

FERMIONS, BOSONS AND POINT PROCESSES

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1. INTRODUCTION

We draw N identical particles on \mathbb{R}^d with some probabilistic rule, and we ask the following question:

What is the probability that there is a particle around \mathbf{x}_1 , a particle around \mathbf{x}_2 , ..., a particle around \mathbf{x}_k .

We disregard where the remaining $(N - k)$ other particles are.

If the positions of the particles are drawn according to some probability measure with density

$$P_x(\mathbf{x}_1, \dots, \mathbf{x}_N),$$

then, the (density of the) previous probability is, up to some normalisation factor, the k -correlation function

$$\rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) := \frac{N!}{(N - k)!} \int_{(\mathbb{R}^d)^k} P_x(\mathbf{x}_1, \dots, \mathbf{x}_N) d\mathbf{x}_{k+1} \cdots d\mathbf{x}_N.$$

We choose this normalisation so that $\int \rho^{(k)} = N!/(N - k)! = k!$. For instance, $\int \rho^{(1)} = N$ is the number of particles in the system, and $\int \rho^{(1)} \mathbb{1}_K$ is the mean number of particles in a subset $K \subset \mathbb{R}^d$.

We assume here that the particles are identical. Mathematically speaking, this means that, for all permutation $\sigma \in \mathcal{S}_N$,

$$P_x(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}) = P_x(\mathbf{x}_1, \dots, \mathbf{x}_N).$$

In particular, we have

$$\rho^{(k)}(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(k)}) = \rho_k(\mathbf{x}_1, \dots, \mathbf{x}_k).$$

The goal of these notes is to describe the correlation functions for systems of N non interacting **quantum** particles. A conference on the topic will take place soon in Lille (<https://dpp-fermions.sciencesconf.org/>).

2. THE WAVE FUNCTION AND THE HAMILTONIAN

In a physical system, we expect a system of N particles to lower their total energy. In these notes, we focus on systems where the particles are **non interacting**, and at zero temperature. For **classical** particles (\neq quantum particles), the energy would be of the form

$$\mathcal{E}(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{k}_1, \dots, \mathbf{k}_N) := \sum_{i=1}^N \left(\frac{1}{2} \mathbf{k}_i^2 + V(\mathbf{x}_i) \right).$$

Here \mathbf{x}_i is the position of the i -th particle, and \mathbf{k}_i is its momentum. The first term correspond to the kinetic energy, and the second term to the potential energy. Here, V is some given fixed external potential.

The problem with classical particles is that the minimum is attained for $\mathbf{k}_i = \mathbf{0}$ and $\mathbf{x}_i \in \text{argmin } V$. The corresponding point process is deterministic: all the N particles will lie in the minimum of V . This model is not satisfactory: if $\min V = -\infty$ (as it is the case for the hydrogen atom $V(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$), the energy is infinite.

A solution to avoid this infinity is to minimise the energy under some constraints. For instance, one could minimise the **mean energy** of the particles, under some constraint on the probability measures. We now minimise

$$\langle \mathcal{E} \rangle := \int_{(\mathbb{R}^d)^N} \left(\sum_{i=1}^N \frac{1}{2} \mathbf{k}_i^2 \right) P_k(\mathbf{k}_1, \dots, \mathbf{k}_N) d\mathbf{k}_1 \cdots d\mathbf{k}_N + \int_{(\mathbb{R}^d)^N} \left(\sum_{i=1}^N V(\mathbf{x}_i) \right) P_x(\mathbf{x}_1, \dots, \mathbf{x}_N) d\mathbf{x}_1 \cdots d\mathbf{x}_N,$$

where $P_k(\mathbf{k}_1, \dots, \mathbf{k}_N)$ is (the density of) the probability that the particle in \mathbf{x}_1 has momentum \mathbf{k}_1 , the particle in \mathbf{x}_2 has momentum \mathbf{k}_2 , and so on. In quantum mechanics, we make the following postulate.

