Feedback Stabilization of Stem Growth

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Abstract

The paper studies a PDE model describing the elongation of a plant stem and its bending as a response to gravity. For a suitable range of parameters in the defining equations, it is proved that a feedback response produces stabilization of growth, in the vertical direction.

1 Introduction

We consider a simple mathematical model describing the growth of the stem of a plant [1, 2]. Here our main interest is how this growth can be stabilized in the vertical direction, by a feedback response to gravity.

Assume that new cells are generated at the tip of the stem, and then they grow in size. Namely, at time $t \geq 0$, the length of the cells born during the time interval $[s,\, s+ds]$ is measured by

$$d\ell \;=\; (1-e^{-\alpha(t-s)})\, ds\,, \tag{1.1}$$

for some constant $\alpha > 0$. The total length of the stem is thus

$$L(t) \;=\; \int_0^t (1-e^{-\alpha(t-s)})\, ds \;=\; t - \frac{1-e^{-\alpha t}}{\alpha}\,. \tag{1.2}$$

At a given time t, the stem is described by a $\mathcal{C}^1$ curve $s \mapsto P(t,s)$ in the plane. For $s \in [0,t]$, the point $P(t,s)$ describes the position of the cell generated at time s.

Moreover, we denote by $\mathbf{k}(t,s)$ the unit tangent vector to the stem at the point $P(t,s)$, so that

$$\mathbf{k}(t,s) \;=\; \frac{P_s(t,s)}{|P_s(t,s)|}\,, \qquad\qquad P_s(t,s) \;\doteq\; \frac{\partial P(t,s)}{\partial s}\,. \tag{1.3}$$

The position of a cell born at time s is thus

$$P(t,s) \ = \ \int_0^s (1-e^{-\alpha(t-s')})\,\mathbf{k}(t,s')\,ds'\,. \tag{1.4}$$

We shall always assume that the curvature vanishes at the tip, so that

$$\frac{\partial}{\partial s}\mathbf{k}(t,s)\bigg|_{s=t} \ = \ 0. \tag{1.5}$$

If there is no response to gravity, then

$$\frac{\partial}{\partial t}\mathbf{k}(t,s) \ = \ 0,$$

and each portion of the stem would grow with a constant direction. Differentiating (1.4) one thus obtains

$$\frac{\partial}{\partial t}P(t,s) \ = \ \int_0^s \alpha e^{-\alpha(t-\sigma)}\,\mathbf{k}(t,\sigma)\,d\sigma\,. \tag{1.6}$$

We seek a model which takes into account a response to gravity, stabilizing growth in the upward direction.

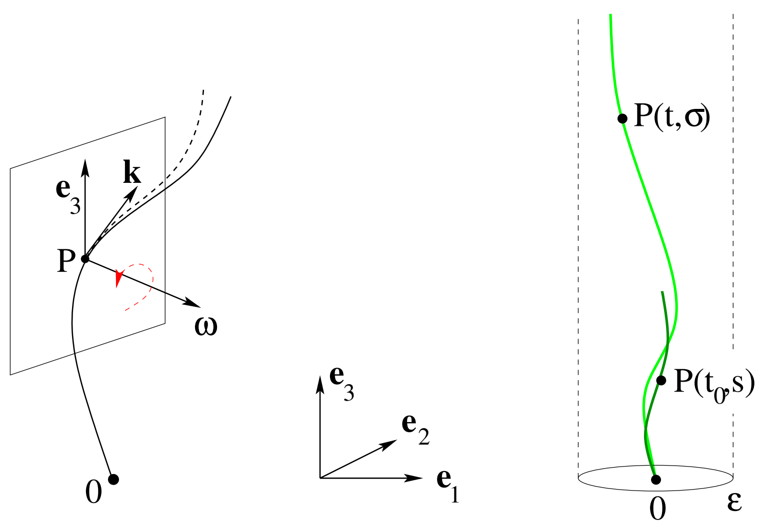

Figure 1: Left: at any point P along the stem, if the tangent vector $\mathbf{k}$ is not vertical, consider the plane spanned by $\mathbf{k}$ and $\mathbf{e}_3$. Then the bending of the stem at P produces an infinitesimal rotation of all the upper portion of the stem, with angular velocity $\omega = \mathbf{k}\times\mathbf{e}_3$. Right: the stability condition introduced in Definition 1. If at the initial time t_0 the stem is almost vertical, then at all times $t \geq t_0$ the stem should remain entirely inside the cylinder where $\sqrt{x_1^2+x_2^2} \leq \varepsilon$.

We assume that, if a portion of the stem is not vertical, growth will be slightly larger on the lower side. This determines a change in the local curvature, affecting the position of the upper section of the stem (Fig. 1, left).

More precisely, let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be the standard orthonormal basis in $I\!R^3$, with $\mathbf{e}_3$ oriented in the upward direction. At every point $P(t,\sigma)$, $\sigma\in[0,t]$, consider the cross product

$$\omega(t,\sigma) \ \doteq \ \mathbf{k}(t,\sigma)\times\mathbf{e}_3.$$

The change in the direction of the stem, in response to gravity, is modeled by

$$\frac{\partial}{\partial t}\mathbf{k}(t,s) \;=\; \int_0^s \mu\, e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\times\mathbf{k}(t,s)\,(1-e^{-\alpha(t-\sigma)})\,d\sigma. \tag{1.7}$$

Notice that, in the above integrand:

- $(1-e^{-\alpha(t-\sigma)})\,d\sigma \;=\; d\ell \;=$ arclength.
- $\omega(t,\sigma) = \mathbf{k}(t,\sigma)\times\mathbf{e}_3$ is an angular velocity, determined by the response to gravity at the point $P(t,\sigma)$. This affects the upper portion of the stem, i.e. all points $P(t,s)$ with $s\in[\sigma,t]$.
- $e^{-\beta(t-s)}$ is a stiffening term. It accounts for the fact that older parts of the stem are more rigid and hence they bend more slowly. On the other hand, $\mu\geq 0$ is a constant that measures the strength of the response to gravity.

Given the position of the stem at some initial time $t_0>0$, to determine the values of $\mathbf{k}(t,s)$ on the domain

$$\mathcal{D} \;\doteq\; \Big\{(t,s)\,;\quad 0\leq s\leq t,\quad t\geq t_0\Big\} \tag{1.8}$$

one can use the integral equation (1.7) together with the boundary conditions

$$\mathbf{k}(t_0,s) \;=\; \overline{\mathbf{k}}(s)\,,\;\; s\in[0,t_0]\,, \qquad\qquad \frac{\partial}{\partial s}\mathbf{k}(t,s)\bigg|_{s=t} = 0\,,\;\; t\geq t_0\,. \tag{1.9}$$

This yields a well posed evolution problem for the unit tangent vector to the stem.

Differentiating (1.4) w.r.t. t and using (1.6) , (1.7) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}P(t,s) - \alpha\int_0^s e^{-\alpha(t-s')}\mathbf{k}(t,s')\,ds' =\\
&\qquad = \int_0^s (1-e^{-\alpha(t-s')})\,\mathbf{k}_t(t,s')\,ds'\\
&\qquad = \int_0^s (1-e^{-\alpha(t-s')})\int_0^{s'} \mu\, e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\times\mathbf{k}(t,s')\,(1-e^{-\alpha(t-\sigma)})\,d\sigma\,ds'\\
&\qquad = \int_0^s \mu\, e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\times\Big(P(t,s)-P(t,\sigma)\Big)\,(1-e^{-\alpha(t-\sigma)})\,d\sigma.
\end{aligned} \tag{1.10}$$

For simplicity, as in [1] our analysis will be concerned with the limit case where $\alpha\to+\infty$, so that the factor $1-e^{-\alpha(t-\sigma)}\equiv 1$ can be omitted. We thus obtain the following evolution equation for points on the stem:

$$\frac{\partial}{\partial t}P(t,s) \;=\; \int_0^s \mu\, e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\times\Big(P(t,s)-P(t,\sigma)\Big)\,d\sigma. \tag{1.11}$$

This is supplemented by the boundary condition (1.5), stating that the curvature vanishes at the tip of the stem.

Numerically computed solutions of (1.11) are shown in Fig. 2. For small values of $\beta>0$, a highly oscillatory behavior is observed. Yet, it appears that some kind of stability is always

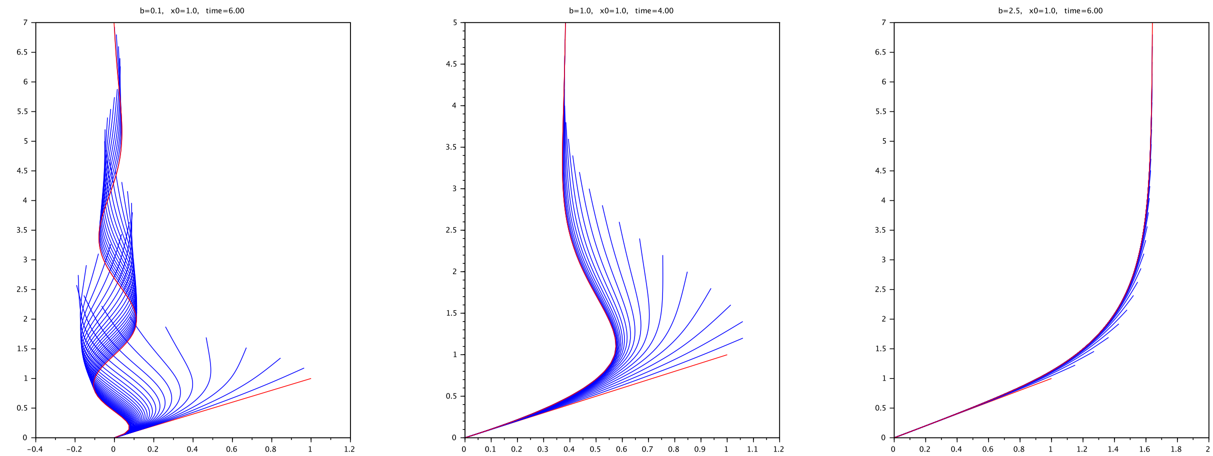

Figure 2: Numerical simulations of stem growth, at discrete times, taking $\mu = 1$ and different stiffness constants. Left: $\beta = 0.1$, center: $\beta = 1.0$, right: $\beta = 2.5$.

achieved. To make this more precise, we introduce a concept of stability for stem growth (Fig. 1, right). Given a point $x = (x_1, x_2, x_3) \in I\!R^3$, its horizontal projection is here defined as $\pi_{hor} x = (x_1, x_2)$.

Definition 1. *We say that the equations of growth (1.11), (1.5) are* **stable in the vertical direction** *if, for every $\varepsilon_0 > 0$ and $t_0 > 0$, there exists $\delta > 0$ such that the following holds. If*

$$\Big|\pi_{hor}\mathbf{k}(t_0, s)\Big| \;\leq\; \delta \quad \text{for all} \quad s \in [0, t_0] \tag{1.12}$$

then

$$\big|\pi_{hor}P(t,s)\big| \;\leq\; \varepsilon_0 \quad \text{and} \quad \big|\pi_{hor}\mathbf{k}(t,s)\big| \;\leq\; \varepsilon_0 \qquad \text{for all } t \geq t_0\,,\;\; s \in [0,t]\,. \tag{1.13}$$

Roughly speaking, if at the initial time t_0 the stem is almost vertical, then at all later times $t \geq t_0$ the stem should remain inside a vertical cylinder with radius ε. Notice that, because of the exponential stiffening term, asymptotic stability cannot be expected. Indeed, as $t \to +\infty$ the stem will not approach a vertical line.

Our main goal is to analyze the equations (1.11), (1.5), and prove that they are indeed stable in the vertical direction, at least for certain values of the parameters μ, β. The proof will be achieved by writing an evolution equation for the first two components of the tangent vector $\mathbf{k} = (k_1, k_2, k_3)$, and proving that these are stable in the space $\mathbf{L}^1(I\!R_+)$ as well as in $\mathbf{L}^\infty(I\!R_+)$.

The remainder of the paper is organized as follows. In Section 2 we derive a linearized version of the growth equations. Section 3 provides a linearized stability analysis in a non-oscillatory regime, with β suitably large. Roughly speaking, this means that if the stem initially bends only on one side, then it will keep bending on the same side for all future times (Fig. 2, right). Here the analysis is based on pointwise estimates, obtained by comparison arguments. In Section 4 we study linearized stability in the oscillatory regime (Fig. 2, left and center). For a somewhat wider range of the stiffening constant $\beta > 0$, linearized stability can now be proved relying on integral estimates. Finally, in Section 5 we prove that the nonlinear system (1.11) is stable in the vertical direction, according to Definition 1, for suitable values of the stiffening constant β.

