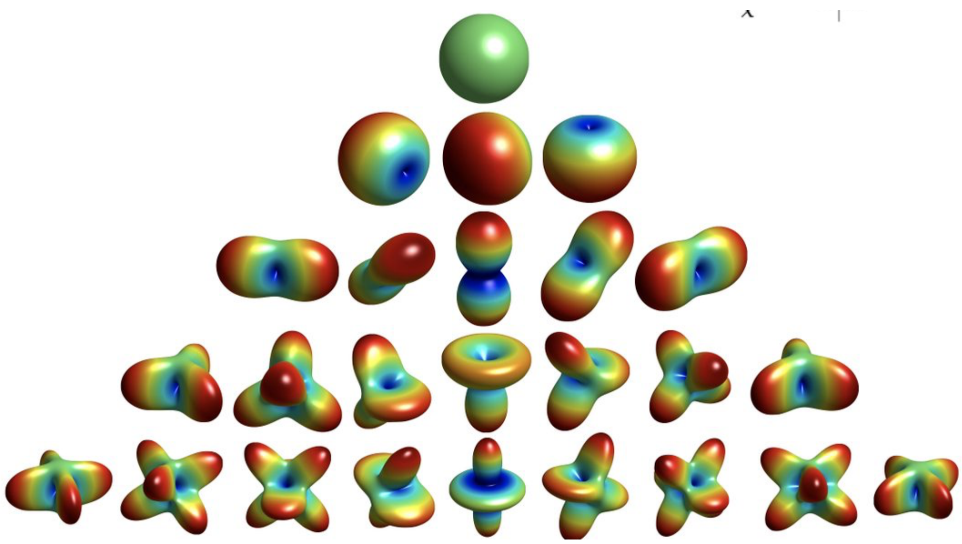
A review of functional analysis tools for PDEs



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INTRODUCTION

The present manuscript contains the notes for a one-week course (15h) given at University Paris-Dauphine, entitled *«A review in functional analysis tool for PDEs»*.

The presentation of the notes, the results, and most of the remarks are taken from the following books:

- *Analysis* by E. Lieb and M. Loss (in English) [LL01];
- *Analyse Fonctionelle* by H. Brezis (in French) [Bre99];
- *Éléments d'analyse fonctionnelle* by F. Hirsh and G. Lacombe (in French, with many exercices) [HL09].

Some other references are

- *Théorie des Distributions* by L. Schwartz (in French, for the chapter on distributions) [Sch66];
- *Partial Differential Equations* by L.C. Evans (in English, very complete, hence quite long) [Eva10];
- *Elliptic partial differential equations of second order* by D. Gilbarg and N.S. Trudinger (in English, when you cannot find the results in the other books) [GT15].

Most of the proofs of the theorems are simplified, and there will be links to the corresponding theorems in these books.

CONTENTS

CHAPTER 1

THE LEBESGUE L^P SPACES

In this Chapter, we define the Lebesgue L^p spaces. We focus on the special case where the measure is the Lebesgue one on $\mathbb{R}^d$.

1.1 Notation and first facts

First, we recall basic facts about measure theory and integration. We refer to [LL01, Chapter 1] for the reader who is not familiar with this theory.

1.1.1 Basics in measure theory

The open ball of $\mathbb{R}^d$ of center $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by

$$\mathcal{B}(x,r) = \left\{ y \in \mathbb{R}^d, \quad \|x-y\|_{\mathbb{R}^d} < r \right\}.$$

The **Borel sigma-algebra** of $\mathbb{R}^d$ is the one generated by the family of all open balls of $\mathbb{R}^d$. There is a *natural* **measure** on this sigma-algebra, called the **Lebesgue measure**, denoted $\mathrm{Leb}_d(A)$, $\mathcal{L}^d(A)$, $|A|$ or $\mathrm{d}x(A)$, which is the one for which

$$\mathrm{Leb}_d\left(\mathcal{B}(x,r)\right) := |\mathcal{B}(x,r)| := \frac{|\mathbb{S}^{d-1}|}{d} r^d, \quad \text{with} \quad |\mathbb{S}^{d-1}| := \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where Γ is the usual Euler's Gamma function. Recall that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, and that $\Gamma(1/2) = \sqrt{\pi}$ while $\Gamma(1) = 1$. This gives the usual well-known formulae

$$\mathrm{Leb}_1(\mathcal{B}(x,r)) = 2r, \quad \mathrm{Leb}_2(\mathcal{B}(x,r)) = \pi r^2, \quad \mathrm{Leb}_3(\mathcal{B}(x,r)) = \frac{4}{3}\pi r^3, \quad \text{etc.}$$

By construction, the Lebesgue measure is translation invariant: $\mathrm{Leb}_d(A) = \mathrm{Leb}_d(A+y)$ for all $y \in \mathbb{R}^d$.

We say that a property $P : \mathbb{R}^d \to \{\mathrm{True}, \mathrm{False}\}$ holds **almost everywhere** (and write **a.e.**) if $P^{-1}\{\mathrm{False}\}$ is (contained in a Borel set) of measure 0.

Example 1.1 (Countable sets have 0 measure)**.** *For all $x \in \mathbb{R}^d$, we have $\mathrm{Leb}_d(\{x\}) = 0$. By countable additivity, we deduce that if $C \in \mathbb{R}^d$ is countable, then C is Borel-measurable, and $\mathrm{Leb}_d(C) = 0$. For instance, since $\mathbb{Q}$ is countable, the assertion «x is irrational» holds almost everywhere.*

In the sequel, Ω always denotes an **non empty open set** of $\mathbb{R}^d$.

We say that a function $f : \Omega \to \mathbb{R}$ is (Borel or Lebesgue)-**measurable** if, for all $\lambda \in \mathbb{R}$, the set

$$\{f > \lambda\} := \{x \in \Omega, \quad f(x) > \lambda\}$$

is Borel-measurable.

Exercice 1.2

Prove that if f is measurable, then for all $\lambda \in \mathbb{R}$, the set $\{f < \lambda\}$ is measurable.
Hint: prove that $\{f \le \lambda + 1/n\}$ is measurable, and that $\{f < \lambda\} = \bigcup_{n \ge 1}\{f \le \lambda + 1/n\}$.

1.1.2 Integrable functions

We now focus on **Integrable** functions.

Definition 1.3 (Lebesgue integration)**.** *A* **positive** *measurable function $f : \Omega \to \mathbb{R}_+$ is (Lebesgue)-integrable if the function $F_f(\lambda) := \mathrm{Leb}_d(\{f > \lambda\})$ is Riemann integrable. In this case, its integral is*

$$\boxed{\int_\Omega f(x)\mathrm{d}x := \int_0^\infty F_f(\lambda)\mathrm{d}\lambda.} \tag{1.1}$$

Remark 1.4. *The function $F_f : \mathbb{R}_+ \to \mathbb{R}_+$ is positive and decreasing, so the Riemann sums always converge (why?). Being integrable only means that the limit is not $+\infty$, that is $\int_\Omega f = \int_0^\infty F_f < \infty$.*

Formally, this formula is easy to understand from Fubini's theorem (see Theorem 1.12 below). Indeed, if $\mathbb{1}(x > 0)$ denotes the Heaviside function, we have

$$\int_0^\infty F_f(\lambda)\mathrm{d}\lambda = \int_0^\infty \left(\int_\Omega \mathbb{1}(f(x) > \lambda)\mathrm{d}x\right)\mathrm{d}\lambda = \int_\Omega \left(\int_0^\infty \mathbb{1}(f(x) > \lambda)\mathrm{d}\lambda\right)\mathrm{d}x = \int_\Omega f(x)\mathrm{d}x.$$

If $f : \Omega \to \mathbb{R}$ is not positive valued, we introduce

$$f_+ := \max\{f, 0\} \quad \text{and} \quad f_- := \max\{-f, 0\}.$$

These two functions are positive valued, and we have $f = f_+ - f_-$ and $|f| = f_+ + f_-$. In this case, we say that f is integrable if f_+ and f_- are both integrable, and we define the integral of f by

$$\int_\Omega f(x)\mathrm{d}x := \int_\Omega f_+(x)\mathrm{d}x - \int_\Omega f_-(x)\mathrm{d}x.$$

Exercice 1.5

Prove that a measurable function f is integrable iff $|f|$ is integrable.

We admit the following result.

Lemma 1.6. *If f is Riemann integrable, then it is Lebesgue integrable, and the two integrals coincide.*

Remark 1.7. *There is a slight change of notation between the Riemann and Lebesgue integral in the one-dimensional case $d = 1$. We can write*

$$\int_{[a,b]} f(x)\mathrm{d}x \quad \textit{(Lebesgue notation)} \qquad \textit{or} \qquad \int_a^b f(x)\mathrm{d}x \quad \textit{(Riemann notation)}.$$

When f is positive, the first integral is always positive, while the second one is positive if $a < b$, and negative if $b < a$.

It is unclear from Definition 1.3 that the integral is linear: $\int(f + g) = \int f + \int g$. It is however the case (although non trivial, see [LL01, Exercice 9 p.37]). So we can use Lebesgue integration *as usual*.

1.1.3 The «powerful» theorems

There are three *powerful* theorems that describe how limits of functions behave with the integral. They are proved for instance in [LL01, Thm 1.6, 1.7 and 1.8].

Theorem 1.8: Monotone Convergence Theorem

If (f_j) is a sequence of measurable functions, **increasing** in the sense $f_j(x) \le f_{j+1}(x)$ a.e., then $f(x) := \lim f_j(x)$ is measurable, and

$$\int_\Omega f(x)\mathrm{d}x = \lim_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

In other words, increasing sequence implies $\int \lim = \lim \int$. The last value can be infinite, in which case f is not integrable.

Remark 1.9 (Lebesgue is «better» than Riemann)**.** *Note the first result, stating that $f = \lim f_j(x)$ is measurable. This is strong statement, which fails in Riemann theory. Recall that the Riemann integral is defined on the set $C^0_{\rm pw}$ of piece-wise continuous functions. However, this set is not closed when taking increasing limits. For instance, label the rationnal numbers by $\mathbb{Q} = \{q_1, q_2, \cdots\}$ (it is countable), and set $f_N(x) = \mathbb{1}(x \in \{q_1, \cdots, q_N\})$. Then f_N is piece-wise continuous (the points where f_N is discontinuous are isolated). We have $f_{N+1} \ge f_N$ and $\lim_{N\to\infty} f_N = \mathbb{1}_{\mathbb{Q}}$, which is not piece-wise continuous.*

Proof. Replacing f_j by $f_j - f_1$, we may assume that $f_j \ge 0$ is positive valued.
We set $F_j := F_{f_j} = \mathrm{Leb}_d(\{f_j > \lambda\})$. Since $f_{j+1}(x) \ge f_j(x)$, we have $F_{j+1}(\lambda) > F_j(\lambda)$. So (F_j) is an increasing family of decreasing functions, which converges point-wise to $F_f(\lambda)$. We leave the rest of the proof to the reader. It is a classical exercise in the theory of Riemann integration. □

Theorem 1.10: Fatou's theorem

Let (f_j) be a sequence of **positive** measurable functions. Then $f := \liminf_{j\to\infty} f_j$ is positive, measurable, and

$$0 \le \int_\Omega f(x)\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

In other words, positivity implies $\int \liminf \le \liminf \int$. A mnemotechnic trick is that *the sum of minima is always lower than the minimum of the sum.*

Proof. Define $g_k(x) := \inf_{j\ge k} f_j(x)$. The sequence g_k is measurable, increasing, with $\lim_{k\to\infty} g_k = \liminf_{j\to\infty} f_j = f$. By the Monotone Convergence Theorem 1.8, we have

$$\int_\Omega f(x)\mathrm{d}x = \lim_{k\to\infty} \int_\Omega g_k(x)\mathrm{d}x.$$

Now, we see that $g_k \le f_j$ for all $k \le j$, so $\int g_k \le \inf_{j\ge k} \int f_j$, and the result follows. □

Finally, we have the *master* Theorem.

Theorem 1.11: Dominated Convergence Theorem

Let (f_j) be a sequence of measurable functions which converges point-wise to f a.e. Assume there is an integrable function G so that $|f_j|(x) \le G(x)$ (**domination**). Then $|f| \le G(x)$ and

$$\int_\Omega f(x)\mathrm{d}x = \lim_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

In other words, domination implies $\lim \int = \int \lim$.

Proof. We only do the proof for positive functions f_j. By Fatou's theorem 1.10, we have

$$\liminf \int f_j \geq \int f.$$

On the other hand, since $G - f_j$ is also a family of positive functions, we have, by Fatou's lemma again

$$\liminf \int G - f_j \geq \int G - f, \quad \text{so} \quad \limsup \int f_j \leq \int f.$$

This proves $\limsup \int f_j \leq \int f \leq \liminf \int f_j$, and the result follows. $\square$

To understand why domination is important, the reader should keep in mind the following three **counterexamples**.

- **The mass goes to infinity**. Let $\psi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^+)$ and $e \in \mathbb{R}^d \setminus \{0\}$. Then $f_j(x) := \psi(x - je)$ converges point-wise to $f = 0$. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.
- **The mass spreads over**. Consider now $f_j(x) = j^{-d}\psi(x/j)$. Again, f_j converges point-wise to $f = 0$ (the convergence is even uniform). However, $\int f_j = \int \psi > 0$, while $\int f = 0$.
- **The mass concentrates**. Take $f_j(x) = j^d \psi(jx)$. Then f_j converge point-wise to f for all $x \neq 0$, so a.e. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.

The following theorem is of different nature, but we include it here, as it is also *powerful*. We skip the proof, which can be found in [LL01, Thm 1.12].

Theorem 1.12: Fubini's theorem

If $f : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{R}^+$ is measurable, then

$$\int_{\mathbb{R}^{d+s}} f(x,y) \mathrm{d}^{d+s}((x,y)) = \int_{\mathbb{R}^s} \left(\int_{\mathbb{R}^d} f(x,y) \mathrm{d}^d x \right) \mathrm{d}^s y = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^s} f(x,y) \mathrm{d}^s y \right) \mathrm{d}^d x.$$

1.2 The L^p spaces

We now introduce the Lebesgue $L^p(\Omega)$ space. We focus on the case $1 \leq p \leq \infty$. The case $p = \infty$ is always a bit *special*.

1.2.1 Definitions

For $1 \leq p < \infty$, we define

$$\mathcal{L}^p(\Omega) := \{ f \text{ measurable from } \Omega \text{ to } \mathbb{C}, \quad \|f\|_{L^p} < \infty \}, \quad \text{where} \quad \boxed{\|f\|_{L^p} := \left(\int_\Omega |f|^p(x) \mathrm{d}x \right)^{1/p},}$$

and for $p = \infty$, we set

$$\mathcal{L}^\infty(\Omega) := \{ f \text{ measurable from } \Omega \text{ to } \mathbb{C}, \text{ bounded a.e.} \}, \quad \boxed{\|f\|_{L^\infty} := \inf\{\lambda \geq 0, \quad \mathrm{Leb}_d(\{|f| > \lambda\}) = 0\}.}$$

We have

- $\|f\|_{L^p} = 0$ iff $f = 0$ almost-everywhere;
- $\|\lambda f\|_{L^p} = |\lambda| \cdot \|f\|_{L^p}$;

- $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ (Minkowski inequality, see Theorem 1.18 below).

This proves that the map $\|\cdot\|$ is a **semi**-norm, in the sense that $\|f\|_{L^p} = 0$ does not imply $f = 0$, but only that $f = 0$ almost everywhere. To cure this problem, we introduce the equivalent relation $f \sim g$ iff $f = g$ almost everywhere, and define $L^p(E) := \mathcal{L}^p(E)/\sim$. In practice, this means that elements of $L^p(E)$ are not functions, but classes of functions. However, we usually say *a function f in* $L^p(E)$, with the convention that f is only defined almost everywhere. For instance, we say $f \in L^p(E)$ is continuous to state that there is a continuous representation of f.

As an example, consider the following result which we admit (see classical textbooks, such as Rudin 1987 for a proof. It uses Hardy-Littlewood maximal function theory). We set $⨍_\Omega f := \frac{1}{|\Omega|}\int_\Omega f$ the average of f on the Ω domain.

Theorem 1.13: Lebesgue differentiation theorem

For all $f \in L^1(\mathbb{R}^d)$, we have

$$f(x) = \lim_{\varepsilon \to 0} ⨍_{\mathcal{B}(x,\varepsilon)} f(y)\mathrm{d}y \quad \text{almost everywhere in } x \in \mathbb{R}^d.$$

In dimension $d = 1$, the result implies that $F(x) := \int_{-\infty}^x f(y)\mathrm{d}y$ is differentiable with $F' = f$, almost everywhere in $x \in \mathbb{R}$. Note that the left-hand side depends on the representation of f, while the right-hand side is independent of the representation. In some sense, the left-hand side selects one representation in the class of f.

1.2.2 The useful inequalities

After the powerful theorems come the powerful inequalities.

Theorem 1.14: Jensen's inequality

Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function, $f : \Omega \to \mathbb{R}$ be measurable, and $\mu : \Omega \to \mathbb{R}^+$ be measurable with $\int_\Omega \mu = 1$, then

$$\int_\Omega J(f)\mu \geq J\left(\int_\Omega f\mu\right).$$

The theorem is also valid if μ is a measure. Taking $f(x) = x$ and $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$ (sum of Dirac masses) with $\sum \lambda_i = 1$, we obtain

$$\sum_{i=1}^n \lambda_i J(x_i) \geq J\left(\sum_{i=1}^n \lambda_i x_i\right),$$

which is a well-known property of convex functions. Jensen's inequality is somehow a $n = \infty$ version of this inequality.

Proof. Assume J differentiable for simplicity. Since J is convex, we have for all $a, b \in \mathbb{R}$,

$$J(a) \geq J(b) + J'(b)(a-b).$$

Taking $b = \int_\Omega f\mu$, and $a = f(x)$ gives

$$J(f(x)) \geq J\left(\int_\Omega f\mu\right) + J'\left(\int_\Omega f\mu\right) \times \left[f(x) - \left(\int_\Omega f\mu\right)\right].$$

We multiply this inequality by $\mu(x)$ (which is positive) and integrate. The term in bracket vanishes since $\int_\Omega \mu = 1$, and the result follows. $\square$

Theorem 1.15: Hölder's inequality

Let $1 \le p, q \le \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \text{or, equivalently,} \quad q = \frac{p}{p-1}.$$

Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$, and

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

In the case $p = q = 2$, we recover the Cauchy-Schwarz inequality $\int fg \le \|f\|_{L^2} \|g\|_{L^2}$. In the sequel, we denote by

$$\boxed{p' := \frac{p}{p-1}} \quad \text{(the } \textbf{dual} \text{ exponent of } p). \tag{1.2}$$

Proof. Without loss of generality, we may assume that f and g are positive. We introduce $G := g/\|g\|_{L^q}$, which satisfies $\|G\|_{L^q} = 1$. We then set $\mu(x) = G^q(x)$, $F(x) := f(x)/G^{q/p}(x)$ and $J(t) := t^p$, which is convex. We apply Jensen's inequality to (J, F, μ), which gives

$$\int_\Omega \left(\frac{f}{G^{q/p}} \right)^p G^q \ge \left(\int_\Omega \frac{f}{G^{q/p}} G^q \right)^p.$$

with (we use that $q(1 - \frac{1}{p}) = 1$ in the last equality)

$$\int_\Omega \left(\frac{f}{G^{q/p}} \right)^p G^q = \int_\Omega f^p, \quad \text{and} \quad \left(\int_\Omega \left(\frac{f}{G^{q/p}} \right) G^q \right)^p = \left(\int_\Omega f G^{q(1-\frac{1}{p})} \right)^p = \left(\int_\Omega fG \right)^p.$$

So

$$\left(\int_\Omega f \frac{g}{\|g\|_{L^q}} \right)^p = \left(\int_\Omega fG \right)^p \le \int_\Omega f^p, \quad \text{hence} \quad \|fg\|_{L^1}^p = \left(\int_\Omega fg \right)^p \le \|f\|_{L^p}^p \|g\|_{L^q}^p.$$

□

The following form of the Hölder's inequality is often used.

Theorem 1.16: Holder's inequality in the general case

Let $1 \le p, q, r \le \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^r(\Omega)$, and

$$\|fg\|_{L^r}(\Omega) \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Proof. We use Hölder's inequality with $F = |f|^r \in L^{p/r}(E)$ and $G = |g|^r \in L^{q/r}(E)$. We have $1/(p/r) + 1/(q/r) = 1$, so $FG \in L^1$, and

$$\|fg\|_{L^r}^r = \int_\Omega |f|^r |g|^r = \|FG\|_{L^1} \le \|F\|_{L^{p/r}} \|G\|_{L^{q/r}} = \left(\int_\Omega |f|^p \right)^{r/p} \left(\int_\Omega |g|^q \right)^{r/q} = \left(\|f\|_{L^p} \cdot \|g\|_{L^q} \right)^r.$$

□

Another useful corollary is the following:

Theorem 1.17: Interpolation

If $1 \le p_1 \le p_2 \le \infty$, and $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, then for all $p \in [p_1, p_2]$, we have $f \in L^p(\Omega)$ with

$$\|f\|_{L^p} \le \|f\|_{L^{p_1}}^{\alpha} \|f\|_{L^{p_2}}^{1-\alpha}, \qquad \text{where } 0 \le \alpha \le 1 \text{ is chosen so that } \frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}.$$

Proof. Write $f = f^{\alpha} f^{(1-\alpha)}$, with $f^{\alpha} \in L^{p_1/\alpha}$ and $f^{(1-\alpha)} \in L^{p_2/(1-\alpha)}$, and apply the previous result. $\square$

Finally, we prove Minkowski's inequality.

Theorem 1.18: Minkowski's inequality

For all $f, g \in L^p(\Omega)$ with $1 \le p \le \infty$, we have

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

In particular, the map $f \mapsto \|f\|_{L^p}$ is convex.

Proof. First, we have the easy bound $|f+g|^p \le (2\max\{f, g\})^p \le |2f|^p + |2g|^p$, so $f + g$ is indeed in $L^p(\Omega)$. Then, we have

$$\int_\Omega |f+g|^p = \int_\Omega |f+g| \cdot |f+g|^{p-1} \le \int_\Omega |f| \cdot |f+g|^{p-1} + \int_\Omega |g| \cdot |f+g|^{p-1}.$$

We use Hölder's inequality with $f \in L^p$ and $|f+g|^{p-1} \in L^q$ with $q = \frac{p}{p-1}$, and get

$$\int_\Omega |f| \cdot |f+g|^{p-1} \le \left(\int_\Omega |f|^p \right)^{1/p} \left(\int_\Omega |f+g|^p \right)^{\frac{p-1}{p}}.$$

This gives $\|f+g\|_{L^p}^p \le (\|f\|_{L^p} + \|g\|_{L^p}) \|f+g\|_{L^p}^{p-1}$, which is also $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$. $\square$

1.2.3 Convolution in Lebesgue spaces

In this section, we take $\Omega = \mathbb{R}^d$ (convolution is only defined on the full space). Let f, g be two complex-valued functions. We define the **convolution** $f * g$ by

$$\boxed{(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y) \mathrm{d}y.}$$

The reader can check that $f * g = g * f$, and that $(f * g) * h = f * (g * h)$: the convolution is **commutative** and **associative**.

Convolution from $L^p \times L^q \to L^r$

Thanks to Hölder's inequality 1.15, we see that for fixed $x \in \mathbb{R}^d$, the integrand defining the convolution is integrable whenever $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $1/p + 1/q = 1$, and we have

$$\forall x \in \mathbb{R}^d, \quad |(f * g)(x)| \le \int_{\mathbb{R}^d} |f(y) g(x-y)| \, \mathrm{d}y \le \|f\|_{L^p} \|g(x - \cdot)\|_{L^q} = \|f\|_{L^p} \|g\|_{L^q}.$$

So the function $f * g$ is bounded, that is $f * g \in L^\infty(\mathbb{R}^d)$, with

$$\|f * g\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^q}, \quad (\text{Case } \tfrac{1}{p} + \tfrac{1}{q} = 1).$$

On the other hand, if $f \in L^1(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, we have by Fubini theorem that $f * g \in L^1(\mathbb{R}^d)$ with

$$\|f * g\|_{L^1} = \|f\|_{L^1} \|g\|_{L^1}.$$

The generalisation of these two (in)-equalities is called Young's inequality (see [LL01, Theorem 4.2]).

Theorem 1.19: Young's inequality, first form

Let $1 \le p, q, s \le \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^s(\mathbb{R}^d)$, and

$$\|f * g\|_{L^s} \le \|f\|_{L^p} \|g\|_{L^q}.$$

One other way to state this theorem is as follows.

Theorem 1.20: Young's inequality, second form

Let $1 \le p, q, r \le \infty$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$. Then the function $(f * g)h$ is in $L^1(\mathbb{R}^d)$, and

$$\left| \iint_{(\mathbb{R}^d)^2} f(x) g(y - x) h(y) \mathrm{d}x \mathrm{d}y \right| \le \|(f * g) h\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

The fact that these two inequalities are equivalent comes from the duality result presented in Theorem 1.27 below. In this second version, the variable r plays the role of the dual variable $s' = s/(s-1)$ of the first version). We prove the second version following [LL01, Theorem 4.2].

