proposition 4 is valid if we suppose only $X \in C^0([0, 1]; C^\infty(\overline{\Omega}))$.*

Proof of Proposition 5. We use Whitney's approximation theorem (see [19, Proposition 3.3.9] and the subsequent remarks). Given $X \in C^0([0, 1]; C^\infty(\overline{\Omega}; \mathbb{R}^2))$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $X^\varepsilon \in C^0([0, 1]; C^\omega(\overline{\Omega}; \mathbb{R}^2))$ such that

$$\|X - X^\varepsilon\|_{C^0([0, 1]; C^k(\overline{\Omega}))} \leq \varepsilon. \quad (92)$$

(As a matter of fact, Whitney's theorem gives this for fixed t ; obtaining this form is just a matter of uniform continuity in time and a partition of unity.) Note that one can conserve the divergence-free character: it suffices to make the approximation at the level of the potential function h where $X = \nabla^\perp h$. One can conserve the condition (24) as well: since this condition is satisfied up to a term of order ε , one can remove a harmonic extension so that h is constant on each connected component of $\partial\Omega$.

Using Gronwall's lemma we infer

$$\|\Phi^X(t, 0, \gamma_0) - \Phi^{X^\varepsilon}(t, 0, \gamma_0)\|_k \leq \|X - X^\varepsilon\|_{C^0([0, 1]; C^k(\overline{\Omega}))} \exp(\|X\|_{L^1(0, 1; C^{k+1}(\overline{\Omega}))}).$$

Hence Proposition 5 follows from Proposition 4.

5 Proofs of Theorems 1 and 2

In this section we establish the two main theorems. We begin by recalling standard results needed in the core of the proof.

5.1 Standard lemmas of bidimensional fluid dynamics

We will use the following classical results mainly due to Wolibner [24] (see also Yudovich [25], Kato [18]). Define the space $LL(U)$ of log-Lipschitz functions on some open set $U \subset \mathbb{R}^2$ as those satisfying

$$\|f\|_{LL} := \|f\|_\infty + \sup_{\substack{x, y \in U \\ 0 < |x - y| \leq 1}} \frac{|f(y) - f(x)|}{|y - x|(1 - \log|y - x|)} < +\infty.$$

Lemma 4. *Consider $T > 0$ and a vector field $y \in L^1([0, T]; C_0^0(U; \mathbb{R}^2))$, such that for some constant D , one has*

$$\|y(t)\|_{L^1((0, T); LL(U))} \leq D. \quad (93)$$

Then the flow of y is uniquely defined and there exist two positive constants $N(T, D)$ and $\delta(T, D)$, such that for any $(s, s', t, t', x, x') \in [0, T]^4 \times U^2$, one has

$$|\Phi^y(t, s, x) - \Phi^y(t', s', x')| \leq N(T, D)(|s - s'|^{\delta(T, D)} + |t - t'|^{\delta(T, D)} + |x - x'|^{\delta(T, D)}). \quad (94)$$

A more precise statement can be found in Chemin's book [4, Théorème 5.2.1] expressed in the case of $U = \mathbb{R}^2$, which is sufficient for our purpose since we deal with compactly supported vector fields. Another central lemma is the following.

Lemma 5. *There exists a positive constant C_{LL} such that the following holds. Consider $\omega \in L^\infty(\Omega; \mathbb{R})$, $\lambda_1, \dots, \lambda_g \in \mathbb{R}$. Then the function y defined in $C^0(\overline{\Omega}; \mathbb{R}^2)$ as the unique solution of the system*

$$\begin{cases} \operatorname{curl} y(x) = \omega(x) \text{ in } \Omega, \\ \operatorname{div} y(x) = 0 \text{ in } \Omega, \\ y(x) \cdot n(x) = 0 \text{ on } \partial\Omega, \\ \int_{\Gamma_i} y(x) \cdot \tau(x) dx = \lambda_i, \text{ for } i = 1 \dots g, \end{cases} \quad (95)$$

satisfies the estimate:

$$\|y\|_{LL(\overline{\Omega})} \leq C_{LL} (\|\omega\|_\infty + |\lambda_1| + \dots + |\lambda_g|). \quad (96)$$

The following lemma can be found in Yudovich's paper [25] (see [25, Lemma 2.2]).

Lemma 6. *Consider the system (95) with $\lambda_1 = \dots = \lambda_g = 0$. For any $p_0 > 1$, one has for some constant $C > 0$: for any $p \geq p_0$,*

$$\|y\|_{L^p(\Omega)} \leq Cp \|\omega\|_{L^p(\Omega)}. \quad (97)$$

Of course, one could introduce non trivial λ_i in Lemma 6 as well.

Finally, we recall that the Euler equation (1) may be restated in vorticity form as

$$\begin{aligned} \partial_t \omega + (u \cdot \nabla) \omega &= 0 \text{ in } [0, T] \times \Omega, \\ \operatorname{div} u &= 0 \text{ in } [0, T] \times \Omega, \\ \operatorname{curl} u &= \omega \text{ in } [0, T] \times \Omega, \\ \int_{\Gamma_i} \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) \cdot \tau(x) dx &= 0 \text{ in } [0, T], \text{ for all } i = 1 \dots g, \end{aligned}$$

the last equation being void if Ω is simply connected, the first one being equivalent to

$$\partial_t \omega + \operatorname{div}(u\omega) = 0 \text{ in } [0, T] \times \Omega.$$