Definition 2.1 (Wave function). *For quantum particles, there is function $\Psi \in L^2(\mathbb{R}^{dN})$, called the wave-function such that*

$$|\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)|^2 = P_x(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad \text{and} \quad |\widehat{\Psi}(\mathbf{k}_1, \dots, \mathbf{k}_N)| = P_k(\mathbf{k}_1, \dots, \mathbf{k}_N),$$

where $\widehat{\Psi}$ is the normalised Fourier transform of Ψ :

$$\widehat{\Psi} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Psi(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}.$$

Remark 2.2. *The wave function Ψ is defined up to a global phase.*

Using the Fourier identity

$$\int_{\mathbb{R}^d} \mathbf{k}_i^2 |\widehat{\Psi}(\mathbf{k}_1, \dots, \mathbf{k}_N)|^2 d\mathbf{k}_1 \dots d\mathbf{k}_N = \int_{\mathbb{R}^d} |\nabla_{\mathbf{x}_i} \Psi|^2(\mathbf{x}_1, \dots, \mathbf{x}_N) d\mathbf{x}_1 \dots d\mathbf{x}_N = \langle \Psi, -\Delta_{\mathbf{x}_i} \Psi \rangle_{L^2((\mathbb{R}^d)^N)},$$

we end up with a minimisation problem on the set of wave-functions Ψ , of the form

$$\min \left\{ \langle \Psi, H \Psi \rangle_{L^2((\mathbb{R}^d)^N)}, \Psi \in L^2((\mathbb{R}^d)^N), \|\Psi\|_{L^2((\mathbb{R}^d)^N)} = 1 \right\}, \quad \text{with} \quad H = \sum_{i=1}^N -\frac{1}{2} \Delta_{\mathbf{x}_i} + V(\mathbf{x}_i).$$

The operator H is called the *Hamiltonian*.

3. BOSONS AND FERMIONS

From the fact that the particles are indistinguishable, we have (we denote by \mathcal{S}_N the symmetric permutation group)

$$\forall \sigma \in \mathcal{S}_N, \quad |\Psi(\mathbf{x}_{\sigma(1)}, \dots, \Psi(\mathbf{x}_{\sigma(N)})| = |\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)|.$$

It is natural to look at the following two cases.

Definition 3.1 (bosons and fermions). *We say that the particles are bosons if Ψ is symmetric:*

$$\forall \sigma \in \mathcal{S}_N, \quad \Psi(\mathbf{x}_{\sigma(1)}, \dots, \Psi(\mathbf{x}_{\sigma(N)}) = \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N).$$

We say that the particles are fermions if Ψ is anti-symmetric

$$\forall \sigma \in \mathcal{S}_N, \quad \Psi(\mathbf{x}_{\sigma(1)}, \dots, \Psi(\mathbf{x}_{\sigma(N)}) = \varepsilon(\sigma) \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad (\text{Pauli principle}),$$

where $\varepsilon(\sigma)$ is the signature of the permutation $\sigma \in \mathcal{S}_N$.

Remark 3.2. *For a fermion, we have $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = 0$ whenever $\mathbf{x}_i = \mathbf{x}_j$ with $i \neq j$. The physical interpretation is that two fermions can never be at the same place. There is natural repulsion between fermions (even though they do not interact via any potential).*

Remark 3.3. *There might be other choices in dimensions 1 and 2. In dimension $d = 2$, one could have an arbitrary phase appearing. These (quasi-) particles are called anyons ("any" for any phase [Wil82]). For $d \geq 3$ however, only bosons and fermions can exist [LM77].*

The spin-statistics theorem states that, in dimension $d = 3$, the nature (boson or fermion) of particles is related to their spins (= intrinsic degree of freedom). Bosons correspond to particles with integer spin, while fermions have half-integer spin. In practice, it turns out that all "elementary" particles that constitute matter (electrons, neutrons, protons, ...) are fermions. Instead, all particles that propagate energy (or information) between electrons are bosons (photons, Higgs, ...).

Remark 3.4. *At a different scale, a set of (indistinguishable) atoms can behave either as fermions or bosons, depending on their internal structure. For instance ${}^4\text{He}$ (2 electrons, 2 neutrons, 2 protons) behave as a boson. This is the boson used in Bose-Einstein condensate experiments. It is the most common form of Helium.*

In the sequel, we denote by $L_a^2((\mathbb{R}^d)^N)$ the set of antisymmetric function

$$L_a^2((\mathbb{R}^d)^N) := \left\{ \Psi \in L^2((\mathbb{R}^d)^N), \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}) = \varepsilon(\sigma) \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) \right\}$$

and by $L_s^2((\mathbb{R}^d)^N)$ the set of symmetric function

$$L_s^2((\mathbb{R}^d)^N) := \left\{ \Psi \in L^2((\mathbb{R}^d)^N), \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}) = \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) \right\}$$

In the sequel, we look for the minimum quantum energy

$$E_{s/a}(N) := \min \left\{ \langle \Psi, H \Psi \rangle_{L^2((\mathbb{R}^d)^N)}, \Psi \in L_{s/a}^2((\mathbb{R}^d)^N), \|\Psi\|_{L^2((\mathbb{R}^d)^N)} = 1 \right\},$$

when the states are restricted to be fermions or bosons. The corresponding minimiser are called *ground states*.