Proof. Without loss of generality, we may assume that f, g, h are positive. Let p', q', r' be the dual powers of p, q, r, see Eqn. (1.2), and set

$$\begin{cases} \alpha(x, y) & := f(x)^{p/r'} g(y-x)^{q/r'} \\ \beta(x, y) & := g(y-x)^{q/p'} h(y)^{r/p'} \\ \gamma(x, y) & := h(y)^{r/q'} f(x)^{p/q'}. \end{cases}$$

We have

$$\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q'} = \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) = 3 - 2 = 1,$$

so we can use Hölder's inequality (on $(\mathbb{R}^d)^2$) and get

$$\int_{(\mathbb{R}^d)^2} \alpha(x, y) \beta(x, y) \gamma(x, y) \mathrm{d}x \mathrm{d}y \le \|\alpha\|_{L^{r'}((\mathbb{R}^d)^2)} \|\beta\|_{L^{p'}((\mathbb{R}^d)^2)} \|\gamma\|_{L^{q'}((\mathbb{R}^d)^2)}.$$

The integrand in the left is also (we focus on the f terms for the computation)

$$\alpha(x, y) \beta(x, y) \gamma(x, y) = f(x)^{\frac{p}{r'} + \frac{p}{q'}} \cdots = f(x)^{p(1 - \frac{1}{p'})} \cdots = f(x) g(y - x) h(y).$$

On the other hand, we have, by Fubini's theorem, that

$$\|\alpha\|_{L^{r'}((\mathbb{R}^d)^2)}^{r'} = \iint_{(\mathbb{R}^d)^2} f(x)^p g(y-x)^q \mathrm{d}x\mathrm{d}y = \|f\|_{L^p}^p \|g\|_{L^q}^q,$$

so indeed $\alpha \in L^{r'}((\mathbb{R}^d)^2)$. Writing similar inequalities for β and γ gives the result. □

1.2.4 Convolution as a smoothing operator

We now prove that, in general, $f * g$ is more regular than f.

Smoothing sequences

Let $j \in C_0^\infty(\mathbb{R}^d)$ be such that j is radial decreasing, with $j(x) = 0$ for all $|x| > 1$, and $\int_{\mathbb{R}^d} j = 1$. The fact that such functions exist is classical. For $\varepsilon > 0$, we set

$$j_\varepsilon(x) := \frac{1}{\varepsilon^d} j\left(\frac{x}{\varepsilon}\right).$$

Since j is compactly supported in $\mathcal{B}(0,1)$, the function j_ε is compactly supported in $\mathcal{B}(0,\varepsilon)$. The scaling is chosen so that $\int_{\mathbb{R}^d} j_\varepsilon = \int_{\mathbb{R}^d} j = 1$. The family (j_ε) is called a **smoothing sequence**, a **mollifier**, or an **approximation of the Dirac** (see Example 2.6 below).

Approximation by smooth functions

In the sequel, we say that a function f is **smooth** if $f \in C^\infty(\mathbb{R}^d)$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we set

$$D^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f.$$

Recall that if f is smooth, the order of the derivatives is irrelevant, thanks to Schwartz' Lemma. The following Theorem shows that convolution smooths functions (see [LL01, Theorem 2.16]).

Theorem 1.21: Convolution smooths functions

Let $1 \le p \le \infty$, and let (j_ε) be a smoothing sequence. For all $f \in L^p(\mathbb{R}^d)$, we set $f_\varepsilon := f * j_\varepsilon$. Then

- f_ε is smooth, and $D^\alpha(f * j_\varepsilon) = f * (D^\alpha j_\varepsilon)$;
- $f_\varepsilon \in L^p(\mathbb{R}^d)$ with $\|f_\varepsilon\|_{L^p} \le \|f\|_{L^p}$;
- if in addition $p < \infty$, then $\|f - f_\varepsilon\|_{L^p} \to 0$ as $\varepsilon \to 0^+$.

Before we give the proof, we emphasise that **the last result is false if** $p = \infty$. Indeed, take f a bounded discontinuous function, and assume that $\|f - f_\varepsilon\|_\infty \to 0$. Then, f would the limit of the continuous functions f_ε for the *uniform* convergence. This would imply that f is continuous (the uniform limit of continuous functions is continuous), a contradiction.

Proof. The inequality $\|f_\varepsilon\|_{L^p} \le \|f\|_{L^p}$ is Young's inequality, together with the fact that $\|j_\varepsilon\|_{L^1} = 1$.

Let us prove the first point. We focus on the case $p = 1$. Let e_i be the i-th canonical vector of $\mathbb{R}^d$. We have

$$\frac{1}{t}\left(f_\varepsilon(x + te_i) - f_\varepsilon(x)\right) = \int_{\mathbb{R}^d} f(y) \left[\frac{1}{t}\left(j_\varepsilon(x - y + te_i) - j_\varepsilon(x-y)\right)\right] \mathrm{d}y.$$

Since j_ε is smooth, the term in bracket converges pointwise to $(\partial_i j_\varepsilon)(x-y)$ as $t \to 0$. In addition, using the mean value theorem, there is c in the segment $[x-y, x-y+te_i]$ so that

$$\left| f(y) \left[\frac{1}{t} \left(j_\varepsilon(x+te_1-y) - j_\varepsilon(x-y) \right) \right] \right| = |f(y)(\partial_i j_\varepsilon)(c)| \le \|(\partial_i j_\varepsilon)\|_{L^\infty} |f(y)|,$$

which is integrable in y, and independent of t. So, by the Dominated Convergence Theorem 1.11, we can take the limit $t \to 0$, and we obtain

$$\partial_i (f * j_\varepsilon) = f * (\partial_i j_\varepsilon).$$

The result follows by induction.

For the last point, we focus again on the case $p = 1$. Consider first the case where $f(x) = \mathbb{1}(x \in A)$, where A is a half-open rectangular set, of the form

$$A = (a_1, b_1] \times \cdots \times (a_d, b_d].$$

Since j_ε is compactly supported in $\mathcal{B}(0, \varepsilon)$ with $\int j = 1$, the function $f_\varepsilon := f * j_\varepsilon$ satisfies

$$\mathbb{1}(x \in A_\varepsilon^-) \le f_\varepsilon(x) \le \mathbb{1}(x \in A_\varepsilon^+), \quad \text{with} \quad A_\varepsilon^\pm := (a_1 \mp \varepsilon, b_1 \pm \varepsilon] \times \cdots \times (a_d \mp \varepsilon, b_d \pm \varepsilon].$$

In particular, we have

$$\|f_\varepsilon - f\|_{L^1} \le \max\{ \|\mathbb{1}(x \in A_\varepsilon^+) - \mathbb{1}(x \in A)\|, \|\mathbb{1}(x \in A) - \mathbb{1}(x \in A_\varepsilon^-)\| \} \approx C\varepsilon \xrightarrow[\varepsilon \to 0]{} 0.$$

So the result holds for $f(x) = \mathbb{1}(x \in A)$. By linearity, it also holds for any finite linear combination of such functions (such combinations are called *really simple functions*). It is a result in measure theory that such combinations are dense in L^1 (see [LL01, Theorem 1.18]). So, by density, the result holds for all $f \in L^1$. $\square$

A direct corollary is the following density result, valid for $p < \infty$ (see [LL01, Lemma 2.19]).

Theorem 1.22: Smooth functions are dense in L^p

For all $1 \le p < \infty$, and all $\Omega \subset \mathbb{R}^d$, the sets $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ are dense in $L^p(\Omega)$.

Again, the result is false in L^∞, see the paragraph after Theorem 1.21.

Proof. We notice that if $f \in L^p(\Omega)$, then the function

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{else}, \end{cases}$$

is in $L^p(\mathbb{R}^d)$. This function is called the **extension of** f. Let $\eta > 0$. By the previous result, $(\widetilde{f}_\varepsilon)$ is a family of smooth functions which converge to $\widetilde{f}$ in $L^p(\mathbb{R}^d)$. Restricting to Ω shows that there is $\varepsilon > 0$ so that $f_\eta := \widetilde{f}_\varepsilon$ satisfies $\|f_\eta - f\|_{L^p(\Omega)} < \eta$. This already proves that $C^\infty(\Omega)$ is dense in $L^p(\Omega)$.

We now prove that we can choose compactly supported functions. The Urysohn's Lemma (see [LL01, Lemma 2.19]) states that there is a sequence of positive compactly supported functions $(\chi_j) \in C_0^\infty(\Omega)$ so that, for all $x \in \Omega$, we have $\chi_{j+1}(x) \ge \chi_j(x)$ and $\lim_{j \to \infty} \chi_j(x) = 1$ (so $\lim_{j \to \infty} \chi_j = \mathbb{1}_\Omega$ for the pointwise convergence). We set $f_j(x) := \chi_j(x) f_\eta(x)$, which is smooth and compactly supported. The sequence $|f_\eta - f_j|^p$ converges point-wise to 0 and is dominated by $|f_\eta|^p$. The Dominated Convergence Theorem 1.11 shows that $\|f_\eta - f_j\|_{L^p} \to 0$ as $j \to \infty$. So, for j large enough, we have $\|f_\eta - f_j\|_{L^p} < \eta$. This proves that $f_j \in C_0^\infty(\Omega)$ satisfies

$$\|f - f_j\|_{L^p} \le \|f - f_\eta\|_{L^p} + \|f_\eta - f_j\|_{L^p} \le 2\eta.$$

$\square$

1.3 L^p spaces as Banach spaces

We now focus on the completion properties of $L^p(\Omega)$ spaces. We first recall some basic notions of Banach spaces. We then focus on the special case of $L^p(\Omega)$ spaces.

1.3.1 Basics in Banach spaces

A normed vectorial space $(E, \|\cdot\|_E)$ is a **Banach space** if it is **complete**, that is if all Cauchy sequences have limits. This is equivalent to

$$\sum_{n\in\mathbb{N}} \|x_n\|_E < \infty \implies \sum_{n\in\mathbb{N}} x_n \text{ converges in } E.$$

A linear form $L : E \to \mathbb{C}$ is **continuous** (or **bounded**) if there is $C \in \mathbb{R}^+$ so that

$$\forall x \in E, \quad |L(x)| \le C\|x\|_E.$$

The set of all continuous forms is called the **dual** of E, and is denoted by E^*. It is a Banach space when equipped with the norm

$$\|L\|_{\mathrm{op}} := \|L\|_{E^*} := \sup\{|L(x)|,\ x \in E,\ \|x\|_E \le 1\} = \sup\left\{\frac{|L(x)|}{\|x\|_E},\ x \in E \setminus \{0\}\right\}.$$

We sometimes write

$$\langle L, x\rangle_{E',E} := L(x).$$

The definition of the operator norm implies the following inequality:

$$\forall x \in E, \quad \forall L \in E^*, \qquad |L(x)| = |\langle L, x\rangle_{E',E}| \le \|L\|_{\mathrm{op}}\|x\|_E. \tag{1.3}$$

We recall the following Theorem, which we use all over these notes (especially the second part of it). The proof can be found in [Bre99, Theorem 1.1]

Theorem 1.23: Hahn-Banach, analytic form

Let $F \subset E$ be a vectorial space in E, and let $L : F \to \mathbb{C}$ be a linear functional on F such that

$$\|L\|_{\mathrm{op,F}} := \sup\{|L(x)|, \quad x \in F,\ \|x\|_E = 1\} < \infty.$$

Then there is an extension $\widetilde{L} \in E^*$ so that $\|\widetilde{L}\|_{\mathrm{op}} = \|L\|_{\mathrm{op,F}}$ and $\widetilde{L} = L$ on F.
If F is **dense**, then this extension is unique.

As a corollary, we record the following (see [Bre99, Corollaire 1.4]).

Theorem 1.24: The norm is a supremum in the dual space

For all $x \in E$, we have

$$\|x\|_E = \sup\{|L(x)|, \quad L \in E^*,\ \|L\|_{\mathrm{op}} \le 1\} = \max\{|L(x)|, \quad L \in E^*,\ \|L\|_{\mathrm{op}} \le 1\}$$

In particular, we do have access to the norm $\|x\|$ by only considering evaluation with operators L in the dual space.

Proof. The inequality (1.3) already implies that $|L(x)| \le \|L\|_{\mathrm{op}}\|x\|$, so

$$\sup\{|L(x)|, \quad L \in E^*,\ \|L\|_{\mathrm{op}} \le 1\} \le \|x\|_E.$$

To conclude, we construct $L_0 \in E^*$ with $\|L_0\|_{\mathrm{op}} = 1$ and $L_0(x) = \|x\|_E$. To do so, we consider the subspace $F = \mathbb{R}x$, and the linear operator $L : F \to \mathbb{R}$ with $L(tx) = t\|x\|_E$. We compute that $\|L\|_{\mathrm{op,F}} = 1$. We extend L to the whole space E with Hahn-Banach theorem, and obtain the linear operator $L_0 : E \to \mathbb{R}$ with $\|L_0\|_{\mathrm{op}} = 1$, and $L_0(x) = \|x\|_E$. □

The **bidual** of E is the dual of the dual, that is $E^{**} := (E^*)^*$. We always have $E \subset E^{**}$ with the identification $E \ni x \mapsto M_x \in E^{**}$, where

$$M_x : E^* \to \mathbb{C}, \quad M_x : L \mapsto L(x).$$

If $E = E^{**}$, we say that E is **reflexive**.

We say that E is **separable** if there is a countable dense set in E. This means that there is a countable family $(x_n)_{n\in\mathbb{N}}$ in E such that, for all $x \in E$ and all $\varepsilon > 0$, there is $n \in \mathbb{N}$ so that $\|x - x_n\|_E < \varepsilon$.

1.3.2 Completion of L^p spaces

We now focus on $L^p(\Omega)$ spaces. We start with the following (see [LL01, Theorem 2.7]).

Theorem 1.25: L^p is complete

Let $1 \le p \le \infty$, and let (f_j) be a Cauchy sequence in $L^p(\Omega)$. Then there is $f \in L^p(\Omega)$ and a subsequence $\phi : \mathbb{N} \to \mathbb{N}$ so that

- $\|f_{\phi(j)} - f\|_{L^p} \to 0$;
- $f_{\phi(j)}(x) \to f(x)$ almost everywhere.

In particular, the space $L^p(\Omega)$ is a Banach space (it is complete).

Proof. We prove the result for $p < \infty$ only. Let (f_j) be a Cauchy sequence in $L^p(\Omega)$. Up to a subsequence, we may assume $\|f_{j+1} - f_j\|_{L^p} \le 1/2^j$ (why?). We introduce

$$F_\ell(x) := |f_1(x)| + \sum_{j=1}^{\ell-1} |f_{j+1}(x) - f_j(x)|.$$

By Minkowski's inequality, F_ℓ is in $L^p(\Omega)$, and

$$\|F_\ell\|_{L^p} \le \|f_1\|_{L^p} + \sum_{j=1}^{\ell-1} \|f_{j+1} - f_j\|_{L^p} \le \|f_1\|_{L^p} + \sum_{j=1}^{\ell-1} \frac{1}{2^j} \le \|f_1\|_{L^p} + 1.$$

The sequence (F_ℓ) is positive and increasing. By the Monotone Convergence Theorem 1.8, the function $F(x) := \lim_{\ell\to\infty} F_\ell(x)$ is in $L^p(\Omega)$, and $\|F\|_{L^p} \le \|f_1\|_{L^p} + 1$.
Next, we notice that (the sum telescopes)

$$f_\ell(x) = f_1(x) + \sum_{k=1}^{\ell-1} \left(f_{j+1}(x) - f_j(x)\right).$$

For a.e. $x \in \Omega$, the sum on the right converges absolutely in $\mathbb{C}$ (adding absolute values, we recover $F_\ell(x)$, which converges to $F(x)$). By completion of $\mathbb{C}$, the sum has a limit as $\ell \to \infty$. We set $f(x) := \lim_{\ell\to\infty} f_\ell(x)$. By definition, the sequence (f_ℓ) converges pointwise to f. In addition, we have the domination $|f_\ell(x)| \le F_\ell(x) \le F(x)$, which is in $L^p(\Omega)$. So, by the Dominated Convergence Theorem 1.11, we have $\|f\|_{L^p} = \lim_{\ell\to\infty} \|f_\ell\|_{L^p} < \infty$. In particular, $f \in L^p(\Omega)$. Finally, the sequence $|f - f_\ell|^p$ converges pointwise to 0, and we have the domination

$$|f - f_\ell|^p \le (|f| + |f_\ell|)^p \le 2^p \left(|f|^p + |f_\ell|^p\right) \le 2^p \left(|f|^p + |F|^p\right),$$

which is integrable. Using again the Dominated Convergence Theorem shows that $\|f - f_\ell\|_{L^p} \to 0$, as wanted. □

1.3.3 Separability of L^p spaces

Theorem 1.26: Separability of L^p

For all $1 \le p < \infty$, the space $L^p(\Omega)$ is separable. The space $L^\infty(\Omega)$ is not separable.

Proof. Consider first $1 \le p < \infty$, and consider the set $\mathcal{A}$ of really simple functions of the form $f = \sum_{j=1}^N f_j \mathbb{1}(x \in A_j)$, with $f_j \in (\mathbb{Q} + \mathrm{i}\mathbb{Q})$ and A_j some rectangles with rational boundaries. The set $\mathcal{A}$ is countable, and dense in $L^p(\Omega)$ for all $1 \le p < \infty$ (see [LL01, Theorem 1.18]). So $L^p(\Omega)$ is separable.

In the case $p = \infty$, let us prove that $L^\infty(\mathbb{R})$ is not countable. We consider the following set of functions. For a subset $Q \subset \mathbb{Z}$, we define

$$f_Q(x) = \begin{cases} 1 & \text{if } \quad \lfloor x \rfloor \in Q \\ 0 & \text{else} \end{cases} .$$

Then $(f_Q)_{Q \subset \mathbb{Z}}$ is an uncountable family, and $Q \neq Q'$ implies $\|f_Q - f_{Q'}\|_\infty = 1$. So $L^\infty(\mathbb{R})$ cannot be separable (why?). We can prove similarly that $L^\infty((-1,1))$ is not separable by considering the functions $g_Q(x) := f_Q(x/(1-x^2))$. □

1.3.4 Duality in L^p spaces

We state the main result, which we will prove later in Section **??** in the case $p = 2$. We refer to [LL01, Theorem 2.14] for the proof in the general case. Recall that the dual exponent of p is $p' = p/(p-1)$ (see Eqn. (1.2)).

Theorem 1.27: The dual of $L^p(\Omega)$

For all $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))^* = L^{p'}(\Omega)$.
(**Case** $p = 1$). The dual space of $L^1(\Omega)$ is $\left(L^1(\Omega)\right)^* = L^\infty(\Omega)$.
(**Case** $p = \infty$). We have the **strict** inclusion $L^1(\Omega) \subsetneq (L^\infty(\Omega))^*$.

For $1 < p < \infty$, the space $L^p(\Omega)$ is reflexive, while $L^1(\Omega)$ and $L^\infty(\Omega)$ **are not** reflexive.

The dual for $L^\infty(\Omega)$ is a subset of measures. We do not elaborate on this point.

We postpone the proof until Section **??** (in the case $p = 2$ only), and just remark that the inclusion $L^{p'}(\Omega) \subset (L^p(\Omega))^*$ comes from Hölder's inequality. Indeed, for all $g \in L^{p'}(\Omega)$, one can consider the linear form $L_g : L^p(\Omega) \to \mathbb{C}$ defined by $L_g(f) := \int_\Omega gf$. Thanks to Hölder's inequality, we have

$$|L_g(f)| \le \int_\Omega |gf| \le \|g\|_{L^{p'}} \|f\|_{L^p}.$$

This proves that $L_g \in (L^p(\Omega))^*$, with $\|L_g\|_{\mathrm{op}} \le \|g\|_{L^{p'}}$. Actually, the result below proves that we have equality $\|L_g\|_{\mathrm{op}} \le \|g\|_{L^{p'}}$, which allows to identify g and L_g.

Theorem 1.28

For all $1 \le p \le \infty$, and all $f \in L^p(\Omega)$, we have

$$\|f\|_{L^p} = \sup \left\{ \int_\Omega fg, \quad g \in L^{p'}(\Omega), \quad \|g\|_{L^{p'}} = 1 \right\}.$$

Proof. This is a consequence of Theorem 1.24 when $p \neq \infty$. However, we can construct explicitly an element g_0 for the optimum, in the case $1 < p < \infty$. Consider $g_0 := \frac{1}{\|f\|_{L^p}^{p-1}} |f|^{p-2}\overline{f}$, so that $|g_0| = \frac{1}{\|f\|_{L^p}^{p-1}} |f|^{p-1}$. Since $f \in L^p(\Omega)$ and since $p' = p/(p-1)$, we have $g_0 \in L^{p'}(\Omega)$ with

$$\|g_0\|_{L^{p'}}^{p'} = \int_\Omega |g_0|^{p'} = \int_\Omega |g_0|^{\frac{p}{p-1}} = \frac{1}{\|f\|_{L^p}^p} \int_\Omega |f|^p = 1.$$

On the other hand, we have

$$\int_\Omega g_0 f = \frac{1}{\|f\|_{L^p}^{p-1}} \int_\Omega |f|^p = \|f\|_{L^p},$$

and the result follows. □

1.4 Topologies of L^p spaces

We now focus on the different topologies of L^p spaces.

1.4.1 Basics in topologies in Banach spaces

Let E be a Banach space. We can consider several topologies on E. The more natural one is the **strong topology**, defined by the following notion of convergence:

$$x_n \xrightarrow[n\to\infty]{} x, \quad \text{iff} \quad \|x_n - x\|_E \to 0. \qquad (\textbf{strong convergence}).$$

We can also define the **weak topology** of E. This one is defined by the following notion:

$$x_n \xhookrightarrow[n\to\infty]{\text{weak}} x, \quad \text{iff} \quad \forall L \in E^*, \quad \langle L, x_n - x\rangle_{E^*,E} \to 0. \qquad (\textbf{weak convergence}).$$

Finally, we sometime use the **weak-$*$ topology**. This only applies if $E = F^*$ is already the dual space of another Banach space F. Then

$$x_n \xhookrightarrow[n\to\infty]{\text{weak-}*} x, \quad \text{iff} \quad \forall f \in F, \quad \langle x_n - x, f\rangle_{F^*,F} \to 0. \qquad (\textbf{weak-* convergence if } E = F^*).$$

The weak-* topology will be used in the space $E = L^\infty(\Omega)$, which is the dual of $F = L^1(\Omega)$: a sequence (f_j) in $L^\infty(\Omega)$ converges weakly-* to $f_* \in L^\infty(\Omega)$ if, for all $g \in L^1(\Omega)$, we have $\int_\Omega g f_j \to \int_\Omega g f_*$.

Theorem 1.29

If $x_n \to x$ strongly in E, then $x_n \to x$ weakly-(*).
If E is reflexive, then weak-* convergence is equivalent to weak convergence.
If $x_n \to x$ for any of these topologies, then (x_n) is bounded in E.
If $x_n \to x$ strongly in E, and $L_n \to L$ weakly-(*) in E^*, then $L_n(x_n) \to L(x)$ in $\mathbb{C}$.

Proof. For the first point, we write that

$$|L(x_n - x)| \le \|L\|_{E^*} \|x_n - x\|_E \xrightarrow[n\to\infty]{} 0.$$

For the second point, we admit that since $E = F^*$ is reflexive, then so if F. This gives $F = F^{**} = E^*$, and the result follows. The third point is a non trivial result which we also admit (see [Bre99, Proposition III.12]). Finally for the last point, (L_n) converges to L, so it is bounded in E^*. This gives

$$|L_n(x_n) - L(x)| \le |L_n(x_n - x)| + |(L_n - L)(x)| \le \left(\max_n \|L_n\|_{E^*}\right) \|x_n - x\|_E + |(L_n - L)(x)|.$$

Let $n \to \infty$, and the result follows. □

1.4.2 Weak convergence which are not strong in L^p

Let $1 \le p < \infty$, and consider the special case $E = L^p(\mathbb{R})$. There are several ways for a subsequence (f_n) to weakly converge to f, and to not strongly converge to f.

- The mass goes to infinity;
- The mass vanishes;
- Oscillations.

We already saw the first two ones (see the counter examples after Theorem 1.11). For oscillations, consider $f_0 \in C_0^\infty(\mathbb{R})$ and set

$$f_n(x) := \mathrm{e}^{inx} f_0(x),$$

which is in $L^p(\mathbb{R})$. For all $\psi \in C_0^\infty(\mathbb{R}) \subset L^{p'}$, we have

$$\langle f_n, \psi \rangle_{L^p, L^{p'}} = \int f_n \psi = \int_{\mathbb{R}} \mathrm{e}^{inx} \psi(x) f_0(x) \mathrm{d}x.$$

We recognise the Fourier transform of the smooth function $f_0\psi$. By the Riemann-Lebesgue theorem, this integral goes to 0 as $n \to \infty$. This proves that, for all $\psi \in C_0^\infty(\mathbb{R})$, we have $\int \psi f_n \to 0$ as $n \to \infty$. By density of $C_0^\infty(\mathbb{R})$ in $L^{p'}(\mathbb{R})$, we deduce that $f_n \to 0$ weakly in $L^p(\mathbb{R})$. However, we always have $\|f_n\|_{L^p} = \|f_0\|_{L^p}$, so the convergence is not strong.

1.4.3 Banach-Alaoglu theorem in L^p

The importance of the weak (or weak-*) topology comes from the following Theorem (see [LL01, Theorem 2.18]). We state it in the case of the L^p spaces, but it can be generalised to any reflexive separable Banach space E (with the same proof).

Theorem 1.30: Banach-Alaoglu theorem (in $L^p(\Omega)$)

Let $1 < p < \infty$. If (f_n) is a bounded sequence in $L^p(\Omega)$, then there is a subsequence $\phi(n)$ and an element $f \in L^p(\Omega)$ so that $(f_{\phi(n)})$ weakly converges to f.
Case $p = \infty$. If (f_n) is a bounded sequence in $L^\infty(\Omega)$, then there is a subsequence $\phi(n)$ and an element $f \in L^\infty(\Omega)$ so that $(f_{\phi(n)})$ weakly-* converges to f.

In other words, bounded sequences have weak-limits up to subsequences. We also say that the unit ball of L^p is (relatively) compact for the weak topology.
The theorem **fails** if one replaces the weak convergence with the strong one. For instance, let (f_j) is a sequence which converges weakly to some f, but which does not converge strongly (see the previous counterexamples), then (f_j) is bounded by Theorem 1.29, but does not converge strongly.