5.2 Proof of Theorem 1

Idea of the proof. The main idea is the following. First construct a solution \bar{u} starting from u_0 “without control on the boundary” on the whole interval $[0, T]$. In fact, this is not quite possible since $u_0 \cdot n$ is not necessarily zero; but this should be seen as a detail — we will bring the boundary condition $\bar{u} \cdot n$ to zero sufficiently fast for our purpose. Then consider

$$\tilde{\gamma}_0 := \Phi^{\bar{u}}(T, 0, \gamma_0). \quad (98)$$

Now fix $\varepsilon > 0$ and $k \in \mathbb{N}$. Deduce from Theorem 3 a function $\theta \in C_0^\infty((0, 1); C^\infty(\overline{\Omega}))$ such that the flow of $\nabla \theta$ drives approximately $\tilde{\gamma}_0$ to γ_1 , precisely satisfying (15) to (18) where $\tilde{\gamma}_0$ replaces γ_0 . Now given $\nu < \min(1, T/2)$, we introduce the solution u^ν of the Euler equation given by \bar{u} on the time interval $[0, T - \nu]$, and on the time interval $[T - \nu, T]$ as the solution of (1) with the control in normal velocity (4) chosen on Σ as

$$u^\nu(t, x) \cdot n(x) = \frac{1}{\nu} \nabla \theta \left(\frac{t - T + \nu}{\nu}, x \right),$$

while the control in vorticity (5) is essentially chosen in order to keep the regularity of the solution. This follows an idea of Coron from [5, 6]: if we act very fast, by a rescaling argument, we will show that the flow during $[T - \nu, T]$ converges toward the flow without vorticity (and without circulation on Γ_i), that is, the potential one. This is tightly connected to Coron's *return method*, for which we refer to [7]. Precisely our goal will be to prove that

$$\|\gamma_1 - \Phi^{u^\nu}(T, 0, \gamma_0)\|_{C^k(\mathbb{S}^1)} \leq \varepsilon + O(\nu), \quad (99)$$

which will establish Theorem 1.

Construction of \bar{u} and u^ν . Let us explain simultaneously the construction of \bar{u} and the one of u^ν once the function θ is fixed in $C_0^\infty((0, 1); C^\infty(\bar{\Omega}))$ (we recall that we are not quite in the situation of [26]). We mainly proceed as in [6]. Include $\bar{\Omega}$ in some ball $B(0, R)$ of \mathbb{R}^2 . Introduce a linear extension operator π from $\bar{\Omega}$ to $B(0, R)$, which sends continuously functions of class $C^{j, \alpha}(\bar{\Omega})$ to functions in $C_0^{j, \alpha}(B(0, R))$ for all $j \in \mathbb{N}$ and $\alpha \in [0, 1)$ and also $LL(\Omega)$ to $LL(B(0, R))$ (the existence of such an operator follows for instance from [16, Corollary 1.3.7, p. 138]). For E one of these spaces, call $c_\pi(E)$ a constant such that

$$\|\pi(f)\|_{E(B(0, R))} \leq c_\pi(E) \|f\|_{E(\Omega)}.$$

Introduce $\rho \in C^\infty([0, 1]; [0, 1])$ such that

$$\begin{cases} \rho(t) = 1 & \text{in } [0, 1/3], \\ \rho(t) = 0 & \text{in } [2/3, 1]. \end{cases} \quad (100)$$

Suppose that we are given $\theta \in C_0^\infty((0, 1); C^\infty(\bar{\Omega}))$ and extend it in time by 0 on \mathbb{R} . Then u^ν (and \bar{u}) will be obtained as the fixed point of the following scheme. Fix $\kappa \in \{0, 1\}$ (the function u^ν corresponding to $\kappa = 1$, the function \bar{u} to the case $\kappa = 0$).

We consider $\mu < \min(1, T/2)$ intended to be small. Given $\ell \in \mathbb{N}$ and $\alpha > 0$, we associate to $\omega \in C^0([0, T]; C^{\ell, \alpha}(\bar{\Omega}))$ the function $y \in C^0([0, T]; C^{\ell+1, \alpha}(\bar{\Omega}))$ (see e.g. [18]) as the unique solution of the following elliptic system:

$$\begin{cases} \operatorname{curl} y = \omega & \text{in } [0, T] \times \Omega, \\ \operatorname{div} y = 0 & \text{in } [0, T] \times \Omega, \\ y \cdot n = \rho\left(\frac{t}{\mu}\right) u_0(x) \cdot n(x) + \frac{\kappa}{\nu} \nabla \theta\left(\frac{t-T+\nu}{\nu}, x\right) \cdot n(x) & \text{on } [0, T] \times \partial\Omega, \\ \int_{\Gamma_i} y(0, x) \cdot \tau(x) dx = \int_{\Gamma_i} u_0(x) \cdot \tau(x) dx & \text{for all } i = 1 \dots g, \\ \int_{\Gamma_i} \left(\frac{\partial y}{\partial t} + y \cdot \nabla y\right) \cdot \tau(x) dx = 0 & \text{in } [0, T], \text{ for all } i = 1 \dots g. \end{cases} \quad (101)$$

Note that the last one can be restated by integrating by parts as

$$\frac{d}{dt} \int_{\Gamma_i} y(t, x) \cdot \tau(x) dx = - \int_{\Gamma_i} \omega(y, n) \text{ in } [0, T], \text{ for all } i = 1 \dots g. \quad (102)$$

Now $\pi(y)$ determines a flow in $B(0, R)$, and we define the operator \mathcal{T} which maps ω to $\mathcal{T}(\omega)$ defined as the restriction to $[0, T] \times \bar{\Omega}$ of $\tilde{\omega}$ defined on $[0, T] \times B(0, R)$ by

$$\tilde{\omega}(t, x) := (\pi\omega_0) \circ \Phi^{\pi(y)}(0, t, x), \quad (103)$$

where $\omega_0(x) := \operatorname{curl} u_0(x)$ in $\bar{\Omega}$. Let us now prove that \mathcal{T} has a fixed point.