It is worth noting that, following a standard approach [4, 8], one first proves the asymptotic stability of a linearized system, and then shows that stability remains valid in the presence of a small nonlinear perturbation. For our equation (1.7), however, asymptotic stability in $\mathbf{L}^\infty$ or $\mathbf{L}^1$ never occurs. For this reason, a more careful analysis is needed. The required estimates will be obtained by representing the solution of the nonlinear equation as a fixed point of a suitable transformation, which maps a particular set of functions (depending on the initial datum) into itself.

Readers who are interested in a general description of plant development from a biological point of view are referred to [5].

2 The linearized equations

Taking the limit $\alpha \to +\infty$, (1.7) reduces to

$$\frac{\partial}{\partial t}\mathbf{k}(t,s) \;=\; \int_0^s \mu\, e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\times\mathbf{k}(t,s)\,d\sigma. \tag{2.1}$$

Remark 1 (coordinate rescaling). Let $\mathbf{k}=\mathbf{k}(t,s)$ be a solution to (2.1). Consider the variable rescaling

$$t \;=\; \lambda\tau\,, \qquad s\;=\;\lambda\xi, \qquad \tilde{\mathbf{k}}(\tau,\xi)\;=\;\mathbf{k}(t,s).$$

Then the rescaled function $\tilde{\mathbf{k}}$ satisfies

$$\begin{aligned}\tilde{\mathbf{k}}_\tau(\tau,\xi) \;&=\; \lambda\mathbf{k}_t(t,s) \;=\; \lambda\mu\left(\int_0^s e^{-\beta(t-\sigma)}\,(\mathbf{k}(t,\sigma)\times\mathbf{e}_3)\,d\sigma\right)\times\mathbf{k}(t,s)\\ &=\; \lambda^2\mu\left(\int_0^\xi e^{-\beta\lambda(\tau-\eta)}\,(\tilde{\mathbf{k}}(\tau,\eta)\times\mathbf{e}_3)\,d\eta\right)\times\tilde{\mathbf{k}}(\tau,\xi)\end{aligned} \tag{2.2}$$

where we performed the change $\eta=\lambda\sigma$ in the variable of integration. By a variable rescaling, it is thus not restrictive to assume $\mu=1$. Of course, if $\mu\neq 1$, we need to replace β by $\beta\mu^{-1/2}$.

Set $\mathbf{k}=(k_1,k_2,k_3)$. From (2.1) with $\mu=1$ we obtain

$$\begin{aligned}\mathbf{k}_t(t,s) \;&=\; \left(\int_0^s e^{-\beta(t-\sigma)}\mathbf{k}(t,\sigma)\times\mathbf{e}_3\,d\sigma\right)\times\mathbf{k}(t,s)\\ &=\; -\int_0^s e^{-\beta(t-\sigma)}\Big(k_1(t,\sigma)\mathbf{e}_2-k_2(t,\sigma)\mathbf{e}_1\Big)d\sigma\\ &\qquad\qquad \times\Big[\mathbf{e}_3+k_1(t,s)\mathbf{e}_1+k_2(t,s)\mathbf{e}_2+(k_3(t,s)-1)\mathbf{e}_3\Big].\end{aligned} \tag{2.3}$$

A stem growing exactly in the vertical direction corresponds to $\mathbf{k}(t,s)\equiv(0,0,1)$. Linearizing around this trivial solution one obtains

$$\begin{cases} k_{1,t}(t,s) &= -\displaystyle\int_0^s e^{-\beta(t-\sigma)} k_1(t,\sigma)\,d\sigma + Q_1(t,s), \\ k_{2,t}(t,s) &= -\displaystyle\int_0^s e^{-\beta(t-\sigma)} k_2(t,\sigma)\,d\sigma + Q_2(t,s), \\ k_{3,t}(t,s) &= Q_3(t,s), \end{cases} \tag{2.4}$$

where Q_1, Q_2, Q_3 denote quadratic terms. More precisely,

$$\begin{cases} Q_i(t,s) &= (1-k_3(t,s)) \displaystyle\int_0^s e^{-\beta(t-\sigma)} k_i(t,\sigma)\,d\sigma, \qquad i=1,2, \\ Q_3(t,s) &= k_1(t,s) \displaystyle\int_0^s e^{-\beta(t-\sigma)} k_1(t,\sigma)\,d\sigma + k_2(t,s) \displaystyle\int_0^s e^{-\beta(t-\sigma)} k_1(t,\sigma)\,d\sigma\,. \end{cases} \tag{2.5}$$

Notice that in the linearized equations the three components are decoupled. Setting $\theta = k_1$ or $\theta = k_2$, we thus focus on the scalar integro-differential equation

$$\frac{\partial}{\partial t}\theta(t,s) \;=\; -\int_0^s e^{-\beta(t-\sigma)}\theta(t,\sigma)\,d\sigma, \tag{2.6}$$

with boundary condition

$$\theta_s(t,s)\Big|_{s=t} \;=\; 0. \tag{2.7}$$

Introducing the variable $u(t,x) \;=\; \theta(t,t-x)$, the equation (2.6) becomes

$$u_t + u_x \;=\; -\int_x^\infty e^{-\beta y} u(y)\,dy \qquad \text{for} \quad x>0, \tag{2.8}$$

with Neumann boundary condition at $x=0$

$$u_x(t,0) \;=\; 0. \tag{2.9}$$

A major portion of our analysis will focus on the stability of the linear system (2.8) with boundary condition (2.9). Notice that this linear evolution equation does not generate a continuous semigroup on $\mathbf{L}^1([0,+\infty[)$. Indeed, for a sequence of smooth initial data such that

$$u_n(0,x) \;=\; \bar u_n(x) \;\doteq\; \begin{cases} 1 & \text{if } \; 0\le x\le n^{-1}, \\ 0 & \text{if } \; x>2n^{-1}, \end{cases} \tag{2.10}$$

the corresponding sequence of solutions $u_n(t,\cdot)$ is smooth but does not converge to zero in $\mathbf{L}^1([0,+\infty[)$, for any $t>0$.

To achieve continuity of the flow, one needs to use the norm $\|u\| \;=\; |u(0)| + \|u\|_{\mathbf{L}^1([0,\infty[)}$. Equivalently, one can consider the evolution equation

$$u_t + u_x \;=\; -\int_{\max\{0,x\}}^\infty e^{-\beta y} u(y)\,dy\,, \tag{2.11}$$

on the space

$$X \;\doteq\; \Big\{ u\in\mathbf{L}^1\big([-1,+\infty[\big)\,; \quad u(x)=u(0) \quad \text{for all } \; x\in[-1,0]\Big\}. \tag{2.12}$$

We regard X as closed subspace of $\mathbf{L}^1([-1,+\infty[)$, with the same norm. In the following, for an initial datum

$$u(0,\cdot) \;=\; \bar u \;\in\; X\,, \tag{2.13}$$

we shall denote by

$$t \;\mapsto\; u(t,\cdot) \;\doteq\; S_t \bar u \tag{2.14}$$

the corresponding solution to (2.8)-(2.9), or equivalently (2.11). On the other hand, the solution of (2.8) with Dirichelet boundary condition

$$u(t,0) \;=\; 0 \tag{2.15}$$

will be denoted by

$$t \;\mapsto\; \tilde u(t,\cdot) \;=\; \widetilde S_t \bar u\,. \tag{2.16}$$

One can still regard (2.8), (2.15) as an evolution equation on the space X in (2.12), where $\tilde u$ now satisfies

$$\tilde u_t + \tilde u_x \;=\; \begin{cases} -\displaystyle\int_x^\infty e^{-\beta y} \tilde u(y)\,dy & \text{if} \quad x>0\,,\\[2mm] \qquad 0 & \text{if} \quad x\in[-1,0]. \end{cases} \tag{2.17}$$

The existence and uniqueness of these two solutions $u, \tilde u$ can be proved by standard techniques [6, 7, 8], relying on the contraction mapping principle.

We remark that, when the boundary condition (2.15) is used, the constant value of the initial datum $\bar u(x)$ for $x\in[-1,0]$ is irrelevant. However, this value does play a role when the Neumann condition (2.9) is used.

There is a close relation between the solutions u and $\tilde u$ in (2.14) and (2.16). Indeed, call $U=U(t,x)$ the particular solution of (2.11) with initial data

$$U(0,x) \;=\; \begin{cases} 1 & \text{if} \quad x\in[-1,0],\\ 0 & \text{if} \quad x>0. \end{cases} \tag{2.18}$$

Comparing (2.17) with (2.11), one derives the representation formula

$$u(t,x) \;=\; \widetilde u(t,x) + \bar u(0)\cdot U(t,x) + \int_0^t \left(\int_0^\infty e^{-\beta y}\, \widetilde u(s,y)\,dy\right) U(t-s,x)\,ds\,. \tag{2.19}$$

We shall refer to the function $U(\cdot,\cdot)$ in (2.18)-(2.19) as the **fundamental solution** of (2.11). In the next section we will prove that, if β is sufficiently large, then U remains always positive. In this case, which we call the "non-oscillatory regime", the proof of linearized stability can be achieved by a simple argument. In Sections 4 and 5 we shall consider smaller values of β, so that the function U can change sign. This we call the "oscillatory regime". The motivation for these names becomes apparent, looking at Figure 2.

3 Linearized stability in the non-oscillatory regime

In this section we consider the case where the stiffening constant $\beta>0$ in (1.11) is large. Our first result shows that in this case the fundamental solution U remains always positive.

Lemma 3.1. *Assume that the stiffening constant satisfies* $\beta^4 - \beta^3 \geq 4$. *Then the solution* U *of (2.11), (2.18) is non-negative, i.e.* $U(t,x) \geq 0$ *for all* $t \geq 0, x \geq 0$. *Moreover, its norm satisfies the uniform bound*

$$\|U(t,\cdot)\|_{\mathbf{L}^1([0,+\infty[)} \leq M \doteq 1 + \frac{\beta}{1-e^{-\beta}} \qquad \textit{for all} \quad t \geq 0. \tag{3.1}$$

Proof. 1. Integrating along characteristics, it is clear that

$$U(t,t) = 1, \qquad U(t,x) = 0 \qquad \text{for all } x > t\,. \tag{3.2}$$

Moreover, the map $x \mapsto U(t,x)$ is Lipschitz continuous on $[-1,t]$ and constant for $x \in [-1,0]$.

2. Assume that $U(t,x) \geq 0$ for all $(t,x) \in [0,T] \times [-1,+\infty[$. We begin by showing that, for all $t \in [0,T]$ and $0 < x < t$, one has

$$U_x \geq 0\,, \qquad U_t \leq 0, \tag{3.3}$$

$$U_{xx} \geq 0, \qquad U_{xt} \leq 0. \tag{3.4}$$

(i) Differentiating (2.11) w.r.t. x we obtain

$$U_{xt} + U_{xx} = e^{-\beta x} U\,. \tag{3.5}$$

Integrating along characteristics and using the Neumann boundary condition, for $0 < x < t$ we obtain

$$U_x(t,x) = \int_{t-x}^{t} e^{-\beta(x-t+s)} U(s, x-t+s)\, ds = \int_0^x e^{-\beta y} U(t-x+y, y)\, dy \geq 0. \tag{3.6}$$

This proves the first inequality in (3.3).

(ii) In turn, the inequality $U_t \geq 0$ is an immediate consequence of the equation (2.11).

(iii) To prove the first inequality in (3.4), fix $t \in [0,T]$ and let $0 < x_1 < x_2 < t$. By (3.6) it follows

$$\begin{aligned} U_x(t,x_1) &= \int_0^{x_1} e^{-\beta y} U(t-x_1+y, y)\, dy \\ &\leq \int_0^{x_1} e^{-\beta y} U(t-x_2+y,\, y)\, dy + \int_{x_1}^{x_2} e^{-\beta y} U(t-x_2+y,\, y)\, dy = U_x(t,x_2), \end{aligned} \tag{3.7}$$

showing that the map $x \mapsto U_x(t,x)$ is nondecreasing for $0 < x < t$.