4. BOSONIC CASE

We assume in this section that our particles are *bosons*. We are therefore looking for the minimal eigenvalue of H , on the vectorial space of symmetric functions. We assume in the sequel that

$$h := -\frac{1}{2}\Delta + V$$

is compact resolvent and bounded from below (say $V(\mathbf{x}) = |\mathbf{x}|^2$). Let $\lambda_1 < \lambda_2 < \dots$ be its eigenvalues (we assume for simplicity that they are all distinct), and let u_1, u_2, \dots be a corresponding orthonormal basis of eigenvectors.

Lemma 4.1. *The family of functions*

$$(u_{i_1} \otimes u_{i_2} \otimes u_{i_N})(\mathbf{x}_1, \dots, \mathbf{x}_N) := u_{i_1}(\mathbf{x}_1) u_{i_2}(\mathbf{x}_2) \dots u_{i_N}(\mathbf{x}_N), \quad (i_1, \dots, i_N) \in (\mathbb{N}^*)^N$$

form an orthonormal basis of $L^2((\mathbb{R}^d)^N)$. In addition, we have

$$\langle (u_{j_1} \otimes u_{j_2} \otimes u_{j_N}), H(u_{i_1} \otimes u_{i_2} \otimes u_{i_N}) \rangle = (\lambda_{i_1} + \dots + \lambda_{i_N}) \delta_{i_1 j_1} \dots \delta_{i_N j_N}.$$

From this Lemma and the fact that λ_1 is the smallest eigenvalue (we assume that $\lambda_1 < \lambda_2$), we directly see that the minimum energy is

$$E_s(N) = N\lambda_1,$$

and that the corresponding minimiser (or ground state) is

$$\Psi_s := \otimes^N u_1, \quad \text{or} \quad \Psi_s(\mathbf{x}_1, \dots, \mathbf{x}_N) = u_1(\mathbf{x}_1) \dots u_1(\mathbf{x}_N).$$

In particular, the probability P_x becomes

$$P_x(\mathbf{x}_1, \dots, \mathbf{x}_N) = |u_1|^2(\mathbf{x}_1) \dots |u_1|^2(\mathbf{x}_N).$$

This is the case of independent particles: the probability to find a particle at \mathbf{x} does not depend on the position of the other particles. This probability is simply given by the first eigenvector of $-\Delta + V$.

5. FERMIONIC CASE

We now look at the fermionic case. Let $\Pi_a : L^2((\mathbb{R}^d)^N) \rightarrow L_a^2((\mathbb{R}^d)^N)$ be the projector on antisymmetric functions:

$$(\Pi_a \Psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \varepsilon(\sigma) \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}),$$

and let

$$u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_N} := (\sqrt{N!}) \Pi_a (u_{i_1} \otimes u_{i_2} \otimes \dots \otimes u_{i_N}),$$

where the normalisation is chosen so that $\|u_{i_1} \wedge \dots \wedge u_{i_n}\|_{L^2} = 1$ or 0 (see later). This is also

$$u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_N} = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} \varepsilon(\sigma) u_{i_1}(\mathbf{x}_{\sigma(1)}) \dots u_{i_N}(\mathbf{x}_{\sigma(N)}) = \frac{1}{\sqrt{N!}} \det (u_{i_k}(\mathbf{x}_j))_{1 \leq k, j \leq N}.$$

Such states are sometimes called *Slater determinant*. If there is $i_k = i_l$ with $k \neq l$, then the corresponding state satisfies $u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_N} = 0$. Actually,

$$L_a^2((\mathbb{R}^d)^N) = \text{Vect} \{u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_N}, \quad 1 \leq i_1 < i_2 < \cdots < i_N\}.$$

We deduce that the minimum $E_a(N)$ is

$$E_a(N) = \lambda_1 + \lambda_2 + \cdots + \lambda_N,$$

and that the corresponding minimiser is

$$\Psi_a = u_1 \wedge u_2 \wedge \cdots \wedge u_N, \quad \text{or} \quad \Psi_a(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \frac{1}{\sqrt{N!}} \det (u_i(\mathbf{x}_j))_{1 \leq k, j \leq N}.$$

In quantum chemistry/physics, the functions u_k are called *orbitals*. Two different fermions must occupy two different orbitals. The fermionic ground-state corresponds to the case where the first N orbitals are occupied.