This theorem also **fails** for $p = 1$. Indeed, consider for instance a smoothing sequence (j_ε) as in Section 1.2.4. The sequence point-wise converges to 0 a.e., so the weak-limit, if exists, can only be 0 (why?). However, taking the constant function $1 \in L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))^*$, we have

$$\langle j_\varepsilon, 1 \rangle_{L^1, L^\infty} = \int_{\mathbb{R}} j_\varepsilon = \|j_\varepsilon\|_{L^1} = 1,$$

and the sequence does not go to 0 as $\varepsilon \to 0$.

Proof. We set $K := \sup_{n \in \mathbb{N}} \|f_n\|_{L^p}$. The dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$ with $1 < p' < \infty$, and $L^{p'}(\Omega)$ is separable. Let (g_j) be a dense countable family in $L^{p'}(\Omega)$. We apply a Cantor diagonal argument to the family (g_j).

• The sequence $g_1(f_n) := \int_\Omega g_1 f_n$ satisfies $|g_1(f_n)| \le \|f_n\|_p \|g_1\|_{p'} \le K\|g_1\|_{p'}$, hence is bounded in $\mathbb{C}$, so there is a subsequence ϕ_1 so that $g_1(f_{\phi_1(n)})$ converges to some $C_1 \in \mathbb{C}$. In addition, $|C_1| \le K\|g_1\|_{L^{p'}}$.

• The sequence $g_2(f_{\phi_1(n)}) := \int_\Omega g_2 f_{\phi_1(n)}$ is bounded in $\mathbb{C}$, so there is a subsequence ϕ_2 so that $g_2(f_{\phi_1(\phi_2(n))})$ converges to some $C_2 \in \mathbb{C}$. In addition, $|C_2| \le K \|g_2\|_{L^{p'}}$.
We go on, and construct a family of subsequences ϕ_j so that $g_j\left(f_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_j(n)}\right)$ converges to some $C_j \in \mathbb{C}$ as $n \to \infty$ with $|C_j| \le K \|g_j\|_{L^{p'}}$. Finally, we set

$$\phi(n) := \phi_1(\phi_2 \cdots (\phi_n(n))).$$

By construction, for all $j \in \mathbb{N}$, the sequence $\int g_j f_{\phi(n)}$ converges to $C_j \in \mathbb{C}$ as $n \to \infty$. We now introduce the functional $L : L^{p'}(\Omega) \to \mathbb{C}$ by

$$\forall g \in L^{p'}(\Omega), \quad L(g) := \lim_{n\to\infty} \int_\Omega g f_{\phi(n)}.$$

This functional is linear, and, on the dense set (g_j), we have

$$|L(g_j)| = |C_j| \le K \|g_j\|_{L^{p'}}.$$

By density, we deduce that L is a continuous linear form on $L^{p'}(\Omega)$, with $\|L\|_{\mathrm{op}} \le K$. In other words, $L \in (L^{p'}(\Omega))^* = L^p(\Omega)$. So, by Theorem 1.27, there is $f \in L^p(\Omega)$ so that $L(g) = \int_\Omega fg$. This proves that

$$\forall g \in (L^p(\Omega))^*, \quad \lim_{n\to\infty} \int_\Omega g f_{\phi(n)} = \int_\Omega g f,$$

hence $f_{\phi(n)}$ weakly converges to f in $L^p(\Omega)$.

The proof in the case $p = \infty$ is similar. This time, we use that $L^1(\Omega)$ is separable and that $L^\infty(\Omega) = L^1(\Omega)^*$. □

1.5 Lower-semi continuity and convexity

Let E be a Banach space. We say that a map $J : E \to \mathbb{R}$ is **lower-semi-continuous** (l.s.c.) if

$$x_n \to x \quad \text{implies} \quad J(x) \le \liminf_{n\to\infty} J(x_n).$$

The convergence $x_n \to x$ can be interpreted with several topologies: a function J can be **strongly l.s.c.**, or **weakly l.s.c.** (or even **weakly-* lsc**). If a map J is weakly l.s.c., then it is strongly l.s.c. .

Theorem 1.31: Lower semi-continuity for convex functions

Let E be a Banach space. If $J : E \to \mathbb{R}$ is convex, then J is strongly l.s.c. iff J is weakly l.s.c. . In particular, the map $\|\cdot\|_E$ is weakly l.s.c.: If $x_n \to x$ weakly in E, then $\|x\|_E \le \liminf \|x_n\|_E$.

Proof. We admit that if J is convex and strongly l.s.c., then, there is a continuous linear operator L_x (called **support plane**) so that

$$\forall y \in E, \quad J(y) \ge J(x) + L_x(y - x).$$

(Think of L_x as the differential $DJ(x)$). In particular, if (x_n) weakly converges to x, we have

$$J(x_n) \ge J(x) + L_x(x_n - x), \quad \text{hence} \quad \liminf_{n\to\infty} J(x_n) \ge \liminf_{n\to\infty} \left(J(x) + L_x(x_n - x)\right) = J(x).$$

The norm map $\|\cdot\|$ is convex (this comes from the triangle inequality), and, *by definition*, it is strongly continuous, hence strongly l.s.c. So $\|\cdot\|_E$ is weakly l.s.c. □

In the case $E = L^p(\Omega)$, we have a stronger result (see [LL01, Theorem 2.11]).

Theorem 1.32

In the case $1 < p < \infty$, if $f_n \to f$ weakly in $L^p(\Omega)$, then $\|f\|_{L^p} \le \liminf \|f_n\|_{L^p}$.
In addition, if $\|f\|_{L^p} = \lim \|f_n\|_{L^p}$, then the convergence is strong.

Proof. The first point is the previous Theorem in the case $E = L^p(\Omega)$. We prove the second point only in the case $p = 2$. If $\|f_n\|_{L^2} \to \|f\|_{L^2}$, then we have

$$\|f - f_n\|_{L^2}^2 = \|f\|_{L^2}^2 + \|f_n\|_{L^2}^2 - 2\mathrm{Re}\int_\Omega \overline{f} f_n \xrightarrow[n\to\infty]{} \|f\|_{L^2}^2 + \|f\|_{L^2}^2 - 2\mathrm{Re}\int_\Omega \overline{f} f = 0.$$

□

1.6 Additional exercices

Exercice 1.33

Let $1 \le p < \infty$, and let $f \in L^p(\mathbb{R}^d)$. Prove that for all $\varepsilon > 0$, there is $R > 0$ so that

$$\int_{\mathcal{B}(0,R)^c} |f(x)|^p \mathrm{d}x < \varepsilon.$$

Exercice 1.34

Let $1 \le p < \infty$, and let $f \in L^p(\mathbb{R}^d)$. Prove that for all $\varepsilon > 0$, there is $h^* > 0$ so that

$$\forall h \in \mathcal{B}(0, h^*), \quad \int_{\mathbb{R}^d} |f(x-h) - f(x)|^p \, \mathrm{d}x < \varepsilon.$$

Exercice 1.35

Let $f \in C_0^\infty(\Omega)$. For $1 \le p < \infty$, we denote by $\alpha := 1/p \in (0,1)$. Prove that the following map in convex on $(0,1]$:

$$\alpha \mapsto \log\left(\|f\|_{L^{\frac{1}{\alpha}}}\right).$$

Exercice 1.36

Let (f_j) be a sequence in $L^p(\Omega)$ which converges pointwise a.e. to f, and which converges weakly to g in $L^p(\Omega)$. Prove that $f = g$.

CHAPTER 2

DISTRIBUTIONS

In this chapter, we introduce the set of distributions $\mathcal{D}'(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$. Some references are [LL01, Chapter 6], and the original book by Schwartz [Sch66]. In the references [Bre99; Eva10], only the Sobolev spaces $W^{m,p}(\Omega)$ are defined (not all distributions).

A distribution is a weak notion of "functions", which we can differentiate any number of times. This allows to prove that some equations have solutions "*in the distributional sense*". These solutions may not be functions (although they will be in most cases).

2.1 Distributions

2.1.1 Definition and examples

We denote by $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ the set of smooth functions with compact support in $\Omega \subset \mathbb{R}^d$, with the following notion of convergence/topology: A sequence $(\phi_n) \in \mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ if:

- there is a *fixed* compact set $K \subset \Omega$ so that the support of $\phi_n - \phi$ is contained in K for all n.
- for all $\alpha \in \mathbb{N}^d$, we have,
$$\sup_{x \in K} |D^\alpha \phi_n(x) - D^\alpha \phi(x)| \xrightarrow[n\to\infty]{} 0.$$

Theorem 2.1

If $\phi \in \mathcal{D}(\Omega)$, then $\phi * j_\varepsilon$ converges to ϕ in $\mathcal{D}(\Omega)$ as $\varepsilon \to 0$.

Proof. Let $\phi \in \mathcal{D}(\Omega)$ with support $K_\phi \subset \Omega$, and let $\delta := \mathrm{dist}(K, \partial\Omega) > 0$. For all $0 < \varepsilon < \delta$, the smooth functions $\phi * j_\varepsilon$ all have support in $K_\phi + \mathcal{B}(0,\delta) =: K \subset \Omega$. For $x \in K$, we have, using that $\int j_\varepsilon = 1$ and the change of variable $z = y/\varepsilon$,

$$\begin{aligned} |\phi * j_\varepsilon(x) - \phi(x)| &= \left| \int_{\mathbb{R}^d} j_\varepsilon(y)\,[\phi(x-y) - \phi(x)]\,\mathrm{d}y \right| \le \int_{\mathbb{R}^d} j_\varepsilon(y)\,|\phi(x-y) - \phi(x)|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} j(z)\,|\phi(x+\varepsilon z) - \phi(x)|\,\mathrm{d}z \le \varepsilon \|\nabla\phi\|_\infty \int_{\mathbb{R}^d} j(z) z \mathrm{d}z, \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. This proves the uniform convergence $\sup_{x\in K} \|\phi * j_\varepsilon - \phi\|_\infty \to 0$. Similarly, $D^\alpha(\phi * j_\varepsilon) = (D^\alpha \phi) * j_\varepsilon$ converges uniformly to $D^\alpha\phi$ on K. Hence $\phi * j_\varepsilon \to \phi$ in $\mathcal{D}(\mathbb{R}^d)$.

□

Definition 2.2 (Distribution). *A* **distribution** T *is a linear map from* $\mathcal{D}(\Omega) \to \mathbb{C}$, *which is continuous in the following sense: for all* $\phi \in \mathcal{D}(\Omega)$ *and all sequences* (ϕ_n) *that converges to* ϕ *in* $\mathcal{D}(\Omega)$, *we have*

$$T(\phi_n) \to T(\phi).$$

The set of distributions is denoted by $\mathcal{D}'(\Omega)$. We also write

$$\langle T, \phi \rangle_{\mathcal{D}',\mathcal{D}} := T(\phi).$$

Example 2.3 (Dirac mass). *The map* $\delta_0 : \phi \mapsto \phi(0)$ *is a distribution, called the* Dirac mass.

Locally integrable functions

For $1 \le p \le \infty$, we set

$$L^p_{\mathrm{loc}}(\Omega) := \{ f \quad \text{measurable on } \Omega \text{ such that, for all compact } K \subset \Omega, \quad f \in L^p(K) \}. \qquad (2.1)$$

The sets $L^p_{\mathrm{loc}}(\Omega)$ **are not normed spaces**. Of course, $L^p(\Omega) \subset L^p_{\mathrm{loc}}(\Omega)$. In addition, by Hölder's inequality, and since K is a bounded set, we always have

$$\| f \|_{L^1(K)} \le \int_K |f| = \int_K 1|f| \le \left(\int_K 1^{p'} \right)^{1/p'} \left(\int_K |f|^p \right)^{\frac{1}{p}} = |K|^{1/p'} \| f \|_{L^p(K)}.$$

This proves that

$$L^p(\Omega) \subset L^p_{\mathrm{loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega),$$

so the space $L^1_{\mathrm{loc}}(\Omega)$ contains all the $L^p(\Omega)$ spaces: if a result is true for all $f \in L^1_{\mathrm{loc}}(\Omega)$, it is also true for $f \in L^p_{\mathrm{loc}}(\Omega)$ and for $f \in L^p(\Omega)$.

Exercice 2.4

Prove that if $1 \le p \le q \le \infty$, then $L^q_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$.

The following theorem shows that one can distinguish L^1_{loc} functions (hence all L^p functions) among distributions.

Theorem 2.5: L^1_{loc} functions are determined by distributions

If $f \in L^1_{\mathrm{loc}}(\Omega)$, then $T_f : \phi \mapsto \int_\Omega f\phi$ is a distribution.
If $f, g \in L^1_{\mathrm{loc}}(\Omega)$, then $T_f = T_g$ in $\mathcal{D}'(\Omega)$ iff $f = g$ a.e.

Proof. For the first part, we write that

$$|T_f(\phi_n) - T_f(\phi)| = \left| \int_K f\,[\phi_n - \phi] \right| \le \| f \|_{L^1(K)} \sup_K |\phi_n - \phi| \to 0.$$

We now prove the second part. Consider the inner neighbourhood of Ω,

$$\Omega_\delta := \{ x \in \Omega, \ \forall y \in \mathcal{B}(0,\delta), \ x + y \in \Omega \}.$$

Let (j_ε) be a smoothing sequence. We fix $x \in \Omega_\delta$, and for $0 < \varepsilon < \delta$, we set $\phi(y) := j_\varepsilon(x - y) \in \mathcal{D}(\Omega)$. Then

$$T_f(\phi) = \int_\Omega f(y)\phi(y)\mathrm{d}y = \int_\Omega f(y) j_\varepsilon(x-y)\mathrm{d}y = \int_{\mathbb{R}^d} f(y) j_\varepsilon(x-y)\mathrm{d}y = f * j_\varepsilon(x).$$

So, if $T_f = T_g$, we have $f * j_\varepsilon = g * j_\varepsilon$ on Ω_δ. Taking $\varepsilon \to 0$ gives $f = g$ a.e. in Ω_δ by Theorem 1.21. Finally, taking $\delta \to 0$ gives $f = g$ a.e. on Ω. $\square$

Convergence of distributions

We say that a sequence (T_n) of distributions converges to T **in the distributional sense**, or in $\mathcal{D}'(\Omega)$, if, for all $\phi \in \mathcal{D}(\Omega)$, we have $T_n(\phi) \to T(\phi)$.

Example 2.6. *If (j_ε) is a smoothing sequence, then $j_\varepsilon \to \delta_0$ in $\mathcal{D}'(\Omega)$. Indeed, if $\phi \in \mathcal{D}(\Omega)$, then, for $x \in \Omega_\delta$ and $0 < \varepsilon < \delta$, we have, as in Theorem 2.1.*

$$|T_{j_\varepsilon}(\phi) - \delta_0(\phi)| = \left| \int_{\mathbb{R}^d} j_\varepsilon(x) \left(\phi(x) - \phi(0)\right) \mathrm{d}x \right| \le \int_{\mathbb{R}^d} j(z) \left|\phi(\varepsilon z) - \phi(0)\right| \mathrm{d}z \le \varepsilon \|\nabla \phi\|_\infty \int_{\mathbb{R}^d} j(z) z \mathrm{d}z,$$

which goes to 0 as $\varepsilon \to 0$. This holds for all $\phi \in \mathcal{D}(\Omega)$, so $T_{j_\varepsilon} \to \delta_0$, or equivalently, $j_\varepsilon \to \delta_0$ in $\mathcal{D}'(\Omega)$.

2.1.2 Operations on distributions

Mimicking what happens for smooth functions, we can define many operations for distributions.

Derivatives

If $f \in C^1(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, we have the integration by part formula (there are no boundary terms since ϕ is compactly supported)

$$\int_\Omega (\partial_{x_i} f)\, \phi = - \int_\Omega f\, (\partial_{x_i} \phi)\, .$$

We extend this property and *define* the derivative of a distribution as follows.

Definition 2.7 (Derivative of a distribution). *If $T \in \mathcal{D}'(\Omega)$, the D^α derivative of T is the distribution noted $D^\alpha T$ and defined by*

$$\left\langle D^\alpha T, \phi \right\rangle_{\mathcal{D}',\mathcal{D}} := (-1)^{|\alpha|} \left\langle T, D^\alpha \phi \right\rangle_{\mathcal{D}',\mathcal{D}} .$$

Remark 2.8. *Since ϕ is smooth, we may use Schwarz' Lemma on ϕ, and deduce that*[1] $\partial^2_{xy} T = \partial^2_{yx} T$.

Theorem 2.9

If (T_m) converges to T in $\mathcal{D}'(\Omega)$, then for all $\alpha \in \mathbb{N}^d$, $(D^\alpha T_m)$ converges to $D^\alpha T$ in $\mathcal{D}'(\Omega)$.

In other words, all derivatives to T_m converge to the associate derivative of T.

Proof. For all $\phi \in \mathcal{D}(\Omega)$, we have $D^\alpha \phi \in \mathcal{D}(\Omega)$, so

$$\left\langle D^\alpha T_m, \phi \right\rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|} \left\langle T_m, D^\alpha \phi \right\rangle_{\mathcal{D}',\mathcal{D}} \xrightarrow[m \to \infty]{} (-1)^{|\alpha|} \left\langle T, D^\alpha \phi \right\rangle_{\mathcal{D}',\mathcal{D}} = \left\langle D^\alpha T, \phi \right\rangle .$$

$\square$

Multiplication by a smooth function

Let g be a $C^\infty(\Omega)$ function (not necessarily compactly supported). If $\phi \in \mathcal{D}(\Omega)$, then $g\phi \in \mathcal{D}(\Omega)$ as well, so we can define the distribution $gT = Tg$ by

$$\langle gT, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle Tg, \phi \rangle_{\mathcal{D}',\mathcal{D}} := \langle T, g\phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

Again, one can check that gT is indeed a distribution.

Exercice 2.10

Prove that $\partial_{x_i}(gT) = (\partial_{x_i} g)T + g(\partial_{x_i} T)$ in $\mathcal{D}'(\Omega)$.

[1] To quote a teacher of mine: "*students like distributions, since they are infinitely differentiable, and the order of derivatives does not matter*".

Convolution

We focus here on the case $\Omega = \mathbb{R}^d$, although one can generalise to any $\Omega \subset \mathbb{R}^d$ by first checking the support of functions (the convolution of two compactly supported function is compactly supported).

For all f, g, ϕ, we have, by the change of variable $(x, z) = (x, y - x)$, that

$$\int_{\mathbb{R}^d} (f * g)\phi := \iint_{(\mathbb{R}^d)^2} f(x)g(y-x)\phi(y)\mathrm{d}y\mathrm{d}x = \iint_{(\mathbb{R}^d)^2} f(x)\phi(z)g(z-x)\mathrm{d}x\mathrm{d}z = \int_{\mathbb{R}^d} f(\widetilde{g} * \phi),$$

where we set $\widetilde{g}(x) := g(-x)$ (the reflection). In addition, if $\phi \in \mathcal{D}(\mathbb{R}^d)$ and g is an $L^1(\mathbb{R}^d)$ function with compact support, then $\tilde{g} * \phi \in \mathcal{D}(\mathbb{R}^d)$ (see Theorem 1.21). This suggests to define, for all $g \in L^1(\mathbb{R}^d)$ with compact support, the distribution $g * T = T * g$ by

$$\langle T * g, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle g * T, \phi \rangle_{\mathcal{D}',\mathcal{D}} := \langle T, \phi * \widetilde{g} \rangle_{\mathcal{D}',\mathcal{D}}.$$

Again, one can check that $g * T$ is indeed a distribution.

Exercice 2.11

Prove that $\partial_{x_i}(g * T) = (\partial_{x_i} g) * T = g * (\partial_{x_i} T)$ in $\mathcal{D}'(\mathbb{R}^d)$.

Theorem 2.12: Convolution smooths distributions

Let (j_ε) be a smoothing sequence. Then the distribution $T_\varepsilon := T * j_\varepsilon$ converges to T in $\mathcal{D}'(\mathbb{R}^d)$. In addition, T_ε can be identified with a $C^\infty(\mathbb{R}^d)$ function, and $D^\alpha T_\varepsilon = T * (D^\alpha j_\varepsilon)$.

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^d)$ with support K, we have, using that $\widetilde{j_\varepsilon} = j_\varepsilon$,

$$T_\varepsilon(\phi) - T(\phi) = \langle T, \phi * j_\varepsilon - \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

We proved in Theorem 2.1 that $\phi * j_\varepsilon \to \phi$ in $\mathcal{D}$. So, by definition of a distribution, we have $T_\varepsilon(\phi) \to T(\phi)$. Since this holds for all $\phi \in \mathcal{D}$, we deduce that $T_\varepsilon \to T$ in $\mathcal{D}'$.

We now prove that T_ε can be seen as a smooth function. Assume first that T is a smooth function. Then

$$t_\varepsilon(x) := T * j_\varepsilon(x) = \int_\Omega T(y) j_\varepsilon(x-y)\mathrm{d}y = \langle T, j_\varepsilon(x - \cdot) \rangle_{\mathcal{D}',\mathcal{D}}.$$

and, similarly,

$$D^\alpha t_\varepsilon(x) = \langle T, (D^\alpha j_\varepsilon)(x - \cdot) \rangle_{\mathcal{D}',\mathcal{D}}.$$

The reader can check that these manipulations are still valid when T is any distribution, in the sense that indeed $T * j_\varepsilon = t_\varepsilon$ in $\mathcal{D}'(\mathbb{R}^d)$. In addition, t_ε is infinitely differentiable. $\square$

Exercice 2.13

Prove that $(D^\alpha T) * j_\varepsilon = T * (D^\alpha j_\varepsilon) = D^\alpha (T * j_\varepsilon)$.

A similar result holds in $\Omega \subset \mathbb{R}^d$ by using techniques similar to the proof of Theorem 2.5. We deduce the following.

Theorem 2.14: Density of smooth functions

The set $C^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$: for all $T \in \mathcal{D}'(\Omega)$, there is a sequence $(T_n) \in C^\infty(\Omega)$ so that $T_n \to T$ in $\mathcal{D}'(\Omega)$.

As an example of how to use the last theorem, we state the following.

Theorem 2.15: Constant distributions

Let Ω be connected, and $T \in \mathcal{D}'(\Omega)$ be such that $\nabla T = 0$. Then T is a constant: there is $C \in \mathbb{C}$ so that $T(\phi) = C \int \phi$.

Proof. We introduce $T_\varepsilon := T * j_\varepsilon$, and we have

$$\nabla T_\varepsilon = \nabla(T * j_\varepsilon) = j_\varepsilon * (\nabla T) = 0.$$

Since T_ε is a smooth function with null derivatives, we have $T_\varepsilon = cst(\varepsilon)$. Letting $\varepsilon \to 0$ proves the result. □

2.2 Example: the Poisson's equation

2.2.1 Integration by parts on domains

We recall here the so-called divergence formula. The notion of **domain with boundary of class C^1** will be detailed later in Section 4.3.2. We start with the divergence theorem.

Theorem 2.16: Divergence theorem

Let Ω be an open set of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. For all F of class $C^1(\overline{\Omega}, \mathbb{R}^d)$, we have

$$\int_\Omega \mathrm{div}(F) = \int_{\partial\Omega} F \cdot \nu \, \mathrm{d}\omega,$$

where ν is the outward normal of Ω, and where $\mathrm{d}\omega$ is the surface measure on $\partial\Omega$.

We admit the result, since its proof needs the definitions of surface measure, of smooth domains, and so on. In the sequel however, we use the theorem mainly in the case where Ω is a ball $\Omega = \mathcal{B}(0, x)$, in which case all these notions are easily understood.

One important application of this formula is the second Green's identity.

Theorem 2.17: Second Green's identity

Let Ω be an open set of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. For all $A, B \in C^2(\overline{\Omega}, \mathbb{R})$, we have

$$\int_\Omega (\Delta A)B - A(\Delta B) = \int_{\partial\Omega} (B\nabla A - A\nabla B) \cdot \nu \, \mathrm{d}\omega.$$

Proof. Take $F = (\nabla A)B - A(\nabla B)$ and apply the Divergence Theorem 2.16. □

2.2.2 Green's functions and the Poisson's equation

We define the following Green's functions. Recall that $d \in \mathbb{N}$ is the dimension.

$$\begin{cases} G_0(x) := -\frac{1}{2}|x|, & \text{if} \quad d = 1, \\ G_0(x) := -\frac{1}{2\pi}\ln(|x|), & \text{if} \quad d = 2, \\ G_0(x) := \frac{1}{4\pi}\frac{1}{|x|}, & \text{if} \quad d = 3, \\ G_0(x) := \dfrac{1}{(d-2)|\mathbb{S}^{d-1}|}\dfrac{1}{|x|^{d-2}}, & \text{if} \quad d > 3 \end{cases}$$

(Actually, the last formula is also valid for $d = 1$ and $d = 3$).

Theorem 2.18: Distributional Laplacian of Green's functions

We have $-\Delta G_0 = \delta_0$ in $\mathcal{D}'(\mathbb{R}^d)$, where δ_0 is the Dirac mass at 0.