(iv) To prove the second inequality in (3.4), fix $0 < x < t_1 < t_2$. Since $U_t \leq 0$, we have

$$U_x(t_2,x) = \int_0^x e^{-\beta y} U(t_2-x+y, y)\, dy \leq \int_0^x e^{-\beta y} U(t_1-x+y, y)\, dy = U_x(t_1,x).$$

3. By (3.4), the function $U(t,\cdot)$ is convex on the interval $x \in [0,t]$. Hence

$$U(t,0) \;\leq\; U(t,x) \;\leq\; \frac{x}{t} + \frac{t-x}{t} U(t,0) \;\leq\; U(t,0) + \frac{x}{t}\,. \tag{3.8}$$

Inserting (3.8) in (2.11) one obtains

$$U_t(0,x) \;\geq\; -\int_0^t e^{-\beta y} \left(U(t,0) + \frac{y}{t} \right) dy. \tag{3.9}$$

From (3.6) and the fact that U decreases along characteristics, it now follows

$$U_x(t,x) \;=\; \int_0^x e^{-\beta y} U(t-x+y,y)\, dy \;\leq\; \int_0^x e^{-\beta y} U(t-x,0)\, dy \;\leq\; \frac{1}{\beta} U(t-x,0).$$

Hence

$$U(t,x) \;\leq\; U(t,0) + \frac{1}{\beta} \int_0^x U(t-y,0)\, dy\,. \tag{3.10}$$

4. Call $Z(t) \doteq U(t,0)$. By (3.10) the scalar function Z satisfies the differential inequality

$$\begin{aligned}
\dot Z(t) \;&=\; -\int_0^t e^{-\beta x} U(t,x) dx \;\geq\; -\int_0^t e^{-\beta x} \left(Z(t) + \frac{1}{\beta} \int_0^x Z(t-y)\, dy \right) dx \\
&\geq\; -\frac{1}{\beta} Z(t) - \frac{1}{\beta} \int_0^t Z(t-y) \left(\int_y^t e^{-\beta x} dx \right) dy \\
&\geq\; -\frac{1}{\beta} Z(t) - \frac{1}{\beta^2} \int_0^t e^{-\beta y} Z(t-y)\, dy\,.
\end{aligned} \tag{3.11}$$

Introducing the variable

$$I(t) \;\doteq\; \int_0^t e^{-\beta(t-y)} Z(y)\, dy\,,$$

by (3.11) we obtain the system of differential inequalities

$$\begin{cases} \dot Z(t) \;\geq\; -\beta^{-1} Z(t) - \beta^{-2} I(t), \\ \dot I(t) \;=\; Z(t) - \beta I(t), \end{cases} \qquad \begin{cases} Z(0) \;=\; 1, \\ I(0) \;=\; 0, \end{cases} \tag{3.12}$$

where the upper dot denotes a derivative w.r.t. time. This implies

$$\frac{d}{dt} \left(\frac{Z(t)}{I(t)} \right) \;=\; \frac{\dot Z(t) I(t) - Z(t) \dot I(t)}{I^2(t)} \;\geq\; \frac{-\beta^{-2} I^2(t) - (1+\beta^{-1}) Z^2(t) + \beta I(t) Z(t)}{I^2(t)}\,. \tag{3.13}$$

Recalling the assumption $\beta^4 - \beta^3 - 4 \geq 0$, when $Z/I = \beta/2$ we have

$$\frac{d}{dt} \left(\frac{Z}{I} \right) \;\geq\; \frac{-\beta^{-2} I^2 - (1+\beta^{-1}) Z^2 + \beta I Z}{I^2} \;=\; -\frac{1}{\beta^2} - \left(1 + \frac{1}{\beta} \right) \frac{\beta^2}{4} + \frac{\beta^2}{2} \;=\; \frac{\beta^4 - \beta^3 - 4}{4\beta^2} \;\geq\; 0\,.$$

As a consequence, if $Z(\tau) \geq (\beta/2) I(\tau)$, then $Z(t) \geq (\beta/2) I(t)$ for all $t \geq \tau$. The initial data in (3.12) imply

$$I(t) \;\leq\; \frac{2}{\beta} Z(t) \qquad\qquad \text{for all } t \geq 0. \tag{3.14}$$

Inserting this in the first inequality in (3.12) we obtain

$$\dot{Z}(t) \;\geq\; -\left(\frac{1}{\beta}+\frac{2}{\beta^3}\right) Z(t).$$

This yields the lower bound

$$Z(t) \;\geq\; \exp\left\{-\frac{\beta^2+2}{\beta^3}\,t\right\}.$$

By the first inequality in (3.3), this implies

$$U(t,x) \;\geq\; \exp\left\{-\frac{\beta^2+2}{\beta^3}\,t\right\} \tag{3.15}$$

for all $t \in [0,T]$ and $0 \leq x \leq t$. The above analysis shows that, if $U \geq 0$ on the domain $\{(t,x)\,;\;\; t\in[0,T],\; 0\leq x\leq t\}$, then U satisfies the strictly positive lower bound (3.15). Since the lower bound of $U(t,\cdot)$ on $[0,t]$ depends continuously on t, we conclude that U can never take negative values.

5. Next, to establish an upper bound for Z we observe that for $t \geq 1$ one has

$$\dot{Z}(t) \;=\; U_t(t,0) \;\leq\; -\int_0^1 e^{-\beta y}U(t,y)\,dy \;\leq\; -\int_0^1 e^{-\beta y}\,dy\cdot U(t,0) \;\leq\; -\frac{1-e^{-\beta}}{\beta}Z(t).$$

Setting $\gamma = \frac{1-e^{-\beta}}{\beta}$, we thus have

$$Z(t) \;\leq\; e^{-\gamma(t-1)}Z(1) \;\leq\; e^{-\gamma(t-1)} \qquad \text{for}\;\; t\geq 1.$$

6. By (3.3) we trivially have

$$0 \;\leq\; U(t,0) \;\leq\; 1, \qquad\qquad \text{for all } t\geq 0, \tag{3.16}$$

$$\begin{cases} U(t,x) \;\in\; [0,1] & \text{if}\;\; x\in[0,t],\\ U(t,x) \;=\; 0 & \text{if}\;\; x>t, \end{cases} \qquad \text{for all } t\in[0,1]. \tag{3.17}$$

Moreover, for $t>1$ an upper bound for the norm $\|U(t,\cdot)\|_{\mathbf{L}^1([0,+\infty[)}$ is now obtained from

$$\int_0^t U(t,x)\,dx \;\leq\; \int_0^t Z(t-x)\,dx \;\leq\; 1+\int_0^{t-1} e^{-\gamma(t-x-1)}\,dx \;\leq\; 1+\frac{1}{\gamma}\,. \tag{3.18}$$

□

Based on the representation formula (2.19), we now prove the stability of the linear semigroup S, in the non-oscillatory regime. We recall that S is defined on the space $X\subset\mathbf{L}^1([-1,+\infty[)$ introduced at (2.12).

Theorem 3.1. *Assume $\beta^4-\beta^3\geq 4$. Then the semigroup S defined at (2.11)–(2.14) is stable.*

Proof. 1. Notice that the assumption implies $\beta > 1$, hence we can choose $1 < \gamma < \beta$. Fix an initial datum $\bar{u} \in X$ and let $\widetilde{u}$ be the corresponding solution of (2.8) with Dirichelet boundary conditions (2.15). Consider the weighted integral

$$J(t) \doteq \int_0^\infty e^{-\gamma y} |\widetilde{u}(t,y)| \, dy. \tag{3.19}$$

Differentiating (3.19) w.r.t. time and using (2.8) one obtains

$$\dot{J}(t) \leq -\gamma J(t) + \int_0^\infty e^{-\beta y} \big|\widetilde{u}(t,y)\big| \left(\int_0^y e^{-\gamma \xi} \, d\xi \right) dy \leq \left(-\gamma + \frac{1}{\gamma} \right) J(t) \,.$$

Setting $\kappa \doteq \gamma - (1/\gamma) > 0$ one obtains

$$J(t) \leq e^{-\kappa t} J(0).$$

In turn this yields a uniform bound on $\big\|\widetilde{u}(t,\cdot)\big\|_{\mathbf{L}^1}$, namely

$$\begin{aligned} \big\|\widetilde{u}(t,\cdot)\|_{\mathbf{L}^1} - \|\bar{u}\|_{\mathbf{L}^1} &\leq \int_0^t \int_0^\infty y e^{-\beta y} \big|\widetilde{u}(s,y)\big| \, dy \, ds \leq C \cdot \int_0^t \int_0^\infty e^{-\gamma y} \big|\widetilde{u}(s,y)\big| \, dy \, ds \\ &= C \cdot \int_0^t J(s) \, ds \leq \frac{C}{\kappa} J(0) \,. \end{aligned} \tag{3.20}$$

Here C is a constant depending only on γ and β.

2. Next, let $u = u(t,x)$ be the solution to (2.11) with the same initial datum $\bar{u}$. Recalling the representation formula (2.19) and the bound (3.1), we conclude

$$\begin{aligned} &\big\|u(t,\cdot)\big\|_{\mathbf{L}^1([-1,\infty[)} - \big\|\bar{u}\big\|_{\mathbf{L}^1([-1,\infty[)} \\ &\leq \big\|\widetilde{u}(t,\cdot)\big\|_{\mathbf{L}^1([0,\infty[)} - \big\|\bar{u}\big\|_{\mathbf{L}^1([0,\infty[)} + \big|\bar{u}(0)\big| \cdot \big\|U(t,\cdot)\big\|_{\mathbf{L}^1([-1,\infty[)} \\ &\qquad + \int_0^t \left(\int_0^{+\infty} e^{-\beta y} \big|\widetilde{u}(s,y)\big| \, dy \right) ds \cdot \max_{\tau \geq 0} \big\|U(\tau,\cdot)\big\|_{\mathbf{L}^1([-1,\infty[)} \\ &\leq \frac{C}{\kappa} J(0) + M \big|\bar{u}(0)\big| + M \int_0^t J(s) \, ds \leq \frac{C+M}{\kappa} J(0) + M \big|\bar{u}(0)\big|. \end{aligned} \tag{3.21}$$

Since $J(0) \leq \big\|\bar{u}\big\|_{\mathbf{L}^1([0,\infty[)}$, we conclude that, for every $t \geq 0$,

$$\big\|u(t,\cdot)\big\|_{\mathbf{L}^1([-1,\infty[)} \leq \left(\frac{C}{\kappa} + \frac{M}{\kappa} + M + 1 \right) \cdot \big\|\bar{u}\big\|_{\mathbf{L}^1([-1,\infty[)} \,.$$

□

4 Stability in the oscillatory regime

We consider again the linear equation (2.8) with Neumann boundary condition (2.9) at $x = 0$. We shall use the equivalent formulation (2.11) on the space X at (2.12).

Solutions $u = u(t,x)$ of (2.11) will be considered, with an arbitrary initial data

$$u(0,\cdot) \;=\; u_0 \;\in\; X\,. \tag{4.1}$$

Our goal is to obtain a priori estimates on the $\mathbf{L}^1$ norm of $u(t,\cdot)$, uniformly valid for all $t > 0$. The next theorem shows that the stability result in Theorem 3.1 remains valid for somewhat smaller values of the stiffening constant β. The proof is entirely different, and we believe it has independent interest.

Theorem 4.1. *The semigroup S generated by (2.11), on the space X at (2.12), is stable for all $\beta \geq \beta^* \doteq (48 + \sqrt{9504})/160$.*

We remark that Theorem 3.1 yields stability for $\beta \geq \beta^\dagger \approx 1.7485$, while Theorem 4.1 extends the stability result for all $\beta \geq \beta^* \approx 0.9093$. It remains an open question whether linearized stability holds for $0 \leq \beta < \beta^*$. The remainder of this section is devoted to a proof of Theorem 4.1.

