Minnaert resonance in bubbly media

David Gontier

CEREMADE, Université Paris-Dauphine

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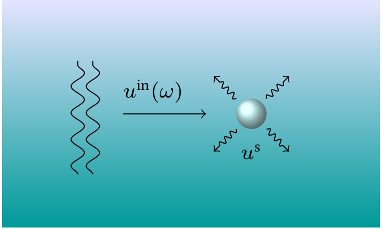
Joint work with Habib Ammari, Brian Fitzpatrick, Hyundae Lee, Hai Zhang.

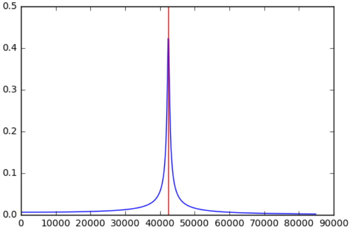
We want to understand the propagation of sound in bubbly water.

Experiment



Results

The function $|u^s/u^{\text{in}}|(\omega)$:



There exists a resonant angular frequency ω_M.
Noticed for the first time by M. Minnaert (1933 : *On musical air-bubbles and the sound of running water*).

$$\omega_M = \sqrt{\frac{3\rho_b}{\rho}} \frac{v_b}{R} \qquad \text{(Minnaert resonance).}$$

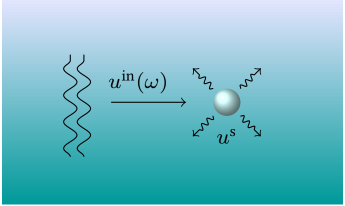
- ρ_b is the density of air (inside the bubble), and ρ the density of water,
- v_b is the speed of sound in the air.
- R is the radius of the bubble.

Example

For a bubble of radius 0.5 mm, this gives $\omega_M = 42000$ Hz (audible), and a wavelength (in water) $\lambda_M = 0.22$ m.

Goal of this talk: understand the previous formula, and extend it.

Our model



- Air bubble: domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega$ of class C^2,
- ρ_b (resp. ρ) the density of air (resp. water),
- v_b (resp. v) the speed of sound in the air (resp. water),
- $u(\mathbf{x})$ the pressure at $\mathbf{x} \in \mathbb{R}^3$,
- $\rho^{-1}u(\mathbf{x}) \sim$ velocity flow at $\mathbf{x} \in \mathbb{R}^3$.

Let ω be the angular frequency of the incident wave u^{in} and introduce

$$k_b = k(\omega) := \frac{\omega}{v_b} \quad \text{and} \quad k := \frac{\omega}{v}. \qquad \text{(wave numbers)}$$

Wave equation (d'Alembert equations) in frequency domain.

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \left(\Delta + k_b^2\right) u = 0 & \text{in} \quad \Omega, \\ u_+ = u_- & \text{on} \quad \partial\Omega, \qquad \text{(continuity of the pressure)} \\ \frac{1}{\rho}\frac{\partial u}{\partial \nu}\Big|_+ = \frac{1}{\rho_b}\frac{\partial u}{\partial \nu}\Big|_- & \text{on} \quad \partial\Omega, \qquad \text{(continuity of the velocity flow)} \\ u^{\text{s}} := u - u^{\text{in}} & \text{satisfies the Sommerfeld radiation condition.} \end{cases}$$

Regime?

We are looking for a resonance mode whose wavelength is much bigger than the size of the bubble:

$$\boxed{\text{Limit 1: } \omega \to 0 \iff k \text{ (and } k_b) \to 0.}$$

The only solution of the limit equation ($k = k_b = 0$), with $u^{\text{in}} \equiv 0$, is $u \equiv 0$. We need something else!

Order of magnitude: $\rho_b = 1.225$ kg.m^{-3} and $\rho = 1000$ kg.m^{-3}, hence $\delta := \dfrac{\rho_b}{\rho} \ll 1$ (contrast).

$$\boxed{\text{Limit 2: } \delta \to 0.}$$

Limit equation (with $u^{\text{in}} \equiv 0$)

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \Delta u = 0 & \text{in} \quad \Omega, \\ u_+ = u_- & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on} \quad \partial\Omega, \\ u & \text{satisfies the Sommerfeld radiation condition.} \end{cases}$$

The inside and outside problems are decoupled:

- 1) Solve the internal (Neumann) problem ($u|_\Omega = 1$),
- 2) Solve the external (Dirichlet) problem.

There exists a non-trivial solution $\Longrightarrow$ resonant mode.

Goal: Track this mode for small k and small δ.