We denote by $\mathcal{R}_\nu^T[\nabla\theta]$ the rescaled function for $\nabla\theta$, that is

$$\mathcal{R}_\nu^T[\nabla\theta](t, x) := \frac{1}{\nu} \nabla \theta\left(\frac{t-T+\nu}{\nu}, x\right) \text{ on } \mathbb{R} \times \bar{\Omega}.$$

We introduce $E[u_0] \in C^\infty([0, T] \times \bar{\Omega})$ as the solution of

$$\begin{cases} \operatorname{curl} E[u_0] = 0 & \text{in } [0, T] \times \Omega, \\ \operatorname{div} E[u_0] = 0 & \text{in } [0, T] \times \Omega, \\ E[u_0] \cdot n = \rho\left(\frac{t}{\mu}\right) u_0 \cdot n & \text{on } [0, T] \times \partial\Omega, \\ \int_{\Gamma_i} E[u_0] \cdot \tau d\sigma = \int_{\Gamma_i} u_0 \cdot \tau d\sigma & \text{for all } i = 1 \dots g. \end{cases} \quad (104)$$

Fix

$$D := C_{LL} \left(c_\pi(C^0) \|\omega_0\|_\infty \left\{ 1 + |\partial\Omega| \left[\mu \|u_0\|_{C^0(\bar{\Omega})} + \|\nabla\theta\|_{L^1((0, 1), C^0(\bar{\Omega}))} \right] \right\} + \int_{\partial\Omega} |u_0(x) \cdot \tau(x)| dx \right),$$

where C_{LL} was introduced in Lemma 5 and $|\partial\Omega|$ is the length of $\partial\Omega$. Define from Lemma 4 the two constants

$$\delta = \delta\left(T, c_\pi(LL)[TD + \|E[u_0]\|_{L^1((0,T);LL(\Omega))} + \|\nabla\theta\|_{L^1((0,1);LL(\Omega))}]\right)$$

$$\text{and } N = N\left(T, c_\pi(LL)[TD + \|E[u_0]\|_{L^1((0,T);LL(\Omega))} + \|\nabla\theta\|_{L^1((0,1);LL(\Omega))}]\right).$$

Without loss of generality, we assume that $\delta \leq \alpha$.

Let us now prove that the operator \mathcal{T} sends the convex set

$$\mathcal{C} := \left\{ \omega \in L^\infty([0, T]; C^\delta(\Omega)) / \|\omega(t)\|_\infty \leq c_\pi(C^0)\|\omega_0\|_\infty \text{ and } \|\omega(t)\|_\delta \leq c_\pi(C^\delta)N\|\omega_0\|_\delta \text{ uniformly in } t \right\}, \quad (105)$$

into itself. It follows from (101) (see also (102)) that whatever $\omega \in \mathcal{C}$, we have

$$\left| \int_{\Gamma_i} y \cdot \tau - \int_{\Gamma_i} u_0 \cdot \tau \right| \leq c_\pi(C^0)\|\omega_0\|_\infty |\Gamma_i| [\mu\|u_0\|_{C^0(\bar{\Omega})} + \|\nabla\theta\|_{L^1((0,1),C^0(\bar{\Omega}))}], \quad (106)$$

so that it is clear with Lemma 5 that for any $\omega \in \mathcal{C}$,

$$\|y(t) - E[u_0] - \mathcal{R}_v^T[\nabla\theta]\|_{LL(\Omega)} \leq D \text{ uniformly in } t.$$

Hence the flow $\Phi^{\pi(y)}$ satisfies

$$\|\Phi^{\pi(y)}\|_{C^\delta([0,T]^2 \times B(0,R))} \leq N,$$

which with (103) proves that \mathcal{T} sends \mathcal{C} into itself.

Equip \mathcal{C} with the $L^\infty((0, T) \times \Omega)$ topology for which it is closed. Then \mathcal{T} is continuous for this topology: if $\omega_n \rightarrow \omega$, then we easily infer $y_n \rightarrow y$ uniformly, and deduce the uniform convergence of the flows by Gronwall's lemma (since (101) and (105) give a uniform Lipschitz bound for y_n and y), and finally we deduce the uniform convergence of $\hat{\omega}_n$. Finally $\mathcal{T}(\mathcal{C})$ is relatively compact, because we clearly have compactness in space, and the compactness in time follows from (103) since it implies that for $\omega \in \mathcal{T}(\mathcal{C})$

$$\frac{\partial\omega}{\partial t} \text{ is bounded in } L^\infty((0, T); W^{-1,\infty}(\Omega)). \quad (107)$$

The relative compactness of $\mathcal{T}(\mathcal{C})$ then follows by interpolation.

Hence it follows from Schauder's fixed point theorem that \mathcal{T} admits a fixed point. From (101), this fixed point gives us a solution of (1) with initial condition (3) and desired boundary condition. That this solution y is in $C^\infty([0, T] \times \bar{\Omega})$ follows from a bootstrap argument and standard considerations on elliptic estimates and regularity of flows in Hölder spaces.

Uniqueness. We follow [26]. Let us establish the uniqueness of the solution of the initial-boundary problem. Consider u_1 and u_2 two solutions with same initial and boundary data. Introduce $\omega_i := \text{curl } u_i$, $\hat{\omega} := \omega_1 - \omega_2$, $\lambda_i^j := \int_{\Gamma_j} u_i \cdot d\tau$ and $\hat{\lambda}^j := \lambda_1^j - \lambda_2^j$. We have

$$\partial_t \hat{\omega} + u_1 \cdot \nabla \hat{\omega} + \hat{u} \cdot \nabla \omega_2 = 0 \text{ in } [0, T] \times \bar{\Omega}. \quad (108)$$

By multiplying by $\hat{\omega}$ and integrating by parts, we get

$$\frac{d}{dt} \int_{\Omega} |\hat{\omega}|^2 dx + \int_{\partial\Omega} (u_1 \cdot n) \hat{\omega}^2 dx + 2 \int_{\Omega} \hat{\omega} (\hat{u} \cdot \nabla \omega_2) dx = 0. \quad (109)$$