Proof. We only do the proof in the case $d = 3$ for clarity. The function G_0 is smooth away from $x = 0$, and satisfies

$$\forall x \neq 0, \quad \nabla G_0(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}, \quad \Delta G_0(x) = 0. \tag{2.2}$$

It remains to check what happens at $x = 0$. First, G_0 is locally integrable since, for all $a > 0$, we have, using radial coordinates,

$$\int_{\mathcal{B}(0,a)} |G_0| = \frac{1}{4\pi}\int_{\mathcal{B}(0,a)} \frac{1}{|x|}\mathrm{d}x = \frac{1}{4\pi}|\mathbb{S}^2|\int_0^a \frac{1}{r}r^2\mathrm{d}r = \frac{a^2}{2}.$$

Take $\phi \in \mathcal{D}(\Omega)$ with support contained in $\mathcal{B}(0,R)$. We divide the ball $\mathcal{B}(0,R)$ into two sets, namely a small ball $\mathcal{D}(0,a)$, and the annulus $\{x \in \mathbb{R}^d, a \leq |x| \leq R\}$. The last equality shows that the integral $\int G_0(-\Delta\phi)$ on $\mathcal{B}(0,a)$ goes to 0 as $a \to 0$. We now evaluate the integral on the annulus. On this set, G_0 is smooth with $\Delta G_0 = 0$. The second Green's identity with $A = \phi$ and $B = G_0$ gives

$$\int_{a \leq |x| \leq R} (-\Delta\phi)G_0 = -\int_{|x|=a} [(\nabla\phi)G_0 - \phi(\nabla G_0)] \cdot \nu \, \mathrm{d}\omega.$$

Let us prove that the first part goes to 0 as $a \to 0$. We have (note that $|a\mathbb{S}^2| = 4\pi a^2$)

$$\left| \int_{|x|=a} (\nabla\phi)G_0 \cdot \nu \, \mathrm{d}\omega \right| \leq \left(\max_{\mathbb{R}^d} \|\nabla\phi\|\right) \int_{a\mathbb{S}^2} |G_0|\mathrm{d}\omega = \left(\max_{\mathbb{R}^d} \|\nabla\phi\|\right) \frac{|a\mathbb{S}^2|}{4\pi a} = \left(\max_{\mathbb{R}^d} \|\nabla\phi\|\right) a \xrightarrow[a\to 0]{} 0.$$

We now focus on the term $\phi(\nabla G_0)$. Using the explicit formula for ∇G_0 and that $\nu(x) = -\frac{x}{|x|}$, we get

$$\int_{|x|=a} \phi(\nabla G_0) \cdot \nu \mathrm{d}\omega = \frac{1}{4\pi a^2}\int_{a\mathbb{S}^2} \phi(\omega)\mathrm{d}\omega \xrightarrow[a\to 0]{} \phi(0),$$

where we used that ϕ is continuous in the last line, and where we recognised the average of ϕ on the sphere $a\mathbb{S}^2$. Hence, we proved that

$$\langle -\Delta G_0, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \phi(0) = \langle \delta_0, \phi\rangle_{\mathcal{D}',\mathcal{D}}.$$

Since this holds for all $\phi \in \mathcal{D}(\mathbb{R}^d)$, we get $-\Delta G_0 = \delta_0$ in the distributional sense, as wanted. □

We can now prove the general case.

Theorem 2.19: The Poisson equation in the whole space

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be such that $|f| * G_0$ is well-defined a.e.. Then $u := f * G_0$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and satisfies

$$-\Delta u = f, \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Formally, this follows from the computation

$$-\Delta\,(f * G_0) = f * (-\Delta G_0) = f * \delta_0 = f.$$

However, this line does not make sense if f is not smooth. We refer to [LL01, Theorem 6.21] for the full proof.

As we can see, the weakening of the notion of functions provides a simple answer for the existence of u solution to $-\Delta u = f$ on the whole space $\mathbb{R}^d$. In the sequel, we will prove that, if f is regular, then so is u. For now, let us just mention the following.

Lemma 2.20. *Assume $d \geq 3$. Let $q \geq \frac{d}{2}$, and assume that*

$$f \in L^{q-\varepsilon}(\mathbb{R}^d) \cap L^{q+\varepsilon}(\mathbb{R}^d) \quad \text{for some} \quad \varepsilon > 0.$$

Then $u = f * G_0$ is in $L^r(\mathbb{R}^d)$, where $r \geq 1$ is defined by

$$\frac{1}{q} - \frac{2}{d} =: \frac{1}{r}.$$

This lemma is a prototype! It can be generalised in many different ways.

Proof. Write

$$G_0(x) = G_0(x)\mathbb{1}(x < R) + G_0(x)\mathbb{1}(x \geq R).$$

Since $G_0(x) \approx |x|^{d-2}$, the first part is in $L^p(\mathbb{R}^d)$ for all $p < p_0 := \frac{d}{d-2}$, while the second part is in $L^p(\mathbb{R}^d)$ for all $p > p_0$. The result then follows from Young's inequality 1.19, and using that

$$\frac{1}{q} + \frac{d-2}{d} = 1 + \frac{1}{r}.$$

□

2.3 Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega)$

2.3.1 Definition

We proved that $L^p(\Omega) \subset L^p_{\rm loc}(\Omega) \subset L^1_{\rm loc}(\Omega) \subset \mathcal{D}'(\Omega)$. In particular, we can consider the distributional derivatives of L^p functions. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the **Sobolev spaces**

$$W^{m,p}(\Omega) := \left\{ f \in L^p(\Omega), \quad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \leq m, \quad D^\alpha f \in L^p(\Omega) \right\},$$

and the corresponding norm

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

For instance, we have

$$\|f\|_{W^{1,2}(\Omega)} = \left(\|f\|^2_{L^2} + \|\partial_{x_1} f\|^2_{L^2} + \cdots + \|\partial_{x_d} f\|^2_{L^2}\right)^{1/2} = \left(\|f\|^2_{L^2} + \|\nabla f\|^2_{(L^2)^d} \right)^{1/2}.$$

Note that $W^{0,p}(\Omega) = L^p(\Omega)$. When $p = 2$, we set

$$H^m(\Omega) := W^{m,2}(\Omega).$$

The H^m spaces will play a special role, since they will be Hilbert spaces when equipped with the natural inner product

$$\langle f, g \rangle_{H^m(\Omega)} := \int_\Omega \overline{f} g + \sum_{|\alpha| \leq m} \int_\Omega \overline{D^\alpha f} \, D^\alpha g.$$

2.3.2 Completion of Sobolev spaces

As one can expect, Sobolev spaces are complete (see [Bre99, Proposition VIII.1]).

Theorem 2.21: The Sobolev spaces $W^{m,p}(\Omega)$ are complete.

For all $m \in \mathbb{N}$ and all $1 \le p \le \infty$, the set $W^{m,p}(\Omega)$ is a Banach space. It is separable if $p < \infty$, and reflexive if $1 < p < \infty$.
In particular, $H^m(\Omega)$ is a (separable) Hilbert space.

Proof. Let (f_j) be a Cauchy sequence in $W^{m,p}(\Omega)$. Then (f_j) is a Cauchy sequence in $L^p(\Omega)$, and, for all $|\alpha| \le m$, $(D^\alpha f_j)$ is a Cauchy sequence in $L^p(\Omega)$. The space $L^p(\Omega)$ being complete, there are $f \in L^p(\Omega)$ and $f_\alpha \in L^p(\Omega)$, so that

$$f_j \xrightarrow{L^p} f, \quad D^\alpha f_j \xrightarrow{L^p} f_\alpha.$$

It remains to prove that $f_\alpha = D^\alpha f$. Since we have convergence in L^p, we also have convergence in the distributional sense, that is $f_j \to f$ in $\mathcal{D}'(\Omega)$. In particular, by Theorem 2.9, we must have $D^\alpha f_j \to D^\alpha f$ in $\mathcal{D}'(\Omega)$. By uniqueness of the limit in $\mathcal{D}'(\Omega)$, we indeed have $f_\alpha = D^\alpha f$. This proves that $W^{m,p}(\Omega)$ is complete.

The map $f \in W^{m,p}(\Omega) \mapsto (D^\alpha f)_{|\alpha| \le m} \in (L^p(\Omega))^N$ with $N := \sharp\{\alpha \in \mathbb{N}^d, |\alpha| \le m\}$ is an isometry. So $W^{m,p}(\Omega)$ can be identified with a closed vectorial space of $(L^p(\Omega))^N$. In particular, $W^{m,p}(\Omega)$ is separable for $p < \infty$, and it is reflexive for $1 < p < \infty$. □

The reader might ask what is the dual space of $W^{1,p}(\Omega)$. We refer to [LL01, Theorem 6.24] and to the discussion in [Bre99, p.174] for this difficult (and not so interesting) question.

We also record the following (see [Eva10, Section 5.3.2]).

Theorem 2.22: Meyers-Serrin

Assume Ω is bounded. For all $m \in \mathbb{N}$ and all $1 \le p < \infty$, the set $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

We warn that the functions in $C^\infty(\Omega)$ are not necessarily smooth up to the boundary, and may explode at the boundary. Later in Theorem 4.10, we enunciate a much stronger result.

2.4 Sobolev spaces $W_0^{m,p}(\Omega)$ and $H_0^m(\Omega)$

2.4.1 Definition

We now study whether $C^\infty(\Omega)$ and/or $C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ is dense in $W^{m,p}(\Omega)$. Surprisingly, these two sets are usually **not** dense in Ω. In what follows, we define

$$W_0^{m,p}(\Omega) := \overline{C_0^\infty(\Omega)} \quad \text{the closure of } C_0^\infty(\Omega) \text{ for the norm of } W^{m,p}(\Omega).$$

Similarly, we set $H_0^m(\Omega) := W_0^{m,2}(\Omega)$. Since $W_0^{m,p}(\Omega)$ is a closed linear space of the Banach space $W^{m,p}(\Omega)$, it is a Banach space for the same norm.

Loosely speaking $H_0^1(\Omega)$ is the set of functions of $H^1(\Omega)$ that *vanish* at the boundary $\partial\Omega$. However, since the measure of $\partial\Omega$ is usually null, this only has an *affective* meaning.

Theorem 2.23: $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$

In the case $\Omega = \mathbb{R}^d$, we have equality $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$ for $1 \le p < \infty$.
In particular, $C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ and $C_0^\infty(\mathbb{R}^d)$ are dense in $W^{m,p}(\mathbb{R}^d)$ for $1 \le p < \infty$.

Proof. Let us first prove that $C^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$. Let $f \in W^{m,p}(\mathbb{R}^d)$, and set $f_\varepsilon := f * j_\varepsilon$ for a smoothing sequence j_ε. By Theorem 2.14, the functions f_ε are smooth. For all $|\alpha| \le m$, the function $D^\alpha f$ is in $L^p(\mathbb{R}^d)$, and we have $D^\alpha(f_\varepsilon) = (D^\alpha f) * j_\varepsilon$. By Theorem 1.21, we deduce that

$$\forall |\alpha| \le m, \quad \|(D^\alpha f) * j_\varepsilon - D^\alpha f\|_{L^p} \xrightarrow[\varepsilon \to 0]{} 0.$$

This already proves that $C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$.

For the second part, we take $f \in C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$, and set $f_n := \chi(x/n) f$, where χ is a smooth cut-off function satisfying $\chi(x) = 1$ for $|x| \le 1$. Then, $f_n \in C_0^\infty(\mathbb{R}^d)$. By the Dominated Convergence Theorem 1.11, we have $\|f_n - f\|_{L^p} \to 0$. Moreover, we have

$$\|\nabla f_n - \nabla f\|_{L^p} = \left\| \frac{1}{n} (\nabla \chi) \left(\frac{x}{n} \right) f + (\nabla f) \left[\chi(x/n) - 1 \right] \right\|_{L^p} \le \frac{1}{n} \|\nabla \chi\|_\infty \|f\|_{L^p} + \|(\nabla f)[\chi(x/n) - 1]\|_{L^p},$$

and the last term goes to 0 with the Dominated Convergence Theorem again. So $\|\nabla f_n - \nabla f\|_{L^p} \to 0$ as well. We go on with all derivatives, which proves that $\|D^\alpha f_n - D^\alpha\|_{L^p} \to 0$ for all $|\alpha| \le m$. This shows that $f_n \to f$ in $W^{m,p}(\mathbb{R}^d)$. □

For $1 < p < \infty$, the dual space of $W_0^{m,p}(\Omega)$ is noted $W^{-m,p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, as in (1.2). By density of $\mathcal{D}(\Omega)$ in $W_0^{m,p}(\Omega)$, $f \in W^{-m,p'}(\Omega) \subset \mathcal{D}'(\Omega)$ iff there is a constant $C > 0$ so that

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \left| \int_\Omega f \phi \right| := |\langle f, \phi \rangle_{\mathcal{D}', \mathcal{D}}| \le C \|\phi\|_{W^{m,p}}.$$

2.4.2 Poincaré's inequalities

In the case where Ω is **bounded**, there are several important inequalities related to $W_0^{1,p}(\Omega)$.

Theorem 2.24: Poincaré's inequality

Let Ω be a bounded open set in $\mathbb{R}^d$ and let $1 \le p < \infty$. There is a constant $C = C(\Omega, p)$ so that, for all $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)}.$$

Proof. We denote by L the diameter of Ω. By density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, it is enough to prove the result for $u \in C_0^\infty(\Omega)$. Let $x \in \Omega$, and consider a point $a \in \partial\Omega$, so that $a_1 = x_1$ (same first coordinate). On the segment $[a, x]$, we have the point-wise bound

$$\begin{aligned} |u(x)| = |u(x) - u(a)| &\le \int_a^x |\partial_{x_1} u|(s, x_2, \cdots, x_N) \mathrm{d}s \le \int_a^{a+L} |\nabla u|(s, x_2, \cdots, x_N) \mathrm{d}s \\ &\le \left(\int_a^{a+L} 1^{p'} \mathrm{d}s \right)^{1/p'} \left(\int_a^{a+L} |\nabla u|^p (s, x_2, \cdots x_N) \mathrm{d}s \right)^{1/p} = L^{1/p'} \left(\int_a^{a+L} |\nabla u|^p (s, x_2, \cdots x_N) \mathrm{d}s \right)^{1/p} \end{aligned}$$

where we used Hölder's inequality in the last line. We take the p power and integrate. This gives, using Fubini, and the fact that the $\mathrm{d}x_1$ integration can be performed on a segment of size L.

$$\begin{aligned} \int_\Omega |u|^p &\le L^{p/p'} \int_\Omega \left(\int_a^{a+L} |\nabla u|^p (s, x_2, \cdots x_N) \mathrm{d}s \right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N \\ &\le L^{1+p/p'} \int |\nabla u|^p (s, x_2, \cdots x_N) \mathrm{d}s \mathrm{d}x_2 \cdots \mathrm{d}x_N = L^{1+p/p'} \int_\Omega |\nabla u|^p. \end{aligned}$$

□

Remark 2.25. *Poincaré's inequality is also valid if Ω is bounded only in 1-direction.*

Corollary 2.26. *If Ω is bounded, then the constant function $f(x) = C$ with $C \neq 0$* **is not** *in $W_0^{1,p}(\Omega)$: we cannot approximate constant functions by $C_0^\infty(\Omega)$ functions, for the $W^{1,p}(\Omega)$ norm.*

A consequence of Poincaré's inequality is that the map $u \mapsto \|\nabla u\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the usual $W^{1,p}(\Omega)$ norm. Indeed, we have

$$\|\nabla u\|_{L^p} \le \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{1/p} = \|u\|_{W^{1,p}} \le \left(C\|\nabla u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{1/p} = (C+1)^{1/p}\|\nabla u\|_{L^p}.$$

If $u \notin W_0^{m,p}(\Omega)$, we have a similar result, that we state for completeness. The proof uses the Rellich's Theorem 4.14 (see below).

Theorem 2.27: Poincaré-Wirtinger's inequality

Let Ω be a bounded **connected** open set of $\mathbb{R}^d$, with boundary $\partial\Omega$ of class C^1, and let $1 \le p < \infty$. There is a constant $C = C(\Omega, p)$ so that, for all $u \in W^{1,p}(\Omega)$, we have

$$\left\| u - \fint_\Omega u \right\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)},$$

where we set $\displaystyle\fint_\Omega u := \frac{1}{|\Omega|}\int_\Omega u$ the average of u.

Proof. Assume otherwise, and let $u_n \in W^{1,p}(\Omega)$ be such that

$$\left\| u_n - \fint_\Omega u_n \right\|_{L^p} \ge n\|\nabla u_n\|_{L^p}.$$

We set $w_n := \frac{u_n - \fint u_n}{\|u_n - \fint u_n\|_{L^p}}$, so that $\fint w_n = 0$ and

$$1 = \|w_n\|_{L^p} \ge n\|\nabla w_n\|_{L^p}.$$

The sequence (w_n) is bounded in $W^{1,p}(\Omega)$. By the Banach-Alaoglu theorem applied in the reflexive separable Banach space $W^{m,p}(\Omega)$, it converges weakly to some $w_* \in W^{1,p}(\Omega)$ up to a subsequence.

In particular, ∇w_n converges weakly to ∇w_* in L^p, and since the L^p norm is weakly lsc (see Theorem 1.32), we have $\|\nabla w_*\| \le \liminf \|\nabla w_n\| = 0$, so $\nabla w_* = 0$. By Theorem 2.15 and the fact that Ω is connected, we deduce that w_* is constant. In addition, also by weak-convergence, we have

$$0 = \int_\Omega w_n = \langle w_n, 1\rangle_{L^p, L^{p'}} \xrightarrow[n\to\infty]{} \langle w_*, 1\rangle_{L^p, L^{p'}} = \int_\Omega w_*.$$

This proves that $w_* = 0$.

However, by the Rellich's Theorem 4.14 (that we prove below), the sequence (w_n) also strongly converges to w_* in $L^p(\Omega)$, so we also have $\|w_*\|_{L^p} = \lim \|w_n\|_{L^p} = 1$, a contradiction. □

CHAPTER 3

HILBERT SPACES, AND LAX-MILGRAM THEOREM

In this chapter, we recall the basic theory of (separable) Hilbert spaces. We prove the Lax-Milgram theorem, and provide some examples of applications.

3.1 Hilbert spaces

A Hilbert space is a Banach space with an inner (or sesquilinear) product. Our convention is that the inner product is linear on the right, and antilinear on the left. The natural Hilbert space we will be working with is the $L^2(\Omega)$ space, with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega \overline{f} g.$$

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space. We recall the Cauchy-Schwarz inequality $|\langle f, g \rangle| \le \|f\| \cdot \|g\|$, and the parallelogram equality

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

A **countable orthonormal basis** of $\mathcal{H}$ is an orthonormal family of vectors $(e_1, e_2, \cdots)$ such that, for any $x \in \mathcal{H}$ there are complex coefficients $(x_i)_{i \in \mathbb{N}}$ so that

$$x = \sum_{i=1}^\infty x_i e_i, \quad \text{in the sense that} \quad \left\| x - \sum_{i=1}^N x_i e_i \right\|_{\mathcal{H}} \xrightarrow[N \to \infty]{} 0.$$

Taking the inner product with e_i gives $x_i = \langle x, e_i \rangle_{\mathcal{H}}$. Similarly, taking the inner product of x with itself shows the Parseval identity $\|x\|^2 = \sum |x_i|^2$. So

$$\boxed{x = \sum_{i=1}^\infty \langle x, e_i \rangle e_i. \quad \text{and} \quad \|x\|_{\mathcal{H}}^2 = \sum_{i=1}^\infty |\langle x, e_i \rangle|^2.}$$

Theorem 3.1. *If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{H}$ admits a countable basis.*

Proof. Let (x_n) be a dense set in $\mathcal{H}$, and consider $F_k := \mathrm{Ran}\{x_1, \cdots, x_k\}$, of dimension at most k. First erase all k so that $\dim F_k = \dim F_{k-1}$ to obtain a new sequence F_k so that $\dim F_k = k$. Then perform a Gram-Schmidt algorithm on the sequence F_k. □

In practice, all *interesting* Hilbert spaces are separable, and we only focus on the theory for separable Hilbert space. In what follows $\mathcal{H}$ is a separable Hilbert space.

3.2 Lax-Milgram theorem

Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bilinear (or sesquilinear) map. We say that a is **continuous** if there is $\beta > 0$ so that

$$\forall u, v \in \mathcal{H}, \quad |a(u,v)| \le \beta \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

We say that a is **coercive** if there is $\alpha > 0$ so that

$$\forall u \in \mathcal{H}, \quad a(u,u) \ge \alpha \|u\|_{\mathcal{H}}^2.$$

Finally, we say that a is **symmetric** (or **hermitian**) if

$$\forall u, v \in \mathcal{H}, \quad a(u,v) = \overline{a(v,u)}.$$

Coercivity is only meaningful when $a(u,u)$ is real-valued for all $u \in \mathcal{H}$. When a is symmetric, then $a(u,u)$ is always real-valued, so these two notions work well together.

Theorem 3.2: Lax-Milgram

Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bilinear form which is continuous and coercive, and let $L : \mathcal{H} \to \mathbb{C}$ be a continuous linear map. Then there is unique $u \in \mathcal{H}$ so that

$$\forall f \in \mathcal{H}, \quad a(u,f) = L(f).$$

In addition, if $\mathcal{H}$ is a real Hilbert space and a is symmetric, then u is the unique minimiser of $J : \mathcal{H} \to \mathbb{R}$ defined by

$$J(v) := \frac{1}{2} a(v,v) - L(v).$$

Note that the J map is real-valued only in real Hilbert space. In complex Hilbert space, we have $J(\lambda v) = \frac{|\lambda|^2}{2} a(v,v) - \lambda L(v)$, which is complex valued in general, due to the second term.

Before we give the proof, let us re-state this theorem in the case $\mathcal{H} = \mathbb{C}^n$ (finite dimension case). In this case, $a(\cdot,\cdot)$ and $L(\cdot)$ are of the form

$$a(u,v) := \langle Au, v \rangle_{\mathbb{C}^n}, \quad \text{and} \quad L(f) = \langle b, f \rangle_{\mathbb{C}^n}$$

for some $A \in \mathcal{M}_n(\mathbb{C})$ and $b \in \mathbb{C}^n$. In this case, $a(\cdot,\cdot)$ is always continuous (why?), and coercivity implies that A is injective. The equation $a(u,f) = L(f)$ is then equivalent to $Au = b$. Since A is injective, it is invertible, and we find $u = A^{-1}b$.

In the real case, if a is symmetric, then A is a symmetric matrix. Coercivity then implies $A > 0$. In particular, the map $J : \mathbb{R}^n \to \mathbb{R}$ defined by

$$J(f) := \frac{1}{2} \langle Af, f \rangle_{\mathbb{R}^d} - \langle b, f \rangle_{\mathbb{R}^d}$$

is strictly convex and coercive, hence admits a unique minimiser u_*. Solving $\nabla J(u) = 0$ proves that $u_* = A^{-1}b$.

The Lax-Milgram Theorem is somehow a generalisation for the invertibility of an operator.

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be a basis of $\mathcal{H}$, and consider the finite vectorial space

$$E_n := \mathrm{Vect}\{e_1, \cdots, e_n\}.$$

We consider the approximate problem with the restriction $a : E_n \times E_n \to \mathbb{R}$ and $L : E_n \to \mathbb{R}$. By the previous argument in finite dimension, there is unique $u_n \in E_n$ so that

$$\forall f \in E_n, \quad a(u_n, f) = L(f).$$

In addition, we have the *a priori* estimate

$$\alpha \|u_n\|_{\mathcal{H}}^2 \le a(u_n, u_n) = L(u_n) \le \|L\|_{\mathrm{op}} \|u_n\|_{\mathcal{H}}, \quad \text{so} \quad \|u_n\|_{\mathcal{H}} \le \alpha^{-1} \|L\|_{\mathrm{op}}.$$

The sequence (u_n) is bounded in the (separable) Hilbert space $\mathcal{H}$. By the Banach-Alaoglu Theorem 1.30, there is a subsequence $\phi(n)$, and an element $u \in \mathcal{H}$ so that $u_{\phi(n)} \to u$ weakly in $\mathcal{H}$. The weak-convergence already proves that

$$\forall f \in \mathcal{H}, \quad a(u, f) = L(f).$$

Let us prove uniqueness. If u_1, u_2 solves the equation, then

$$\begin{aligned} \alpha \|u_1 - u_2\|_{\mathcal{H}}^2 &\le a(u_1 - u_2, u_1 - u_2) = a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \\ &= L(u_1 - u_2) - L(u_1 - u_2) = 0. \end{aligned}$$

So there is a unique solution. In particular, the whole sequence (u_n) converges (weakly) to u.

Assume now that $\mathcal{H}$ is a real Hilbert space, and that a is symmetric. The function $J(f) := \frac{1}{2} a(f,f) - L(f)$ is (strongly) continuous by definition. It is strictly convex and coercive, so has a unique minimum $u_* \in \mathcal{H}$. Since J is differentiable, we must have $D_u J(u^*) = 0$, which gives

$$\forall f \in \mathcal{H}, \quad a(u_*, f) = L(f).$$

This proves that $u_* = u$.
Actually, if a is symmetric, then the convergence of (u_n) to u is strong. Indeed, we have

$$\begin{aligned} \alpha \|u - u_n\|_{\mathcal{H}}^2 &\le a(u - u_n, u - u_n) = a(u, u - 2u_n) + a(u_n, u_n) \\ &= L(u - 2u_n) + L(u_n) = L(u - u_n) \to 0. \end{aligned}$$

□

3.2.1 Application: Riesz' theorem

As a special case of Lax-Milgram theorem, we record the infamous Riesz' theorem. First, for $v \in \mathcal{H}$, we set $L_v : f \mapsto \langle v, f \rangle_{\mathcal{H}}$. We have $|L_v(f)| \le \|v\|_{\mathcal{H}} \|f\|_{\mathcal{H}}$ by Cauchy-Schwarz, so $L_v \in \mathcal{H}^*$. In addition, taking $f = v$ shows that $\|L_v\|_{\mathcal{H}^*} = \|v\|_{\mathcal{H}}$. This proves that $\mathcal{H} \subset \mathcal{H}^*$. The next result shows that we have equality.

Theorem 3.3: Riesz' theorem

For all $L \in \mathcal{H}^*$, there is a unique $v \in \mathcal{H}$ so that $L = L_v$. In particular, $\mathcal{H} \approx \mathcal{H}^*$.

In particular, we have $(L^2(\Omega))^* = L^2(\Omega)$ (this is the case $p = 2$ in Theorem 1.27).