Layer potentials and Fredholm theory

(3d) Green's function for Helmholtz: solution to $(\Delta + k^2)G^k = \delta_0$.

$$G^k(\mathbf{x}, \mathbf{y}) := G^k(\mathbf{x} - \mathbf{y}) := \frac{-1}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}.$$

Single layer potential

$$\forall \Psi \in C^\infty(\partial\Omega),\ \forall \mathbf{x} \in \mathbb{R}^3, \quad \widetilde{\mathcal{S}^k}[\Psi](\mathbf{x}) := \int_{\partial\Omega} G^k(\mathbf{x} - \mathbf{y})\Psi(\mathbf{y})d\sigma(\mathbf{y}).$$

Dirichlet-to-Neumann operator

$$\forall \Psi \in C^\infty(\partial\Omega),\ \forall \mathbf{x} \in \partial\Omega, \quad \mathcal{K}^{k,*}[\Psi](\mathbf{x}) := \int_{\partial\Omega} \frac{\partial G^k}{\partial \nu_{\mathbf{x}}}(\mathbf{x} - \mathbf{y})\Psi(\mathbf{y})d\sigma(\mathbf{y}).$$

Hilbert spaces

$$L^2 := L^2(\partial\Omega), \quad H^{-1/2} := H^{-1/2}(\partial\Omega), \quad H^{1/2} := H^{1/2}(\partial D).$$

Proposition (The operators are well-defined)

i) *The operators $\widetilde{\mathcal{S}^k}$ are bounded from $H^{-1/2}$ to $H^1_{\mathrm{loc}}(\mathbb{R}^3)$.*

ii) *The operators $\mathcal{S}^k := \widetilde{\mathcal{S}^k}\big|_{\partial\Omega}$ are bounded $H^{-1/2}$ to $H^{1/2}$.*

iii) *The operators $\mathcal{K}^{k,*}$ are compact (hence bounded) from $H^{-1/2}$ to $H^{-1/2}$.*

Proposition (Second properties)

Let $\psi \in H^{-1/2}$*, and* $u = \widetilde{\mathcal{S}^k}[\psi] \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$*. Then*

i) $(\Delta + k^2)u = 0$ *in* Ω *and in* $\mathbb{R}^3 \setminus \overline{\Omega}$ *(+ Sommerfeld radiation conditions);*

ii) u *is the (unique) solution to the Dirichlet problem* $(\Delta + k^2)u = 0$ *and* $u|_{\partial\Omega} = \mathcal{S}^k[\psi]$*;*

iii) *jump formula:*

$$\partial_\nu u\big|_\pm = \left(\mathcal{K}^{k,*} \pm \frac{1}{2}\right)[\psi].$$

The scattering problem can be encoded at the boundary of the bubble.

Ansazt

$$u = \begin{cases} u^{\mathrm{in}} + \widetilde{\mathcal{S}^k}[\psi] & \text{on} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \widetilde{\mathcal{S}^{k_b}}[\psi_b] & \text{on} \quad \Omega. \end{cases}$$

Initial problem

Problem with operators

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \left(\Delta + k_b^2\right) u = 0 & \text{in} \quad \Omega, \\ u_+ = u_- & \text{on} \quad \partial\Omega, \\ \delta \frac{\partial u}{\partial \nu}\big|_+ = \frac{\partial u}{\partial \nu}\big|_- & \text{on} \quad \partial\Omega. \end{cases} \iff \underbrace{\begin{pmatrix} \mathcal{S}^{k_b} & -\mathcal{S}^k \\ \mathcal{K}^{k_b,*} - \frac{1}{2} & -\delta\left(\frac{1}{2} + \mathcal{K}^{k,*}\right) \end{pmatrix}}_{\mathcal{A}(\omega,\delta)} \cdot \begin{pmatrix} \psi_b \\ \psi \end{pmatrix} = \begin{pmatrix} u^{\mathrm{in}}|_+ \\ \delta \frac{\partial u^{\mathrm{in}}}{\partial \nu}|_+ \end{pmatrix}.$$

Definition (Resonant mode)

We say that the pair (ω, δ) is a resonant mode if $\mathcal{A}(\omega, \delta)$ is non invertible.

The unperturbed operator $\mathcal{A}(0,0)$.

$$\mathcal{A}(0,0) := \begin{pmatrix} \mathcal{S} & -\mathcal{S} \\ \mathcal{K}^* - \frac{1}{2} & 0 \end{pmatrix} \quad : \quad H^{-1/2} \times H^{-1/2} \to H^{1/2} \times H^{-1/2}.$$

Resonant mode?

$$\mathcal{A}(0,0) \begin{pmatrix} \psi_b \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \iff \quad \begin{cases} \mathcal{S}[\psi_b - \psi] = 0, \\ \left(\mathcal{K}^* - \dfrac{1}{2}\right)[\psi_b] = 0. \end{cases}$$

Lemma (Classical results)

i) *The operator* $\mathcal{S} : H^{-1/2} \to H^{1/2}$ *is a bounded invertible operator with bounded inverse.*

ii) *The operator* $\mathcal{K}^*$ *is compact on* $H^{-1/2}$ *and* $\sigma(\mathcal{K}^*) \subset (-1/2, 1/2]$*. Moreover,*

$$\operatorname{Ker}\left(\mathcal{K}^* - \frac{1}{2}\right) = \operatorname{Vect}\{\,\phi_e\,\}, \quad \textit{where} \quad \phi_e := \mathcal{S}^{-1}[\mathbb{1}_{\partial\Omega}] \in H^{-1/2}.$$

Remark: $u_e := \widetilde{\mathcal{S}}[\phi_e]$ satisfies $\Delta u_e = 0$, and $u_e|_{\partial\Omega} = 1$, hence $\widetilde{\mathcal{S}}[\phi_e] = 1$ in Ω.