The corresponding density is

$$P_x(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \left(\frac{1}{\sqrt{N!}} \det (u_i(\mathbf{x}_j))_{1 \leq i, j \leq N} \right)^2.$$

We now compute the correlation functions for a point process with such law P_x . In order to do so, we need an important tool.

5.1. The one-body density matrix. We define the one-body density matrix $\gamma(\mathbf{x}, \mathbf{y}) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ with

$$\gamma(\mathbf{x}, \mathbf{y}) := N \int_{(\mathbb{R}^d)^{N-1}} \Psi_a(\mathbf{x}, \mathbf{x}_2, \cdots) \overline{\Psi_a(\mathbf{y}, \mathbf{x}_2, \cdots)} d\mathbf{x}_2 \cdots d\mathbf{x}_N.$$

Notice that $\rho^{(1)}(\mathbf{x}) = \gamma(\mathbf{x}, \mathbf{x})$. It turns out that γ has a simple expression.

Lemma 5.1. *We have*

$$\gamma(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^N u_i(\mathbf{x}) \overline{u_i(\mathbf{y})}.$$

Proof. We compute

$$\begin{aligned} \gamma(\mathbf{x}, \mathbf{y}) &= \frac{N}{N!} \sum_{\sigma \in \mathcal{S}_N} \sum_{\sigma' \in \mathcal{S}_N} \varepsilon(\sigma) \varepsilon(\sigma') u_{\sigma(1)}(\mathbf{x}) \overline{u_{\sigma'(1)}(\mathbf{y})} \times \\ &\quad \left(\int_{\mathbb{R}^d} u_{\sigma(2)}(\mathbf{x}_2) \overline{u_{\sigma'(2)}(\mathbf{x}_2)} d\mathbf{x}_2 \right) \cdots \left(\int_{\mathbb{R}^d} u_{\sigma(N)}(\mathbf{x}_N) \overline{u_{\sigma'(N)}(\mathbf{x}_N)} d\mathbf{x}_N \right). \end{aligned}$$

By orthonormality of the functions, each integral is 0 if $\sigma(i) \neq \sigma'(i)$, and is 1 else. We deduce that the only non-null terms are the ones for which $\sigma(2) = \sigma'(2)$, $\sigma(3) = \sigma'(3)$ and so on. This implies $\sigma = \sigma'$. We therefore obtain

$$\gamma(\mathbf{x}, \mathbf{y}) = \frac{1}{(N-1)!} \sum_{\sigma \in \mathcal{S}_N} u_{\sigma(1)}(\mathbf{x}) \overline{u_{\sigma(1)}(\mathbf{y})} = \sum_{i=1}^N u_i(\mathbf{x}) \overline{u_i(\mathbf{y})}.$$

□

We can see $\gamma(\mathbf{x}, \mathbf{y})$ as the kernel of an operator γ acting on the one-particle space $L^2(\mathbb{R}^d)$. The corresponding operator corresponds to the orthogonal projection on $\text{Vect}\{u_1, \cdots, u_N\}$, that is the projection on the occupied orbitals. In particular, γ is a self-adjoint rank- N projector. It satisfies $0 \leq \gamma \leq 1$.

5.2. Correlation functions and the determinantal point process. From the previous expression for γ , we directly deduce that

$$\rho^{(1)}(\mathbf{x}) = \sum_{i=1}^N |u_i|^2(\mathbf{x}).$$

Actually, we can generalise this formula, for all correlation functions. We now compute the k -correlation function $\rho^{(k)}(\mathbf{x})$. The main result of these notes is the following.