4.1 Estimates on $u(t,0)$

We use the notation $U_0(t) \doteq u(t,0)$. For $\gamma > 0$, we write

$$\mathcal{J}_\gamma(t) \;\doteq\; \int_0^\infty e^{-\gamma x} u(t,x)\,dx.$$

Differentiating w.r.t. time, one obtains

$$\begin{aligned}
\mathcal{J}_\gamma'(t) &= \int_0^\infty e^{-\gamma x}\left[-\partial_x u(t,x) - \int_x^\infty e^{-\beta y} u(t,y)\,dy\right] dx \\
&= U_0(t) - \gamma \mathcal{J}_\gamma(t) - \int_0^\infty \int_x^\infty e^{-\gamma x - \beta y} u(t,y)\,dy\,dx, \\
&= U_0(t) - \gamma \mathcal{J}_\gamma(t) - \int_0^\infty e^{-\beta y} u(t,y) \left(\int_0^y e^{-\gamma x}\,dx\right) dy, \\
&= U_0(t) - \gamma \mathcal{J}_\gamma(t) - \frac{1}{\gamma}\mathcal{J}_\beta(t) + \frac{1}{\gamma}\mathcal{J}_{\gamma+\beta}(t).
\end{aligned}$$

In particular we have for all $n \geq 1$:

$$\mathcal{J}_{n\beta}' = U_0 - n\beta \mathcal{J}_{n\beta} - \frac{1}{n\beta}\mathcal{J}_\beta + \frac{1}{n\beta}\mathcal{J}_{(n+1)\beta}. \tag{4.2}$$

By (2.8) and (2.9) it follows

$$U_0' \;=\; -\,\mathcal{J}_\beta\,. \tag{4.3}$$

Hence, for $n = 1$ we have

$$U_0'' + \left(\beta + \frac{1}{\beta}\right) U_0' + U_0 \;=\; -\,\frac{\mathcal{J}_{2\beta}}{\beta}. \tag{4.4}$$

Next, we would like to express $\mathcal{J}_{2\beta}$ in terms of U_0 and U_0'. For $\alpha > 0$, consider the convolution operator

$$I_\alpha[f](t) \;\doteq\; \int_0^t e^{-\alpha(t-s)} f(s)\,ds. \tag{4.5}$$

Notice that $I_\alpha[f] = f * \rho_\alpha$ is obtained by taking the convolution with the kernel $\rho_\alpha(t) = \mathbf{1}_{I\!R_+}(t)e^{-\alpha t}$. In particular, recalling that for any $a, b \in \mathbf{L}^1(I\!R_+)$ one has

$$\|a * b\|_{\mathbf{L}^1} \;\leq\; \|a\|_{\mathbf{L}^1}\|b\|_{\mathbf{L}^1}, \tag{4.6}$$

we see that I_α is a bounded linear operator from $\mathbf{L}^1(I\!R_+)$ into itself:

$$\Big\|I_\alpha[f]\Big\|_{\mathbf{L}^1(I\!R_+)} \;\leq\; \frac{1}{\alpha}\|f\|_{\mathbf{L}^1(I\!R_+)}\,. \tag{4.7}$$

We shall also use the weighted Lebesgue space $\mathbf{L}^1_\gamma(I\!R_+)$, with norm

$$\|f\|_{\mathbf{L}^1_\gamma(I\!R_+)} \;\doteq\; \int_0^\infty e^{\gamma x}|f(x)|\,dx\,. \tag{4.8}$$

Relying on the identity

$$\int_{I\!R_+} e^{-\gamma t}\left|\int_0^t a(t-s)b(s)\,ds\right|\,dt \;=\; \int_{I\!R_+}\left|\int_0^t e^{-\gamma(t-s)}a(t-s)e^{-\gamma s}b(s)\,ds\right|\,dt,$$

valid for any two functions $a, b \in \mathbf{L}^1(I\!R_+)$, we deduce that the same inequality (4.6) holds for the weighted $\mathbf{L}^1$ norm:

$$\|a * b\|_{\mathbf{L}^1_\gamma} \;\leq\; \|a\|_{\mathbf{L}^1_\gamma}\|b\|_{\mathbf{L}^1_\gamma}. \tag{4.9}$$

Thus, for any $\alpha > \gamma$ one has

$$\Big\|I_\alpha[f]\Big\|_{\mathbf{L}^1_\gamma(I\!R_+)} \;\leq\; \frac{1}{\alpha-\gamma}\|f\|_{\mathbf{L}^1_\gamma(I\!R_+)}. \tag{4.10}$$

Finally for $\alpha > 0$ we will denote by $\mathbf{e}_\alpha : I\!R_+ \mapsto I\!R_+$ the function

$$\mathbf{e}_\alpha(x) \;\doteq\; e^{-\alpha x}. \tag{4.11}$$

Integrating (4.2) one obtains

$$\mathcal{J}_{n\beta}(t) \;=\; \mathcal{J}_{n\beta}(0)e^{-n\beta t} + I_{n\beta}[U_0](t) - \frac{1}{n\beta}I_{n\beta}[\mathcal{J}_\beta](t) + \frac{1}{n\beta}I_{n\beta}[\mathcal{J}_{(n+1)\beta}](t), \tag{4.12}$$

which, in the case $n = 2$, yields

$$\mathcal{J}_{2\beta}(t) \;=\; \mathcal{J}_{2\beta}(0)e^{-2\beta t} + I_{2\beta}[U_0](t) - \frac{1}{2\beta}I_{2\beta}[\mathcal{J}_\beta](t) + \frac{1}{2\beta}I_{2\beta}[\mathcal{J}_{3\beta}](t). \tag{4.13}$$

Relying on (4.12), (4.13), and proceeding by induction on $n \geq 2$, we obtain

$$\mathcal{J}_{2\beta}(t) \;=\; \sum_{k=2}^n f_k(t) + \sum_{k=2}^n \big(A_k[U_0](t) - B_k[\mathcal{J}_\beta](t)\big) + \frac{1}{n!\beta^{n-1}}I_{2\beta}\circ\cdots\circ I_{n\beta}[\mathcal{J}_{(n+1)\beta}](t), \tag{4.14}$$

with

$$f_k(t) \;\doteq\; \begin{cases} \mathcal{J}_{2\beta}(0)\,e^{-2\beta\,t} & \text{if } k=2, \\ \mathcal{J}_{k\beta}(0)\dfrac{I_{2\beta}\circ\cdots\circ I_{(k-1)\beta}[\mathbf{e}_{k\beta}](t)}{(k-1)!\,\beta^{k-2}} & \text{otherwise,} \end{cases} \tag{4.15}$$

$$A_k[U_0](t) \;\doteq\; \frac{I_{2\beta}\circ\cdots\circ I_{k\beta}[U_0](t)}{(k-1)!\,\beta^{k-2}}\,, \tag{4.16}$$

$$B_k[\mathcal{J}_\beta](t) \doteq \frac{I_{2\beta}\circ\cdots\circ I_{k\beta}[\mathcal{J}_\beta](t)}{k!\,\beta^{k-1}}. \tag{4.17}$$

The series with general term f_k converges in $\mathbf{L}^1(I\!\!R_+)$ to

$$f \doteq \sum_{k=2}^{\infty} f_k. \tag{4.18}$$

The series with general term A_k and B_k, $k \geq 2$, converge to

$$A \doteq \sum_{k=2}^{\infty} A_k \qquad \text{and} \qquad B \doteq \sum_{k=2}^{\infty} B_k, \tag{4.19}$$

in the space $\mathcal{B}(\mathbf{L}^1(I\!\!R_+))$ of bounded linear operators from $\mathbf{L}^1(I\!\!R_+)$ into itself. In fact, thanks to (4.7) one has the bounds

$$\|f\|_{\mathbf{L}^1} \leq \sum_{k=2}^{\infty} \frac{1}{(k-1)!\,k!\,\beta^{2k-3}}\|u(0,\cdot)\|_{\mathbf{L}^1}, \tag{4.20}$$

$$\|A\|_{\mathcal{B}(\mathbf{L}^1(I\!\!R_+))} \leq \sum_{k=2}^{\infty} \frac{1}{(k-1)!\,k!\,\beta^{2k-3}}, \qquad \|B\|_{\mathcal{B}(\mathbf{L}^1(I\!\!R_+))} \leq \sum_{k=2}^{\infty} \frac{1}{(k!)^2\,\beta^{2k-2}}. \tag{4.21}$$

Concerning the last term in the right hand side of (4.14), a similar argument yields that, for any $T > 0$,

$$\left\|\frac{1}{n!\beta^{n-1}} I_{2\beta}\circ\cdots\circ I_{n\beta}[\mathcal{J}_{(n+1)\beta}]\right\|_{\mathbf{L}^1([0,T])} \leq \frac{1}{(n!)^2\,\beta^{2n-1}}\|\mathcal{J}_{(n+1)\beta}\|_{\mathbf{L}^1([0,T])}.$$

For any $T > 0$, observing that $\|\mathcal{J}_{(n+1)\beta}\|_{\mathbf{L}^1([0,T])} \leq \|u\|_{\mathbf{L}^1([0,T]\times I\!\!R_+)}$ and letting $n \to +\infty$ in (4.14), one deduces that, whenever $u \in \mathbf{L}^1_{loc}(I\!\!R_+; \mathbf{L}^1(I\!\!R_+))$, the function $\mathcal{J}_{2\beta}$ admits the representation

$$\mathcal{J}_{2\beta} = f + A[U_0] - B[\mathcal{J}_\beta]. \tag{4.22}$$

Here f, A, B are the functions defined at (4.15)–(4.19). Consequently, the equation (4.4) can be written as

$$\begin{pmatrix} U_0 \\ U_0' \end{pmatrix}' = M \begin{pmatrix} U_0 \\ U_0' \end{pmatrix} - \frac{1}{\beta}\begin{pmatrix} 0 \\ f \end{pmatrix} - \frac{1}{\beta}\begin{pmatrix} 0 \\ A[U_0] - B[\mathcal{J}_\beta] \end{pmatrix}, \tag{4.23}$$

with

$$M \doteq \begin{pmatrix} 0 & 1 \\ -1 & -(\beta + \frac{1}{\beta}) \end{pmatrix}. \tag{4.24}$$

Recalling (4.3), we thus have

$$\begin{pmatrix} U_0 \\ U_0' \end{pmatrix}(t) = \widetilde{\mathcal{T}}\begin{pmatrix} U_0 \\ U_0' \end{pmatrix}(t), \tag{4.25}$$

with

$$\begin{aligned} \widetilde{\mathcal{T}}\begin{pmatrix} V_0 \\ V_0' \end{pmatrix}(t) \doteq{} & \exp(tM)\begin{pmatrix} U_0(0) \\ U_0'(0) \end{pmatrix} - \frac{1}{\beta}\int_0^t \exp((t-s)M)\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \\ & -\frac{1}{\beta}\int_0^t \exp((t-s)M)\begin{pmatrix} 0 \\ A[V_0](s) + B[V_0'](s) \end{pmatrix} ds. \end{aligned} \tag{4.26}$$

Notice that the matrix M has negative eigenvalues $-\beta$ and $-1/\beta$.

Next, consider the space $\mathbf{L}^1(I\!R_+)\times\mathbf{L}^1(I\!R_+)$ with norm $\|(f,g)\| \doteq \max\big\{\|f\|_{\mathbf{L}^1}, \|g\|_{\mathbf{L}^1}\big\}$. We will show that, for $\beta \geq 1$ and even for some $\beta < 1$ sufficiently close to 1, the operator $\widetilde{\mathcal{T}}$ defined in (4.26) is contractive. For this purpose, it is of course sufficient to prove the contractivity of the linear part, defined by

$$\mathcal{T}\begin{pmatrix} V_0 \\ V_0' \end{pmatrix}(t) \;\doteq\; -\frac{1}{\beta}\int_0^t \exp((t-s)M)\begin{pmatrix} 0 \\ A[V_0](s)+B[V_0'(s)] \end{pmatrix} ds\,. \tag{4.27}$$

Lemma 4.1. *There exists a continuous function $\kappa:(0,+\infty)\to(0,+\infty)$ such that:*

(i) For any $\beta > 0$, $\|\mathcal{T}\|_{\mathcal{B}(\mathbf{L}^1\times\mathbf{L}^1)} \leq \kappa(\beta)$.

(ii) For all $\beta > \beta^* \doteq (48+\sqrt{9504})/160 \approx 0.9093$, one has $\kappa(\beta) < 1$.

Proof. For any $\beta \neq 1$ we have

$$\exp(tM) \;=\; \frac{1}{\beta^2-1}\begin{pmatrix} \beta^2 e^{-t/\beta} - e^{-\beta t} & \beta e^{-t/\beta} - \beta e^{-\beta t} \\ \beta e^{-\beta t} - \beta e^{-t/\beta} & e^{-t/\beta} - \beta^2 e^{-\beta t} \end{pmatrix},$$

and for $\beta = 1$

$$\exp(tM) \;=\; \begin{pmatrix} e^{-t}(t+1) & te^{-t} \\ -te^{-t} & -(t-1)e^{-t} \end{pmatrix}.$$

The mapping $(t,\beta)\mapsto \exp(tM)$ is smooth w.r.t. both variables $t\geq 0$ and $\beta > 0$. One has

$$\int_0^t \exp((t-s)M)\begin{pmatrix} 0 \\ A[V_0](s)+B[V_0'(s)] \end{pmatrix} ds \;=\; \begin{pmatrix} (A[V_0](s)+B[V_0'(s)]) * m_{12} \\ (A[V_0](s)+B[V_0'(s)]) * m_{22} \end{pmatrix},$$

where, for $t > 0$,

$$m_{12}(t) \;\doteq\; \begin{cases} \frac{\beta}{\beta^2-1}(e^{-t/\beta} - e^{-\beta t}) & \text{if } \beta \neq 1, \\ te^{-t} & \text{if } \beta = 1, \end{cases}$$

$$m_{22} \;\doteq\; \begin{cases} \frac{1}{\beta^2-1}(e^{-t/\beta} - \beta^2 e^{-\beta t}) & \text{if } \beta \neq 1, \\ -(t-1)e^{-t} & \text{if } \beta = 1, \end{cases}$$

and where the functions m_{12}, m_{22} and $A[V_0](s)+B[V_0'(s)]$ are defined to be zero for $s\leq 0$.