Conclusion

$$\operatorname{Ker} \mathcal{A}(0,0) = \operatorname{Vect}\left\{ \begin{pmatrix} \phi_e \\ \phi_e \end{pmatrix} \right\}.$$

Interlude: Complex analysis

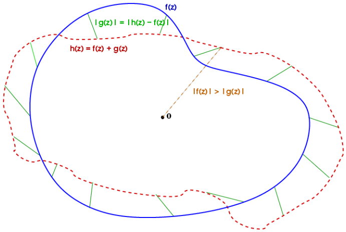
If $f(z)$ is analytic with $f(\lambda) = 0$ and $f(z) \neq 0$ for all $z \in \mathcal{B}(\lambda, r) \setminus \{\lambda\}$, then

$$\frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{f'(z)}{f(z)} \mathrm{d}z = \sharp \{\text{zeros of } f \text{ in } \mathcal{B}(\lambda, r)\} = 1, \quad \text{and} \quad \frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{f'(z)}{f(z)} z\mathrm{d}z = \lambda.$$

Theorem (Rouché's Theorem)

Let f be as before. Then, for all g analytic such that $|\frac{g}{f}| < 1$ on $\mathscr{C}(\lambda, r)$, it holds that $f + g$ has a unique zero λ_{f+g} in $\mathcal{B}(\lambda, r)$, and

$$\lambda_{f+g} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{(f+g)'(z)}{(f+g)(z)} z\mathrm{d}z.$$



Complex analysis: operator version

If $A(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is an analytic map of Fredholm operators (of index 0) such that

- For all $z \in \mathcal{B}(\lambda, r) \setminus \{\lambda\}$, $\quad \dim \mathrm{Ker} A(z) = \dim \mathrm{Ker} A^*(z) = 0$;
- $\dim \mathrm{Ker} A(\lambda) = 1, \qquad$ (hence $\dim \mathrm{Ker} A^*(z) = 1$),

then

$$1 = \frac{1}{2\mathrm{i}\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{A(z)} A'(z) \mathrm{d}z \right] \quad \text{and} \quad \lambda = \frac{1}{2\mathrm{i}\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{A(z)} A'(z) z \mathrm{d}z \right].$$

Remarks

- If $A : \mathcal{H}_1 \to \mathcal{H}_2$, then $A^{-1}A' : \mathcal{H}_1 \to \mathcal{H}_1$. The notion of trace exists.
- The operators A^{-1} and A' may not commute. However, $\mathrm{Tr}_{\mathcal{H}_1}(A^{-1}A') = \mathrm{Tr}_{\mathcal{H}_2}(A'A^{-1})$.

Theorem (Operator version of Rouché: Gohberg-Sigal theorem[1])

For all operator-valued analytic map $B(z) : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\|A^{-1}B\|_{\mathscr{B}(\mathcal{H}_1)} < 1$ on $\mathscr{C}(\lambda, r)$, then the operator $A + B$ is Fredholm of index 0, and there exists a unique point $\lambda_{A+B} \in \mathcal{B}(\lambda, r)$ such that

$$\dim \mathrm{Ker}(A+B)(\lambda_{A+B}) = 1 \qquad (= 0 \textit{ otherwise}).$$

Moreover,

$$\lambda_{A+B} = \frac{1}{2\mathrm{i}\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{(A+B)(z)} (A+B)'(z) z \mathrm{d}z \right].$$

[1] U. Gohberg, E.I. Sigal, Sbornik: Mathematics 13.4 (1971).

In our case

- We see the contrast δ as the complex variable (z), and ω as the perturbation parameter.
- For all ω, $\mathcal{A}(\omega,\cdot)$ is analytic in δ.
- For $\omega = 0$, $\mathcal{A}(0,0)$ is non invertible.

The operators $\mathcal{A}(0,\delta)$

$$\mathcal{A}(0,\delta) := \begin{pmatrix} \mathcal{S} & -\mathcal{S} \\ \mathcal{K}^* - \frac{1}{2} & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 \\ 0 & -\left(\mathcal{K}^* + \frac{1}{2}\right) \end{pmatrix}.$$

Invertible?

$$\mathcal{A}(0,\delta)\begin{pmatrix}\psi_b \\ \psi\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases} \mathcal{S}[\psi_b - \psi] = 0, \\ \left(\mathcal{K}^* - \dfrac{1}{2}\right)[\psi_b] = \delta\left(\mathcal{K}^* + \dfrac{1}{2}\right)[\psi] \end{cases} \iff \begin{cases} \psi = \psi_b, \\ \mathcal{K}^*[\psi] = \dfrac{1}{2}\left(\dfrac{1+\delta}{1-\delta}\right)\psi. \end{cases}$$

It holds that $\frac{1}{2}$ is an <u>isolated</u> eigenvalue of $\mathcal{K}^*$.

We deduce that there exists $\delta^* > 0$ such that

$$\forall \delta \in \mathbb{C},\ |\delta| \le \delta^*,\ \delta \neq 0, \quad \mathrm{Ker}\,\mathcal{A}(0,\delta) = \left\{ \begin{pmatrix}0\\0\end{pmatrix} \right\}.$$

Key point

$\mathcal{A}(0,\delta)$ is non invertible only for $\delta = 0$ in $\mathcal{B}(0,\delta^*)$.

Fredholm? We need the adjoint of $\mathcal{K}^* : H^{-1/2} \to H^{-1/2}$.