Proof. Consider $a(u, v) = \langle u, v \rangle_{\mathcal{H}}$ the scalar product of $\mathcal{H}$. We have $\|u\|^2 = a(u, u)$, so a is coercive, and Cauchy-Schwarz inequality gives $|a(u, v)| \le \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$, so a is continuous. Lax-Milgram theorem applies, so there is $v \in \mathcal{H}$ so that $a(u, v) = L(v)$, which is Riesz' theorem. □

Example 3.4. *We take $\mathcal{H} = (H^1(\mathbb{R}), \| \cdot \|_{H^1})$, with the inner product*

$$\langle f, g \rangle_{\mathcal{H}} := \int_{\mathbb{R}} \overline{f} g + \overline{f'} g'.$$

We will see later in Theorem 4.5 that the linear map $\delta_0 : f \mapsto f(0)$ is continuous on $H^1(\mathbb{R})$. By the Riesz' theorem, there is $f_0 \in \mathcal{H}$ so that $\langle f, f_0 \rangle_{\mathcal{H}} = f(0)$. A computation reveals that

$$f_0(x) := \frac{1}{2} \mathrm{e}^{-|x|}.$$

Indeed, we have

$$\langle f_0, f\rangle_{H^1} = \frac{1}{2}\int_{\mathbb{R}} f(x)\mathrm{e}^{-|x|}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} f'(x)(-\mathrm{sgn}x)\mathrm{e}^{-|x|}\mathrm{d}x.$$

With an integration by part, the can compute the last integral in $\mathbb{R}^+$ *with*

$$-\frac{1}{2}\int_{\mathbb{R}^+} f'(x)\mathrm{e}^{-x}\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^+} f(x)\mathrm{e}^{-x}\mathrm{d}x - \frac{1}{2}\left[f(x)\mathrm{e}^{-x}\right]_0^{\infty} = -\frac{1}{2}\int_{\mathbb{R}^+} f(x)\mathrm{e}^{-x}\mathrm{d}x + \frac{1}{2}f(0).$$

and the result follows.

3.2.2 An operator interpretation

Going back to Lax Milgram theorem, we note the following. By the Riesz theorem, there is $f \in \mathcal{H}$ so that $L(v) = \langle f, v\rangle_{\mathcal{H}}$. Also, for all $u \in \mathcal{H}$, the map $v \mapsto a(u, v)$ is continuous linear, so by the Riesz' theorem again, there if an element denoted $Au \in \mathcal{H}$ so that $a(u, v) = \langle Au, v\rangle_{\mathcal{H}}$. The map $A : \mathcal{H} \to \mathcal{H}$ is linear. The Lax-Milgram theorem finds a solution to the equation

$$Au = f.$$

It states that if the operator A is continuous and coercive, then it is invertible (or bijective).

3.3 Some examples of applications

3.3.1 The Laplace equation

Let $f \in L^2(\mathbb{R}^d, \mathbb{C})$. We would like to **solve** the equation

$$\boxed{-\Delta u + u = f, \quad \text{on} \quad \mathbb{R}^d.}$$

Assume first that we find a **strong solution**, that is a solution $u \in C^2(\mathbb{R}^d, \mathbb{C})$. Then multiplying the equation by $\phi \in \mathcal{D}(\mathbb{R}^d)$ and integrating by parts would give

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla \phi + \int_{\mathbb{R}^d} u\phi = \int_{\mathbb{R}^d} f\phi.$$

This suggests to recast the problem in the following Lax-Milgram form. Take $\mathcal{H} = H^1(\mathbb{R}^d)$ with its natural inner product, and consider

$$a(u, v) := \int_{\mathbb{R}^d} \overline{u}v + \int_{\mathbb{R}^d} \overline{\nabla u} \cdot \nabla v.$$

(We recognise the usual inner product of $H^1(\mathbb{R}^d)$). This map is bilinear, continuous and coercive. In addition, it is symmetric. Now, consider the linear form $L : \mathcal{H} \to \mathbb{R}$ defined by

$$L(v) := \int_{\mathbb{R}^d} \overline{f}v.$$

Since $f \in L^2(\mathbb{R}^d)$, we have by Cauchy-Schwarz

$$|L(v)| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H^1}.$$

So L is continuous. We can apply the Lax-Milgram theorem, and deduce that there is a unique $u \in H^1(\mathbb{R}^d)$ so that

$$\forall v \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \overline{u}v + \int_{\mathbb{R}^d} \overline{\nabla u} \cdot \nabla v = \int_{\mathbb{R}^d} \overline{f}v.$$

It is the unique solution in $H^1(\mathbb{R}^d)$. Taking $v \in \mathcal{D}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ proves that u is a **weak solution**: a solution in the distributional sense.

Remark 3.5 (There is no uniqueness in $\mathcal{D}'(\mathbb{R}^d)$)**.** *The solution in not unique in $\mathcal{D}'(\mathbb{R}^d)$. For instance, in dimension $d = 1$, the* kernel *of the operator $u \mapsto -u'' + u$ is not empty, since*

$$-u'' + u = 0 \quad \textit{iff} \quad u(x) = \alpha \mathrm{e}^{x} + \beta \mathrm{e}^{-x} \quad \textit{for some } \alpha, \beta \in \mathbb{C}.$$

All these functions are L^1_{loc}, so are distributions, hence solve $-u'' + u = 0$ in $\mathcal{D}'(\mathbb{R})$. However, they all explode exponentially fast to $+\infty$ or $-\infty$, except the solution with $\alpha = \beta = 0$. The Lax-Milgram solution somehow selects *the only integrable solution.*

We have $-\Delta u = f - u$ in $\mathcal{D}'(\mathbb{R}^d)$. In addition, we have $u \in H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, so $f - u \in L^2(\mathbb{R}^d)$. So the equation $-\Delta u + u = f$ actually holds in $L^2(\mathbb{R}^d)$ (this is much better than $\mathcal{D}'(\mathbb{R}^d)$).

One can wonder whether $u \in H^2(\mathbb{R}^d)$. We already know that $u \in L^2(\mathbb{R}^d)$ and $\Delta u \in L^2(\mathbb{R}^d)$. The fact that all crossed derivatives are also in $L^2(\mathbb{R}^d)$ comes from the following Theorem.

Theorem 3.6

If $u \in L^2(\mathbb{R}^d)$ is such that $\Delta u \in L^2(\mathbb{R}^d)$, then $u \in H^2(\mathbb{R}^d)$. In addition, for all $1 \le i, j \le d$, we have

$$\sum_{i=1}^{d} \|\partial_i u\|_{L^2}^2 \le \|u\|_{L^2} \|\Delta u\|_{L^2}, \quad \text{and} \quad \sum_{i,j=1}^{d} \|\partial_{ij}^2 u\|_{L^2}^2 = \|\Delta u\|_{L^2}^2.$$

In particular, the norm $\|u\| := \|u\|_{L^2} + \|\Delta u\|_{L^2}$ is equivalent to the usual $H^2(\mathbb{R}^d)$ norm.

Proof. The standard proof uses the Fourier transform (see below). Let us give an alternative proof. Consider first $u \in C_0^\infty(\mathbb{R}^d)$. In this case, one can use Schwarz' Lemma and integrate by part. For instance, we have

$$\sum_{i=1}^{d} \|\partial_i u\|_{L^2}^2 = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \overline{(\partial_i u)} (\partial_i u) = \int_{\mathbb{R}^d} u \left(- \sum_{i=1}^{d} \partial_{ii}^2 u \right) \overset{C.S.}{\le} \|u\|_{L^2} \|\Delta u\|_{L^2}$$

Similarly,

$$\sum_{i,j=1}^{d} \|\partial_{ij}^2 u\|_{L^2}^2 = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left(\partial_{ij}^2 u\right) \left(\partial_{ij}^2 u\right) = \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} \partial_{ii}^2 u \right) \left(\sum_{j=1}^{d} \partial_{jj}^2 u \right) = \|\Delta u\|_{L^2}^2.$$

So the result holds for $u \in C_0^\infty(\mathbb{R}^d)$. Using convolution with smoothing sequences and cut-off functions, one can prove that $C_0^\infty(\mathbb{R}^d)$ is dense in $\{u \in L^2, \ \Delta u \in L^2\}$ for the norm $\|u\| := \|u\|_{L^2} + \|\Delta u\|_{L^2}$. The result then follows by density. □

3.3.2 The Dirichlet equation in a bounded domain

Let $\Omega \subset \mathbb{R}^d$ be a **bounded**, and let $f \in L^2(\Omega)$. We would like to solve the equation

$$\boxed{\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}}$$

The second condition, called **Dirichlet boundary conditions**, is a shortcut notation! Indeed, u is a distribution, so $u(x)$ has no meaning *a priori*. The correct meaning is

$$(u = 0 \quad \text{on} \quad \partial\Omega) \quad \text{means} \quad u \in H_0^1(\Omega).$$

Without the second condition, there is trivially an infinity of solutions, since if u is solution, then so is $u + cst$.

As before, we recast the problem in a Lax-Milgram form. We set $\mathcal{H} = H_0^1(\Omega)$, with the $H^1(\Omega)$ inner product. We set

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v.$$

This is clearly a bounded bilinear form in $H_0^1(\Omega)$. Thanks to the Poincaré's inequality (see Theorem 2.24 and the remark afterwards), it is also coercive, since

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (C^2 + 1)\|\nabla u\|_{L^2}^2 = (C^2 + 1)a(u, u).$$

We set $L(v) := \int_\Omega fv$. Using again Cauchy-Schwarz and Poincaré inequality, we have

$$|L(v)| \leq \int_\Omega |fv| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2} C\|v\|_{H_0^1},$$

so L is indeed continuous. Applying Lax-Milgram proves that there is unique solution $u \in H_0^1(\Omega)$ to the equation $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

Since $f \in L^2(\Omega)$, we have $-\Delta u \in L^2(\Omega)$, and $\|\Delta u\|_{L^2} = \|f\|_{L^2}$. Unfortunately, this is not enough to conclude that $u \in H^2(\Omega)$, as the following example shows.

Example 3.7. *Consider $\Omega = (0, 1)^3$ in $d = 3$, and the function $u(x) := \frac{1}{4\pi}\frac{1}{|x|}$ (this is the 3d Green's function defined in Section 2.2.2). We have $u \in L^2(\Omega)$, and $\Delta u = 0$ in Ω (this is because $0 \notin \Omega$). However, we have (see (2.2))*

$$|\nabla u|(x) = \frac{1}{4\pi}\frac{1}{|x|^2},$$

which is not square integrable at the origin. So this function satisfies $u, \Delta u \in L^2(\Omega)$ but $\nabla u \notin L^2(\Omega)$.

In the previous example, the function u is not in $H_0^1(\Omega)$. It turns out that we can indeed gain smoothness if Ω has a boundary which is regular enough. We record the following result (whose proof is quite complex).

Theorem 3.8: Elliptic regularity in smooth domain

Assume Ω has boundary $\partial\Omega$ of class C^2. Then any $u \in H_0^1(\Omega)$ with $-\Delta u \in L^2(\Omega)$ is in $H^2(\Omega)$. In addition, there is $C = C(\Omega)$ independent of u so that $\|u\|_{H^2(\Omega)} \leq C\|\Delta u\|_{L^2}$.

This proves that if Ω is regular enough, then the solution of $-\Delta u = f$ satisfies $u \in H_0^1(\Omega) \cap H^2(\Omega)$ with $\|u\|_{H^2} \leq C\|f\|_{L^2}$.

3.3.3 Neumann problem on bounded domain

We want to solve the Neumann problem

$$\boxed{\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases}}$$

where $\partial_\nu u = \nabla u \cdot \nu$ is the **normal derivative** on $\partial\Omega$ (we assume that this one is well-defined). Assume first that u is a strong solution, in $C^2(\overline{\Omega})$. By the divergence Theorem 2.16, we have, for $\psi \in C^\infty(\overline{\Omega})$ that

$$\int_\Omega (-\Delta u)\psi = \int_\Omega \nabla u \cdot \nabla\psi - \int_{\partial\Omega} (\partial_\nu u)\psi = \int_\Omega \nabla u \cdot \nabla\psi.$$

This suggests to consider this time $\mathcal{H} = H^1(\Omega)$ and $a(u,v) = \langle u, v\rangle_{H^1}$. Again, applying Lax-Milgram we deduce that there is a unique solution $u \in H^1(\Omega)$ so that

$$\forall v \in H^1(\Omega), \quad \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv. \tag{3.1}$$

Taking $v \in \mathcal{D}(\Omega)$ shows that $u - \Delta u = f$ in the distributional sense. In particular, since $u \in H^1(\Omega) \subset L^2(\Omega)$, we have $-\Delta u = f - u \in L^2(\Omega)$. Again, we cannot directly conclude that u is smoother than $H^1(\Omega)$.

Remark 3.9. *One could also have considered the Hilbert space $\widetilde{\mathcal{H}} = H^1_0(\Omega)$. Since the $H^1(\Omega)$ and $H^1_0(\Omega)$ norm are equivalent on $H^1_0(\Omega)$, one concludes that there is a unique $\widetilde{u} \in H^1_0(\Omega)$ so that (compare with (3.1))*

$$\forall v \in H^1_0(\Omega), \quad \int_\Omega \widetilde{u}v + \int_\Omega \nabla \widetilde{u} \cdot \nabla v = \int_\Omega fv.$$

*Note that u and $\widetilde{u}$ are different. Taking $v = \phi \in \mathcal{D}(\Omega)$ shows that both u and $\widetilde{u}$ solve the equation $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$, so there is no uniqueness in $\mathcal{D}'(\Omega)$. The solution $\widetilde{u} \in H^1_0(\Omega)$ corresponds to the Dirichlet boundary condition for this problem. In some sense, «*boundary conditions select a particular solution*».*

3.3.4 A less trivial example

We come back to the Poisson equation (we focus on the case $d = 3$ for simplicity), and study it from a Lax-Milgram point of view. This time, we want to solve

$$-\Delta u = f \quad \text{in} \quad \mathbb{R}^3, \quad \text{where} \quad f \in L^{6/5}(\mathbb{R}^3).$$

Testing against $\phi \in \mathcal{D}'(\Omega)$ gives the Lax-Milgram form

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi = \int_{\mathbb{R}^3} f\phi.$$

So we are looking for an Hilbert space $\mathcal{H}$ in which the bilinear (symmetric) form a is both continuous and coercive. In order to do so, we introduce *homogeneous* Sobolev norm

$$\|u\|_{\dot{H}^1} := \|\nabla u\|_{L^2}.$$

This is a norm on $C^\infty_0(\mathbb{R}^3)$ (if $\nabla u = 0$, then u is a constant, but since it is compactly supported, this constant can only be 0). We denote the closure of $C^\infty_0(\mathbb{R}^3)$ for this norm by

$$\dot{H}^1(\mathbb{R}^3) := \overline{C^\infty_0(\mathbb{R}^3)}^{\|\cdot\|_{\dot{H}^1}}. \tag{3.2}$$

By construction, this is a Banach space. It is an Hilbert space with the inner product $\langle u, v\rangle := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v$. We will prove below, using Sobolev embedding, that $u \in \dot{H}^1(\mathbb{R}^3)$ implies $u \in L^6(\mathbb{R}^3)$ with $\|u\|_{L^6} \le C_S \|\nabla u\|_{L^2}$, so

$$\dot{H}^1(\mathbb{R}^3) = \left\{ u \in L^6(\mathbb{R}^3), \quad \nabla u \in L^2(\mathbb{R}^3) \right\}.$$

We can now take $\mathcal{H} = \dot{H}^1(\mathbb{R}^3)$. In this Hilbert space, the bilinear form a is clearly continuous and coercive. In addition, since $f \in L^{6/5}(\mathbb{R}^3) = \left(L^6(\mathbb{R}^3)\right)^*$, we have

$$\forall v \in \dot{H}^1(\mathbb{R}^3), \quad \left| \int_{\mathbb{R}^3} fv \right| \le \|f\|_{L^{6/5}} \|v\|_{L^6} \le C_S \|f\|_{L^{6/5}} \|v\|_{\dot{H}^1},$$

and the linear map $v \mapsto \int fv$ is bounded. We can apply the Lax-Milgram theorem, and deduce that there is a unique $u \in \dot{H}^1(\mathbb{R}^3)$ so that

$$\forall v \in \dot{H}^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla u \cdot \nabla v = \int_{\mathbb{R}^3} fv.$$

Taking $v \in \mathcal{D}(\Omega)$ shows that u is a distributional solution of $-\Delta u = f$.

CHAPTER 4

COMPLEMENTS ON SOBOLEV SPACES

In this section, we discuss embeddings of the form $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$. As we will see, the theory is not so difficult if $\Omega = \mathbb{R}^d$ is the whole space, and there might be difficulties if $\Omega \neq \mathbb{R}^d$, because of its boundary.

4.1 Basics in operator theory

We recall here some basic notions for operator theory.

A **bounded operator** from F to E is a linear map $A : F \to E$ so that there is $C \in \mathbb{R}$ with

$$\forall x \in F, \quad \|Ax\|_E \leq C\|x\|_F.$$

The smallest C satisfying this property is the **operator norm** of A, so

$$\|A\|_{\mathrm{op}} = \sup\left\{\|Ax\|_E, \quad x \in F, \ \|x\|_F = 1\right\}.$$

A bounded operator $A : F \to E$ is **compact** if

$$A\left(\mathcal{B}_F(0,1)\right) \quad \text{is (relatively) compact in } E.$$

If A is compact and if (x_n) converges weakly to x_* in E, then Ax_n converges strongly to Ax_* in F.

We say that F is **embedded** in E if $F \subset E$ and if the injection map $i : F \to E$ defined by $i(x) = x$ is bounded. This means that there is $C \geq 0$ so that, for all $x \in F$, we have $\|x\|_E \leq C\|x\|_F$. We say that F is **compactly embedded** in E if the injection map is compact. In this case, if $x_n \to x$ weakly in F, then $x_n \to x$ strongly in E.

4.2 Sobolev embeddings

In this section, we focus on the so-called Sobolev embeddings. This set of inequalities states that if $u \in W^{m,p}(\mathbb{R}^d)$, then $u \in L^q(\mathbb{R}^d)$ as well, for some $q > p$. In other words, *regularity* implies *integrability*.

4.2.1 Sobolev embeddings on the whole space

We begin with the case $\Omega = \mathbb{R}^d$, see [Bre99, Theorem IX.9].

Theorem 4.1: Gagliardo, Nirenberg, Sobolev's inequality

For all $1 \le p < \frac{d}{m}$, there is a constant $C = C(m,p,d)$ so that, for all $u \in C_0^\infty(\mathbb{R}^d)$, we have

$$\|u\|_{L^q} \le C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{d}.$$

This theorem (and the next ones) provides an inequality for $u \in C_0^\infty(\mathbb{R}^d)$. However, by density, one can extend the result in a larger Banach space. One way to do this is to use the *homogeneous* Sobolev spaces, as in (3.2)). We introduce, for $u \in C_0^\infty(\mathbb{R}^d)$, the homogeneous norm

$$\|u\|_{\dot{W}^{m,p}} := \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p}.$$

(note the dot, and the fact that we only consider $|\alpha| = m$). If $\|u\| = 0$, then $D^\alpha u = 0$ for all $|\alpha| = m$, so u is a polynomial, but since u is compactly supported, u is the null function. This proves that $\| \cdot \|_{\dot{W}^{m,p}}$ is indeed a norm on C_0^∞. We then complete to obtain a Banach space, by setting

$$\dot{W}^{m,p}(\mathbb{R}^d) := \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{\dot{W}^{m,p}}}.$$

We restate the previous theorem as follows.

Theorem 4.2: Gagliardo, Nirenberg, Sobolev's inequality, version 2

For all $1 \le p < \frac{d}{m}$, we have $\dot{W}^{m,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ with $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$. More specifically, there is a constant $C = C(m,p,d)$ so that, for all $u \in \dot{W}^{m,p}(\mathbb{R}^d)$, we have $u \in L^q(\mathbb{R}^d)$ with

$$\|u\|_{L^q} \le C \|u\|_{\dot{W}^{m,p}}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{d}.$$

It is easy to see that $W^{m,p}(\mathbb{R}^d) \hookrightarrow \dot{W}^{m,p}(\mathbb{R}^d)$ (why?). We deduce that $W^{m,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ with continuous embedding. In the case $m = 1$, the corresponding exponent q is written p^*, so

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d) \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}, \quad \text{that is} \quad \boxed{p^* := \frac{dp}{d-p}.}$$

Before we prove this theorem, let us note that the exponent q is *natural*. Indeed, consider the scaling $u_\lambda(x) = u(\lambda x)$. We compute

$$\|u_\lambda\|_{L^q} = \left(\int_{\mathbb{R}^d} |u(\lambda x)|^q \frac{\mathrm{d}(\lambda x)}{\lambda^d} \right)^{1/q} = \lambda^{-\frac{d}{q}} \|u\|_{L^q}.$$

On the other hand, we have, for $|\alpha| = m$,

$$\|D^\alpha u_\lambda\|_{L^p} = \left(\int_{\mathbb{R}^d} |\lambda^m (D^\alpha u)(\lambda x)|^p \frac{\mathrm{d}(\lambda x)}{\lambda^d} \right)^{1/p} = \lambda^{m - \frac{d}{p}} \|D^\alpha u\|_{L^p}.$$

Since the Sobolev inequality must be valid for all values of $\lambda > 0$, the homogeneity in λ must be similar, so $-\frac{d}{q} = m - \frac{d}{p}$, which is also $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$.

Proof. By induction, it is enough to prove the result for $m = 1$ only. We prove the result in the case $d = 2$, and refer to [Bre99, Theorem IX.9] for a proof in all dimensions.
We start with $p = 1$. Since $u \in C_0^\infty(\mathbb{R}^d)$ is compactly supported, we have

$$|u(x_1, x_2)| \le \int_{-\infty}^{x_1} |\partial_{x_1} u(s, x_2)| \mathrm{d}s \le \int_{-\infty}^{\infty} |\partial_{x_1} u(s, x_2)| \mathrm{d}s =: v_1(x_2).$$

Similarly, we have with similar notation that $|u(x_1,x_2)| \le v_2(x_1)$. So

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |u(x_1,x_2)|^2 \mathrm{d}x_1\mathrm{d}x_2 \le \int_{\mathbb{R}^2} |v_1(x_2)| \cdot |v_2(x_1)| \mathrm{d}x_1\mathrm{d}x_2 \\ &= \|v_1\|_{L^1(\mathbb{R})}\|v_2\|_{L^1(\mathbb{R})} = \|\partial_{x_1}u\|_{L^1(\mathbb{R}^2)}\|\partial_{x_2}u\|_{L^1(\mathbb{R}^2)} \le \|\nabla u\|^2_{L^1(\mathbb{R}^2)}.\end{aligned}$$

This proves the result in the case $p=1$ and $d=2$. For the case $1 \le p < 2$, we apply the result to the function $u_t := |u|^{t-1}u$. This function satisfies $\nabla u_t = t|u|^{t-1}\nabla u$. This gives

$$\|u\|^t_{L^{2t}} = \|u_t\|_{L^2} \le \|\nabla u_t\|_{L^1} = t\||u|^{t-1}\nabla u\|_{L^1} \le t\||u|^{t-1}\|_{L^{p'}}\|\nabla u\|_{L^p} = t\|u\|^{t-1}_{L^{(t-1)p'}}\|\nabla u\|_{L^p},$$

We choose t so that $2t = (t-1)p' = (t-1)\frac{p}{p-1}$, that is $t = \frac{p}{2-p}$, and we get, as wanted

$$\|u\|_{L^{\frac{2p}{2-p}}} \le \frac{p}{2-p}\|\nabla u\|_{L^p}.$$

□

Since $u \in W^{m,p} \subset L^p$ and $u \in L^q$, we obtain with Theorem 1.17 that $u \in L^r(\mathbb{R}^d)$ for all $r \in [p,q]$.

Remark 4.3 (Bounded domains)**.** *The same result applies trivially in $C_0^\infty(\Omega)$ for any domain $\Omega \subset \mathbb{R}^d$. One can define similarly the homogeneous space $\dot{W}^{m,p}(\Omega)$. Actually, Poincaré's inequality shows that $\dot{W}^{m,p}(\Omega) = W_0^{m,p}(\Omega)$ (with equivalent norms). So in this case, we have instead $\dot{W}^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)$ (this inclusion is opposite than in the case $\Omega = \mathbb{R}^d$). Note that the function $f = \mathbb{1}_\Omega$ is in $W^{m,p}(\Omega)$, but not in $\dot{W}^{m,p}(\Omega)$, so $\dot{W}^{m,p}(\Omega)$ is a strict closed subspace of $W^{m,p}(\Omega)$.*

4.2.2 Morrey's embedding in the whole space

The Sobolev inequality proves non trivial embeddings for $p < \frac{d}{m}$. On the other side $p > \frac{d}{m}$, we have different embeddings, see [Bre99, Theorem IX.12, and Corollary IX.13].

For $k \in \mathbb{N}$ and $0 < \theta \le 1$, we define $C^{k,\theta}$**-Hölder continuous** norm of a smooth function $u \in C_0^\infty(\Omega)$ by

$$\|u\|_{C^{k,\theta}} := \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\theta}.$$

This means that $D^\alpha u \in C^{0,\theta}$ for all $|\alpha| = k$ (so we only need to focus on the $k=0$ case). If $\|u\|_{C^{k,\theta}} = 0$, then $D^\alpha u$ is constant for all $|\alpha| = k$, so u is a polynomial. But since u is compactly supported, $u = 0$. This proves that $\|\cdot\|_{C^{k,\theta}}$ is a norm on $C_0^\infty(\Omega)$. We complete the space and get the space of $C^{k,\theta}$**-Hölder continuous** functions

$$\forall k \in \mathbb{N}, \quad \forall 0 < \theta \le 1, \quad C^{k,\theta}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{C^{k,\theta}}}.$$

For instance, $C^{0,1}$ is the set of Lipschitz functions. It is not difficult to see that if $u \in C^{0,\theta}$, then u is continuous. If $\theta = 0$, we set $C^{k,0}(\Omega) := C_0^k(\Omega)$ the usual set of k-continuously differentiable functions vanishing at the boundary. If $u \in C^{0,\theta}$, then we have

$$\forall x,y \in \Omega, \qquad |u(x) - u(y)| \le \|u\|_{C^{0,\theta}}|x-y|^\theta.$$

Exercice 4.4

Prove that if u satisfies $|u(x) - u(y)| \le C|x-y|^\theta$ for some $\theta > 1$, then u is constant.