When $(U,U') = \mathcal{T}(\tilde U, \tilde U')$, one has

$$\|U\|_{\mathbf{L}^1} \;\leq\; \|A\|_{\mathcal{B}(\mathbf{L}^1)}\|m_{12}\|_{\mathbf{L}^1}\|\tilde U\|_{\mathbf{L}^1} + \|B\|_{\mathcal{B}(\mathbf{L}^1)}\|m_{12}\|_{\mathbf{L}^1}\|\tilde U'\|_{\mathbf{L}^1},$$
$$\|U'\|_{\mathbf{L}^1} \;\leq\; \|A\|_{\mathcal{B}(\mathbf{L}^1)}\|m_{22}\|_{\mathbf{L}^1}\|\tilde U\|_{\mathbf{L}^1} + \|B\|_{\mathcal{B}(\mathbf{L}^1)}\|m_{22}\|_{\mathbf{L}^1}\|\tilde U'\|_{\mathbf{L}^1}.$$

Consequently

$$\|\mathcal{T}\|_{\mathcal{B}(\mathbf{L}^1\times\mathbf{L}^1)} \;\leq\; \max\Big\{\|m_{12}\|_{\mathbf{L}^1}, \|m_{22}\|_{\mathbf{L}^1}\Big\}\big(\|A\|_{\mathcal{B}(\mathbf{L}^1)} + \|B\|_{\mathcal{B}(\mathbf{L}^1)}\big).$$

Defining

$$\kappa(\beta) \doteq \max(\|m_{12}\|_{\mathbf{L}^1}, \|m_{22}\|_{\mathbf{L}^1}) \left(\sum_{k=2}^{\infty} \frac{1}{(k-1)!\,k!\,\beta^{2k-3}} + \sum_{k=2}^{\infty} \frac{1}{(k!)^2\,\beta^{2k-2}} \right), \tag{4.28}$$

by (4.21) we obtain part (i) of the Lemma.

Next, for $\beta \neq 1$ one has

$$\|m_{12}\|_{\mathbf{L}^1} = \frac{\beta}{\beta^2-1} \int_0^{\infty} (e^{-t/\beta} - \beta^2 e^{-\beta t})\,dt = 1,$$

$$\|m_{22}\|_{\mathbf{L}^1} = \frac{1}{\beta^2-1} \left(\int_0^{t^*} (\beta^2 e^{-\beta t} - e^{-t/\beta})\,dt + \int_{t^*}^{\infty} (e^{-t/\beta} - \beta^2 e^{-\beta t})\,dt \right),$$

where $t^* \doteq 2\ln\beta/(\beta - 1/\beta)$ is chosen so that $e^{-t^*/\beta} = \beta^2 e^{-\beta t^*}$. Hence

$$\|m_{22}\|_{\mathbf{L}^1} = \frac{2\beta^2}{\beta^2-1}(e^{-t^*/\beta} - e^{-\beta t^*}) = 2\exp\left(\frac{(\beta^2+1)\ln\beta}{1-\beta^2} \right) < \frac{2}{e} \qquad \text{for } \beta \neq 1.$$

On the other hand, if $\beta = 1$ one has $\|m_{12}\|_{\mathbf{L}^1} = 1$ and $\|m_{22}\|_{\mathbf{L}^1} = 2/e$.

We now observe that, if $u_k \doteq \dfrac{1}{(k-1)!\,k!}$ and $\tilde{u}_k \doteq \dfrac{1}{(k!)^2}$, then for every k one has

$$\frac{u_{k+1}}{u_k} \leq \frac{1}{6}, \qquad \frac{\tilde{u}_{k+1}}{\tilde{u}_k} \leq \frac{1}{9}.$$

Using the above inequalities, for every $\beta \geq 1$ we obtain

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)!\,k!\,\beta^{2k-3}} \leq \frac{3}{5\beta}, \qquad \sum_{k=2}^{\infty} \frac{1}{(k!)^2\,\beta^{2k-2}} \leq \frac{9}{32\beta^2}.$$

Recalling (4.21), we conclude

$$\|A\|_{\mathcal{B}(\mathbf{L}^1(I\!R_+))} + \|B\|_{\mathcal{B}(\mathbf{L}^1(I\!R_+))} \leq \frac{3}{5\beta} + \frac{9}{32\beta^2}. \tag{4.29}$$

An elementary computation now shows that the right hand side is < 1 provided that $\beta > \beta^*$.
□

As a consequence of the above lemma, for any $\beta > \beta^*$ we obtain that (4.25) has a unique solution in $\mathbf{L}^1(I\!R_+) \times \mathbf{L}^1(I\!R_+)$. (Actually, our analysis shows that, for any $\beta > 0$, (4.25) has a unique solution. However, when $\beta \leq \beta^*$ the first component of this solution may only lie in the space $\mathbf{L}^1_\gamma(I\!R_+)$ defined at (4.8), for some $\gamma < 0$.) Moreover, due to the contraction property, the norm of this solution is measured by $\|\widetilde{\mathcal{T}}(0,0)\|_{\mathbf{L}^1 \times \mathbf{L}^1}$. Therefore, for some $C = C(\beta)$, one has

$$\|U_0\|_{\mathbf{L}^1} + \|U_0'\|_{\mathbf{L}^1} \leq C(\|U_L\|_{\mathbf{L}^1} + \|f\|_{\mathbf{L}^1}),$$

where

$$U_L(t) \doteq \exp(tM) \begin{pmatrix} U_0(0) \\ U_0'(0) \end{pmatrix}.$$

Finally, observing that

$$\big|U_0'(0)\big| \;=\; \big|\mathcal{J}_\beta(0)\big| \;\le\; \big\|u(0,\cdot)\big\|_{\mathbf{L}^1}, \tag{4.30}$$

recalling that the matrix M has negative eigenvalues and using (4.3), (4.20), we deduce that for all $\beta > \beta^*$ there is some $C(\beta) > 0$ such that

$$\|U_0\|_{\mathbf{L}^1} + \|\mathcal{J}_\beta\|_{\mathbf{L}^1} \;\le\; C(\beta)\,\Big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1}\Big). \tag{4.31}$$

In the same way, we can obtain uniform estimates on $\mathcal{J}_{j\beta},\;\; j \ge 2$, as well. Indeed, it suffices to replace (4.14) with

$$\mathcal{J}_{j\beta} \;=\; \sum_{k=j}^{n}\Big(f_{j,k} + A_{j,k}[U_0] - B_{j,k}[\mathcal{J}_\beta]\Big) + \frac{(j-1)!}{n!\beta^{n-j+1}} I_{j\beta}\circ\cdots\circ I_{n\beta}[\mathcal{J}_{(n+1)\beta}], \tag{4.32}$$

where

$$f_{j,k} \;=\; \mathcal{J}_{k\beta}(0)\frac{(j-1)!\, I_{j\beta}\circ\cdots\circ I_{k\beta}[\mathbf{e}_{k\beta}]}{(k-1)!\,\beta^{k-j}}\,,$$

$$A_{j,k}[U_0] \;=\; \frac{(j-1)!\, I_{j\beta}\circ\cdots\circ I_{k\beta}[U_0]}{(k-1)!\,\beta^{k-j}}\,, \qquad\qquad B_{j,k}[\mathcal{J}_\beta] \;=\; \frac{(j-1)!\, I_{j\beta}\circ\cdots\circ I_{k\beta}[\mathcal{J}_\beta]}{k!\,\beta^{k-j+1}}\,.$$

We thus obtain

$$\mathcal{J}_{j\beta} \;=\; \sum_{k=j}^{\infty} f_{j,k} + \sum_{k=j}^{\infty} A_{j,k}[U_0] - \sum_{k=j}^{\infty} B_{j,k}[\mathcal{J}_\beta],$$

with

$$\begin{aligned}
\Big\|\sum_{k=j}^{\infty} f_{j,k}\Big\|_{\mathbf{L}^1(I\!R_+)} \;&\le\; \sum_{k=j}^{\infty} |\mathcal{J}_{k\beta}(0)|\frac{(j-1)!^2}{(k-1)!\,k!\,\beta^{2k-2j+1}}\\
&\le\; \sum_{m=0}^{\infty}\frac{1}{(m!)^2\,\beta^{2m+1}}\|u(0,\cdot)\|_{\mathbf{L}^1},
\end{aligned}$$

and similarly

$$\Big\|\sum_{k=j}^{\infty} A_{j,k}\Big\|_{\mathcal{B}(\mathbf{L}^1(I\!R_+))} \le \sum_{m=0}^{\infty}\frac{1}{(m!)^2\,\beta^{2m+1}}\,, \qquad \Big\|\sum_{k=j}^{\infty} B_{j,k}\Big\|_{\mathcal{B}(\mathbf{L}^1(I\!R_+))} \le \sum_{m=0}^{\infty}\frac{1}{(m!)^2\,\beta^{2m+2}}.$$

To obtain the above estimates, we used the identity $(j-1)!/(k-1)! \le 1/(k-j)!$ and made the change of variable $m = k - j$. In turn, this yields

$$\|\mathcal{J}_{j\beta}\|_{\mathbf{L}^1} \;\le\; C(\beta)\big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1}\big), \qquad j \ge 2\,, \tag{4.33}$$

for some constant $C(\beta)$ independent of j (which may differ from the above $C(\beta)$ used in (4.31), a convention that we use from now on). We underline that we do not need to reduce the range of β in this argument.

4.2 Exponential decay

The above estimates can be slightly improved, choosing some $\varepsilon > 0$ and working in the weighted space $L^1_\varepsilon(I\!R_+)$, with norm defined as in (4.8). This will imply the exponential decay of the solutions.

Proposition 4.1. *For any $\beta > \beta^*$, there exists $\varepsilon > 0$ and $C > 0$ such that*

$$\|U_0\|_{\mathbf{L}^1_\varepsilon} + \max_{j\geq 1} \|\mathcal{J}_{j\beta}\|_{\mathbf{L}^1_\varepsilon} \;\leq\; C\Big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1}\Big). \tag{4.34}$$

Proof. For any $\beta > \beta^*$, we claim that there exists $\varepsilon > 0$ such that operator $\widetilde{\mathcal{T}}$ (or equivalently $\mathcal{T}$) is still a contraction on $\mathbf{L}^1_\varepsilon \times \mathbf{L}^1_\varepsilon$. Indeed, using (4.10) repeatedly (instead of (4.7)), one can replace (4.21) with the statement that, for $\varepsilon < 2\beta$, A and B are continuous operators on $\mathbf{L}^1_\varepsilon$, with norms

$$\begin{aligned} \|A\|_{\mathcal{L}(\mathbf{L}^1_\varepsilon(I\!R_+))} \;&\leq\; \sum_{k=2}^\infty \frac{1}{(k-1)!\,\beta^{k-2}\,(2\beta-\varepsilon)\cdots(k\beta-\varepsilon)}, \\ \|B\|_{\mathcal{L}(\mathbf{L}^1_\varepsilon(I\!R_+))} \;&\leq\; \sum_{k=2}^\infty \frac{1}{k!\,\beta^{k-1}\,(2\beta-\varepsilon)\cdots(k\beta-\varepsilon)}. \end{aligned}$$

Moreover, in place of (4.20), we can estimate f by

$$\|f\|_{\mathbf{L}^1_\varepsilon} \;\leq\; \sum_{k=2}^\infty \frac{1}{k!\,\beta^{k-2}\,(2\beta-\varepsilon)\cdots(k\beta-\varepsilon)(k\beta-\varepsilon)} \|u(0,\cdot)\|_{\mathbf{L}^1}\,.$$

Relying on (4.9), it follows that $\mathcal{T}$ is continuous in $\mathbf{L}^1_\varepsilon$ for $\varepsilon < \min(\beta, 1/\beta)$, with

$$\begin{aligned} \|\mathcal{T}\|_{\mathcal{L}(\mathbf{L}^1_\varepsilon\times\mathbf{L}^1_\varepsilon)} \;\leq\; & \max\Big\{\|m_{12}\|_{\mathbf{L}^1_\varepsilon}, \|m_{22}\|_{\mathbf{L}^1_\varepsilon}\Big\} \\ & \cdot\left(\sum_{k=2}^\infty \frac{1}{(k-1)!\,\beta^{k-2}\,(2\beta-\varepsilon)\cdots(k\beta-\varepsilon)} + \sum_{k=2}^\infty \frac{1}{k!\,\beta^{k-1}\,(2\beta-\varepsilon)\cdots(k\beta-\varepsilon)}\right). \end{aligned}$$

For $\beta > 0$ and $\varepsilon \geq 0$, we define $\tilde\kappa(\beta,\varepsilon)$ to be the right-hand side in the above formula. We observe that this is a continuous function of (β,ε) and that $\tilde\kappa(\beta,0) = \kappa(\beta)$, with κ defined as in (4.28). Together with Lemma 4.1, this proves the existence of $\varepsilon = \varepsilon(\beta) > 0$ such that the operator $\mathcal{T}$ is a contraction on $\mathbf{L}^1_\varepsilon \times \mathbf{L}^1_\varepsilon$.