Problem: The inner product of $H^{-1/2}$ is not explicit.

Lemma (classical, new definition of Hilbert spaces)

i) *Let $\psi \in H^{-1/2}$ and set $u := \widetilde{\mathcal{S}}[\Psi]$. We have*

$$\langle \psi, -\mathcal{S}[\psi] \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega \cup (\mathbb{R}^3 \setminus \overline{\Omega})} |\nabla u|^2.$$

ii) *The space $\mathcal{H}^- := H^{-1/2}$ is a Hilbert space (equivalent to $H^{-1/2}$) when endowed with the norm*

$$\|\psi\|^2_{\mathcal{H}^-} := \langle \psi, -\mathcal{S}[\psi] \rangle_{H^{-1/2}, H^{1/2}}.$$

iii) *The space $\mathcal{H}^+ := H^{1/2}$ is a Hilbert space (equivalent to $H^{1/2}$) when endowed with the norm*

$$\|\phi\|^2_{\mathcal{H}^+} := \langle -\mathcal{S}^{-1}[\phi], \phi \rangle_{H^{-1/2}, H^{1/2}}.$$

iv) *The operator $\mathcal{S}$ is unitary from $\mathcal{H}^-$ to $\mathcal{H}^+$. In particular, $\mathcal{S}^* = \mathcal{S}^{-1}$.*

v) *(Calderón's identity) The operator $\mathcal{K}^*$ is compact self-adjoint on $\mathcal{H}^-$.*

Fact

$$\mathrm{Ker}\mathcal{A}(0,0)^* = \left\{ \begin{pmatrix} 0 \\ \phi_e \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{Ker}\mathcal{A}(0, \delta \neq 0)^* = \{0\}.$$

We can apply Gohberg-Sigal theorem!

We consider $\omega \neq 0$ as a perturbation of the $\omega = 0$ case.

Green's function (bis)

$$G^k(\mathbf{x}) : -\frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|} = -\frac{1}{4\pi|\mathbf{x}|}\left(1 + \mathrm{i}k|\mathbf{x}| + \frac{(\mathrm{i}k|\mathbf{x}|)^2}{2} + \cdots\right) = G^0(\mathbf{x}) + kG_1(\mathbf{x}) + k^2 G_2(\mathbf{x}) + \cdots$$

Single-layer potential (bis)

$$\mathcal{S}^k[\psi](\mathbf{x}) = \int_D \left(G^0 + kG_1 + \cdots\right)(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}) = \mathcal{S}[\psi](\mathbf{x}) + k\mathcal{S}_1[\psi](\mathbf{x}) + \cdots$$

Dirichlet-to-Neumann (bis)

$$\mathcal{K}^{k,*} = \mathcal{K}^* + k\mathcal{K}_1 + k^2\mathcal{K}_2 + \cdots$$

The operator $\mathcal{A}_\omega$

$$\mathcal{A}_\omega := \mathcal{A}(\omega,\cdot) = \mathcal{A}_0 + \omega\mathcal{A}_1 + \omega^2\mathcal{A}_2 + \cdots$$

Theorem (H. Ammari, DG, B. Fitzpatrick, H. Lee, H. Zhang)

For ω small enough, there exists a unique $\delta_\omega \in \mathcal{B}(0,\delta)$ such that $\mathcal{A}(\omega,\delta_\omega)$ is non invertible. Moreover, the map $\omega \to \delta_\omega$ is analytic, and

$$\begin{aligned}\delta_\omega &= \frac{1}{2\mathrm{i}\pi}\mathrm{Tr}_{\mathcal{H}^{--}}\left[\oint_{\mathscr{C}(0,\delta)} \frac{1}{\mathcal{A}_\omega(\delta)}\frac{\partial\mathcal{A}_\omega}{\partial\delta}(\delta)\delta\mathrm{d}\delta\right] \\ &= \left(\frac{|\Omega|}{v_b^2\mathrm{Cap}_\Omega}\right)\omega^2 + \left(\frac{\mathrm{i}|\Omega|}{4\pi v_b^2 v}\right)\omega^3 + O(\omega^4).\end{aligned}$$

Remark: The result holds for all shapes of bubbles.

$$\boxed{\delta_\omega = \left(\frac{|\Omega|}{v_b^2 \mathrm{Cap}_\Omega}\right)\omega^2 + \left(\frac{\mathrm{i}|\Omega|}{4\pi v_b^2 v}\right)\omega^3 + O(\omega^4).}$$

Capacity

$$\mathrm{Cap}_\Omega := \|\phi_e\|^2_{\mathcal{H}^-} = \|\mathbb{1}_{\partial\Omega}\|^2_{\mathcal{H}^+} \quad = \langle -S^{-1}\left[\mathbb{1}_{\partial\Omega}\right], \mathbb{1}_{\partial\Omega}\rangle^2_{H^{-1/2},H^{1/2}} \quad (>0).$$

For the sphere S_R of radius R, $\mathrm{Cap}_{S_R} = 4\pi R$.