Theorem 4.5: Morrey's embedding in the whole space

If $m \geq 1$ and $p > \frac{d}{m}$, then $W^{m,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. In addition, set

$$k := \left\lfloor m - \frac{d}{p} \right\rfloor, \quad \text{and} \quad \theta := m - \frac{d}{p} - k \in [0,1).$$

If $\theta \neq 0$, then there is a constant $C = C(m,p,d)$ so that,

$$\forall u \in C_0^\infty(\mathbb{R}^d), \quad \|u\|_{C^{k,\theta}} \leq C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p}.$$

In particular, we have $W^{m,p}(\mathbb{R}^d) \hookrightarrow \dot{W}^{m,p}(\mathbb{R}^d) \hookrightarrow C^{k,\theta}(\mathbb{R}^d)$.

The most important case is the case $m = 1$. Then we always have $k = 0$ and $0 < \theta < 1$. We deduce that u is $C^{0,\theta}$, that is

$$\forall x, y \in \mathbb{R}^d, \quad |u(x) - u(y)| \leq C|x-y|^\theta \|\nabla u\|_{L^p}$$

In particular, u is continuous (in the sense "*there exists a continuous representation of* u").

Proof. The full proof of Theorem 4.5 is quite complex, so we admit it. The case $d = 1$ however is simple to prove. By induction, we only need to consider the case $m = 1$. We write that

$$|u(x) - u(y)| \leq \int_{[x,y]} |u'(s)| \mathrm{d}s \leq \left(\int_{[x,y]} |u'|^p(s) \mathrm{d}s \right)^{1/p} \left(\int_{[x,y]} 1^{p'} \mathrm{d}s \right)^{1/p'} \leq \|u'\|_{L^p} |x-y|^{1/p'}.$$

This proves the result with $\theta = \frac{1}{p'} = 1 - \frac{1}{p}$, as wanted. □

The reader may ask what happens at the critical point $p = \frac{d}{m}$. The answer is unfortunately not so easy. If $u \in W^{m,\frac{d}{m}}(\mathbb{R}^d)$, then

- u always belong to all $L^r(\mathbb{R}^d)$ space, for all $p \leq r < \infty$;
- sometimes, u also belongs to $C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The last case happens for instance in the case $m = 1$ and $d = 1$ [Bre99, Theorem VIII.7].

Theorem 4.6: Critical case in dimension 1.

For all $1 \leq p \leq \infty$, we have $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$.

Proof. The proof for $p > 1$ comes from Morrey's embedding. Let us prove the result for $p = 1$. We have, for $u \in C_0^\infty(\mathbb{R})$,

$$|u(x)| \leq \int_{(-\infty,x]} |u'(s)| \mathrm{d}s \leq \|u'\|_{L^1}.$$

So $\|u\|_\infty \leq \|u'\|_{L^1}$. This proves the result for all $u \in C_0^\infty(\mathbb{R})$, hence for all $u \in W^{1,1}(\mathbb{R})$ by density.

Let us prove that $u \in W^{1,1}(\mathbb{R})$ is also continuous. Let (u_n) a sequence of $C_0^\infty(\mathbb{R})$ functions so that $\|u_n - u\|_{W^{1,p}} \to 0$. Then we also have $\|u_n - u\|_{L^\infty} \to 0$. So u is the limit of the continuous functions u_n for the uniform convergence, hence u is continuous. □

4.3 Extension operators

We now focus on the case of bounded set $\Omega \subset \mathbb{R}^d$. We would like to have Sobolev and Morrey embeddings in $W^{m,p}(\Omega)$. As we will see, this is possible if Ω has a boundary $\partial\Omega$ which is regular enough. In this case, we will gain compactness.

Let $\Omega \subset \mathbb{R}^d$. An **extension** of $u \in W^{m,p}(\Omega)$ is a function $\widetilde{u} \in W^{m,p}(\mathbb{R}^d)$ so that $\widetilde{u}(x) = u(x)$ a.e. in Ω. An **extension operator** is a bounded linear operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$ so that Eu is an extension of u for all $u \in W^{m,p}(\Omega)$. Here, bounded means that there is $C > 0$ so that

$$\forall u \in W^{m,p}(\Omega), \quad \|Eu\|_{W^{m,p}(\mathbb{R}^d)} \le C\|u\|_{W^{m,p}(\Omega)}.$$

For instance, in the case $L^p(\Omega)$ corresponding to $m = 0$, we can define

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ 0 & \text{else} \end{cases} \quad \text{which satisfies} \quad \|\widetilde{u}\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p(\Omega)}.$$

So the map $E : u \mapsto \widetilde{u}$ is an extension operator of $L^p(\Omega)$, with norm $\|E\|_{\mathrm{op}} = 1$. However, the function $\widetilde{u}$ has discontinuities at the boundary $\partial\Omega$, hence it is not smooth, and such construction will not work for general $W^{m,p}(\Omega)$.

4.3.1 Extension operator on half-space

We start with the case of the half-space $\Omega = \mathbb{R}^d_+ := \mathbb{R}^{d-1} \times \mathbb{R}^+$, with boundary $\partial\Omega = \mathbb{R}^d_0 := \mathbb{R}^{d-1} \times \{0\}$. First, we prove the following (see [Eva10, Chapter 5.3.3]).

Theorem 4.7: Density of smooth functions, up to the boundary

In the case $\Omega = \mathbb{R}^d_+$, $C^\infty(\overline{\Omega}) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for all $m \in \mathbb{N}$ and all $1 \le p < \infty$.

Comparing this result with the Meyer-Serrin Theorem 2.22, we now consider functions which are smooth up to the boundary $\partial\Omega$, in the sense that each $D^\alpha u$ has a continuous extension on $\partial\Omega$.

Proof. We introduce the translation operator

$$\tau_h(u)(x_1, \cdots, x_{d-1}, x_d) := u(x_1, \cdots, x_{d-1}, x_d + h),$$

and we set $u_\varepsilon := \tau_\varepsilon(u) * j_\varepsilon = \tau_\varepsilon(u * j_\varepsilon)$, so that, for $x \in \mathbb{R}^d_+$, and since j_ε is compactly supported in $\mathcal{B}(0, \varepsilon)$,

$$u_\varepsilon(x) := \int_{y \in \mathcal{B}(0,\varepsilon)} j_\varepsilon(y) u(x_1 - y_1, \cdots x_{d-1} - y_{d-1}, x_d - y_d + \varepsilon) \mathrm{d}x.$$

We have translated u downwards so that the convolution is well-defined everywhere in $\mathbb{R}^d_+$. By the properties of convolutions, we have $u_\varepsilon \in C^\infty(\overline{\Omega})$. In addition, we have

$$\begin{aligned} \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p} &\le \|D^\alpha u_\varepsilon - D^\alpha(\tau_\varepsilon u)\|_{L^p} + \|D^\alpha(\tau_\varepsilon u) - D^\alpha u\|_{L^p} \\ &\le \|\left(D^\alpha \tau_\varepsilon u\right) * j_\varepsilon - D^\alpha \tau_\varepsilon u\|_{L^p} + \|\tau_\varepsilon(D^\alpha u) - D^\alpha u\|_{L^p}. \end{aligned}$$

The first term is goes to zero by the properties of smoothing sequences, and the second goes to zero since translations are continuous in L^p. ☐

Remark 4.8. *The same proof shows that $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, if Ω is bounded with $\partial\Omega$ of class C^1. This time, one needs to locally translate u along the normal direction.*

Theorem 4.9: Extension on half space

For all $m \in \mathbb{N}$ and all $1 \le p < \infty$, there is an extension operator $E : W^{m,p}(\mathbb{R}^d_+) \to W^{m,p}(\mathbb{R}^d)$.

Proof. We only do the proof in the case $d = 1$ and $m = 1$ to highlight the main ideas. We refer to [Eva10, Chapter 5.4] for the general case.
For $u \in W^{1,p}(\mathbb{R}^+)$, we define the **first order reflection**

$$\bar{u}(x) = \begin{cases} u(x) & \text{if} \quad x > 0, \\ -3u(-x) + 4u(-\frac{x}{2}) & \text{else.} \end{cases}$$

By density, it is enough to prove the result for $u \in C^1([0, \infty))$. We claim that $\bar{u}$ is of class C^1 as well. First, we check that $\bar{u}(0^-) = -3u(0) + 4u(0) = u(0) = \bar{u}(0^+)$, so $\bar{u}$ is continuous at $x = 0$. Next, we have

$$\forall x < 0, \quad \bar{u}'(x) = 3u'(-x) - 2u'(-\tfrac{x}{2}), \quad \text{so} \quad \bar{u}'(0^-) = u'(0) = \bar{u}(0^+),$$

so $\bar{u}'$ is also continuous at $x = 0$. This proves as wanted that $\bar{u}$ is indeed C^1. In addition,

$$\begin{aligned} \|\bar{u}\|^p_{W^{1,p}(\mathbb{R})} &= \int_{\mathbb{R}} |\bar{u}|^p + |\bar{u}'|^p \\ &= \int_{\mathbb{R}^+} |u|^p + |u'|^p + \int_{\mathbb{R}^-} |-3u(-x) + 4u(-\tfrac{x}{2})|^p + |3u'(-x) - 2u'(-\tfrac{x}{2})|^p \\ &= C\|u\|^p_{W^{1,p}(\mathbb{R}^+)}, \end{aligned}$$

for some large constant $C > 0$, independent of u. This proves that the map $E : u \mapsto \bar{u}$ is a bounded extension operator.

The reader can check that the same construction works on $\mathbb{R}^d$, by setting

$$\bar{u}(x) = \begin{cases} u(x) & \text{if} \quad x > 0, \\ -3u(x_1, \cdots, x_{d-1}, -x_d) + 4u(x_1, \cdots, x_{d-1}, -\frac{x_d}{2}) & \text{else.} \end{cases}$$

The proof for higher derivatives $m \geq 1$ necessitates higher order reflections. $\square$

4.3.2 Extension operators on domains with smooth boundary

We say that a domain $\Omega \in \mathbb{R}^d$ has a **boundary $\partial\Omega$ of class C^k** if, for all $x \in \partial\Omega$, there is $\varepsilon > 0$, $\delta > 0$, and a C^k diffeomorphism

$$\Psi : \mathcal{B}(x, \varepsilon) \to \mathcal{B}(0, \delta),$$

so that $\Psi(x) = 0$ and $\Psi(\mathcal{B}(x, \varepsilon) \cap \Omega) = \Psi(\mathcal{B}(0, \delta) \cap \mathbb{R}^d_+)$. In particular, this implies that $\Psi(\partial\Omega) = \mathbb{R}^d_0$.

If Ω is a **bounded** domain of class C^1, then its boundary $\partial\Omega$ is closed and bounded, hence compact. First, we state the following (compare with Theorem 2.22).

Theorem 4.10

Assume Ω is bounded with boundary $\partial\Omega$ of class C^1. For all $m \in \mathbb{N}$ and all $1 \leq p < \infty$, the set $C^\infty(\overline{\Omega}) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

We refer to [Eva10, Section 5.3.3] for the full proof. It relies on the fact that the outward normal derivative is well defined whenever $\partial\Omega$ is of class C^1.

Apart from the previous result, we usually *transport* results which are valid for $\Omega = \mathbb{R}^d_+$ to the case Ω with $\partial\Omega$ of class C^k using the diffeomorphisms Ψ. In this case, we construct functions which cannot be smoother than C^k.

Theorem 4.11: Extension operators on smooth domains

If Ω is bounded with $\partial\Omega$ of class C^k, then for all $m \leq k$ and all $1 \leq p \leq \infty$, there is an extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$.

We require the boundary to be at least as regular as the functions inside: $m \leq k$.

Proof. For all $x \in \partial\Omega$, construct $\varepsilon_x > 0$ and a C^k diffeomorphism Ψ_x as above. The collection of open sets $\bigcup_{x\in\partial\Omega} \mathcal{B}(x, \varepsilon_x)$ covers the compact boundary $\partial\Omega$, so there is a finite collection of points $\{x_1, \cdots, x_N\} \in (\partial\Omega)^N$ so that $\bigcup_{i=1}^N \mathcal{B}(x_i, \varepsilon_i)$ covers $\partial\Omega$. We set $\varepsilon_i = \varepsilon_{x_i}$, $\Psi_i := \Psi_{x_i}$ and $U_i := \mathcal{B}(x_i, \varepsilon_i)$. We also consider $U_0 \subset \Omega$ an open set so that $K := \overline{U_0} \subset \Omega$ as well, and so that $\Omega = U_0 \cup U_1 \cup \cdots \cup U_N$.

Next we consider a **partition** of unity subordinated to $(U_0, U_1, \cdots U_N)$, that is a family of smooth functions $\theta_0, \theta_1, \cdots, \theta_N$ so that

- For all $x \in \Omega$, we have $\sum_{i=0}^N \theta_i(x) = 1$;
- The function θ_i is compactly supported with support (strictly) included in U_i.

For $u \in W^{m,p}(\Omega)$ we write $u = \sum_{i=0}^N u_i$ with $u_i := \theta_i u$. This gives

$$\|u\|_{W^{m,p}(\Omega)} = \left\| \sum_{i=0}^N u_i \right\|_{W^{m,p}} \le \sum_{i=0}^N \|u_i\|_{W^{m,p}(\Omega)}, \quad \text{and} \quad \|u_i\|_{W^{m,p}(\Omega)} \le C\|u\|_{W^{m,p}(\Omega)},$$

and since the functions θ_i are smooth and compactly supported, there is $C > 0$ so that $\|D^\alpha \theta_i\|_\infty < C$ for all $|\alpha| < m$. In particular, there is $C' > 0$ independent of u so that

$$\|u_i\|_{W^{m,p}(\Omega)} = \|\theta_i u\|_{W^{m,p}(\Omega)} \le C'\|u\|_{W^{m,p}(\Omega)}.$$

The function u_0 is compactly supported in Ω, so it has a trivial extension $\widetilde{u_0}$ to $W^{m,p}(\mathbb{R}^d)$.

For $1 \le i \le N$, the function u_i is compactly supported in $U_i = \mathcal{B}(x_i, \varepsilon_i)$. We define on $V_i := \mathcal{B}(0, \delta_i) \cap \mathbb{R}^d_+$ the function $v_i(x) := u_i(\Psi_i^{-1}(x))$. Since $\Psi^{-1} \in C^k$ with $k \le m$, we have $v_i \in W^{m,p}(\mathbb{R}^d_+)$. By Theorem 4.9, we can extend v_i on $W^{m,p}(\mathbb{R}^d_+)$ by a function $\bar{v}_i$. In addition, by construction, $\bar{v}_i$ is compactly supported in $\mathcal{B}(0, \delta_i)$. We finally set, for all $x \in U_i$, $\bar{u}_i(x) := \bar{v}_i(\Psi(x))$. Again, since $\Psi \in C^k$, we can check that $\bar{u}_i$ is compactly supported in U_i, and that $u_i \in W^{m,p}(\mathbb{R}^d)$.

Finally, we set $\bar{u} := \sum_{i=0}^N \bar{u}_i \in W^{m,p}(\mathbb{R}^d)$. For all $x \in \Omega$, we have $\bar{u}_i(x) = u_i(x)$, so indeed $\bar{u}(x) = u(x)$. Finally, by construction, the map $u \mapsto \bar{u}$ is linear, and bounded from $W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$. □

Exercice 4.12

Let $U_1, \cdots U_N$ be a collection of disjoint sets covering $\mathbb{R}^d$, in the sense $\bigcup_{i=1}^N U_i = \mathbb{R}^d$. Define

$$\theta_i^\varepsilon(x) := \int_{U_i} j_\varepsilon(x - y)\mathrm{d}y = (\mathbb{1}_{U_i} * j_\varepsilon)(x).$$

Prove that θ_i^ε is smooth, that $\sum_{i=1}^N \theta_i^\varepsilon = 1$ on $\mathbb{R}^d$, and that θ_i^ε is compactly supported in an ε-neighborhood of U_i.

4.3.3 Compact embedding in bounded domains

Let Ω be a bounded domain. If Ω has a smooth enough boundary $\partial\Omega$, we may use an extension operator to have similar theorems than Sobolev and Morrey. We only state a simple version of these theorems [Bre99, Corollaire IX.14].

Theorem 4.13: Embedding theorems in bounded domains

Let Ω is a bounded domain with boundary $\partial\Omega$ of class C^k. Then, for all $m \le k$ and all $1 \le p \le \infty$, the conclusions of Theorems 4.2 and 4.5 hold for all $u \in W^{m,p}(\Omega)$.

We emphasise that the boundary must be at least as smooth as the functions inside.

Now, since Ω is bounded, we actually gain *compactness*.

Theorem 4.14: Rellich Kontrachov Theorem

Let Ω is a bounded domain with boundary $\partial\Omega$ of class C^k, and let $m \leq k$.
- If $1 \leq p < \frac{d}{m}$, then for all $r \in [p, q)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$, the embedding $W^{m,p}(\Omega) \to L^r(\Omega)$ is **compact**.
- If $p \geq \frac{d}{m}$, then for all $r \in [p, \infty)$, the embedding $W^{m,p}(\Omega) \to L^r(\Omega)$ is **compact**.
- If $p > \frac{d}{m}$, the embedding $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$ is **compact**.

Again, we skip the proof (see [Bre99, p. IX.16]). This one is not so difficult, but uses tools (Ascoli theorem) that goes beyond the scope of this course.

This theorem combines nicely with the Banach-Alaoglu Lemma: weak-limits in $W^{m,p}(\Omega)$ becomes strong limits in $L^r(\Omega)$. We will see some applications in the next Chapter.

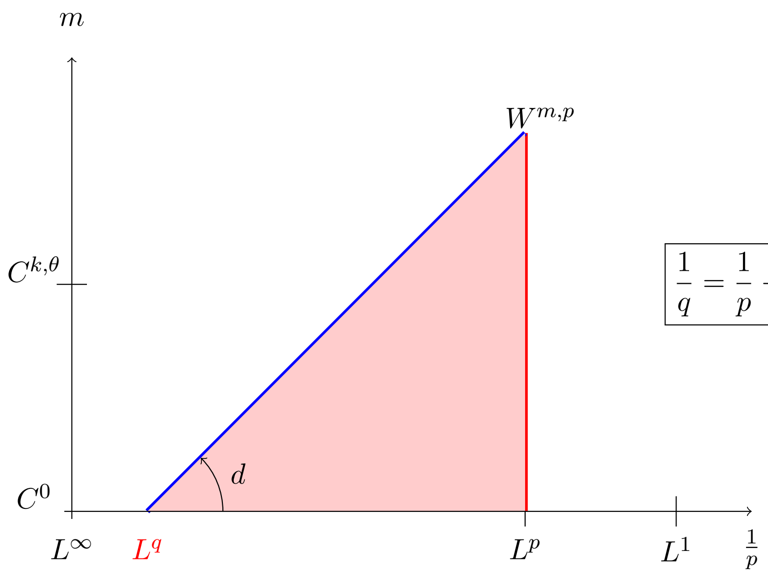

Figure 4.1: Sobolev's and Morrey's embbedings in a single graph. The Lebesgue spaces L^p are at $m = 0$, the Hölder's space $C^{k,\theta}$ are at $\frac{1}{p} = 0$, and the Sobolev's spaces $W^{m,p}$ are in the $(m, \frac{1}{p})$ quarter space. The point $(0, 0)$ represent either L^p for all $p \geq 1$, or L^∞, or C^0, depending on the dimension. If $u \in W^{m,p}$, then u belongs to all Sobolev/Hölder's spaces in the red area (including the critical blue line). If Ω is bounded with smooth boundary, the embeddings are compact in the red area (excluding the critical blue line).

Example 4.15. *The main application is the embedding of $H^1(\Omega)$. We have the following table*

d = 1	d = 2	d = 3	d = 4
$C^{0,1/2}$	$L^r, \forall r \geq 2$,	L^6	L^4

Table 4.1: Embedding of H^1 in low dimensions.

4.4 Trace operators

In this section, we study what are the properties of u restricted to the boundary $\partial\Omega$. The next Theorem can be found in [Eva10, Chapter 5.5].

4.4.1 Trace operators on the half-space

We begin with the half-space $\Omega := \mathbb{R}^d_+$. In this case, the boundary $\partial\Omega = \mathbb{R}^d_0 = \mathbb{R}^{d-1} \times \{0\}$ can be identified with $\mathbb{R}^{d-1}$. In particular, it has a well-defined $(d-1)$-dimensional Lebesgue measure.

Theorem 4.16: Traces in half-space

For all $1 \le p < \infty$ and all $u \in W^{1,p}(\mathbb{R}^d_+)$, the function $u|_{\partial\Omega} : \mathbb{R}^{d-1} \to \mathbb{C}$ belongs to $L^p(\mathbb{R}^{d-1})$.

So the intersection of a $W^{1,p}(\mathbb{R}^d)$ function with a plane is an $L^p(\mathbb{R}^{d-1})$ function.

Proof. Using an extension operator, we can consider $u \in W^{1,p}(\mathbb{R}^d)$ and study its intersection with the plane $\{x_d = 0\}$. Assume first that $u \in C^\infty_0(\mathbb{R}^d)$ is smooth, compactly supported, and positive valued. For all $\underline{x} \in \mathbb{R}^{d-1} = \partial\Omega$, we have

$$\left| u|_{\partial\Omega} \right|^p(\underline{x}) = |u|^p(\underline{x}, 0) \le \int_{[0,\infty)} \left| \partial_{x_d} \left(|u|^p \right) \right| (\underline{x}, s) \mathrm{d}s = p \int_{[0,\infty)} |u|^{p-1} |\partial_{x_d} u| (\underline{x}, s) \mathrm{d}s.$$

Using the inequality $|a|^{p-1}|b| \le |a|^p + |b|^p$ (consider the case $|a| \le |b|$ and $|b| \le |a|$), this gives the point-wise estimate

$$\left| u|_{\partial\Omega} \right|^p(\underline{x}) \le p \int_{[0,\infty)} (|u|^p + |\nabla u|^p)(\underline{x}, s) \mathrm{d}s.$$

Integrating in $\underline{x} \in \partial\Omega$ proves that $\|u\|_{L^p(\partial\Omega)} \le p\|u\|_{W^{1,p}(\mathbb{R}^d)}$. By density of $C^\infty_0(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$, the result holds on the whole space $W^{1,p}(\mathbb{R}^d)$. □

The corresponding map $T : W^{1,p}(\mathbb{R}^d_+) \to L^p(\mathbb{R}^d_0)$ is called the **trace operator**. The following Lemma identifies the kernel of T (see [Eva10, Chapter 5.5] for the proof).

Theorem 4.17

Let $u \in W^{1,p}(\mathbb{R}^d_+)$, we have $Tu = 0$ iff $u \in W^{1,p}_0(\mathbb{R}^d_+)$.

We can wonder whether the map T is *surjective*. Unfortunately, it is not the case, and characterising precisely the image is a difficult task. In the case $p = 2$, the image of T is denoted by $H^{1/2}(\partial\Omega)$. Indeed, this space can be interpretated as a Sobolev space (with fractional exponent) on $\partial\Omega \approx \mathbb{R}^{d-1}$.

As the notation suggests, we indeed have $L^2(\mathbb{R}^{d-1}) \subset H^{1/2}(\mathbb{R}^{d-1}) \subset H^1(\mathbb{R}^d)$, with continuous embeddings. We do not comment more on this space, and just recap the discussion with the following Theorem.

Theorem 4.18

The trace operator $T : H^1(\mathbb{R}^d_+) \to H^{1/2}(\mathbb{R}^{d-1})$ is **surjective**: for any $\gamma \in H^{1/2}(\mathbb{R}^{d-1})$, there is $u \in H^1(\mathbb{R}^d_+)$ so that $Tu = \gamma$.
In addition, if $v \in H^1(\mathbb{R}^d_+)$ is another function such that $Tv = \gamma$, then $v - u \in H^1_0(\mathbb{R}^d_+)$.

4.4.2 Trace operators on bounded domains

Similar results holds in the case where Ω is a bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. In this case, the Lebesgue space $L^p(\partial\Omega)$ is defined in terms of the **surface measure** $\mathrm{d}s$:

$$\|f\|_{L^p(\partial\Omega)} := \int_{\partial\Omega} |f|^p \mathrm{d}s.$$

We do not elaborate more on this point, and just enunciate the main result.

Theorem 4.19

Let $1 \le p < \infty$, and let Ω be a bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. Then the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is bounded. In addition,

- for all $u \in W^{1,p}(\Omega)$, we have $Tu = 0$ iff $u \in W_0^{1,p}(\Omega)$;
- **(Case** $p = 2$**)** T is surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$.

This justifies *a posteriori* the notation "$u = 0$ on $\partial\Omega$" for $u \in W_0^{1,p}(\Omega)$.

As in the previous section, the proof uses localisation and flattening of the boundary to recover the half-space case, see [Eva10, Chapter 5.5].

CHAPTER 5

OPTIMISATION

We this chapter, we introduce the Euler-Lagrange equations. We use it to prove the spectral decomposition of compact operators, and we provide some examples.

5.1 Euler-Lagrange equations

We recall the Implicit Function Theorem. In the sequel, X, Y and Z are Banach spaces.