The second integral can be taken over $\Sigma_t^+ := \{x \in \partial\Omega / u_1(t, \cdot) \cdot n > 0\}$ since it is zero on the remaining of the boundary. For the last one, we use that $\nabla \omega_2$ is bounded in L^∞ , Cauchy-Schwarz inequality, and Lemma 6:

$$\|\hat{u}\|_{H^1(\Omega)} \lesssim \|\hat{\omega}\|_{L^2(\Omega)} + \sum_{j=1}^g |\hat{\lambda}^j|.$$

We deduce

$$\frac{d}{dt} \|\hat{\omega}\|_{L^2}^2 + \int_{\Sigma_t^+} (u_1 \cdot n) \hat{\omega}^2 dx \leq C \|\hat{\omega}\|_{L^2} \left(\|\hat{\omega}\|_{L^2} + \sum_{j=1}^g |\hat{\lambda}^j| \right). \quad (110)$$

Now we use (102) and $u_1 \cdot n = u_2 \cdot n$ on $\partial\Omega$ to deduce that

$$\frac{d}{dt} \hat{\lambda}^j = \int_{\Sigma_t^+ \cap \Gamma_j} (u_1 \cdot n) \hat{\omega} \, dx \leq C \left(\int_{\Sigma_t^+} (u_1 \cdot n) \hat{\omega}^2 \, dx \right)^{1/2}.$$

Denote

$$h^p(t) := \int_0^t \int_{\Sigma_t^+} (u_1 \cdot n) |\hat{\omega}|^p \, dx \, dt \text{ and } j(t) := \|\hat{\omega}\|_{L^2} + h^1(t).$$

Note that by the previous considerations we have

$$\|\hat{\omega}\|_{L^2} \leq C \int_0^t [\|\hat{\omega}\|_{L^2}(\tau) + h^1(\tau)] \, d\tau.$$

We finally integrating (110) we deduce

$$j^2(t) \leq C(\|\hat{\omega}\|_{L^2}^2 + h^2(t)) \leq C \int_0^t \|\hat{\omega}(\tau)\|_{L^2} j(\tau) \, d\tau \leq C \left(\int_0^t j(\tau) \, d\tau \right)^2.$$

Hence the conclusion follows from Gronwall's lemma.

Proof of (99). It remains to explain why this construction is effective. Define \bar{u} as the function in $C^\infty([0, T] \times \bar{\Omega}; \mathbb{R}^2)$ corresponding to the above fixed point when $\kappa = 0$ (this of course does not depend on the function θ). Define $\tilde{\gamma}_0$ by (98). For $\varepsilon > 0$ and $k \in \mathbb{N}$, get from Theorem 3 a function $\theta \in C_0^\infty((0, 1); C^\infty(\bar{\Omega}))$ satisfying (15) to (18) with $\tilde{\gamma}_0$ as the initial curve.

Now construct u^ν as previously with $\kappa = 1$ and the function θ that we have introduced. Now define $\tilde{\gamma}_\nu := \Phi^{u^\nu}(T - \nu, 0, \gamma_0)$. Then by uniqueness of the solution constructed above, $\tilde{\gamma}_\nu := \Phi^{\bar{u}}(T - \nu, 0, \gamma_0)$ and this notation is consistent with the one for $\tilde{\gamma}_0$.

Now, rescale u^ν during $[T - \nu, T]$ by

$$v^\nu(t, x) := \nu u^\nu(T - \nu + \nu t, x) \text{ for } (t, x) \in [0, 1] \times \bar{\Omega}. \quad (111)$$

Of course,

$$\Phi^{v^\nu}(1, 0, \tilde{\gamma}_\nu) = \Phi^{u^\nu}(T, 0, \gamma_0), \quad (112)$$

so we are led to prove

$$\|\Phi^{v^\nu}(1, 0, \tilde{\gamma}_\nu) - \Phi^{\nabla\theta}(1, 0, \tilde{\gamma}_0)\|_{C^k(\mathbb{S}^1)} = O(\nu). \quad (113)$$

Now the error between these two curves comes from the fact that $\tilde{\gamma}_\nu \neq \tilde{\gamma}_0$ and that $v^\nu \neq \nabla\theta$. But in both cases, the error is small. It is indeed a consequence of the regularity of \bar{u} that

$$\|\tilde{\gamma}_\nu - \tilde{\gamma}_0\|_{C^k(\mathbb{S}^1)} = O(\nu). \quad (114)$$

Concerning v^ν , we see that $w := v^\nu - \nabla\theta$ satisfies

$$\begin{cases} \operatorname{curl} w = \nu \operatorname{curl} u^\nu(T - \nu + \nu t, x) \text{ in } [0, 1] \times \Omega, \\ \operatorname{div} w = 0 \text{ in } [0, 1] \times \Omega, \\ w \cdot n = 0 \text{ on } [0, 1] \times \partial\Omega, \\ \int_{\Gamma_i} w(t, x) \cdot \tau(x) \, dx = \nu \int_{\Gamma_i} u^\nu(T - \nu + \nu t, x) \cdot \tau(x) \, d\sigma \text{ for all } i = 1 \dots g. \end{cases} \quad (115)$$

But it follows from the construction of u^ν and from the estimates given in (105) that $\operatorname{curl} u^\nu$ is bounded in $C^\delta(\bar{\Omega})$; the same bootstrap argument as previously shows that $\operatorname{curl} u^\nu$ is bounded in $C^{k-1, \alpha}(\bar{\Omega})$ independently of ν . It follows also from (106) that the circulations of u^ν remain bounded as $\nu \rightarrow 0^+$. It follows that

$$\|v^\nu - \nabla\theta\|_{C^{k, \alpha}(\bar{\Omega})} = O(\nu). \quad (116)$$

Hence (113) follows from (114), (116) and a standard Gronwall's argument (since $\nabla\theta$ is a fixed smooth vector field). Note that once (116) is proven, (10) is a consequence of (17), taking ν suitably small. This finishes the proof of Theorem 1.