Inverse formula: $\delta \to \omega_\delta$

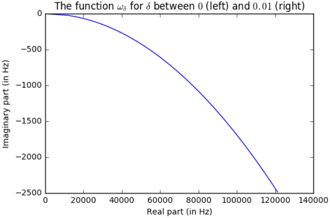
$$\omega_\delta = \left(\frac{\mathrm{Cap}_\Omega v_b^2}{|\Omega|}\right)^{1/2}\sqrt{\delta} - \mathrm{i}\left(\frac{\mathrm{Cap}_\Omega^2 v_b^2}{8\pi v|\Omega|}\right)\delta + O(\delta^{3/2}).$$

Leading order

For a sphere, $\omega_\delta = \omega_M$. We recover Minnaert's result.

Second order:

Purely imaginary $\Longrightarrow$ Dissipative term $\equiv$ Radiative damping.



Remarks

- The resonance is very close to the real-line, even in physical situations.
- We obtain a resonance phenomenon, and a damping effect, from **ab initio** principles.
- It corresponds to the so-called breathing mode.

The point scatterer approximation

What happens to a (fix) pressure wave (fix ω) with a small bubble $\Omega^\varepsilon = \varepsilon\Omega$ as $\varepsilon \to 0$? How much is the resonant mode excited?

Initial problem

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega^\varepsilon}, \\ \left(\Delta + k_b^2\right) u = 0 & \text{in} \quad \Omega^\varepsilon, \\ u_+ = u_- & \text{on} \quad \partial\Omega^\varepsilon, \\ \delta \frac{\partial u}{\partial \nu}\big|_+ = \frac{\partial u}{\partial \nu}\big|_- & \text{on} \quad \partial\Omega^\varepsilon, \\ u^s = u - u^{\mathrm{in}} & \text{satisfies Sommerfeld.} \end{cases}$$

Regime?

$\boxed{\text{Limit 1: } \varepsilon \to 0}$ $\boxed{\text{Limit 2: } \delta \to 0.}$

Idea: We know that $\varepsilon_M \approx \sqrt{\delta}$. Fix $\mu > 0$, and

$$\boxed{\text{Limit: } \varepsilon \to 0 \text{ and } \delta = \mu\varepsilon^2.}$$

Minnaert resonance

$$\mu_M := \frac{|\Omega| k_b^2}{\mathrm{Cap}_\Omega}.$$

Theorem (H. Ammari, DG, B. Fitzpatrick, H. Lee, H. Zhang)

If $\mathbf{0} \in D$, then the solution $u^\varepsilon := u[\varepsilon, \delta = \mu\varepsilon^2]$ satisfies

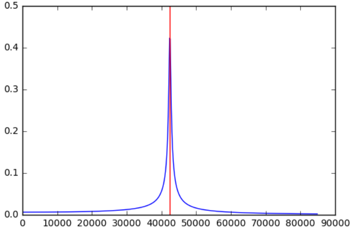
$$u^\varepsilon(\mathbf{x}) = u^{\mathrm{in}}(\mathbf{x}) + \begin{cases} \varepsilon \left(\dfrac{\mathrm{Cap}_\Omega}{1 - \frac{\mu_M}{\mu}} u^{\mathrm{in}}(\mathbf{0}) \right) G^k(\mathbf{x}) + O_\mu(\varepsilon^2) & \textit{if} \quad \mu \neq \mu_M, \\ \left(\mathrm{i} \dfrac{4\pi}{k} u^{\mathrm{in}}(\mathbf{0}) \right) G^k(\mathbf{x}) + O(\varepsilon) & \textit{if} \quad \mu = \mu_M. \end{cases}$$

Loosely speaking, $u^s = u - u^{\text{in}}$ satisfies

$$u^s(\mathbf{x}) \approx u^{\text{in}}(\mathbf{0}) g_s(\omega) G^k(\mathbf{x} - \mathbf{0}), \quad \text{with} \quad g_s(\omega) := \frac{\mathrm{Cap}_\Omega}{\left(1 - \frac{\omega^2}{\omega_M^2}\right) - \mathrm{i}\frac{\mathrm{Cap}_\Omega \omega}{4\pi v}} \quad \text{(response function).}$$

Example For a bubble of radius 0.5 mm, we get

The function $g_s(\omega)$ for ω between 0 and $2\omega_M$.



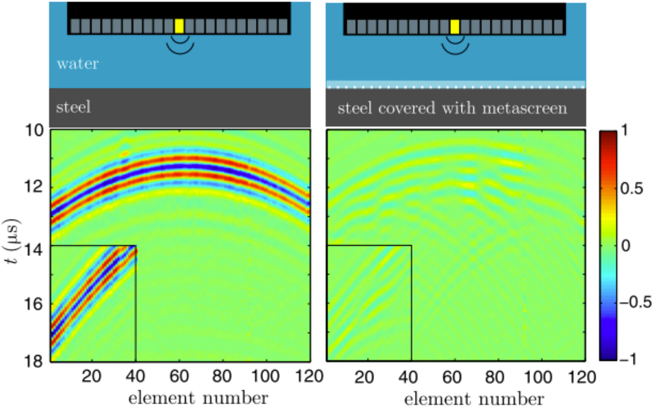
Remarks

- The imaginary part in the denominator of g_s is the radiative damping.
- The poles of g_s are in the lower half complex plane: from Titchmarsh's theorem, g_s is a causal response function.
- Monopole point scatterer: We only use the value $u^{\text{in}}(\mathbf{0})$, and not $\nabla u^{\text{in}}(\mathbf{0})$ (dipole scatterer).
- We recover the expression found in [1] for g_s.