Theorem 5.1: Implicit Function Theorem

Let $F : X \times Y \to Z$ be a function of class C^k with $k \geq 1$, and let $(x_0, y_0) \in X \times Y$ be such that $F(x_0, y_0) = 0$. Assume that $D_y F(x_0, y_0)$ is invertible (as a linear operator from Y to Z). Then,

- there is a neighbourhood U_x of x_0 in X, and a neighbourhood U_y of y_0 in Y,
- there is a (unique) map $\Psi : U_x \to U_y$, which is of class C^k,

so that

$$\forall x, y \in U_x \times U_y, \quad F(x, y) = 0 \quad \text{iff} \quad y = \Psi(x).$$

In particular, we have $F(x, \Psi(x)) = 0$ for all $x \in U_x$, and

$$D_x \Psi(x_0) = - \left[D_y F(x_0, y_0) \right]^{-1} D_x F(x_0, y_0).$$

This theorem states that, under the *mild* condition $D_y F$ invertible, a solution to $F(x_0, y_0) = 0$ belongs to a unique *branch of solutions*. We do not prove this Theorem, as it is classical.

Our main application of this theorem concerns the properties of optimisers for problems under constraints. We are interested in problems of the form

$$\inf \left\{ F(x), \quad x \in X, \ G(x) = 0 \right\},$$

where $F : X \to \mathbb{R}$ (always real-valued, otherwise *optimisation* is not possible), and $G : X \to Y$ is a set of constraints. For our purpose, we usually need one constraint, so we assume $G : X \to \mathbb{R}$. Recall that an optimisation problem is **well-posed** if the infimum is a minimum: an optimiser exists.

Theorem 5.2: Euler-Lagrange equations

Let F and G be functions of class C^1 from X to $\mathbb{R}$. Assume that $x_* \in X$ is an optimiser of the minimisation problem

$$\inf\left\{F(x), \quad x \in X,\ G(x) = 0\right\},$$

Assume in addition that $D_x G(x_*) \neq 0$. Then there is $\lambda \in \mathbb{R}$ so that

$$D_x F(x_*) = \lambda D_x G(x_*) \qquad \text{(Euler-Lagrange equations).}$$

The number $\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**.

Proof. Since $D_x G(x_*) \neq 0$, there is $e_1 \in X$ so that $D_x G(x_*) \cdot e_1 = 1$. We set $E_1 := \mathrm{Vect}\{e_1\}$, and consider a complement V of E_1 in E, that is $E = V \oplus E_1$. We write $x = (\underline{x}, x_1) =: (\underline{x}, y)$ in this decomposition. Note that y is a real number, we write ∂_y for D_y to emphasise this point. We have $\partial_y G(\underline{x_*}, y_*) = D_x G(x_*) \cdot e_1 = 1$ by construction, so we can apply the Implicit Function Theorem to G. We deduce that there is C^1 map $\Psi : V \to E_1$ so that, locally around x_*, we have

$$G(\underline{x}, y) = 0 \quad \text{iff} \quad y = \Psi(\underline{x}).$$

In addition, since $\partial_y G = 1$, we have

$$\Psi'(\underline{x_*}) = -D_{\underline{x}} G(x_*), \quad \text{so that} \quad D_x G(x_*) = \begin{pmatrix} -\Psi'(\underline{x_*}) \\ 1 \end{pmatrix}$$

We found a parametrisation of the constraint $G(x) = 0$ around x_*. In particular, $\underline{x_*}$ is the minimum of the function $\underline{x} \mapsto F(\underline{x}, \Psi(\underline{x}))$. We deduce that

$$D_{\underline{x}} F(x_*) + \partial_y F(x_*) \Psi'(\underline{x_*}) = 0, \quad \text{so that} \quad D_x F(x_*) = \begin{pmatrix} D_{\underline{x}} F(x_*) \\ \partial_y F(x_*) \end{pmatrix} = \partial_y F(x_*) \begin{pmatrix} -\Psi'(\underline{x_*}) \\ 1 \end{pmatrix},$$

and the result follows. Actually, we proved that $\lambda = \partial_y F(x_*)$. $\square$

5.1.1 Application, defocusing NLS in a bounded domain

As an application, we would like to find a non trivial solution (λ, u) to the following non-linear Schrödinger (NLS) problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$ of class C^1. We consider the problem

$$\boxed{\begin{cases} -\Delta u + u^3 = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \quad \partial\Omega. \end{cases}}$$

As usual, the second line means $u \in H_0^1(\Omega)$. We introduce the functional $F : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4} \int_\Omega |u|^4.$$

Note that if $F(u) < \infty$, then we have $u \in L^4(\Omega)$ as well, so we may restrict ourself to the Banach space

$$H_0^1(\Omega) \cap L^4(\Omega), \quad \text{with norm} \quad \|u\| = \|\nabla u\|_{L^2} + \|u\|_{L^4}.$$

We consider the minimisation problem

$$J := \inf\left\{F(u), \quad u \in H_0^1(\Omega) \cap L^4(\Omega), \quad \|u\|_{L^2} = 1\right\}.$$

The functional F is positive, hence bounded from below. Let $(u_n) \in H_0^1(\Omega) \cap L^4(\Omega)$ be a minimising sequence for this problem, that is $\|u_n\|_{L^2} = 1$ and $F(u_n) \to J$.

Existence of a minimum. The sequence $F(u_n)$ converges, hence is bounded. So the sequence (u_n) is bounded in $H_0^1(\Omega)$ and in $L^4(\Omega)$. According to the Banach–Alaoglu theorem, there is $u_1 \in H_0^1(\Omega)$ and $u_2 \in L^4(\Omega)$ so that

$$u_n \to u_1 \quad \text{weakly in } H_0^1(\Omega), \quad \text{and} \quad u_n \to u_2 \quad \text{weakly in } L^4(\Omega).$$

In particular, $u_n \to u_1$ and $u_n \to u_2$ in the distributional sense $\mathcal{D}'(\Omega)$, so, by uniqueness of the limit, $u_1 = u_2 =: u_*$. (We could also have used the Banach–Alaoglu theorem directly in the Banach space $H_0^1(\Omega) \cap L^4(\Omega)$ and obtain that $u_n \to u_*$ weakly in $H_0^1(\Omega) \cap L^4(\Omega)$).

Let us prove that $\|u_*\|_{L^2} = 1$. By Rellich's theorem 4.14, and since Ω is bounded with C^1 boundary, the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is **compact**, so (u_n) converges strongly to u_* in $L^2(\Omega)$. We deduce that $\|u_*\|_{L^2} = \lim_{n\to\infty} \|u_n\|_{L^2} = 1$.

We now prove that $F(u_*) = J$. The function F is convex, and strongly continuous from $H_0^1(\Omega) \cap L^4(\Omega) \to \mathbb{R}$. So, by Theorem 1.31, it is weakly lower semi continuous. In particular,

$$F(u_*) \le \liminf_{n\to\infty} F(u_n) = J.$$

So u_* is a minimiser of the problem.

Euler-Lagrange equations. We now derive the Euler-Lagrange equations. First, we note that

$$F(u_* + h) = F(u_*) + \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u_*^3 h + O\left(\|h\|_{H_0^1}^2 + \|h\|_{L^4}^2 \right).$$

We deduce that $F : H_0^1(\Omega) \cap L^4(\Omega) \to \mathbb{R}$ is differentiable, and that its derivative $D_u F(u_*)$ is the linear map from $H_0^1(\Omega) \cap L^4(\Omega)$ to $\mathbb{R}$ defined by

$$D_u F(u_*) : h \mapsto \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u_*^3 h.$$

Similarly, the map $N(u) := \int_{\mathbb{R}^d} u^2$ is differentiable on $H_0^1(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$, and $D_u N(u^*) : h \mapsto 2 \int_{\mathbb{R}^d} uh$. The Euler-Lagrange equations shows that there is $\lambda \in \mathbb{R}$ so that

$$\forall h \in H_0^1(\Omega) \cap L^4(\Omega), \quad \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u_*^3 h = \lambda \int_\Omega u_* h.$$

We first take $h = u$, and see that

$$\int_\Omega |\nabla u_*|^2 + \int_\Omega |u_*|^4 = \lambda \int_\Omega |u_*|^2.$$

This proves that $\lambda > 0$.

On the other hand, taking $h \in \mathcal{D}(\Omega)$ shows that $-\Delta u_* + u_*^3 = \lambda u_*$ in $\mathcal{D}'(\Omega)$. Since $\|u_*\|_{L^2} = 1$, we have $u_* \neq 0$. We conclude that u_* is a non trivial solution to the NLS equation.

Property of the solution. We have found a non trivial solution of the NLS problem by minimising a functional F. There might be other solutions, but we will not explore this direction here. The solution that we constructed has many good properties. Let us prove for instance that it is positive. We use the following result (see [LL01, Theorem 6.17]).

Theorem 5.3

If $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$, and $\big|\nabla |u|\big| \le |\nabla u|$ a.e.

Proof. We have the inequality

$$\left|2|u|\nabla|u|\right| = \left|\nabla|u|^2\right| = 2\left|\mathrm{Re}\left(\overline{u}\nabla u\right)\right| \leq 2|u|\cdot|\nabla u|.$$

On the set $\{x\in\Omega,\ u(x)\neq 0\}$, we can divide by $2|u|$, which gives the result. To handle the set $\{u=0\}$, we refer to [LL01, Theorem 6.17]. $\square$

Using Theorem 5.3 in our case, we find that $F(|u|)\leq F(u)$. So, since u_* is a minimiser, so is $|u_*|$. In particular, $|u_*|$ is a **positive** non trivial solution of the NLS equation. (Actually, by strict convexity of F, we must have $u_* = \pm|u_*|$).

5.1.2 Application, focusing NLS in a bounded domain

This time, we want to solve

$$\boxed{\begin{cases} -\Delta u - u^3 = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}}$$

Compared to the previous section, there is a minus sign in front of the non-linear term u^3. This makes the analysis much harder.

As before, we introduce the functional $G : H_0^1(\Omega)\to\mathbb{R}$ defined by

$$G(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega |u|^4,$$

and the minimisation problem

$$K := \inf\left\{G(u),\quad u\in H_0^1(\Omega),\quad \|u\|_{L^2}=1\right\}.$$

Due the minus sign, it is no longer obvious that G is bounded from below. Actually, it is unclear that $u\in L^4(\Omega)$ at this point. According to Sobolev's embedding, we have $H_0^1(\Omega)\approx H^1(\Omega)=W^{1,2}(\Omega)\hookrightarrow L^{p*}(\Omega)$ with $p^* = \dfrac{2d}{d-2}$. So $u\in H_0^1(\Omega)$ implies $u\in L^4(\Omega)$ only if $d\leq 4$. In addition, it is unclear that the L^4 norm of u is controlled by its H_0^1 norm.

The functional G is not bounded from below in dimension $d\geq 3$.
Without loss of generality, we assume $0\in\Omega$. Consider the family

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^{d/2}}\psi\left(\frac{x}{\varepsilon}\right),$$

with ψ a smooth compactly supported function with $\|\psi\|_{L^2}=1$. For $\varepsilon>0$ small enough, the support of ψ_ε is contained in Ω, so $\psi_\varepsilon\in H_0^1(\Omega)$. We have

$$\int_\Omega |\nabla\psi_\varepsilon|^2 = \int_\Omega \frac{1}{\varepsilon^2}\frac{1}{\varepsilon^d}\psi^2\left(\frac{x}{\varepsilon}\right)\mathrm{d}\left(\frac{x}{\varepsilon}\right)\varepsilon^d = \frac{1}{\varepsilon^2}\int_\Omega |\nabla\psi|^2.$$

and, for $p\geq 1$,

$$\int_\Omega |\psi_\varepsilon|^p = \int_\Omega \frac{1}{\varepsilon^{\frac{dp}{2}}}\psi^p\left(\frac{x}{\varepsilon}\right)\mathrm{d}\left(\frac{x}{\varepsilon}\right)\varepsilon^d = \frac{1}{\varepsilon^{d(\frac{p}{2}-1)}}\int_\Omega |\psi|^p.$$

Note that for $p=2$, we recover that $\|\psi_\varepsilon\|_{L^2}=\|\psi\|_{L^2}=1$, so ψ_ε is a valid test function for the minimization problem. We obtain, with $p=4$,

$$G(\psi_\varepsilon) = \frac{1}{\varepsilon^2}\left(\frac{1}{2}\int_\Omega |\nabla\psi|^2\right) - \frac{1}{\varepsilon^d}\left(\frac{1}{4}\int_\Omega |\psi|^4\right).$$

In the limit $\varepsilon \to 0$, it diverges to $-\infty$ in the case $d \geq 3$. In the case $d = 2$, it also diverges to $-\infty$ if $G(\psi) < 0$ (note that this depends on the choice of the constants in the functional G. Whether there is $\psi \in H_0^1(\mathcal{B}(0,1))$ with $G(\psi) < 0$ is linked to the best constants in Sobolev's embedding inequalities).

This scaling argument shows that we have a decent chance to find an optimizer for G only in dimension $d = 1$. So we restrict to this case in what follows.

The functional G is bounded from below in dimension $d = 1$.
In dimension $d = 1$, Sobolev's embedding states that there a constant $S > 0$ so that

$$\forall u \in H_0^1(\Omega), \quad u \in L^\infty(\Omega), \quad \text{with} \quad \|u\|_{L^\infty} \leq S \|\nabla u\|_{L^2}.$$

Recall also that we focus on $u \in H_0^1(\Omega)$ with $\|u\|_{L^2} = 1$. According to Theorem 1.17, we deduce that for such u, we have $u \in L^p(\Omega)$ for all $2 \leq p \leq \infty$. In particular, $u \in L^4(\Omega)$, and since $\frac{1}{4} = \frac{\alpha}{2} + \frac{1-\alpha}{\infty}$ for $\alpha = 1/2$, we have

$$\forall u \in H_0^1(\Omega) \text{ with } \|u\|_{L^2} = 1, \quad \|u\|_{L^4} \leq \|u\|_{L^2}^{1/2} \|u\|_{L^\infty}^{1/2} \leq S^{1/2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}.$$

In particular, for all $u \in H_0^1(\Omega)$ with $\|u\|_{L^2} = 1$, we have

$$G(u) \geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{S^2}{4} \|\nabla u\|_{L^2}^2 \geq \left(\frac{1}{2} - \frac{S^2}{4} \right) \|\nabla u\|^2.$$

Unfortunately, this is not enough to conclude! The constant might be negative (this depends on the choice of the constants in the function G). Going back to our Sobolev embeddings (see Figure 4.1), we see that the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is not sharp in dimension 1. According to this figure, we also have $H^{1/2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, but since we did not introduce the Sobolev space $H^{1/2}(\mathbb{R})$, we will use the following inequality.

Theorem 5.4: Gagliardo–Niremberg inequality in dimension $d = 1$

Let $\Omega \subset \mathbb{R}$ be open. For all $u \in H_0^1(\Omega)$, we have $u \in L^\infty(\Omega)$, and

$$\|u\|_\infty \leq \sqrt{2} \|u'\|_2^{1/2} \|u\|_2^{1/2}.$$

Proof. For u a smooth compactly supported function, we have

$$|u(x)|^2 = 2 \left| \int_{-\infty}^x u'(s) u(s) \mathrm{d}s \right| = 2 \int_{-\infty}^\infty |u'(s)| \cdot |u(s)| \mathrm{d}s \leq 2 \|u'\|_2 \|u\|_2,$$

where we used Cauchy-Schwarz in the last inequality. We conclude by density. □

With this inequality, we get the better inequality

$$G(u) \geq \frac{1}{2} \|u'\|_{L^2}^2 - \frac{1}{2} \|u'\|_{L^2} \geq \inf_{X > 0} \left(\frac{1}{2} X^2 - \frac{1}{2} X \right) > -\infty. \tag{5.1}$$

This time, we can conclude that G is bounded from below.

Existence of a minimiser. We consider a minimising sequence (u_n) for G, that is $u_n \in H_0^1(\Omega)$ with $\|u_n\|_{L^2} = 1$, and $\lim G(u_n) = \inf G = K$. The sequence $G(u_n)$ is bounded. Using again the inequality (5.1), we see that $\|\nabla u_n\|_{L^2}$ is also a bounded sequence, *i.e.* (u_n) is bounded in $H_0^1(\Omega)$.

By the Banach-Alaoglu theorem, there is $u_* \in H_0^1(\Omega)$ so that $u_n \to u_*$ weakly in $H_0^1(\Omega)$. As before, using the convexity of $u \mapsto \|\nabla u\|^2$, this implies that

$$\|\nabla u_*\|_{L^2}^2 \leq \liminf_{n \to \infty} \|\nabla u_n\|_{L^2}^2.$$

Unfortunately, the term $-\frac{1}{4}\|u\|_{L^4}$ is not convex (it is concave), so we cannot handle this term as in the previous section. However, thanks to Rellich's Theorem 4.14, the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $2 \le p < 6$. In particular,

$$\|u_*\|_{L^4} = \lim_{n\to\infty} \|u_n\|_{L^4}, \quad \text{and} \quad \|u_*\|_{L^2} = \lim_{n\to\infty} \|u_n\|_{L^2} = 1.$$

This proves that $G(u_*) \le \liminf G(u_n) = K$, and since $\|u_*\|_{L^2} = 1$, u_* is a minimiser.

Euler-Lagrange equation. Now we can repeat the arguments of the previous section, and see that there is $\lambda \in \mathbb{R}$ so that $u_* \in H^1_0(\Omega)$ is a distributional solution of

$$-\Delta u - u^3 = \lambda u \quad \in \Omega.$$

5.2 Spectral decomposition of compact symmetric operators

In this section, we prove the spectral decomposition of symmetric compact operators. We refer to [Bre99, Chapitre VI] for a full presentation, and extension in the non-symmetric case.

5.2.1 Basic notions in operator theory

Let $\mathcal{H}$ be a (separable) Hilbert space. Recall that $A : \mathcal{H} \to \mathcal{H}$ is a **bounded operator** if A is linear, and if there is $C \ge 0$ so that $\|Ax\|_{\mathcal{H}} \le C\|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$, and that A is **compact** if $A\mathcal{B}$ is relatively compact in $\mathcal{H}$, where $\mathcal{B} = \{x \in \mathcal{H}, \ \|x\| = 1\}$ is the unit ball of $\mathcal{H}$.

The **adjoint** of $A : \mathcal{H} \to \mathcal{H}$ is the application $A^* : \mathcal{H} \to \mathcal{H}$ so that

$$\forall x, y \in \mathcal{H}, \quad \langle x, Ay\rangle_{\mathcal{H}} = \langle A^*x, y\rangle_{\mathcal{H}}.$$

The operator A is **symmetric** if $A = A^*$.

We will be interested in the operator $\lambda - A := \lambda\mathbb{I}_{\mathcal{H}} - A$ for $\lambda \in \mathbb{C}$. The **resolvent set** of A is the set $\rho(A) \subset \mathbb{C}$ defined by

$$\rho(A) := \{\lambda \in \mathbb{C}, \quad \lambda - A \quad \text{if bijective from } \mathcal{H} \text{ to } \mathcal{H}\}.$$

If $\lambda \in \rho(A)$, then $(\lambda - A)^{-1}$ is a bounded operator (this result, known as the Banach-Steinhaus theorem is non trivial, see [Bre99, Corollaire II.6]).

The **spectrum** of A is the complement of $\rho(A)$, that is $\sigma(A) := \mathbb{C} \setminus \rho(A)$. The set $\rho(A)$ is open in $\mathbb{C}$, and the set $\sigma(A)$ is closed in $\mathbb{C}$.

A number $\lambda \in \mathbb{C}$ is an **eigenvalue** of A if $\ker\{\lambda - A\} \neq \{0\}$. In this case any $u \in \ker\{\lambda - A\}$ is a corresponding **eigenvector**. The **multiplicity** of λ is $\dim\ker\{\lambda - A\} \in \mathbb{N} \cup \{\infty\}$.

5.2.2 Decomposition of compact symmetric operators

Let A be a symmetric compact operator on $\mathcal{H}$. For all $x \in \mathcal{H}$, we have

$$\langle x, Ax\rangle = \langle Ax, x\rangle = \overline{\langle x, Ax\rangle},$$

so $\langle x, Ax\rangle$ is always a real number. We set

$$m := \inf\{\langle x, Ax\rangle_{\mathcal{H}}, \ \|x\|_{\mathcal{H}} = 1\}, \quad \text{and} \quad M := \sup\{\langle x, Ax\rangle_{\mathcal{H}}, \ \|x\|_{\mathcal{H}} = 1\}.$$

Theorem 5.5

If $\lambda \neq 0$ is a non-null eigenvalue of A, it is of finite multiplicity.
The numbers m and M are eigenvalues of A.

Proof. Let us prove that for all $\lambda \neq 0$, $E_\lambda := \ker\{\lambda - A\}$ is finite dimensional. For all $x \in E_\lambda$, we have $\lambda x = Ax$, so the unit ball of E_λ satisfies $\mathcal{B}_{E_\lambda} \subset \lambda^{-1} A(\mathcal{B}_E)$, hence is relatively compact. By the Riesz' lemma (see [Bre99, Theorem VI.5]), E_λ is finite dimensional.

Let us prove that m is an eigenvalue (the proof for M is similar). The problem defining m is an optimisation problem under constraint. Let (x_n) be a minimising sequence for this problem, so $\|x_n\| = 1$ and $\langle x_n, Ax_n \rangle \to m$. Since (x_n) is bounded in the reflexive Banach space $\mathcal{H}$, we can apply the Banach-Alaogly theorem. There is a subsequence, still noted n and an element $x_* \in \mathcal{H}$ so that (x_n) weakly converges to x_* in $\mathcal{H}$. In addition, since A is compact, we have $Ax_n \to Ax_*$ strongly. In particular, by Theorem 1.29, we have $m = \lim_{n\to\infty} \langle x_n, Ax_n \rangle = \langle x_*, Ax_* \rangle$, so $x_* \neq 0$.

The bilinear form $a(x,y) := \langle x, (A-m)y \rangle$ is symmetric and positive. By Cauchy-Schwarz, we have $|a(x,y)|^2 \le a(x,x)a(y,y)$, which gives

$$|\langle x, (A-m)y \rangle|^2 \le \langle x, (A-m)x \rangle \langle y, (A-m)y \rangle.$$

Taking the supremum over all y with $\|y\| = 1$ proves that

$$\|(A-m)x\|^2 \le (M-m)\langle x, (A-m)x \rangle.$$

In particular, we have $\|(A-m)x_n\| \to 0$. Since Ax_n converges strongly to Ax_*, we deduce that mx_n also converges strongly to mx_*. At the limit, we have $Ax^* = mx^*$, which proves that x^* is a non-null eigenvector for the eigenvalue m. □

This allows to prove the following important theorem.

Theorem 5.6: Spectral decomposition of symmetric compact operators

Let $\mathcal{H}$ be a separable Hilbert space, and let A be a symmetric compact operator on $\mathcal{H}$. Then there is a (countable) basis $(e_1, e_2, \cdots)$ of $\mathcal{H}$ where all elements are eigenvectors of A. We can order the basis so that

$$Ae_n = \lambda_n e_n, \quad |\lambda_1| \ge |\lambda_2| \ge \cdots \ge 0.$$

Finally, 0 is the only accumulation point of $(\lambda_n)_{n\in\mathbb{N}}$.

In the sequel, we often use the Dirac notations. An element $x \in \mathcal{H}$ is denoted $|x\rangle$ (ket) if x is seen as an element of $\mathcal{H}$, and $\langle x|$ (bra), if it is seen as an element of the dual $\mathcal{H}* = \mathcal{H}$. This allows to write

$$A = \sum_{n\in\mathbb{N}} \lambda_n |e_n\rangle\langle e_n|, \quad \text{in the sense} \quad Ax = \sum_{n\in\mathbb{N}} \left(\lambda_n \langle e_n, x\rangle_{\mathcal{H}}\right) e_n.$$

We may have $\lambda_n = \lambda_{n+1}$: we repeat the eigenvalues as many times as their multiplicities.

Proof. We do not prove fully this Theorem, and refer to [Bre99, Theorem VI.11] for a complete proof. Let $\lambda_0 := 0$, let (λ_n) be the set of non-null eigenvalues of A (counting multiplicities), and let $E_n := \ker\{\lambda_n - A\}$. First, we notice that if $x \in E_n$ and $y \in E_m$ are normalised vectors, then, since A is symmetric,

$$(\lambda_n - \lambda_m)\langle x, y \rangle = \langle \lambda_n x, y \rangle - \langle x, \lambda_m y \rangle = \langle Ax, y \rangle - \langle x, Ay \rangle = 0,$$

so $\lambda_n \neq \lambda_m$ implies $\langle x, y \rangle = 0$. In other words, the spaces E_n are orthogonal.

Let $F := E_0 \oplus E_1 \oplus E_2 \oplus \cdots$. We have $AF \subset F$, so $AF^\perp \subset F^\perp$ (why?). The operator $A_{F^\perp} : F^\perp \to F^\perp$, that is the operator A *restricted to* $F^\perp$, is a compact symmetric operator. Since $A_{F^\perp}$ cannot have eigenvalues (since otherwise, they would have been considered in F), we deduce by Theorem 5.5 that $F^\perp = 0$. In other words, F is dense in $\mathcal{H}$. It remains to choose a basis in all (finite dimensional) spaces E_n, and the result follows. □

5.2.3 Application: the spectrum of Dirichlet Laplacien in bounded domains

In this section, we study the operator $-\Delta$ as an operator on $\mathcal{H} := L^2(\Omega)$. Unfortunately, this operator is not compact, so we cannot directly apply the result of the previous section. One idea is to consider the operator $(-\Delta)^{-1}$. However, to define such operator, one needs to precise the *boundary conditions*.