5.3 Proof of Theorem 2

The idea of the proof of Theorem 2 is basically the same as the one of Theorem 1. First, we construct the solution \bar{u} “without control”. We will prove that $\tilde{\gamma}_0 := \Phi^{\bar{u}}(T, 0, \gamma_0)$ is a C^∞ curve (in fact, if $u_0.n = 0$ on Σ , a much stronger result has been established by Depauw in [8] — see also Dutrifoy [9]). Then, again, given $\varepsilon > 0$ and $k \in \mathbb{N}$, we introduce $\theta \in C_0^\infty((0, 1); C^\infty(\bar{\Omega}))$ as in Theorem 3 with $\tilde{\gamma}_0$ as initial curve and γ_1 as target, and we construct u^ν as we did previously.

However there are several differences. First, now one uses (12) as the control in vorticity (5) (actually this is rather a simplification). Next, we use Yudovich’s theory [25] to construct the solutions whose vorticity is merely in L^∞ (note that the paper [26] on the initial boundary problem considers more regular solutions). Then the fact that $\tilde{\gamma}_0$ is smooth does no longer follow from the smoothness of the velocity field. And of course, we do no longer have $\text{curl } u \in C^{k-1, \alpha}$, so (14) cannot be a consequence of (116). We must use tools adapted to this particular type of solutions instead.

Construction of \bar{u} and u^ν . We will use the same notations as in the previous section for $B(0, R)$, π , etc. As an additional notation we call χ the extension operator $L^p(\Omega) \rightarrow L^p(B(0, R))$ which extends $f \in L^p(\Omega)$ by 0 on $B(0, R) \setminus \Omega$. As previously, we will explain in the same time the construction of the solution \bar{u} when “no control” is employed (again, we must take $u_0.n$ into account, though), and the solution u^ν based on some function θ which itself is in fact deduced from the latter.

Assume that we have deduced from Theorem 3 a potential flow $\nabla\theta$. Introduce

$$\tilde{D} := C_{LL}\|\omega_0\|_\infty.$$

Consider $\kappa \in \{0, 1\}$ and $\mu, \nu < \min(1, T/2)$, intended to be small. Define $E[u_0]$ by (104). Introduce the convex set

$$\mathcal{C}' := \left\{ y \in C^0([0, T] \times \bar{\Omega}) / \|y - E[u_0] - \kappa \mathcal{R}_\nu^T[\nabla\theta]\|_{L^\infty((0, T), LL(\bar{\Omega}))} \leq \tilde{D} \right\}, \quad (117)$$

which is closed when equipped with the $C^0([0, T] \times \bar{\Omega})$ norm, and nonempty since it contains $E[u_0] + \kappa \mathcal{R}_\nu^T[\nabla\theta]$. Let us describe the new operator \mathcal{S} whose fixed point will give the functions \bar{u} and u^ν (for $\kappa = 0$ and $\kappa = 1$ respectively). To $y \in \mathcal{C}'$, we associate

$$\tilde{\omega}(t, x) := (\chi\omega_0) \circ \Phi^{\pi(y)}(0, t, x). \quad (118)$$

Then we associate $\mathcal{S}(y) = \tilde{y}$ by the system

$$\begin{cases} \text{curl } \tilde{y} = \tilde{\omega} \text{ in } [0, T] \times \Omega, \\ \text{div } \tilde{y} = 0 \text{ in } [0, T] \times \Omega, \\ \tilde{y}(t, x).n(x) = \rho\left(\frac{t}{\mu}\right)u_0(x).n(x) + \frac{\kappa}{\nu}\nabla\theta\left(\frac{t-T+\nu}{\nu}, x\right).n(x) \text{ on } [0, T] \times \partial\Omega, \end{cases} \quad (119)$$

and

$$\frac{d}{dt} \int_{\Gamma_i} \tilde{y}(t, x). \tau(x) dx = 0 \text{ in } [0, T], \quad \text{for all } i = 1 \dots g. \quad (120)$$

When comparing to (101), the reason for replacing (102) by (120) is that, since we expect (10) and (12) to hold, we will have $\omega = 0$ near $\partial\Omega$. In that situation, (120) is of course equivalent to (102).

Now let us prove that \mathcal{S} sends \mathcal{C}' into itself. From (118), it is clear that

$$\|\tilde{\omega}(t, \cdot)\|_\infty \leq \|\omega_0\|_\infty \text{ for all } t.$$

Using (119)-(120) and Lemma 5 we infer that $\mathcal{S}(y) \in \mathcal{C}'$, for any $y \in \mathcal{C}'$. Now $\mathcal{S}(\mathcal{C}')$ is relatively compact in $C^0([0, T] \times \bar{\Omega})$, because of the log-Lipschitz estimate in space given in (117), of (107) being valid for elements of $\mathcal{S}(\mathcal{C}')$ and an interpolation argument. Finally, \mathcal{S} is continuous: suppose that $(y_n) \in (\mathcal{C}')^\mathbb{N}$ converges towards y . We use Gronwall’s lemma with logarithm: for $\varepsilon \in (0, 1)$

$$\text{if } \alpha(t) \leq \varepsilon + \int_0^t C\alpha(1 - \ln(\alpha)),$$

$$\text{then as long as } \alpha \in [0, 1], \text{ one has } \alpha(t) \leq \exp\left(1 - \exp(-Ct + \ln(1 - \ln \varepsilon))\right),$$

which is obtained by comparison of α with the solution of $y' = Cy(1 - \ln(y))$, $y(0) = \varepsilon$. Now we have

$$\frac{d}{dt}(\Phi^{y_n} - \Phi^y)(t, s, x) = \left[y_n(t, \Phi^{y_n}(t, s, x)) - y_n(t, \Phi^y(t, s, x)) \right] + \left[(y_n - y)(t, \Phi^y(t, s, x)) \right].$$

Using the uniform log-Lipschitz estimate on (y_n) , we deduce that Φ^{y_n} converges uniformly towards Φ^y and then by dominated convergence $\tilde{\omega}_n \rightarrow \tilde{\omega}$ in L^p for each time time ($p < \infty$), where $\tilde{\omega}_n$ and $\tilde{\omega}$ are the solutions associated to y_n and y by (118), respectively. Then using Lemma 6 on $y - E[u_0] - \kappa \mathcal{R}_\nu^T[\nabla\theta]$ and then Sobolev imbedding (for $p > 2$), we deduce $\mathcal{S}(y_n) \rightarrow \mathcal{S}(y)$ in $C^0(\bar{\Omega})$ for each time, but this convergence is uniform due to the relative compactness of $\mathcal{S}(C')$.