[1] M. Devaud, Th. Hocquet, J.-C. Bacri, and V. Leroy. Eur. J. Phys., 29(6):1263, 2008.

The periodic case: the periodic Minnaert resonance

Experiment[2]

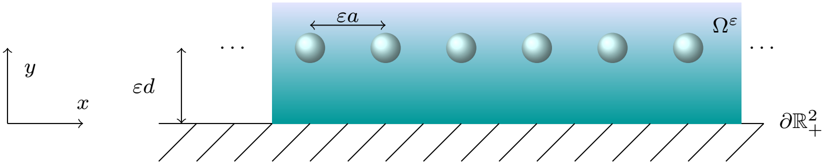


[2] V. Leroy, A. Strybulevych, M. Lanoy, F. Lemoult, A. Tourin, J.H. Page, Phys. Rev. B 91, 020301(R) (2015).

We now set bubbles on a $(d-1)$ dimensional lattice $\mathcal{R}$, on top of a Dirichlet surface.

Bubbles domain

$$\Omega^\varepsilon := \bigcup_{\mathbf{R}\in\mathcal{R}} \varepsilon\left(\Omega + \mathbf{R}\right).$$



In this talk

- two-dimensional: $\mathcal{R} = a\mathbb{Z}$.
- $U^{\rm in}(x,y) := u_0 \mathrm{e}^{-iky}$ with $k > 0$ **fixed** ($\Longrightarrow$ incoming plane-wave orthogonal to the plane).

The problem is $\mathcal{R}$-periodic in the x direction!

Scattering problem

$$\left\{\begin{array}{ll}
\left(\Delta + k^2\right) U^\varepsilon = 0 & \text{on} \quad \mathbb{R}^d_+ \setminus \overline{\Omega^\varepsilon}, \\
\left(\Delta + k_b^2\right) U^\varepsilon = 0 & \text{on} \quad \Omega^\varepsilon, \\
U^\varepsilon|_+ = U^\varepsilon|_- & \text{on} \quad \partial\Omega^\varepsilon, \\
\partial_\nu U^\varepsilon|_- = \delta \partial_\nu U^\varepsilon|_+ & \text{on} \quad \partial\Omega^\varepsilon, \\
U^s := U^\varepsilon - U^{\rm in} & \text{satisfies the outgoing radiation condition,} \\
U^\varepsilon = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \\
U^\varepsilon(x+\varepsilon\mathcal{R}, y) = U(x,y). &
\end{array}\right.$$

$\left.\begin{array}{l} \\ \\ \end{array}\right\}$ boundary conditions

Regime?

$\boxed{\text{Limit 1: } \varepsilon \to 0.}$ $\boxed{\text{Limit 2: } \delta \to 0.}$

Limit problem (after rescaling $u(\mathbf{x}) := U(\mathbf{X}/\varepsilon)$).

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \mathbb{R}^d_+ \setminus \overline{\Omega}, \\ \Delta u = 0 \quad \text{on} \quad \Omega, \\ u|_+ = u|_- \quad \text{on} \quad \partial\Omega, \\ \partial_\nu u|_- = 0 \quad \text{on} \quad \partial\Omega, \\ + \quad \text{boundary conditions.} \end{cases}$$

Again, two decoupled problems: there exists a non trivial solution (with $u|_\Omega = 1$).

Similar to the single bubble case (existence + tracking of the resonance).

Periodic Green's function? Solution to $(\Delta + k^2)G_\sharp^k(\mathbf{x}) = \sum_{l\in\mathbb{Z}} \delta_l(\mathbf{x})$.

For $k = 0$

$$G_\sharp^0(\mathbf{x}) = G_\sharp^0(x,y) = \frac{|y|}{2a} - \sum_{l\in\frac{2\pi}{a}\mathbb{Z}^*} \frac{1}{2al}\mathrm{e}^{\mathrm{i}lx}\mathrm{e}^{-|ly|}.$$

- **First part:** 1d Green's function $\Longrightarrow$ propagative mode,
- **Second part:** Exponentially decreasing away from the plane ($y \to \infty$) $\Longrightarrow$ evanescent modes.

For small k, $k \neq 0$

$$G_\sharp^k(\mathbf{x}) = G_\sharp^k(x,y) = \frac{\mathrm{e}^{-\mathrm{i}k|y|}}{2\mathrm{i}ka} - \sum_{l\in\frac{2\pi}{a}\mathbb{Z}^*} \frac{1}{2al\sqrt{l^2-k^2}}\mathrm{e}^{2\mathrm{i}\pi\frac{x}{a}}\mathrm{e}^{-\sqrt{l^2-k^2}|y|}.$$

Periodic-Dirichlet Green's function

$$G_+^k(\mathbf{x};\mathbf{x}') = G_+^k(x,y;x',y') = G_\sharp^k(x-x',y-y') - G_\sharp^k(x-x',y+y').$$

Remarks

- «mirror image» to enforce Dirichlet conditions.
- $G_+^k(\mathbf{x};\mathbf{x}')$ is not translation invariant ($G_+^k(\mathbf{x};\mathbf{x}') \neq G_+^k(\mathbf{x}-\mathbf{x}')$).
- $G_\sharp^k(\mathbf{x})$ has a $\frac{1}{k}$ singularity as $k \to 0$ (problematic for our limit $k \to 0$).
- The Dirichlet condition removes this $\frac{1}{k}$ singularity.