Here, we study the **Dirichlet Laplacian** $-\Delta_0$ that we define now. Recall that, in Section 3.3.2, we proved that, if Ω is a domain with boundary of class at least C^2, then

$$\forall f \in L^2(\Omega), \quad \text{there is a unique } u \in H^2(\Omega) \cap H^1_0(\Omega) \text{ so that } \quad -\Delta u = f.$$

In addition, we have $\|u\|_{H^2} \le \|f\|_{L^2}$. Let $\widetilde{A_0} : f \mapsto u$ be the corresponding bounded linear map, from $L^2(\Omega)$ to $H^2(\Omega) \cap H^1_0(\Omega)$. It is invertible, and the **Dirichlet Laplacian** is, by definition, the inverse

$$-\Delta_0 := \widetilde{A_0}^{-1}, \quad \text{from } H^2(\Omega) \cap H^1_0(\Omega) \text{ to } L^2(\Omega).$$

The operator $\widetilde{A_0}$ has different starting and ending spaces. We rather study the operator $A_0 = I\widetilde{A_0}$, where I is the trivial injection map $H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$. In this case, we say that the operator $(-\Delta_0)$ from $L^2(\Omega)$ to $L^2(\Omega)$ has dense domain $H^2(\Omega) \cap H^1_0(\Omega)$.

If Ω has a boundary $\partial\Omega$ of class C^1, then, by Rellich's embedding, the embedding I is compact. In particular, $A_0 : L^2(\Omega) \to L^2(\Omega)$ is compact. Applying Theorem 5.6 gives the following.

Theorem 5.7: Spectrum of the Dirichlet Laplacian

Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. Then, there is a basis $(e_1, e_2, \cdots)$ of $L^2(\Omega)$ so that

$$\forall n \in \mathbb{N}^*, \quad e_n \in H^2(\Omega) \cap H^1_0(\Omega), \quad \text{and} \quad -\Delta_0 e_n = \lambda_n e_n.$$

In addition, we have $0 < \lambda_1 \le \lambda_2 \le \cdots$, and the sequence (λ_n) goes to infinity.

Proof. For the first part, we apply the spectral decomposition to A_0, and deduce that there is a basis $(e_1, e_2, \cdots)$ of $L^2(\Omega)$ so that

$$A_0 = \sum_{n=1}^{\infty} \mu_n |e_n\rangle\langle e_n|.$$

Since A_0 is injective, 0 is not an eigenvalue of A_0, so A_0 is indeed invertible. In addition, we have $A_0 e_n = \lambda_n e_n$, so $e_n \in \operatorname{Im} A_0 = \operatorname{Im} I\widetilde{A_0} \subset \operatorname{Im} I$, that is $e_n \in H^2(\Omega) \cap H^1_0(\Omega)$.

We then set $\lambda_n := 1/\mu_n$, and deduce that

$$(-\Delta_0) = \sum_{n=1}^{\infty} \lambda_n |e_n\rangle\langle e_n|.$$

It remains to prove that $\lambda_n \ge 0$. Since $e_n \in H^2(\Omega) \cap H^1_0(\Omega)$, we can integrate by part, and get

$$\lambda_n = \langle e_n, (-\Delta_0) e_n \rangle_{L^2} = \int_\Omega |\nabla e_n|^2 \ge 0.$$

This concludes the proof. □

One can perform the same analysis for the **Neumann Laplacian** $(-\Delta_N)$. Note however that the constant function 1_Ω is in $H^2(\Omega) \cap H^1(\Omega)$, and $(-\Delta_N)1_\Omega = 0$, so 0 is now an eigenvalue of $(-\Delta_N)$, and $(-\Delta_N)$ is not invertible. One needs to study the operator $A_N^{-1} := (1 - \Delta_N)$ to apply the theory.

CHAPTER 6

FOURIER TRANSFORM

In this section, we review basic facts on the Fourier transform.

6.1 Fourier transform in L^1.

For $f \in L^1(\mathbb{R}^d)$, we define its *Fourier transform*, denoted $\widehat{f}$ or $\mathcal{F}(f)$, by

$$\boxed{\forall \omega \in \mathbb{R}^d, \quad \widehat{f}(\omega) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\omega \cdot x} \mathrm{d}x.} \tag{6.1}$$

The normalisation convention depends on the textbooks. Here, we adopt the usual convention in quantum physics. It is chosen so that $\mathcal{F}$ is a unitary on $L^2(\mathbb{R}^d)$, as we will prove below.

Since $f \in L^1(\mathbb{R}^d)$, we have

$$|\widehat{f}(\omega)| = \left| \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}x \right| \le \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |f| = \frac{1}{(2\pi)^{d/2}} \|f\|_{L^1},$$

so $\widehat{f}$ is bounded, that is $\widehat{f} \in L^\infty(\mathbb{R}^d)$, with $\|\widehat{f}\|_{L^\infty} \le (2\pi)^{-d/2} \|f\|_{L^1}$. The map $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty$ is linear, and we have the usual formulae.

- **Conjugation:** $\widehat{\overline{f}}(\omega) = \overline{\widehat{f}}(-\omega)$.
- **Translations:** If $g(x) := f(x - x_0)$ for some $x_0 \in \mathbb{R}^d$, then $\widehat{g}(\omega) = \mathrm{e}^{-\mathrm{i}\omega x_0} \widehat{f}(\omega)$.
- **Dilation:** If $g(x) := f(\lambda x)$ for some $\lambda > 0$, then $\widehat{g}(\omega) = \frac{1}{\lambda} \widehat{f}\left(\frac{\omega}{\lambda}\right)$.
- **Convolution:** If $f, g \in L^1(\mathbb{R}^d)$, then $f * g \in L^1(\mathbb{R}^d)$ by Young's inequality, and we have, by Fubini's theorem
$$\widehat{(f * g)} = (2\pi)^{d/2} \widehat{f}\widehat{g}.$$
  The factor $(2\pi)^{d/2}$ is a bit annoying, but can be recovered easily. We say that a convolution becomes a multiplication in Fourier space.

Also, the Fourier transform can trade *derivative* and *multiplication by x*. We state it as a Theorem (it can be proved using a simple integration by part). We write $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Theorem 6.1: Fourier transform, regularity and decay at infinity

Let $f \in L^1(\mathbb{R}^d)$ be such that $|x|^n |f|(x) \in L^1(\mathbb{R}^d)$ for some $n \in \mathbb{N}$. Then $\widehat{f}$ is of class C^n, and, for all α with $|\alpha| \le n$,

$$\boxed{D^\alpha(\widehat{f})(\omega) = (-\mathrm{i})^{|\alpha|}\widehat{(x^\alpha f)}(\omega).}$$

Conversely, if f is of class C^n with $|D^\alpha f| \in L^1(\mathbb{R}^d)$ for some $|\alpha| \le n$, then

$$\boxed{\widehat{(D^\alpha f)}(\omega) = (\mathrm{i}\omega)^\alpha \widehat{f}(\omega).}$$

In particular, if a function f is compactly supported, its Fourier transform is C^∞, and if f is C^∞, its Fourier transform decays faster than any polynomial at infinity (in particular $\widehat{f}$ is integrable). Note that

$$\widehat{(-\Delta f)}(\omega) = |\omega|^2 \widehat{f}(\omega).$$

To sum up, the Fourier transforms trades the following:

translation	$\longleftrightarrow$	multiplication by a phase
convolution	$\longleftrightarrow$	multiplication
regularity	$\longleftrightarrow$	decay at infinity.

As an example, we can compute the Fourier transform of a Gaussian function. We set

$$g_a(x) := \left(\frac{a}{\pi}\right)^{d/4} \mathrm{e}^{-\frac{ax^2}{2}}.$$

The normalisation is chosen so that $\int_{\mathbb{R}^d} g_a^2 = 1$ (integral of the Gaussian[1]).

Theorem 6.2: Fourier transform of the Gaussian

For all $a > 0$, we have $\widehat{g_a}(\omega) = g_{\frac{1}{a}}(\omega)$.

In other words, the Fourier transform of the Gaussian is again a Gaussian.

Proof. Since g_a is smooth and decays fast at infinity, so is its Fourier transform. The function g_a satisfies the equation

$$\partial_{x_i} g_a(x) = -a x_i g_a(x).$$

Taking the Fourier transform of this equation shows that

$$\omega_i \widehat{g_a}(\omega) = -a \partial_{\omega_i} \widehat{g_a}(\omega), \quad \text{which is also} \quad \partial_{\omega_i} \widehat{g_a}(\omega) = -\frac{1}{a}\omega_i \widehat{g_a}(\omega).$$

So $\widehat{g_a}$ solves an equation similar to g_a, but with $\frac{1}{a}$ instead of a. We deduce that $\widehat{g_a}$ is of the form $\widehat{g_a} = \lambda g_{\frac{1}{a}}$ for some $\lambda \in \mathbb{R}$. At $\omega = 0$, we have, using the integral of the Gaussian,

$$\widehat{g_a}(0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g_a = \left(\frac{a}{\pi}\right)^{d/4}, \quad \text{and} \quad g_{\frac{1}{a}}(0) = \left(\frac{a}{\pi}\right)^{d/4},$$

so $\lambda = 1$, which concludes the proof. □

[1] Recall that

$$\int_{\mathbb{R}^d} \mathrm{e}^{-ax^2} \mathrm{d}x = \left(\frac{\pi}{a}\right)^{d/2}.$$

Theorem 6.3: Riemann-Lebesgue Theorem

If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f}$ is continuous, and goes to 0 at infinity. In addition, we have

$$\|\widehat{f}\|_{L^\infty} \le \frac{1}{(2\pi)^{d/2}} \|f\|_{L^1}.$$

Proof. Let $\omega_n \to \omega$. We introduce

$$g_n(x) := f(x)\mathrm{e}^{-\mathrm{i}\omega_n \cdot x} \quad \text{and} \quad g(x) := f(x)\mathrm{e}^{-\mathrm{i}\omega\cdot x}.$$

The functions g_n converge pointwise to g. In addition, we have the domination $|g_n| \le |f|$ which is integrable. So, by the dominated convergence theorem, we have $\int g_n \to \int g$, which is also $\widehat{f}(\omega_n) \to \widehat{f}(\omega)$. So $\widehat{f}$ is continuous.
We already proved the last inequality. It remains to prove that $\widehat{f}$ goes to 0 at infinity. Assume first that f is $C_0^\infty(\mathbb{R}^d)$. Using Theorem 6.1, we have, for $\omega \neq 0$,

$$\left|\widehat{f}(\omega)\right| = \frac{1}{|\omega|^2}\left|\widehat{(-\Delta f)}(\omega)\right| \le \frac{1}{|\omega|^2}\left\|\widehat{(-\Delta f)}\right\|_{L^\infty} \le \frac{1}{|\omega|^2}\frac{1}{(2\pi)^{d/2}}\|(-\Delta f)\|_{L^1}.$$

So $\widehat{f}(\omega)$ goes to 0 as $\omega \to \infty$, uniformly in $|\omega|$. Consider now $f \in L^1(\mathbb{R}^d)$, and let $\varepsilon > 0$. By density of $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, there is $g \in C_0^\infty(\mathbb{R}^d)$ so that $\|f-g\|_{L^1} < \varepsilon$. For this g, there is $R > 0$ so that, for all $|\omega| > R$, we have $|\widehat{g}|(\omega) < \varepsilon$. This gives

$$\forall |\omega| > R, \quad \left|\widehat{f}(\omega)\right| \le \left|(\widehat{f}-\widehat{g})(\omega)\right| + |\widehat{g}(\omega)| \le \|\widehat{f}-\widehat{g}\|_{L^\infty} + \varepsilon \le \left(\frac{1}{(2\pi)^{d/2}} + 1\right)\varepsilon,$$

and the result follows.

□

Theorem 6.4: Inverse Fourier transform

Let $f \in L^1(\mathbb{R}^d)$ be such that $\widehat{f} \in L^1(\mathbb{R}^d)$ as well. Then f and $\widehat{f}$ are continuous, and

$$\forall x \in \mathbb{R}^d, \quad f(x) = \mathcal{F}^*(\widehat{f}) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}} \widehat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega.$$

Proof. The usual proof uses convolution with a Gaussian function (see [LL01, Theorem 5.5]). We take here another route to emphasise the link with Fourier series. We give a proof in dimension $d = 1$, and for $f \in C_0^\infty(\mathbb{R})$. Let $L > 0$ be so that the support of f is included in $[-L, L]$. Consider $\widetilde{f}$ the $2L$-periodic function which equals f on $(-L, L)$, that is

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}} f(x - 2kL).$$

The function $\widetilde{f}$ is smooth and $2L$-periodic, so we can consider its Fourier **series**. Its n-th Fourier coefficient is

$$c_n(\widetilde{f}) = \frac{1}{\sqrt{2L}}\int_{-L}^{L}\widetilde{f}(x)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{2L}x}\mathrm{d}x = \frac{1}{\sqrt{2L}}\int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}\frac{\pi}{L}nx}\mathrm{d}x = \frac{\sqrt{\pi}}{\sqrt{L}}\widehat{f}\left(n\frac{\pi}{L}\right).$$

Since $\tilde{f}$ is continuous, Dirichlet's theorem shows that $\widetilde{f}$ is point-wise equal to the Fourier series, that is

$$\sum_{k\in\mathbb{Z}} f(x-2kL) = \widetilde{f}(x) = \frac{1}{\sqrt{2L}}\sum_{n\in\mathbb{Z}} c_n(\widetilde{f})\mathrm{e}^{\mathrm{i}\frac{\pi}{L}nx} = \frac{1}{\sqrt{2\pi}}\left[\frac{\pi}{L}\sum_{n\in\mathbb{Z}}\widehat{f}\left(n\frac{\pi}{L}\right)\mathrm{e}^{\mathrm{i}\frac{\pi}{L}nx}\right]. \tag{6.2}$$

Recall that $\widehat{f}$ is continuous by Riemann-Lebesgue, and that we assume it to be integrable. We let $L \to \infty$, and recognise a Riemann sum, with steps $\frac{\pi}{L}$. At the limit, we obtain as wanted

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}\omega.$$

□

Theorem 6.5: Parseval identity

If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $\widehat{f} \in L^2(\mathbb{R}^d)$, and we have

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

Proof. Again, the usual proof is via Gaussian convolution (see [LL01, Theorem 5.3]). Let us continue the previous proof, and write the Parseval equality (for Fourier series) for the periodic function $\widetilde{f}$ considered previously. Using that the support of $f(x - 2kL)$ are all disjoints, we obtain

$$\int_{\mathbb{R}} |f|^2(x) \mathrm{d}x = \int_{-L}^{L} \left| \widetilde{f}(x) \right|^2 \mathrm{d}x = \sum_{n \in \mathbb{Z}} |c_n(\widetilde{f})|^2 = \frac{\pi}{L} \sum_{n \in \mathbb{Z}} \left| \widehat{f}\left(n \frac{\pi}{L} \right) \right|^2.$$

Again, we recognise a Riemann sum, with step $\frac{\pi}{L}$. We let $L \to \infty$, and we obtain the continuous Parseval formula

$$\int_{\mathbb{R}} |f|^2(x) \mathrm{d}x = \int_{\mathbb{R}} |\widehat{f}|^2(\omega) \mathrm{d}\omega. \tag{6.3}$$

□

6.2 Fourier transform in L^2

If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then the Parseval formula shows that $\widetilde{f} \in L^2(\mathbb{R}^d)$, and that

$$\mathcal{F} : L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad \text{satisfies} \quad \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}.$$

So $\mathcal{F}$ is bounded as a map from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Since $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we can extend $\mathcal{F}$ *by density* on the whole space $L^2(\mathbb{R}^d)$.

Theorem 6.6: Fourier transform in $L^2(\mathbb{R}^d)$

The map $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary: for all $f \in L^2(\mathbb{R}^d)$, we have $\|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}$. For all $f, g \in L^2(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \overline{f}(x) g(x) \mathrm{d}x = \int_{\mathbb{R}^d} \overline{\widehat{f}}(\omega) \widehat{g}(\omega) \mathrm{d}\omega, \quad \text{that is} \quad \langle f, g \rangle_{L^2} = \langle \mathcal{F}(f), \mathcal{F}(g) \rangle_{L^2}.$$

Proof. We already proved the first part. Let us prove the second. First, by Hölder's inequality (or Cauchy-Schwarz), the functions $\overline{f} g$ and $\overline{\widehat{f}} \widehat{g}$ are indeed in $L^1(\mathbb{R}^d)$. In addition, we have the polarisation formula

$$\mathrm{Re}(\overline{f} g) = \frac{1}{2}(|f + g|^2 - |f|^2 - |g|^2).$$

Integrating and using that $\mathcal{F}$ is unitary shows that the real part are equal. Multiplying f by i shows that the imaginary part are equal as well. □

The integral formula (6.1) is no longer valid for functions $f \in L^2(\mathbb{R}^d) \setminus L^1(\mathbb{R}^d)$, but it is common practice to still write it, to emphasise what normalization we are working with.

6.3 Fourier transform for distributions

6.3.1 Definition

We introduce the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d), \quad \forall n \in \mathbb{N}, \ |x|^n f \in L^1(\mathbb{R}^d) \right\}.$$

A function f is in $\mathcal{S}(\mathbb{R}^d)$ if it is smooth, and decays faster than any polynomial. Note that $\mathcal{D}(\mathbb{R}^d) \subsetneq \mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{F}$ trades *regularity* and *decay at infinity*, it maps $\mathcal{S}(\mathbb{R}^d)$ to itself.

One can repeat all the arguments for distribution, and introduce the «dual» space $\mathcal{S}'(\mathbb{R}^d)$ (loosely speaking, we replace $\mathcal{D}(\mathbb{R}^d)$ by $\mathcal{S}(\mathbb{R}^d)$ in the chapter on distributions).

If $f, g \in \mathcal{S}(\mathbb{R}^d)$ are smooth functions, we have, by Parseval identity

$$\langle \mathcal{F}(T), \phi \rangle_{L^2, L^2} := \langle T, \mathcal{F}^*(\phi) \rangle_{L^2, L^2}.$$

This suggests to define the Fourier transform for distributions in $\mathcal{S}'(\mathbb{R}^d)$ by

$$\forall T \in \mathcal{S}'(\mathbb{R}^d), \quad \langle \mathcal{F}(T), \phi \rangle_{\mathcal{S}', \mathcal{S}} := \langle T, \mathcal{F}^*(\phi) \rangle_{\mathcal{S}', \mathcal{S}}.$$

We do not comment more on this point. Since $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$: we cannot take the Fourier transform of all distributions in $\mathcal{D}'(\mathbb{R}^d)$. However, since we also have $L^1_{\rm loc}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, so we can still take the Fourier transform *the distributional sense* of many functions.

6.3.2 First examples

Let us give some examples.

Theorem 6.7: Fourier transform of the Dirac mass

For all $x \in \mathbb{R}^d$, the Dirac mass δ_x belongs to $\mathcal{S}'(\mathbb{R}^d)$. In addition, we have $\widehat{\delta_x}(\omega) = (2\pi)^{-d/2} \mathrm{e}^{\mathrm{i}\omega x}$, in the sense

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \phi(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\omega) \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}\omega.$$

In particular, $\widehat{\delta_0} = (2\pi)^{-d/2}$ is the constant function.

Proof. This is just the inverse Fourier transform formula in $\mathcal{S}(\mathbb{R}^d)$. □

This allows to compute the Fourier transform of the Laplace Green's functions. Recall that the Green's functions have been introduced in Section 2.2.2, and that they solve $-\Delta G_0 = \delta_0$ in the distributional sense.

Theorem 6.8: Fourier transform of the Green's functions

Let G_0 be the Green's function on $\mathbb{R}^d$ defined in Section 2.2.2. Then $G_0 \in \mathcal{S}'(\mathbb{R}^d)$, and

$$\widehat{G_0}(\omega) = \frac{1}{(2\pi)^{d/2}} \frac{1}{|\omega|^2}.$$

Proof. The equation $-\Delta G_0 = \delta_0$ reads, in Fourier, $|\omega|^2 \widehat{G_0}(\omega) = (2\pi)^{-d/2}$, and the result follows. □

We can also compute the Fourier transform of the *Dirac Comb*. Introduce, for $T > 0$,

$$\mathrm{Comb}_T(x) := \sum_{k \in \mathbb{Z}^d} \delta(x - kT),$$

which is a periodic version of the Dirac mass.

Theorem 6.9: Poisson summation formula

The Dirac comb Comb_T belongs to $\mathcal{S}'(\mathbb{R}^d)$. In addition, we have

$$\widehat{\mathrm{Comb}_T} = \left(\frac{\sqrt{2\pi}}{T}\right)^d \mathrm{Comb}_{\frac{2\pi}{T}},$$

in the sense

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \sum_{n \in \mathbb{Z}^d} \widehat{\phi}(nT) = \left(\frac{\sqrt{2\pi}}{T}\right)^d \sum_{k \in \mathbb{Z}^d} \phi\left(k\frac{2\pi}{T}\right).$$

Proof. We prove the result in dimension 1 (the proof is similar in higher dimensions). Applying the Fourier series identity (6.2) at $x = 0$ and $L = \pi/T$ gives

$$\langle \mathrm{Comb}_{2\pi/\mathrm{T}}, \phi \rangle = \sum_{k \in \mathbb{Z}} \phi\left(k\frac{2\pi}{T}\right) = \widetilde{\phi}(0) = \frac{T}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \widehat{\phi}(nT) = \frac{T}{\sqrt{2\pi}} \langle \mathrm{Comb}_T, \widehat{\phi} \rangle = \frac{T}{\sqrt{2\pi}} \langle \widehat{\mathrm{Comb}_T}, \phi \rangle.$$

In the last equality, we used the definition of the Fourier transform for distribution, and that Comb_T is symmetric. Identifying gives the result. □

6.3.3 Characterisation of the Sobolev space $H^s(\mathbb{R}^d)$ using Fourier transform

Recall that

$$H^k(\mathbb{R}^d) := W^{k,2}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d), \quad \forall |\alpha| \le k, \quad D^\alpha f \in L^2(\mathbb{R}^d) \right\}.$$

Using that $\widehat{D^\alpha f}(\omega) = (\mathrm{i}\omega)^\alpha \widehat{f}(\omega)$ and that $D^\alpha f \in L^2(\mathbb{R}^d)$ iff $\widehat{D^\alpha f} \in L^2(\mathbb{R}^d)$, we see that

$$H^k(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d), \quad \forall |\alpha| \le k, \quad \omega^\alpha \widehat{f} \in L^2(\mathbb{R}^d) \right\}.$$

In addition, the H^k norm can be written in Fourier with

$$\| f \|_{H^k}^2 = \sum_{|\alpha| \le k} \| D^\alpha f \|_{L^2}^2 = \sum_{|\alpha| \le k} \| \omega^\alpha \widehat{f} \|_{L^2}^2 = \int_{\mathbb{R}^d} \left(\sum_{|\alpha| \le k} |\omega^\alpha|^2 \right) |\widehat{f}|^2(\omega) \mathrm{d}\omega.$$

The reader can check that there are constants $0 < c < C < \infty$ so that, for all $\omega \in \mathbb{R}^d$,

$$c\left(1 + |\omega|^{2k}\right) \le \sum_{|\alpha| \le k} |\omega^\alpha|^2 \le C\left(1 + |\omega|^{2k}\right).$$

This shows that the norm

$$\| f \|_{\widetilde{H^k}}^2 := \int_{\mathbb{R}^d} (1 + |\omega|^{2k}) |\widehat{f}|^2(\omega) \mathrm{d}\omega.$$

is equivalent to «usual» H^k norm. This new norm is somehow simpler to use, as it only involves the L^2 norm of f and of $(-\Delta)^k f$. This gives an alternative proof of Theorem 3.6. Let us state a more general theorem (we emphasise that this theorem is only valid in the full space $\mathbb{R}^d$).

Theorem 6.10

If $f \in L^2(\mathbb{R}^d)$ is such that $(-\Delta f) \in H^k(\mathbb{R}^d)$ for some $k \ge 0$, then $f \in H^{k+2}(\mathbb{R}^d)$.

TODO: introduce $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, and prove the trace theorem in half space.

6.3.4 Application: the heat kernel

We define the **heat kernel** on $\mathbb{R}^d$

$$\boxed{\mathcal{G}_t(x) := \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{|x|^2}{4t}}.}$$

A computation reveals that, for all $t > 0$, we have

$$\Delta \mathcal{G}_t = \partial_t \mathcal{G}_t \quad \left(= \mathcal{G}_t(x) \times \left[-\frac{d}{t} + \frac{|x|^2}{4t^2} \right] \right).$$

In addition, we have

$$\lim_{t \to 0^+} \mathcal{G}_t = \delta_0, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

One way to see this goes as follows. We take the Fourier transform of $\mathcal{G}_t$. Since it is Gaussian, we have

$$\widehat{\mathcal{G}_t}(\omega) = \frac{1}{(4\pi t)^{d/2}} (2t)^{d/2} \mathrm{e}^{-|\omega|^2 t} \xrightarrow[t \to 0]{} \frac{1}{(2\pi)^{d/2}} = \widehat{\delta_0}.$$

While this is not a proof (pointwise convergence is not $\mathcal{S}'(\mathbb{R}^d)$ convergence), it strongly indicates that the result is correct, and indeed it is.

For $f \in L^2(\mathbb{R}^d)$, we set

$$F(t,x) := \left(\mathrm{e}^{t\Delta} f \right)(x) := \mathcal{G}_t * f(x) = \int_{\mathbb{R}^d} \mathcal{G}_t(x-y) f(y) \mathrm{d}y.$$

As for the Green's function, we can check that F is solution to the heat equation

$$\begin{cases} \Delta_x F(t,x) = \partial_t F(t,x) \\ F(t=0,x) = f(x). \end{cases}$$

For all $t > 0$, the function $\mathcal{G}_t$ is smooth (in $\mathcal{S}(\mathbb{R}^d)$). We deduce that $F(t, \cdot)$ is $C^\infty(\mathbb{R}^d)$ for all $t > 0$ (convolution by a smooth function). The heat kernel smooths functions at all (strictly) positive time.

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