By Schauder's theorem, we get a fixed point of the equation which as previously will be a solution of (1), provided that we prove that as claimed, we have $\omega = 0$ near the boundary. We will call this solution \bar{u} for $\kappa = 0$. Again, we introduce $\tilde{\gamma}_0$ by (98). Assuming that $\tilde{\gamma}_0$ is smooth (as we will prove later), we introduce θ coming from Theorem 3 and driving approximately $\tilde{\gamma}_0$ to γ_1 . We define u^ν as the above fixed point computed with $\kappa = 1$.

Distance of the patch from the boundary. Let us show the central property that for μ (independent of θ) and ν (depending on θ) suitably small, the flows of the solutions \bar{u} and u^ν satisfy (10), that to be more precise that for some $\underline{d} > 0$, one has

$$\forall t \in [0, T], \quad d(\Phi^{\bar{u}}(t, 0, \gamma_0), \partial\Omega) \geq \underline{d} \text{ and } d(\Phi^{u^\nu}(t, 0, \gamma_0), \partial\Omega) \geq \underline{d}. \quad (121)$$

We notice that independently of $\mu \in [0, \min(1, T/2))$ we have a uniform bound on $E[u_0]$, let us say $\|E[u_0]\|_\infty \leq C\|u_0\|_\infty$. This involves that we have a uniform bound for the sup norm of \bar{u} and u^ν for t in $[0, \mu]$. Hence putting $\bar{d} := d(\gamma_0, \partial\Omega)$ and taking

$$\mu < \frac{\bar{d}}{2(\bar{D} + C\|u_0\|_\infty)}, \quad (122)$$

we are sure that during the time interval $[0, \mu]$, $d(\Phi^{\bar{u}}(t, 0, \gamma_0), \partial\Omega) \geq \bar{d}/2$, and the same for u^ν . Hence (121) is established for what concerns \bar{u} , since afterwards one has $\bar{u}.n = 0$ on $\partial\Omega$ (we recall that the flow is unique at this level of regularity). Using (121), the uniqueness of \bar{u} is proven in the next paragraph; this involves in particular that $\bar{u}(t, \cdot) = u^\nu(t, \cdot)$ for $t \in [0, T - \nu]$.

For what concerns u^ν , (121) is still to be proven for the time interval $[T - \nu, T]$. But here we use that $\nabla\theta$ satisfies (17), and define $w(t, x) = v^\nu(t, x) - \nabla\theta(t, x)$ in $[0, 1] \times \bar{\Omega}$, where v^ν is again defined by (111). This function w satisfies

$$\begin{cases} \operatorname{curl} w = \nu \operatorname{curl} u^\nu(T - \nu + \nu t, x) \text{ in } [0, 1] \times \Omega, \\ \operatorname{div} w = 0 \text{ in } [0, 1] \times \Omega, \\ w.n = 0 \text{ on } [0, T] \times \partial\Omega, \\ \int_{\Gamma_i} w(t, x). \tau(x) dx = \nu \int_{\Gamma_i} u_0. \tau(x) d\sigma \text{ for all } i = 1 \dots g \text{ and } t \in [0, 1]. \end{cases} \quad (123)$$

Hence it follows that $\|w\|_{C^0([0, 1] \times \bar{\Omega})} = O(\nu)$. On another side the uniform bound on \bar{u} in $[0, T]$ proves that $\|\Phi^{\bar{u}}(T - \nu, 0, \gamma_0) - \tilde{\gamma}_0\|_\infty = O(\nu)$. Hence by (112) and a Gronwall argument (since $\nabla\theta$ is smooth) we get (121) by choosing ν sufficiently small depending on θ , say $\nu \leq \nu_0(\theta)$. Moreover, one can find $\underline{d} > 0$ such that (121) applies.

Uniqueness. Call Γ the union of the connected components of $\partial\Sigma$ meeting Σ . Let us prove that the above solutions are unique among those satisfying

$$\operatorname{curl} u = 0 \text{ for all } t \in [0, T] \text{ and } x \in \Omega \text{ such that } d(x, \partial\Gamma) \leq \underline{d}, \quad (124)$$

and which are moreover $L^\infty((0, T), LL(\Omega))$. Given two such solutions u_1 and u_2 , we consider $\hat{u} := u_1 - u_2$ and write

$$\partial_t \hat{u} + u_1. \nabla \hat{u} + \hat{u}. \nabla u_2 = 0.$$

Multiplying by \hat{u} , integrating in Ω and integrating by parts yields

$$\frac{d}{dt} \int_{\Omega} \hat{u}^2 dx = 2 \int_{\Omega} \hat{u}. (\hat{u}. \nabla u_2) dx + \int_{\partial\Omega} (u_1.n) \hat{u}^2 dx. \quad (125)$$

For what concerns the first term in the right hand side, we follow [25]. We use Lemma 6 to deduce that for some constant $C > 0$ independent of $p \geq 2$, we have

$$\|\nabla u_2\|_{L^p} \leq Cp\|\omega_2\|_{L^p} + C(\|u_2 \cdot n\|_{C^{1,\alpha}} + \int_{\partial\Omega} |u_2(x) \cdot \tau(x)| dx), \quad (126)$$

by decomposing u_2 between a part with homogeneous boundary condition and vorticity ω_2 , and a part with non-homogeneous boundary conditions and no vorticity. Hence

$$\|\nabla u_2\|_{L^p} \leq pC(\omega_0) + C(u_0, \nabla\theta) \leq p\tilde{C}(u_0, \theta), \quad (127)$$