Layer potentials

As before: $\widetilde{\mathcal{S}_{+}^{k}}, \mathcal{S}_{+}^{k}, \mathcal{K}_{+}^{k}$, and $\phi_{e,+} := [\mathcal{S}_{+}]^{-1}(\mathbb{1}_{\partial\Omega})$.

Periodic capacity

$$\mathrm{Cap}_{\Omega,\mathcal{R}}^{+} := \left\langle \mathbb{1}_{\partial\Omega}, [-\mathcal{S}_{+}]^{-1}(\mathbb{1}_{\partial\Omega}) \right\rangle \quad = -\int_{\partial\Omega} \phi_{e,+}(\mathbf{x}')\mathrm{d}\sigma(\mathbf{x}').$$

Periodic Minnaert resonance

$$\boxed{\omega_M^{+} := \left(\frac{\mathrm{Cap}_{\Omega,\mathcal{R}}^{+} v_b^2}{|\Omega|} \right)^{1/2} \sqrt{\delta}.}$$

Remarks:

- Similar expression: we had $\omega_M = \left(\frac{\mathrm{Cap}_{\Omega} v_b^2}{|\Omega|} \right)^{1/2} \sqrt{\delta}.$
- The periodic capacity depends on the lattice.

Meta-surfaces and high-contrast homogenisation

How much is the resonant mode excited when we send a (fix) incoming wave U^{in}?

Following the one bubble case, we expect a «meta-surface approximation» of the form:

$$U^s(y) = \underbrace{U^0(y=0)}_{\text{Solution without bubbles}} \quad \underbrace{g_{s,+}(\omega)}_{\text{Response function}} \quad \underbrace{G_+^k(y; y'=0)}_{\text{Green's function between plane and } y} .$$

Problem

Without bubbles, the solution is

$$U^0(x,y) = U^{\text{in}}(x,y) - U^{\text{in}}(x,-y) = u_0 \mathrm{e}^{-\mathrm{i}ky} - u_0 \mathrm{e}^{\mathrm{i}ky} = -2\mathrm{i}u_0 \sin(ky).$$

In particular, $U^0(x, y=0) = 0$: the monopole mode is not excited at first order.

Solution

Need next order: dipole approximation ($U^0(x, y \approx 0) \approx -2\mathrm{i}u_0 ky$).

Monopole approximation	Dipole approximation
Internal problem $\mathbb{1}_\Omega$ solution to $\Delta \mathbb{1}_\Omega = 0$ and $\partial_\nu \mathbb{1}_\Omega = 0$.	**Internal problem** y solution to $\Delta y = 0$ and $\partial_\nu y = \nu_y$.
External problem α_{monop} solution to $\left\{\begin{array}{ll} \Delta \alpha_{\text{monop}} = 0 & \text{on } \mathbb{R}^d_+ \setminus \overline{\Omega}, \\ \alpha_{\text{monop}}\|_+ = \mathbb{1}_{\partial\Omega} & \text{on } \partial\Omega, \\ + \text{ Boundary conditions.} & \end{array}\right.$	**External problem** α_{dip} solution to $\left\{\begin{array}{ll} \Delta \alpha_{\text{dip}} = 0 & \text{on } \mathbb{R}^d_+ \setminus \overline{\Omega}, \\ \alpha_{\text{dip}}\|_+ = y_{\partial\Omega} & \text{on } \partial\Omega, \\ + \text{ Boundary conditions.} & \end{array}\right.$

Monopole and dipole solutions (bis)

$$\alpha_{\text{monop}} = \widetilde{\mathcal{S}_+}\left(\phi_{e,+}\right) \quad = \widetilde{\mathcal{S}_+}\left(\left[\mathcal{S}_+\right]^{-1}\left(\mathbb{1}_{\partial\Omega}\right)\right), \quad \text{and} \quad \alpha_{\text{dip}} = \widetilde{\mathcal{S}_+}\left(\left[\mathcal{S}_+\right]^{-1}\left(y_{\partial\Omega}\right)\right).$$

Asymptotics of the Green's function ($y \to \infty$)

$$G_+^0(\mathbf{x};\mathbf{x}') = -\frac{y'}{a} \quad + \text{evanescent modes}.$$

Asymptotics ($y \to \infty$)

$$\alpha_{\text{monop}}(x,y) = \alpha_{\text{monop}}^\infty + \text{ evanescent modes}, \quad \text{with} \quad \alpha_{\text{monop}}^\infty := -\int_{\partial\Omega} \frac{y'}{a}\phi_{e,+}(\mathbf{x}')\mathrm{d}\sigma(\mathbf{x}'),$$

$$\alpha_{\text{dip}}(x,y) = \alpha_{\text{dip}}^\infty + \text{ evanescent modes}.$$

Regime? We expect $\delta \approx \varepsilon^2$. Fix $\mu > 0$, and

$$\boxed{\text{Limit: } \varepsilon \to 0 \text{ and } \delta = \mu\varepsilon^2.}$$