Then we can argue as in (4.31)-(4.33) and obtain the estimate (4.34). □

4.3 Estimates on $u(t,x)$

All the previous analysis was concerned with the function $U_0(t) = u(t,0)$, where $u = u(t,x)$ is a solution to (2.8). To derive estimates on $u(t,x)$ for $x > 0$ we use similar arguments, along characteristics. Given $\gamma > 0$ and $\tau \in I\!R$, for all $t \geq \max\{\tau, 0\}$ we define

$$\mathcal{J}^\tau_\gamma(t) \;\doteq\; \int_{t-\tau}^\infty e^{-\gamma x} u(t,x)\,dx. \tag{4.35}$$

By (2.8) one has

$$\frac{d}{dt}u(t,t-\tau) \;=\; (u_t+u_x)(t,t-\tau) \;=\; -\,\mathcal{J}^\tau_\beta(t).$$

Hence $\mathcal{J}^\tau_\beta$ is related to the characteristic issued from $(t,x)=(\tau,0)$ when $\tau\geq 0$ and to the one issued from $(t,x)=(0,|\tau|)$ when $\tau\leq 0$.

Differentiating (4.35) w.r.t. t we obtain

$$\begin{aligned}
(\mathcal{J}^\tau_\gamma)'(t) &= -e^{-\gamma(t-\tau)}u(t,t-\tau)+\int_{t-\tau}^\infty e^{-\gamma x}\left[-\partial_x u(t,x)-\int_x^\infty e^{-\beta y}u(t,y)\,dy\right]dx\\
&= -\gamma\mathcal{J}^\tau_\gamma(t)-\int_{t-\tau}^\infty\int_x^\infty e^{-\gamma x-\beta y}u(t,y)\,dy\,dx\\
&= -\gamma\mathcal{J}^\tau_\gamma(t)-\int_{t-\tau}^\infty\left(\int_{t-\tau}^y e^{-\gamma x-\beta y}u(t,y)\,dx\right)dy\\
&= -\gamma\mathcal{J}^\tau_\gamma(t)-\frac{e^{-\gamma(t-\tau)}}{\gamma}\mathcal{J}^\tau_\beta(t)+\frac{1}{\gamma}\mathcal{J}^\tau_{\gamma+\beta}(t).
\end{aligned}$$

In particular, for $n\geq 1$ and $t\geq\max\{\tau,0\}$ one has

$$(\mathcal{J}^\tau_{n\beta})'(t)+n\beta\mathcal{J}^\tau_{n\beta}(t) \;=\; -\,e^{-n\beta(t-\tau)}\frac{\mathcal{J}^\tau_\beta(t)}{n\beta}+\frac{\mathcal{J}^\tau_{(n+1)\beta}(t)}{n\beta}\,. \tag{4.36}$$

To obtain estimates on $\mathcal{J}^\tau_{n\beta}$, we treat the cases $\tau\geq 0$ and $\tau\leq 0$ separately.

Case 1: $\tau\geq 0$. In this case we deduce from (4.36) that

$$\mathcal{J}^\tau_{n\beta} \;=\; \mathcal{J}^\tau_{n\beta}(\tau)\mathbf{e}^\tau_{n\beta}-\frac{\tilde{I}^\tau_{n\beta}[\mathcal{J}^\tau_\beta]}{n\beta}+\frac{I^\tau_{n\beta}[\mathcal{J}^\tau_{(n+1)\beta}]}{n\beta}, \tag{4.37}$$

where, for $\alpha>0$ and $t\geq\tau$, we define

$$\mathbf{e}^\tau_\alpha(t) \;\doteq\; e^{-\alpha(t-\tau)}, \tag{4.38}$$

$$I^\tau_\alpha[f](t) \;\doteq\; \int_\tau^t e^{-\alpha(t-s)}f(s)\,ds, \qquad \tilde{I}^\tau_\alpha[f](t) \;\doteq\; I_\alpha[e^\tau_\alpha\, f](t) \;=\; e^{-\alpha(t-\tau)}\int_\tau^t f(s)\,ds. \tag{4.39}$$

An important fact is that $\tilde{I}^\tau_\alpha$ is a compact operator on $\mathbf{L}^1([\tau,+\infty))$. Notice also that, by (4.35),

$$\mathcal{J}^\tau_{n\beta}(\tau) \;=\; \mathcal{J}_{n\beta}(\tau). \tag{4.40}$$

Using induction we obtain that, for all $n\geq 1$,

$$\mathcal{J}^\tau_\beta \;=\; \sum_{k=1}^n f^\tau_k \;-\; \sum_{k=1}^n \tilde{A}^\tau_k[\mathcal{J}^\tau_\beta] \;+\; \frac{1}{n!\beta^n}I^\tau_\beta\circ\cdots\circ I^\tau_{n\beta}[\mathcal{J}^\tau_{(n+1)\beta}],$$

where

$$f^\tau_k \;\doteq\; \begin{cases}\mathcal{J}_\beta(\tau)\,\mathbf{e}^\tau_\beta & \text{if } k=1,\\[2mm] \mathcal{J}_{k\beta}(\tau)\dfrac{I^\tau_\beta\circ\cdots\circ I^\tau_{(k-1)\beta}[\mathbf{e}^\tau_{k\beta}]}{(k-1)!\beta^{k-1}} & \text{otherwise,}\end{cases} \tag{4.41}$$

$$\tilde{A}_k^\tau[\mathcal{J}_\beta^\tau] \doteq \frac{I_\beta^\tau \circ \cdots \circ I_{(k-1)\beta}^\tau \circ \tilde{I}_{k\beta}^\tau[\mathcal{J}_\beta^\tau]}{(k-1)!\beta^k}. \tag{4.42}$$

The series with general terms $\tilde{A}_k^\tau$ and f_k^τ converge normally in $\mathcal{L}(\mathbf{L}^1([\tau,+\infty)))$ and $\mathbf{L}^1([\tau,+\infty))$ respectively to

$$\tilde{A}^\tau = \sum_{k=1}^\infty A_k^\tau \qquad \text{and} \qquad f^\tau = \sum_{k=1}^\infty f_k^\tau ,$$

with

$$\|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))} \le \sum_{k=1}^\infty \frac{|\mathcal{J}_{k\beta}(\tau)|}{k!\,(k-1)!\,\beta^{2k-1}}. \tag{4.43}$$

For $t \ge \tau$, we can now obtain $\mathcal{J}_\beta^\tau$ as a fixed point in $\mathbf{L}^1([\tau,+\infty))$ of

$$\mathcal{J}_\beta^\tau = -\tilde{A}^\tau[\mathcal{J}_\beta^\tau] + f^\tau.$$

The main difference with Subsection 4.1 is that here the operator $\widetilde{A}^\tau$ is compact, being a strong limit of compact operators. Hence $\mathrm{Id} + \tilde{A}^\tau$ is a Fredholm operator. That its kernel is trivial is a direct consequence of Gronwall's lemma. Indeed, one has a bound of the form

$$\big|\tilde{A}^\tau(f)\big|(t) \le C\int_\tau^t \left(\max_{\xi\in[\tau,s]} |f(\xi)|\right) ds.$$

Therefore $\mathrm{Id} + \tilde{A}^\tau$ is invertible. Moreover the norm $\|(\mathrm{Id} + \tilde{A}^\tau)^{-1}\|_{\mathcal{L}(\mathbf{L}^1([\tau,+\infty)))}$ is independent of τ because, as seen from (4.42), for different τ and τ', the operator $\tilde{A}^{\tau'}$ is obtained from $\tilde{A}^\tau$ by a simple translation. As a consequence we deduce that, for $\tau \ge 0$,

$$\|\mathcal{J}_\beta^\tau\|_{\mathbf{L}^1([\tau,+\infty))} \le C(\beta)\|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))}. \tag{4.44}$$

We underline that we did not reduce the range of β in this step either.

Case 2: $\tau \le 0$. In this case, instead of (4.37), we deduce from (4.36) that for all $t \ge 0$

$$\mathcal{J}_{n\beta}^\tau = \mathcal{J}_{n\beta}^\tau(0)\mathbf{e}_{n\beta} - \frac{\tilde{I}_{n\beta}^0[\mathcal{J}_\beta^\tau]}{n\beta} + \frac{I_{n\beta}\left[\mathcal{J}_{(n+1)\beta}^\tau\right]}{n\beta}.$$

The operators $I_{j\beta}$ and $\tilde{I}_{k\beta}^0$ were defined in (4.5) and in (4.39) respectively. By induction we obtain that, for all $n \ge 1$,

$$\mathcal{J}_\beta^\tau = \sum_{k=1}^n f_k^\tau - \sum_{k=1}^n \tilde{A}_k[\mathcal{J}_\beta^\tau] + \frac{1}{n!\beta^n} I_\beta \circ \cdots \circ I_{n\beta}[\mathcal{J}_{(n+1)\beta}^\tau],$$

with

$$f_k^\tau \doteq \mathcal{J}_{k\beta}^\tau(0)\frac{I_\beta \circ \cdots \circ I_{(k-1)\beta}[e_{k\beta}]}{(k-1)!\beta^{k-1}}, \tag{4.45}$$

$$\tilde{A}_k[\mathcal{J}_\beta^\tau] \doteq \frac{I_\beta \circ \cdots \circ I_{(k-1)\beta} \circ \tilde{I}_{k\beta}^0[\mathcal{J}_\beta^\tau]}{(k-1)!\beta^{k-1}}.$$

We notice that the above quantities are continuous at $\tau = 0$.

Defining $\tilde{A} \doteq \sum_{k=1}^{\infty} \tilde{A}_k$ in $\mathcal{L}(\mathbf{L}^1(I\!R_+))$ and $f^\tau \doteq \sum_{k=1}^{\infty} f_k^\tau$ in $\mathbf{L}^1(I\!R_+)$, with f_k^τ as in (4.45), and arguing in a similar way as in (4.44), we obtain

$$\begin{aligned}\|\mathcal{J}_\beta^\tau\|_{\mathbf{L}^1(I\!R_+)} &\leq C(\beta)\|f^\tau\|_{\mathbf{L}^1(I\!R_+)} \\ &\leq C(\beta)\|u(0,\cdot)\|_{\mathbf{L}^1(I\!R_+)} e^{-\beta|\tau|},\end{aligned} \tag{4.46}$$

for any $\tau \leq 0$. Notice that here the last inequality follows from (4.35) and (4.45).

Going back to (2.11) we see that, for all $t \geq 0$,

$$\frac{d}{dt}\|u(t,\cdot)\|_{\mathbf{L}^1} \leq |u(t,0)| + \int_0^\infty \left| \int_x^\infty e^{-\beta y} u(t,y)\, dy \right| dx.$$

For $t \geq 0$ and $x \geq 0$ one has

$$\int_x^\infty e^{-\beta y} u(t,y)\, dy = \mathcal{J}_\beta^{t-x}(t),$$

hence

$$\frac{d}{dt}\|u(t,\cdot)\|_{\mathbf{L}^1(I\!R_+)} \leq |u(t,0)| + \int_{-\infty}^t |\mathcal{J}_\beta^\tau(t)|\, d\tau.$$

Using (4.44) and (4.46) we deduce

$$\|u(t,\cdot)\|_{\mathbf{L}^1(I\!R_+)} \leq C(\beta)\left(\|U_0\|_{\mathbf{L}^1(I\!R_+)} + \|u(0,\cdot)\|_{\mathbf{L}^1(I\!R_+)} + \int_0^\infty \|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))}\, d\tau\right),$$

for a constant $C(\beta)$ uniformly valid for all $t \geq 0$. Using (4.43) we have

$$\int_0^\infty \|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))}\, d\tau \leq C \sum_{k=2}^\infty \frac{1}{k!\,(k-1)!\,\beta^{2k-1}} \int_0^\infty |\mathcal{J}_{k\beta}(\tau)|\, d\tau.$$

Recalling (4.31) and (4.33), we finally obtain an estimate on u, uniformly valid for all $t \geq 0$:

$$\|u(t,\cdot)\|_{\mathbf{L}^1} \leq C(\beta)\big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1}\big). \tag{4.47}$$

This completes the proof of Theorem 2. □

Thanks to Proposition 4.1, we can also prove the following exponential decay estimate.