We infer

$$\begin{aligned} \left| \int_{\Omega} \hat{u} \cdot (\hat{u} \cdot \nabla u_2) dx \right| &\leq p\tilde{C}(u_0, \theta) \left(\int_{\Omega} |\hat{u}(t, \cdot)|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq p\tilde{C}(u_0, \theta) \|\hat{u}\|_{L^\infty}^{\frac{2}{p}} \|\hat{u}\|_{L^2}^{\frac{2p-1}{p}}. \end{aligned}$$

The norm $\|\hat{u}\|_{L^\infty}$ can be bounded in terms of ω_0 and the boundary conditions, so we deduce that for $p \geq 2$

$$\left| \int_{\Omega} \hat{u} \cdot (\hat{u} \cdot \nabla u_2) dx \right| \leq p\check{C}(u_0, \theta) \|\hat{u}\|_{L^2}^{\frac{2p-1}{p}}.$$

Concerning the second term in the right-hand side of (125), we introduce $\Lambda \in C^\infty(\overline{\Omega}; \mathbb{R})$ such that $\Lambda = 1$ near Γ and $\Lambda = 0$ in $\{x \in \overline{\Omega} / d(x, \Gamma) \geq \underline{d}\}$. Using (124), we deduce the following equation satisfied by $\Lambda\hat{u}$:

$$\begin{cases} \operatorname{curl}(\Lambda\hat{u}) = \hat{u} \cdot \nabla^\perp \Lambda \text{ in } (0, T) \times \Omega, \\ \operatorname{div}(\Lambda\hat{u}) = \hat{u} \cdot \nabla \Lambda \text{ in } (0, T) \times \Omega, \\ \Lambda\hat{u} \cdot n = 0 \text{ on } (0, T) \times \partial\Omega, \\ \int_{\Gamma_i} \Lambda(x)\hat{u}(t, x) \cdot \tau(x) dx = 0 \text{ for all } i = 1 \dots g. \end{cases}$$

Using Lemma 6 for $p = 2$ we deduce

$$\|\Lambda\hat{u}\|_{H^1(\Omega)} \leq C(\underline{d}) \leq C(\underline{d}) \|\hat{u}\|_{L^2(\Omega)}.$$

Consequently we get by using a trace estimate that

$$\frac{d}{dt} \int_{\Omega} \hat{u}^2 dx \leq \hat{C}(p) \|\hat{u}(t, \cdot)\|_{L^2}^{\frac{2p-1}{p}} + \|\hat{u}(t, \cdot)\|_{L^2}^2$$

Finally, calling $\delta(t) := \|\hat{u}(t, \cdot)\|_{L^2(\Omega)}^2$, we deduce

$$\delta'(t) \leq \hat{C}(p)\delta(t)^{1-1/p} + \delta(t).$$

For small times, one has $\delta(t) < 1/e^2$, then one chooses $p = -\ln \delta(t)$, which yields

$$\delta'(t) \leq C(u_0, \theta)\delta(t)|\ln \delta(t)|,$$

which proves the uniqueness.

Proof of the relevance of u^ν . Call as previously $\tilde{\gamma}_\nu := \Phi^{\bar{u}}(T - \nu, 0, \gamma_0) = \Phi^{u^\nu}(T - \nu, 0, \gamma_0)$, the last equality coming from the uniqueness of the solution. Let us prove that

$$\|\Phi^{u^\nu}(T, T - \nu, \tilde{\gamma}_\nu) - \Phi^{\nabla\theta}(1, 0, \tilde{\gamma}_0)\|_{C^k(\mathbb{S}^1)} = O(\nu), \quad (128)$$

which includes the non trivial fact that these curves actually belong to C^k (they are in fact in C^∞). This will establish Theorem 2.

For that, we will rely on an approach considering the contour dynamics of the curve γ (see [1]). By the Biot-Savart law, the velocity field V_γ generated in \mathbb{R}^2 by a vortex patch on $\operatorname{Int}_\gamma$:

$$\operatorname{curl} V_\gamma = \mathbf{1}_{\operatorname{Int}(\gamma)} \text{ and } \operatorname{div} V_\gamma = 0 \text{ in } \mathbb{R}^2,$$

and decaying at infinity is given by

$$V_\gamma(x) = \frac{1}{2\pi} \int_{\text{Int}(\gamma)} \nabla^\perp \log|x-y| dy = -\frac{1}{2\pi} \int_0^{2\pi} \log|x-\gamma(\sigma)| \tau(\sigma) d\sigma, \quad (129)$$

where τ is the tangent on γ . In our case, the contour dynamics has to take into account the presence of the boundary $\partial\Omega$ and the control $u.n$. Define the following ‘‘correction’’ w_γ of V_γ , for $\gamma \subset \Omega$, as the solution of the following Dirichlet problem:

$$\begin{cases} \Delta w_\gamma = 0 \text{ in } \Omega, \\ w_\gamma(x) = \frac{1}{2\pi} \int_{\text{Int}(\gamma)} \log|x-y| dy \text{ on } \partial\Omega, \end{cases} \quad (130)$$

so that $W_\gamma := V_\gamma - \nabla^\perp w_\gamma$ satisfies $\text{curl} W_\gamma = \mathbf{1}_{\text{Int}(\gamma)}$, $\text{div} W_\gamma = 0$ in Ω and $W_\gamma.n = 0$ on $\partial\Omega$.