Resonance

$$\mu_M := \frac{|\Omega| k_b^2}{\text{Cap}_{\Omega,\mathcal{R}}^+}.$$

Norm? Strip $S_a := \mathbb{R} \times (a,\infty)$ and

$$\|f\|_{W^{1,\infty}(S_a)} := \sup_{\mathbf{X}\in S_a} |f|(\mathbf{X}) + \sup_{\mathbf{X}\in S_a} |\nabla f|(\mathbf{X}).$$

Theorem (H. Ammari, DG, B. Fitzpatrick, H. Lee, H. Zhang)

1) If $\mu \neq \mu_M$, then $U^\varepsilon := U[\varepsilon, \delta = \mu\varepsilon^2]$ satisfies uniformly in $W^{1,\infty}(S_{\varepsilon L})$

$$U^\varepsilon(\mathbf{X}) = U^0(\mathbf{X}) + \varepsilon\left(U_1(\mathbf{X}) + U_{\mathrm{BL}}\left(\mathbf{X}, \frac{\mathbf{X}}{\varepsilon}\right)\right) + O_\mu(\varepsilon^2),$$

where $U^0(\mathbf{X}) = -2\mathrm{i}u_0 \sin(kY)$ is the solution without bubbles, and where

$$U_1(\mathbf{X}) := (2\mathrm{i}u_0 k)\mathrm{e}^{\mathrm{i}kY}\left(\alpha_{\mathrm{dip}}^\infty - \frac{K}{1-\frac{\mu_M}{\mu}}\alpha_{\mathrm{monop}}^\infty\right), \quad \textit{with} \quad K := \frac{\alpha_{\mathrm{monop}}^\infty a}{\mathrm{Cap}_{\Omega,\mathcal{R}}^+}.$$

$$U_{\mathrm{BL}}(\mathbf{X},\mathbf{x}) := (2\mathrm{i}u_0 k)\left((\alpha_{\mathrm{dip}}(\mathbf{x}) - \alpha_{\mathrm{dip}}^\infty) - \frac{K}{1-\frac{\mu_M}{\mu}}(\alpha_{\mathrm{monop}}(\mathbf{x}) - \alpha_{\mathrm{monop}}^\infty)\right).$$

Remarks

- Uniform bounds in $S_{\varepsilon L}$ (boundary limit terms U_{BL}).
- $U_{\mathrm{BL}}(\mathbf{X},\mathbf{x})$ is exponentially decreasing as $y \to \infty$.
- The dipole and monopole terms are of same order of magnitude.
- Only the monopole part is resonant (singularity $\mu \to \mu_M$).

Theorem (bis)

2) If $\mu = \mu_M$, then $U^\varepsilon := U[\varepsilon, \delta = \mu_M \varepsilon^2]$ satisfies uniformly in $W^{1,\infty}(S_{\varepsilon L})$

$$U^\varepsilon(\mathbf{X}) = U^0(\mathbf{X}) + \left(U_1(\mathbf{X}) + U_{\mathrm{BL}}\left(\mathbf{X}, \frac{\mathbf{X}}{\varepsilon}\right)\right) + O(\varepsilon),$$

where $U_1(\mathbf{X}) := 2u_0 \mathrm{e}^{\mathrm{i}kY}$ and where

$$U_{\mathrm{BL}}(\mathbf{X}, \mathbf{x}) := (2u_0)\left(\frac{\alpha_{\mathrm{monop}}}{\alpha_{\mathrm{monop}}^\infty}(\mathbf{x}) - 1 \right) \quad \textit{is exponentially decreasing as } y \to \infty.$$

Interpretation

The meta-screen behaves like an acoustic plane with reflection coefficient

$$R(\omega) \approx -1 - 2\left(\frac{\mathrm{i}\omega\eta}{1 - \left(\frac{\omega}{\omega_M^+}\right)^2 - \mathrm{i}\omega\eta^*} \right) \quad \text{with} \quad \eta = \eta^* := \frac{(\alpha_{\mathrm{monop}}^+)^2 a}{v\mathrm{Cap}_{\Omega,\mathcal{R}}^+}.$$

Remarks:

- We recover the radiative damping (η^*).
- If $\omega \ll \omega_M$ or $\omega \gg \omega_M$, then $R(\omega) \approx -1$ (Dirichlet plane) $\sim$ no bubble case.
- If $\omega = \omega_M$, then $R(\omega_M) = 1$ (Neumann plane).
- Considering other source of damping (*e.g.* viscous), and assuming $\eta^* = 2\eta$, we have

$$R(\omega_M) = 0 \quad \text{(absorption plane)}.$$

Conclusions

- Regime $\varepsilon \to 0$ and $\delta \to 0$ such that $\delta \approx \varepsilon^2$ (high-contrast limit[3]).
- Tracking of the resonance through Gohberg-Sigal theory.
- Point scatterer approximation and meta-surfaces from the study of layer potentials.
- Resonance phenomenon as the limit of well-posed and easy-to-study problems.

Bibliography

- H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang, arXiv:1603.03982 (single bubble).
- H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang, arXiv:1608.02733 (meta-surface) (accepted in SIAM J. Appl. Math.).

Thank you for your attention.

[3] cf. also works by G. Bouchitté and D. Felbacq