Proposition 5.2. *We have*

$$\rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \det(\gamma(\mathbf{x}_i, \mathbf{x}_j))_{1 \leq i, j \leq k}.$$

Remark 5.3. *Loosely speaking, the matrix $\gamma(\mathbf{x}_i, \mathbf{x}_j)$ is the Gram matrix of γ is the basis $(\delta_{\mathbf{x}_i})_{1 \leq i \leq k}$. Since γ is a positive operator, so is the corresponding Gram matrix, and the determinant is indeed positive.*

Proof. By definition, we have,

$$\begin{aligned} \rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) &= \frac{1}{(N-k)!} \sum_{\sigma \in \mathcal{S}_N} \sum_{\sigma' \in \mathcal{S}_N} \varepsilon(\sigma) \varepsilon(\sigma') u_{\sigma(1)}(\mathbf{x}_1) \overline{u_{\sigma'(1)}}(\mathbf{x}_1) \cdots u_{\sigma(k)}(\mathbf{x}_k) \overline{u_{\sigma'(k)}}(\mathbf{x}_k) \times \\ &\quad \left(\int_{\mathbb{R}^d} u_{\sigma(k+1)}(\mathbf{x}_{k+1}) \overline{u_{\sigma'(k+1)}}(\mathbf{x}_{k+1}) d\mathbf{x}_{k+1} \right) \cdots \left(\int_{\mathbb{R}^d} u_{\sigma(N)}(\mathbf{x}_N) \overline{u_{\sigma'(N)}}(\mathbf{x}_N) d\mathbf{x}_N \right). \end{aligned}$$

Again, the integrals vanish whenever there is $\sigma(j) \neq \sigma'(j)$ with $j \geq k+1$. The remaining terms correspond to the cases where $I_\sigma = I_{\sigma'}$ as sets, where $I_\sigma := \{\sigma(1), \dots, \sigma(k)\}$. Labelling the sum with the image $I = I_\sigma = I_{\sigma'} = \{i_1 < \dots < i_k\}$, we can see σ, σ' as k -permutation on the image I , and get

$$\begin{aligned} \rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k < N} \sum_{\sigma \in \mathcal{S}_k} \sum_{\sigma' \in \mathcal{S}_k} \varepsilon(\sigma) \varepsilon(\sigma') u_{i_{\sigma(1)}}(\mathbf{x}_1) \overline{u_{i_{\sigma'(1)}}}(\mathbf{x}_1) \cdots u_{i_{\sigma(k)}}(\mathbf{x}_k) \overline{u_{i_{\sigma'(k)}}}(\mathbf{x}_k) \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_k < N} \sum_{\sigma \in \mathcal{S}_k} \sum_{\sigma' \in \mathcal{S}_k} \varepsilon(\sigma) \varepsilon(\sigma') u_{i_1}(\mathbf{x}_{\sigma(1)}) \overline{u_{i_1}}(\mathbf{x}_{\sigma'(1)}) \cdots u_{i_N}(\mathbf{x}_{\sigma(N)}) \overline{u_{i_N}}(\mathbf{x}_{\sigma'(N)}), \end{aligned}$$

where we make the change of variable $\sigma \rightarrow \sigma^{-1}$ and $\sigma' \rightarrow (\sigma')^{-1}$. By setting $\sigma'' = \sigma' \sigma^{-1}$, this is also

$$\begin{aligned} \rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k < N} \sum_{\sigma \in \mathcal{S}_k} \left[\sum_{\sigma'' \in \mathcal{S}_k} \varepsilon(\sigma'') u_{i_1}(\mathbf{x}_{\sigma(1)}) \overline{u_{i_1}}(\mathbf{x}_{\sigma'' \sigma(1)}) \cdots u_{i_N}(\mathbf{x}_{\sigma(N)}) \overline{u_{i_N}}(\mathbf{x}_{\sigma'' \sigma(N)}) \right] \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_k < N} \sum_{\sigma \in \mathcal{S}_k} \det(u_{i_l}(\mathbf{x}_{\sigma(j)}))_{1 \leq l, j \leq k}. \end{aligned}$$