Now the contour motion for $\bar{\gamma}(t) := \Phi^{\bar{v}}(t, 0, \gamma_0)$ and $\gamma^\nu(t) := \Phi^{u^\nu}(t, 0, \gamma_0)$ is given by the equation

$$\partial_t \bar{\gamma}(t, \sigma) = [V_{\bar{\gamma}(t)} - \nabla^\perp w_{\bar{\gamma}(t)} + E[u_0](t)](\bar{\gamma}(t, \sigma)), \quad (131)$$

$$\partial_t \gamma^\nu(t, \sigma) = [V_{\gamma^\nu(t)} - \nabla^\perp w_{\gamma^\nu(t)} + E[u_0](t) + \mathcal{R}_\nu^T[\nabla\theta]](\gamma^\nu(t, \sigma)). \quad (132)$$

(Recall that $E[u_0]$ is given by (104).) We will see that the solutions obtained in this way are the same as the one constructed above. We will use the following result, see [20, Proposition 8.8].

Proposition 6. *The mapping $\gamma \mapsto V_\gamma \circ \gamma$ is a locally Lipschitz continuous operator from*

$$O^M := \left\{ \gamma \in C^{j,\alpha}(\mathbb{S}^1) / |\gamma|_* := \inf_{\theta_1 \neq \theta_2} \frac{|\gamma(\theta_1) - \gamma(\theta_2)|}{|\theta_1 - \theta_2|} \geq 1/M \text{ and } \|\partial_\theta \gamma\|_\infty \leq M \right\} \quad (133)$$

to $C^{j,\alpha}(\mathbb{S}^1; \mathbb{R}^2)$ for $j \geq 1$.

1. We first show that there exists a unique solution $\bar{\gamma}$ of (131) (or (132)) of class $C^{1,\alpha}$ during the time interval $[0, \mu]$. Again we choose μ so that (122) applies. By classical interior elliptic estimates it follows that on $\{x \in \Omega / d(x, \partial\Omega) \geq \underline{d}/2\}$, which contains the curve γ_0 , we have estimates for $w_{\bar{\gamma}}$ in arbitrary norm. The part $E[u_0]$ of the right hand side of (131) is smooth, and the part $\mathcal{R}_\nu^T[\nabla\theta]$ is zero during this time interval. Then it is elementary to see by a Picard fixed point argument that for μ small enough (due to (122) we are sure that the vorticity of the solution stays $\{x \in \Omega / d(x, \partial\Omega) \geq \underline{d}/2\}$ for $t \in [0, \mu]$), (132) determines a unique $t \mapsto \bar{\gamma}(t)$ in O^M for M large enough. Since the solution is obtained as a fixed point of a contractive map, we have in particular the uniqueness of the solution in the class of solutions defined by $\gamma(t) \in C^{1,\alpha}$. It follows from our uniqueness statement that this solution coincides with the one constructed above.

2. That (131) determines then $\bar{\gamma}$ for $[\mu, T]$ in $C^{1,\alpha}$ during the time interval $[\mu, T]$ is known, see Depauw [8, Théorème 2.1], since for this time interval the boundary conditions are homogeneous. Then once this is proven at this level of regularity, the $C^{k,\alpha}$ case follows in a straightforward manner from Chemin’s techniques [2, Section 4], or from the following Proposition [20, Proposition 8.10]:

Proposition 7. *For $j \geq 2$, the mapping $\gamma \mapsto V_\gamma(\gamma)$ defined on the set (133) satisfies that for some constant $C(M, |\gamma|_{j-1,\alpha})$:*

$$\|V_\gamma(\gamma)\|_{C^{j,\alpha}} \leq C(M, |\gamma|_{j-1,\alpha}) \|\gamma\|_{C^{j,\alpha}}. \quad (134)$$

This proves that the curve $\bar{\gamma}$ stays in C^∞ , during the whole time interval $[0, T]$. Hence one can indeed define θ by Theorem 3 with $\bar{\gamma}(T)$ as initial curve. The uniqueness result established previously shows that the corresponding velocity field is \bar{u} and that $\bar{\gamma}(t, \cdot) = \gamma^\nu(t, \cdot)$ for $t \in [0, T - \nu]$.

3. So now we only need to focus on γ^ν during the time interval $[T - \nu, T]$. As previously, we rescale the time, denote $\hat{\gamma}(t, \sigma) := \gamma^\nu(T - \nu + \nu t, \sigma)$. The dynamics now write

$$\partial_t \hat{\gamma}(t, \sigma) = [\nu V_{\hat{\gamma}(t)} - \nu \nabla^\perp w_{\hat{\gamma}(t)} + \nu E[u_0](t) + \nabla\theta(t)](\hat{\gamma}(t, \sigma)). \quad (135)$$

By the same arguments on the regularity of the terms and using (121), we see that all terms but the first one are smooth in the neighborhood of $\hat{\gamma}$, as long as

$$\hat{\gamma}(t, \cdot) \subset \{x \in \Omega / d(x, \partial\Omega) \geq \underline{d}/2\}. \quad (136)$$

For the first one, one may apply Proposition 6. We deduce that for ν small enough and small times in order that (136) applies, one may find $\hat{\gamma}$ as a contractive fixed point in $L^\infty([0, 1]; O^M)$ (with fixed $j = k$). Due to the uniqueness of the solution, the resulting $\hat{\gamma}$ is inside $\{x \in \Omega / d(x, \partial\Omega) \geq \underline{d}\}$ and we can define the solution for the whole time interval $[T - \nu, T]$. Now (128) is a consequence of (135), Proposition 6 and Gronwall's lemma. Again, Proposition 7 proves that γ^ν stays in C^∞ . Finally, that the corresponding velocity field is Lipschitz regular is a consequence of the $C^{1,\alpha}$ regularity of γ , see for instance [4, Proposition 3.2.2] and [20, Proposition 8.12]. The proof is complete.

Acknowledgements. The authors are partially supported by the Agence Nationale de la Recherche, project ContrôleFlux. They would like to thank Jean-Pierre Puel for fruitful discussions.

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