Proposition 4.2. *For any given $\beta > \beta^*$ and $0 < \lambda < 1$, letting $\varepsilon > 0$ be the constant provided by Proposition 4.1, there exists a constant $C(\beta,\varepsilon)$ such that, for every solution u of (2.8) and every $0 < \nu \leq (1-\lambda)\varepsilon$, one has*

$$\|u(t,\cdot)\|_{\mathbf{L}^1([0,\lambda t])} \leq C(\beta,\varepsilon)\Big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1(I\!R_+)}\Big)e^{-\nu t} \qquad \text{for all } t \geq 0. \tag{4.48}$$

Proof. Tracing the solution along characteristics, for any $x \in [0, \lambda t]$ one has

$$u(t,x) = u(t-x,0) - \int_{t-x}^t \mathcal{J}_\beta^{t-x}(s)\, ds.$$

Integrating over $x \in [0, \lambda t]$ we get

$$\begin{aligned}
\|u(t,\cdot)\|_{\mathbf{L}^1([0,\lambda t])} &\leq \|U_0\|_{\mathbf{L}^1([(1-\lambda)t,\,t])} + \int_0^{\lambda t}\int_{t-x}^t |\mathcal{J}_\beta^{t-x}(s)|\,ds\,dx \\
&\leq \|U_0\|_{\mathbf{L}^1([(1-\lambda)t,\,t])} + \int_0^{\lambda t} \|\mathcal{J}_\beta^{t-x}\|_{\mathbf{L}^1([t-x,+\infty[)}\,dx \\
&= \|U_0\|_{\mathbf{L}^1([(1-\lambda)t,\,t])} + \int_{(1-\lambda)t}^t \|\mathcal{J}_\beta^\tau\|_{\mathbf{L}^1([\tau,+\infty[)}\,d\tau.
\end{aligned}$$

We now choose $\varepsilon > 0$ as in Proposition 4.1. Using (4.44) we obtain

$$\begin{aligned}
\|u(t,\cdot)\|_{\mathbf{L}^1([0,\lambda t])} &\leq \|U_0\|_{\mathbf{L}^1([(1-\lambda)t,\,t])} + C\int_{(1-\lambda)t}^t \|f^\tau\|_{\mathbf{L}^1([\tau,+\infty[)}\,d\tau \\
&\leq e^{-\varepsilon(1-\lambda)t}\|U_0\|_{\mathbf{L}^1_\varepsilon((1-\lambda)t,\,t)} + Ce^{-\varepsilon(1-\lambda)t}\int_{(1-\lambda)t}^t e^{\varepsilon\tau}\|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))}\,d\tau.
\end{aligned}$$

Recalling (4.41) and (4.43) we see that

$$\int_{(1-\lambda)t}^t e^{\varepsilon\tau}\|f^\tau\|_{\mathbf{L}^1([\tau,+\infty))}\,d\tau \;\leq\; \sum_{k=1}^\infty \frac{1}{k!\,(k-1)!\,\beta^{2k-1}}\int_{(1-\lambda)t}^t e^{\varepsilon\tau}|\mathcal{J}_{k\beta}(\tau)|\,d\tau.$$

In view of (4.34), this yields the desired conclusion, with $\nu \leq (1-\lambda)\varepsilon$. □

In particular, relying on (4.47), we see that for any fixed interval $[0, M]$ there exist some positive constants C and ν such that

$$\|u(t,\cdot)\|_{\mathbf{L}^1([0,M])} \;\leq\; C(\beta, M)\big(|U_0(0)| + \|u(0,\cdot)\|_{\mathbf{L}^1(I\!\!R_+)}\big)e^{-\nu t}.$$

5 Nonlinear stability

Based on the previous results on the stability of the linearized equation (2.6), in this section we prove the stability of the full nonlinear system (2.4).

Theorem 5.1. *Assume $\beta > \beta^* \doteq (48 + \sqrt{9504})/160$. Then the nonlinear growth equation*

$$\frac{\partial}{\partial t}P(t,s) \;=\; \int_0^s e^{-\beta(t-\sigma)}\,(P_s(t,\sigma)\times \mathbf{e}_3)\times\Big(P(t,s) - P(t,\sigma)\Big)\,d\sigma \tag{5.1}$$

with boundary condition

$$P_{ss}(t,t) \;=\; 0 \qquad\qquad \textit{for all} \quad t > t_0 \tag{5.2}$$

is stable in the vertical direction.

As in Section 2, let $\mathbf{k}(t,s) = (k_1, k_2, k_3)(t,s)$ be the unit tangent vector to the stem, at the point $P(t,s)$. According to Definition 1, the above theorem can be established by proving

Theorem 5.2. *Assume $\beta > \beta^* \doteq (48+\sqrt{9504})/160$. Then, for any given t_0, $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If*

$$\begin{cases} \big|k_1(t_0,x)\big| + \big|k_2(t_0,x)\big| \;\leq\; \delta & \text{if } \; x\in[0,t_0],\\ \big|k_1(t_0,x)\big| = \big|k_2(t_0,x)\big| = 0 & \text{if } \; x > t_0\,, \end{cases} \tag{5.3}$$

then for all $t \geq t_0$ one has the bounds

$$\big\|k_1(t,\cdot)\big\|_{\mathbf{L}^1([0,t])} + \big\|k_2(t,\cdot)\big\|_{\mathbf{L}^1([0,t])} \;\leq\; \varepsilon\,, \tag{5.4}$$

$$\big\|k_1(t,\cdot)\big\|_{\mathbf{L}^\infty([0,t])} + \big\|k_2(t,\cdot)\big\|_{\mathbf{L}^\infty([0,t])} \;\leq\; \varepsilon\,. \tag{5.5}$$

Proof. 1. Let $t_0 > 0$ and $\varepsilon > 0$ be given. In order to use the previous results on linearized stability, we define

$$u_i(t,x) \doteq \begin{cases} k_i(t,t-x) & \text{if} \quad x\in[0,t]\,,\\ 0 & \text{if} \quad x>t\,. \end{cases}$$

As long as the unit vector $(k_1,k_2,k_3) = (u_1,u_2,u_3)$ remains close to $(0,0,1)$, we have

$$k_3 \;=\; \sqrt{1-k_1^2-k_2^2}\,. \tag{5.6}$$

In particular, from $|u_1|, |u_2| < 1/4$ it follows

$$|1-u_3| \;\leq\; |u_1| + |u_2|\,. \tag{5.7}$$

By (2.4)-(2.5) and (1.9), on the domain $\{t > t_0, \quad x > 0\}$ the first two components u_1, u_2 satisfy the equations

$$u_{i,t} + u_{i,x} \;=\; -\int_x^\infty e^{-\beta y} u_i(y)\,dy + g_i(t,x), \qquad\qquad i=1,2, \tag{5.8}$$

$$g_i(t,x) \;\doteq \Big(1-\sqrt{1-u_1^2(t,x)-u_2^2(t,x)}\Big)\int_x^\infty e^{-\beta y} u_i(t,y)\,dy\,. \tag{5.9}$$

with Neumann boundary conditions at $x = 0$

$$u_{i,x}(t,0) \;=\; 0. \tag{5.10}$$

We define $S(t)$ the semigroup associated with the equation (2.11) on the space X defined in (2.12). Then (4.47) yields

$$\|S(t)\overline{u}\|_{L^1(I\!R_+)} \;\leq\; C(\beta)\,\|\overline{u}\|_X \;=\; C(\beta)\big(\|\overline{u}\|_{L^1(I\!R_+)} + |\overline{u}(0)|\big), \tag{5.11}$$

for any given initial data $\overline{u}\in X$.

2. We now observe that all the estimates performed in Section 4 in the spaces $\mathbf{L}^1(I\!R_+)$, $\mathbf{L}^1_\gamma(I\!R_+)$, can be performed in $\mathbf{L}^\infty(I\!R_+)$ and in the weighted Lebesgue space $\mathbf{L}^\infty_\gamma(I\!R_+)$, with norm

$$\|f\|_{\mathbf{L}^\infty_\gamma(I\!R_+)} \;\doteq\; \text{ess-}\sup_{x\in I\!R_+} \; e^{\gamma x}|f(x)|. \tag{5.12}$$

Indeed, we perform convolutions with $\mathbf{L}^1(I\!R_+)$ functions, and the inequalities

$$\|a * b\|_{\mathbf{L}^\infty} \ \leq \ \|a\|_{L^1} \, \|b\|_{\mathbf{L}^\infty}, \qquad\qquad \|a * b\|_{\mathbf{L}^\infty_\gamma} \ \leq \ \|a\|_{\mathbf{L}^1_\gamma} \, \|b\|_{\mathbf{L}^\infty_\gamma} \tag{5.13}$$

are valid in the same way as (4.6) and (4.9), for functions a, b in the corresponding spaces. Hence, in particular, for any $\alpha > \gamma$

$$\Big\| I_\alpha[f] \Big\|_{\mathbf{L}^\infty_\gamma(I\!R_+)} \ \leq \ \frac{1}{\alpha - \gamma} \|f\|_{\mathbf{L}^\infty_\gamma(I\!R_+)} \tag{5.14}$$

holds as (4.10). As a consequence, all estimates of Subsection 4.1 remain valid when replacing $\mathbf{L}^1$ with $\mathbf{L}^\infty$. The same can be done in Subsections 4.2 and 4.3. Notice that $\tilde{I}^\tau_\alpha$ is also compact in $\mathbf{L}^\infty$ (actually, it sends $\mathbf{L}^\infty(I\!R_+)$ in the space C_0 of continuous functions converging to 0 as $x \to +\infty$, and one can use the Ascoli-Arzelà theorem), and consequently $I + \tilde{A}^\tau$ is again a Fredholm operator on $\mathbf{L}^\infty(I\!R_+)$. Thus, viewing again S as an operator acting on the Banach space X at (2.12), we obtain an estimate uniform in t:

$$\big\| S(t)\overline{u} \big\|_{\mathbf{L}^\infty([-1,+\infty))} \ \leq \ C(\beta)\big(\|\overline{u}\|_{\mathbf{L}^\infty(I\!R_+)} + |\overline{u}(0)| \big) \, . \tag{5.15}$$

In particular, this implies

$$\big| S(t)\overline{u}(0) \big| \ \leq \ C(\beta) \|\overline{u}\|_{\mathbf{L}^\infty(I\!R_+)} \qquad \text{for all} \ \ t \geq 0.$$

Moreover, an estimate analogous to (4.34) holds in the weighted norm:

$$\|S(\cdot)\overline{u}_{|x=0}\|_{\mathbf{L}^\infty_\varepsilon(I\!R_+)} \ \leq \ C\Big(|\overline{u}(0)| + \|\overline{u}\|_{\mathbf{L}^\infty(I\!R_+)} \Big) \, . \tag{5.16}$$

As a consequence, an exponential estimate such as (4.48) can be established also the $\mathbf{L}^\infty$ norm:

$$\|S(t)\overline{u}\|_{\mathbf{L}^\infty([-1,\,\lambda t])} \ \leq \ C(\beta, \varepsilon)\Big(|\overline{u}(0)| + \|\overline{u}\|_{\mathbf{L}^\infty(I\!R_+)} \Big) e^{-\nu t} \qquad \text{for all} \ \ t \geq 0. \tag{5.17}$$

Without loss of generality (possibly reducing its value), we can assume that ν in (4.48) and (5.17) satisfies

$$\nu \leq \min\left\{ \frac{\beta}{8}, \frac{\varepsilon}{2} \right\}, \tag{5.18}$$

where ε is the constant in (4.34).

3. We shall construct the solution of the nonlinear system (5.8)-(5.10) as a fixed point of a suitable operator. We will prove that the solution satisfies the claimed stability. To this purpose, we introduce the following functional space:

$$\mathcal{A}_\nu \doteq \Big\{ u \in \mathbf{L}^\infty\Big(I\!R_+; \ (\mathbf{L}^1 \cap \mathbf{L}^\infty)([-1, +\infty)) \Big) \ \Big/ \ \|u\|_{\mathcal{A}_\nu} \doteq \text{ess-}\sup_{t \geq 0} \Big| e^{\nu t} \|u(t, \cdot)\|_{(\mathbf{L}^1 \cap \mathbf{L}^\infty)([-1, t/4])} \Big| < +\infty \Big\}.$$

Next, in connection with any fixed $\overline{u} = (\overline{u}_1, \overline{u}_2) \in X \times X$, consider the closed, bounded set

$$\mathcal{S}^{\overline{u}} \doteq \Big\{ (u_1, u_2) \in \mathcal{A}_\nu \times \mathcal{A}_\nu \ \Big/ \ \max(\|u_i\|_{\mathbf{L}^\infty(I\!R_+; (\mathbf{L}^1 \cap \mathbf{L}^\infty)([-1, +\infty)))}, \|u_i\|_{\mathcal{A}_\nu}) \leq 2C^* \big(\|\overline{u}_i\|_{(\mathbf{L}^1 \cap \mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)| \big) \Big\}. \tag{5.19}$$

Here and throughout the following, the constant C^* denotes the maximum of all constants $C(\beta)$ in (5.11), (5.15), $C(\beta,\varepsilon)$ in (4.48), (5.17) with $\lambda = 1/2$, and C in (4.34), (5.16). Moreover we adopt the notation $\|\cdot\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)} \doteq \max(\|\cdot\|_{\mathbf{L}^1}, \|\cdot\|_{\mathbf{L}^\infty})$.