We now notice that

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} = \frac{1}{k!} \sum_{1 \leq i_1 \leq N} \sum_{1 \leq i_2 \leq N} \cdots \sum_{1 \leq i_k \leq N},$$

since the terms $i_k = i_l$ cancel. Plugging in the previous expression, we deduce that

$$\begin{aligned} \rho^{(k)}(\mathbf{x}_1, \dots, \mathbf{x}_k) &= \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \sum_{\sigma'' \in \mathcal{S}_k} \varepsilon(\sigma'') \left(\sum_{1 \leq i_1 \leq N} u_{i_1}(\mathbf{x}_{\sigma(1)}) \overline{u_{i_1}}(\mathbf{x}_{\sigma'' \sigma(1)}) \right) \cdots \left(\sum_{1 \leq i_k \leq N} u_{i_k}(\mathbf{x}_{\sigma(k)}) \overline{u_{i_k}}(\mathbf{x}_{\sigma'' \sigma(k)}) \right) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \sum_{\sigma' \in \mathcal{S}_k} \varepsilon(\sigma'') \gamma(\mathbf{x}_{\sigma(1)}, \mathbf{x}_{\sigma'' \sigma(1)}) \cdots \gamma(\mathbf{x}_{\sigma(k)}, \mathbf{x}_{\sigma'' \sigma(k)}) \\ &= \frac{k!}{k!} \sum_{\sigma'' \in \mathcal{S}_k} \varepsilon(\sigma'') \gamma(\mathbf{x}_1, \mathbf{x}_{\sigma''(1)}) \cdots \gamma(\mathbf{x}_k, \mathbf{x}_{\sigma''(k)}) \\ &= \det(\gamma(\mathbf{x}_i, \mathbf{x}_j))_{1 \leq i, j \leq k}. \end{aligned}$$

□

Definition 5.4. *A point process where the correlation functions are of the form of Proposition 5.2 is called a determinantal point process.*

6. ONE EXAMPLE: THE FERMIONIC GAS

We now focus on the example of the fermionic gas. We want to describe a infinite number of free fermions in \mathbb{R}^d with mean density ρ of particles. The density $\rho \in \mathbb{R}^+$ is the only parameter in the model. In order to handle infinite number of particles, we perform a thermodynamic limit: We consider N fermions in a torus of size L , denoted by $L\mathbb{T}^d$. The density in this case is NL^{-d} . At the end, we take the limit $N \rightarrow \infty$ and $L \rightarrow \infty$ with $NL^{-d} \rightarrow \rho$.

6.1. The Fermi wavevector. In our model, there is no external potential, and only the kinetic energy. The energy *per unit volume* of the system is

$$\mathcal{E}(L, V) := \frac{1}{L^d} \min \left\{ \left\langle \Psi, \left(\sum_{i=1}^N -\Delta_{\mathbf{x}_i} \right) \Psi \right\rangle, \quad \Psi \in L_a^2((L\mathbb{T}^d)^N) \right\}.$$

In Fourier space, we see that the eigenvalues of $-\Delta$ acting on $L^2(L\mathbb{T}^d)$ are $(p/L)^2$ with $p \in 2\pi\mathbb{Z}^d$, and the corresponding eigenvector is

$$u_{\mathbf{p}}^L(\mathbf{x}) := \frac{e^{i\frac{\mathbf{p}}{L} \cdot \mathbf{x}}}{L^{d/2}} =: e_{\mathbf{k}}(\mathbf{x}) \quad \text{with} \quad \mathbf{k} := \frac{\mathbf{p}}{L}.$$

The ground-state is the fermionic product of the functions $u_{\mathbf{p}}^L$, where the product runs for all \mathbf{p} with small enough module. Up to some boundary effect, we have

$$\Psi_a^L(\mathbf{x}) := \bigwedge_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} e_{\mathbf{k}}.$$

The radius k_F is chosen so that the density of the system converges to ρ at the limit. Actually, since $|e_{\mathbf{k}}| = 1$, the density of the system is

$$\forall \mathbf{x} \in L\mathbb{T}^d, \quad \rho^L(\mathbf{x}) = \sum_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} |e_{\mathbf{k}}(\mathbf{x})|^2 = \frac{1}{L^d} \sum_{\mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d} \mathbb{1}(k \leq k_F),$$

which is independent of $\mathbf{x} \in \mathbb{R}^d$ (as expected). We recognise a Riemann sum. At the limit $L \rightarrow \infty$, we therefore obtain

$$\rho^L \xrightarrow{L \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{1}(k \leq k_F) d\mathbf{k} = \frac{1}{(2\pi)^d} \frac{|\mathbb{S}^{d-1}|}{d} k_F^d.$$

So we choose

$$\boxed{k_F = 2\pi \left(\frac{d}{|\mathbb{S}^{d-1}|} \right)^{1/d} \rho^{1/d}.$$

The number k_F is called the *Fermi wave-vector*, or *Fermi momentum*.

6.2. Energy of the system. The energy *per unit volume* of the system is

$$\mathcal{E}(L, N) = \frac{1}{L^d} \sum_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} k^2 \frac{1}{L^d} \sum_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} k^2 \xrightarrow{L \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} k^2 \mathbb{1}(k \leq k_F) d\mathbf{k} = \frac{1}{(2\pi)^d} \frac{|\mathbb{S}^{d-1}|}{d+2} k_F^{d+2}.$$

In terms of the density, we obtain that the mean-energy of the fermionic gas is

$$\mathcal{E}(\rho) = 4\pi^2 \left(\frac{d}{|\mathbb{S}^{d-1}|} \right)^{2/d} \frac{|\mathbb{S}^{d-1}|}{d+2} \rho^{\frac{d+2}{d}} =: C \rho^{\frac{d+2}{d}}.$$

Remark 6.1. *The energy is not linear with the density! If we double the number of electrons (we multiply the density by 2), then is energy is multiplied by a factor $2^{\frac{d+2}{d}} > 2$. For bosonic system, the energy would scale linearly with the density.*

Remark 6.2. By minimality of Ψ_a , we proved that, for all $\Psi \in L_a^2((L\mathbb{T}^d)^N)$ with $\|\Psi\|_{L^2((L\mathbb{T}^d)^N)} = 1$, we have

$$\sum_{i=1}^N \|\nabla_{\mathbf{x}_i} \Psi\|_{L^2((L\mathbb{T}^d)^N)}^2 \geq C_{\text{TF}} \int_{(L\mathbb{T}^d)} \rho_{\Psi}^{\frac{d+2}{d}}(\mathbf{x}) \quad \text{with} \quad \rho_{\Psi} := N|\Psi|^2.$$

This type of inequality is called **Lieb-Thirring inequality**: the kinetic energy must be greater than some non linear functional of the density.

6.3. One-body density matrix. We now compute the one-body density matrix $\gamma(\mathbf{x}, \mathbf{y})$. We have

$$\gamma^L(\mathbf{x}, \mathbf{y}) = \sum_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} e_{\mathbf{k}}(\mathbf{x}) \overline{e_{\mathbf{k}}(\mathbf{y})} = \frac{1}{L^d} \sum_{\substack{\mathbf{k} \in \mathcal{B}(0, k_F) \\ \mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \xrightarrow{L \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \mathbf{1}(k < k_F) d\mathbf{k}.$$

The corresponding operator γ is the projector (of infinite dimension) that cuts high-frequencies, in the sense that

$$\widehat{\gamma f}(\mathbf{k}) = \widehat{f}(\mathbf{k}) \mathbf{1}(k \leq k_F).$$

In dimension $d = 1$, the limit is (when $d = 1$, we have $k_F = \pi\rho$)

$$\gamma(x, y) := \int_{-k_F}^{k_F} e^{ik(x-y)} dk = \frac{\sin(\pi\rho(x-y))}{\pi(x-y)}.$$

This is sometimes called the **sine kernel**, or the **Dyson sine kernel**. Of course, we recover the fact that $\gamma(x, x) = \rho(x) = \rho$.

This kernel is believed to be *universal*. This means that for any point process with some kind of repulsion, the local behaviour of the points in a suitable limit are described by the previous determinantal process with the Dyson sine kernel (the density ρ is given by the *local density* of the process in this case). We refer to the blog of Terence Tao for a discussion [Tao12]

From the previous discussion, we deduce that the correlations functions $\rho^{(k)}$ of the infinite one-dimensional fermionic gas is described by a determinantal point process of kernel $\gamma(x, y)$. For instance, the two-points correlation function is

$$\rho^{(2)}(x, y) = \det \begin{pmatrix} \gamma(x, x) & \gamma(x, y) \\ \gamma(y, x) & \gamma(y, y) \end{pmatrix} = \rho(x)\rho(y) - |\gamma(x, y)|^2 = \rho^2 \left(1 - \left(\frac{\sin(\pi\rho(x-y))}{\pi\rho(x-y)} \right)^2 \right).$$

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