In particular, if $(u_1,u_2)\in\mathcal{S}^{\overline{u}}$, then the above definition implies

$$|u_i(t,0)| \;\leq\; 2C^*\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!\!R_+)} + |\overline{u}_i(0)|\big)e^{-\nu t} \qquad \text{for a.e. } t>0, \qquad i=1,2\,.$$

Notice that $\mathcal{A}_\nu$ is complete w.r.t. the norm $\max\{\mathbf{L}^\infty(\mathbf{L}^1\cap\mathbf{L}^\infty), \|\cdot\|_{\mathcal{A}_\nu}\}$. Moreover, by (5.7), if $\|\overline{u}_i\|_X$ are sufficiently small, we can assume that

$$\Big|1-\sqrt{1-u_1^2(t,x)-u_2^2(t,x)}\Big| \;\leq\; \big|u_1(t,x)\big|+\big|u_2(t,x)\big| \qquad \text{for a.e. } t>0\,,\ x>-1\,. \tag{5.20}$$

4. An operator

$$F:(u_1,u_2) \;\longmapsto\; (\hat{u}_1,\hat{u}_2) \tag{5.21}$$

mapping $\mathcal{S}^{\overline{u}}$ into itself is constructed as follows.

- Given $(u_1,u_2)\in\mathcal{S}^{\overline{u}}$, we first define the pair (g_1,g_2) as in (5.9).
- We then define $(\hat{u}_1,\hat{u}_2)$ in terms of Duhamel's formula [3, 6, 7] by setting

$$\hat{u}_i \doteq S(t)\overline{u}_i + \int_0^t S(t-s)g_i(s)\,ds \qquad i=1,2\,. \tag{5.22}$$

In the forthcoming steps, we perform estimates relative to the $\mathbf{L}^\infty(\mathbf{L}^1\cap\mathbf{L}^\infty)$-norm, in order to prove that F admits a fixed point.

5 - Estimates on g_i in $\mathbf{L}^1\cap\mathbf{L}^\infty$. Given $(u_1,u_2)\in\mathcal{S}^{\overline{u}}$, the integral $\int_x^\infty e^{-\beta y}u_i(t,y)\,dy$ can be estimate as follows.

- If $x>t/4$, by definition of $\mathcal{S}^{\overline{u}}$ we simply have

$$\begin{aligned}\left|\int_x^\infty e^{-\beta y}u_i(t,y)\,dy\right| &\leq\; e^{-\beta t/4}\|u_i(t,\cdot)\|_{\mathbf{L}^1}\\ &\leq\; 2C^*e^{-\beta t/4}\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!\!R_+)} + |\overline{u}_i(0)|\big).\end{aligned}$$

- If $x<t/4$, we write

$$\int_x^\infty e^{-\beta y}u_i(t,y)\,dy \;=\; \int_x^{t/4} e^{-\beta y}u_i(t,y)\,dy + \int_{t/4}^\infty e^{-\beta y}u_i(t,y)\,dy.$$

  Then, using the definition of the set $\mathcal{S}^{\overline{u}}$, we deduce

$$\begin{aligned}\left|\int_x^\infty e^{-\beta y}u_i(t,y)\,dy\right| &\leq\; 2C^*e^{-\nu t}(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!\!R_+)} + |\overline{u}_i(0)|) + e^{-\beta t/4}\|u_i(t,\cdot)\|_{\mathbf{L}^1(I\!\!R_+)}\\ &\leq\; 2C^*\big(e^{-\nu t}+e^{-\beta t/4}\big)(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!\!R_+)} + |\overline{u}_i(0)|).\end{aligned}$$

In both cases, recalling (5.18) we obtain

$$\left|\int_x^\infty e^{-\beta y}u_i(t,y)\,dy\right| \;\leq\; 4C^* e^{-\nu t}\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big). \tag{5.23}$$

Next, going back to the functions g_i, and using (5.20), we find

$$|g_i(t,x)| \;\leq\; C\big(|u_1(t,x)|+|u_2(t,x)|\big)\left|\int_x^\infty e^{-\beta y}u_i(t,y)\,dy\right|. \tag{5.24}$$

By (5.24), and thanks to the definition of the set $\mathcal{S}^{\overline{u}}$, we now estimate the $(\mathbf{L}^1\cap\mathbf{L}^\infty)$-norm of $g_i(t,\cdot)$ as follows.

- Integrating (5.24) w.r.t. x over the interval $[0,\,t/4]$, for a.e. $t>0$ one obtains

$$\begin{aligned}\|g_i(t,\cdot)\|_{\mathbf{L}^1(0,t/4)} &\leq 8C\big[C^* e^{-\nu t}(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)\big]^2\\ &\leq Ke^{-2\nu t}(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)^2.\end{aligned} \tag{5.25}$$

- Integrating (5.24) for $x\in[t/4,+\infty[$ and recalling(5.18), for a.e. $t>0$ we obtain

$$\begin{aligned}\|g_i(t,\cdot)\|_{\mathbf{L}^1(t/4,+\infty)} &\leq 2C\big[2C^*(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)\big]^2 e^{-\beta t/4}\\ &\leq Ke^{-2\nu t}(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)^2.\end{aligned} \tag{5.26}$$

- Finally, (5.20) together with (5.23) yields, for a.e. $t>0$ and $x>-1$,

$$\begin{aligned}|g_i(t,x)| &\leq C(|u_1(t,x)|+|u_2(t,x)|)\left|\int_0^\infty e^{-\beta y}u_i(t,y)\,dy\right|\\ &\leq Ke^{-2\nu t}\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big)^2.\end{aligned} \tag{5.27}$$

6 - Estimates on $\hat{u}_i$ in $\mathbf{L}^1\cap\mathbf{L}^\infty$. Relying on (4.47) and (5.15), we have

$$\|S(t)\overline{u}_i\|_{\mathbf{L}^1\cap\mathbf{L}^\infty([-1,+\infty))} \leq C^*(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)\,. \tag{5.28}$$

uniformly in t. Hence, using (5.25), (5.26) and (5.27), we deduce

$$\begin{aligned}\left\|\int_0^t S(t-s)g_i(s)\,ds\right\|_{\mathbf{L}^1\cap\mathbf{L}^\infty([-1,+\infty))} &\leq C^*\int_0^t\big(\|g_i(s,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |g_i(s,0)|\big)\,ds\\ &\leq K\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big)^2.\end{aligned} \tag{5.29}$$

In turn, recalling (5.22), from (5.28)-(5.29) we deduce an estimate uniform in t:

$$\big\|\hat{u}_i(t,\cdot)\|_{\mathbf{L}^1\cap\mathbf{L}^\infty([-1,+\infty))} \;\leq\; \big(C^*+2K(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|)\big)\times\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|\big). \tag{5.30}$$

7 - Estimates on $\hat{u}_i$ in $\mathcal{A}_\nu$. Observe first that (4.48), (5.17) with $\lambda=1/4$ give us directly

$$\|S(\cdot)\overline{u}_i\|_{\mathbf{L}^1\cap\mathbf{L}^\infty((0,t/4))} \;\leq\; C^*\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big). \tag{5.31}$$

We estimate the second term on the right hand side of (5.22) by writing

$$\left\| \int_0^t S(t-s) g_i(s)\, ds \right\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,t/4)} \;\leq\; A + B\,, \tag{5.32}$$

where

$$A \doteq \left\| \int_0^{t/2} S(t-s) g_i(s)\, ds \right\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,t/4)}, \qquad B \doteq \left\| \int_{t/2}^{t} S(t-s) g_i(s)\, ds \right\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,t/4)},$$

The two terms A, B are estimated as follows.

- Using again (4.48) and (5.17) with $\lambda = 1/2$, one obtains

$$\begin{aligned} A \;&\leq\; \int_0^{t/2} \| S(t-s) g_i(s)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,t/4)}\, ds \\ &\leq\; \int_0^{t/2} \| S(t-s) g_i(s)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,(t-s)/2)}\, ds \\ &\leq\; \int_0^{t/2} C^* e^{-\nu(t-s)} \big(\|g_i(s,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |g_i(s,0)|\big)\, ds. \end{aligned}$$

  Now (5.25), (5.26) and (5.27) imply that, for almost every $s > 0$, there holds

$$\|g_i(s,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |g_i(s,0)| \leq 2K(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|)^2 e^{-2\nu s}.$$

  Therefore

$$A \;\leq\; K'(\|\overline{u}_i\|_{L^1\cap L^\infty} + |\overline{u}_i(0)|)^2 e^{-\nu t}.$$

- Relying on (5.28) together with (5.25)–(5.27), we obtain

$$\begin{aligned} B \;&\leq\; \int_{t/2}^{t} \| S(t-s) g_i(s)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}\, ds \\ &\leq\; C^* \int_{t/2}^{t} \big(\|g_i(s,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |g_i(s,0)|\big)\, ds \\ &\leq\; 2K\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big)^2 \int_{t/2}^{t} e^{-2\nu s}\, ds \\ &\leq\; 2K\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big)^2 e^{-\nu t}. \end{aligned}$$

Hence, recalling again (5.22), we deduce from (5.31)-(5.32) the uniform estimate in t:

$$\begin{aligned} &\|\hat{u}_i(t,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(0,t/4)} \\ &\quad \leq\; e^{-\nu t}\big(C^* + 2K(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + \overline{u}_i(0)|)\big) \times \big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big)\,. \end{aligned} \tag{5.33}$$

8 - Estimates on $\hat{u}_i$ at $x = 0$.

- Relying on (5.16) and recalling (5.18), we deduce

$$\begin{aligned} \big|S(t)\overline{u}_{i|x=0}\big| \;&\leq\; C^* e^{-\varepsilon t}\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + |\overline{u}_i(0)|\big) \\ &\leq\; C^* e^{-\nu t}\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)} + \overline{u}_i(0)|\big) \end{aligned}$$

- Concerning the term involving g_i, using (5.17) together with (5.25), (5.26) and (5.27), we obtain

$$\begin{aligned}\left|\int_0^t S(t-s)g_i(s)\,ds\right|_{|x=0} &\leq C^*\int_0^t e^{-\nu(t-s)}\big(\|g_i(s,\cdot)\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|g_i(s,0)|\big)\,ds\\ &\leq 2K\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+\overline{u}_i(0)|\big)^2\int_0^t e^{-\nu(t-s)}e^{-2\nu s}\,ds\\ &\leq 2K\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|\big)^2 e^{-\nu t}.\end{aligned}$$

Hence, relying again on Duhamel's formula (5.22), we obtain

$$\big|\hat{u}_i(t,0)\big| \leq e^{-\nu t}\big(C^*+2K(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|)\big)\times\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|\big). \tag{5.34}$$

Together with (5.33), this yields

$$\|\hat{u}_i\|_{\mathcal{A}_\nu}\leq\big(C^*+2K(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|)\big)\times\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|\big)\,. \tag{5.35}$$

9 - Conclusion of the proof. For a given initial data $\mathbf{k}(t_0,\cdot)$, the local existence and uniqueness of a solution to the equations (2.3) follows from classical theory [6, 8]. Equivalently, in terms of the variables (u_1,u_2,u_3), the fixed point of the transformation F in (5.21) must be unique.

On the other hand, putting together all the above estimates we see that

$$F(\mathcal{S}^{\overline{u}})\ \subset\ \mathcal{S}^{\overline{u}}, \tag{5.36}$$

provided that the norms $\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|$, $i=1,2$, are small enough. Given $t_0,\varepsilon>0$, we now choose $\delta>0$ such that, if the initial datum $(\bar{u}_1,\bar{u}_2)$ satisfies (5.3), then (5.36) holds together with

$$2C^*\big(\|\overline{u}_i\|_{(\mathbf{L}^1\cap\mathbf{L}^\infty)(I\!R_+)}+|\overline{u}_i(0)|\big)\ \leq\ \varepsilon\,,\qquad\qquad i=1,2\,. \tag{5.37}$$

Now consider any initial data $(\bar{u}_1,\bar{u}_2)$ satisfying (5.3). Since the unique solution of (5.8)–(5.10) provides a fixed point of F, we conclude that this solution remains in $\mathcal{S}^{\overline{u}}$. In particular, by the definition (5.19), for every $t\geq t_0$ we have

$$\big\|u_i(t,\cdot)\big\|_{\mathbf{L}^1(I\!R_+)}\ \leq\ \varepsilon\,,\qquad \big\|u_i(t,\cdot)\big\|_{\mathbf{L}^\infty(I\!R_+)}\ \leq\ \varepsilon\,,\qquad\qquad i=1,2.$$

Going back to the original variables k_1,k_2, this proves Theorem 5.2. □

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